Electromagnetics

<Chap. 7> Time-varying fields and Maxwell's Equations

Section 7.1 ~ 7.7

(1st class of **week 2**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 7 | Contents for 1st class of week 2

Sec 1. Introduction

Sec 2. Faraday's law of electromagnetic induction

Sec 3. Maxwell's equations

Sec 4. (Electric and Magnetic) Potential Functions

Sec 5. Electromagnetic boundary condition

	Electrostatics	Magnetostatics
Source	Static electric charges	Steady-state currents
Governing Equations	$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \end{cases}$	$\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} \end{cases}$
Constitutive relation *(for a simple medium)*	$\boldsymbol{D} = \varepsilon \boldsymbol{E}$	$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$
Characteristics	• Only functions of space: $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{H}(x, y, z)$ • Not a function of time • Independently defined! • Special forms of Maxwell's equations	

Time-varying electromagnetics
time-varying currents
$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \end{cases}$ $\begin{cases} \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \end{cases}$
Same
• Functions of both time and space • ***E*** and ***B*** are mutually dependent • a particular solution to Maxwell's eqns: electromagnetic waves propagating with the speed of light

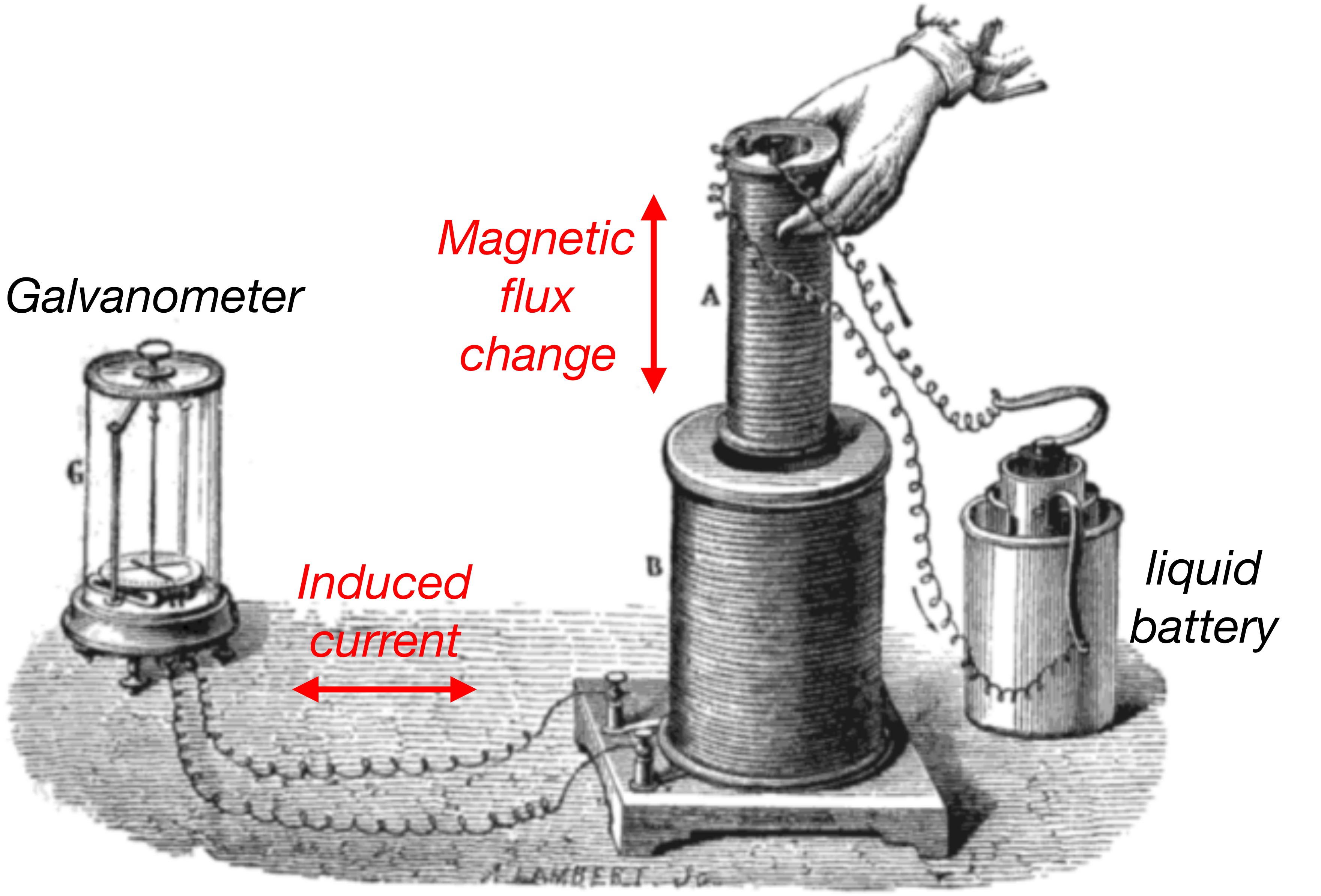
Maxwell's Equations

Chap. 7 | Faraday's law of Electromagnetic Induction

Michael Faraday
(1791~1867)

Faraday's experiment

: a current was induced in a conducting loop when the magnetic flux linking the loop changes



Faraday's experiment
(1831)

Fundamental postulate

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \rightarrow \qquad \boldsymbol{E} \neq -\nabla V$$

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s}$$

For a stationary circuit in a time-varying *B*

$$\left[\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} \triangleq v\right] = \left[-\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{s} = -\frac{d\Phi}{dt}\right]$$

$$\boxed{\therefore v = -\frac{d\Phi}{dt}}$$

Faraday's law of electromagnetic induction

: ***Electromotive force (emf)*** induced in a closed circuit C = ***Negative time rate of change of magnetic flux*** across surface S

Lenz's law

: *Induced emf* results in a current flowing in a direction ***opposing the change of the linking magnetic flux***

Electromotive force

: ***an electrical energy*** provided by an external source such as a battery or a generator. A device converting other forms of energy into electrical energy provides *emf* as its output.

e.g.) battery, solar cell, and so on

Img src: Wikipedia

Moving circuit in a time-varying *B* (1/3)

Lorentz's Equation

- When a charge *q* moves with a velocity ***u*** in a region where both ***E*** and ***B*** exist, the electromagnetic force ***F*** on *q* is

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)$$

To an observer "moving with *q*"

- There is no motion and ***F*** on *q* can be interpreted as caused by ***E'*** such that

$$\boldsymbol{E}' = \frac{\boldsymbol{F}}{q} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} \quad \text{or} \quad \boldsymbol{E} = \boldsymbol{E}' - \boldsymbol{u} \times \boldsymbol{B}$$

- Now, since

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} \quad (\mathrm{V}) \quad \left(\because \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\right)$$

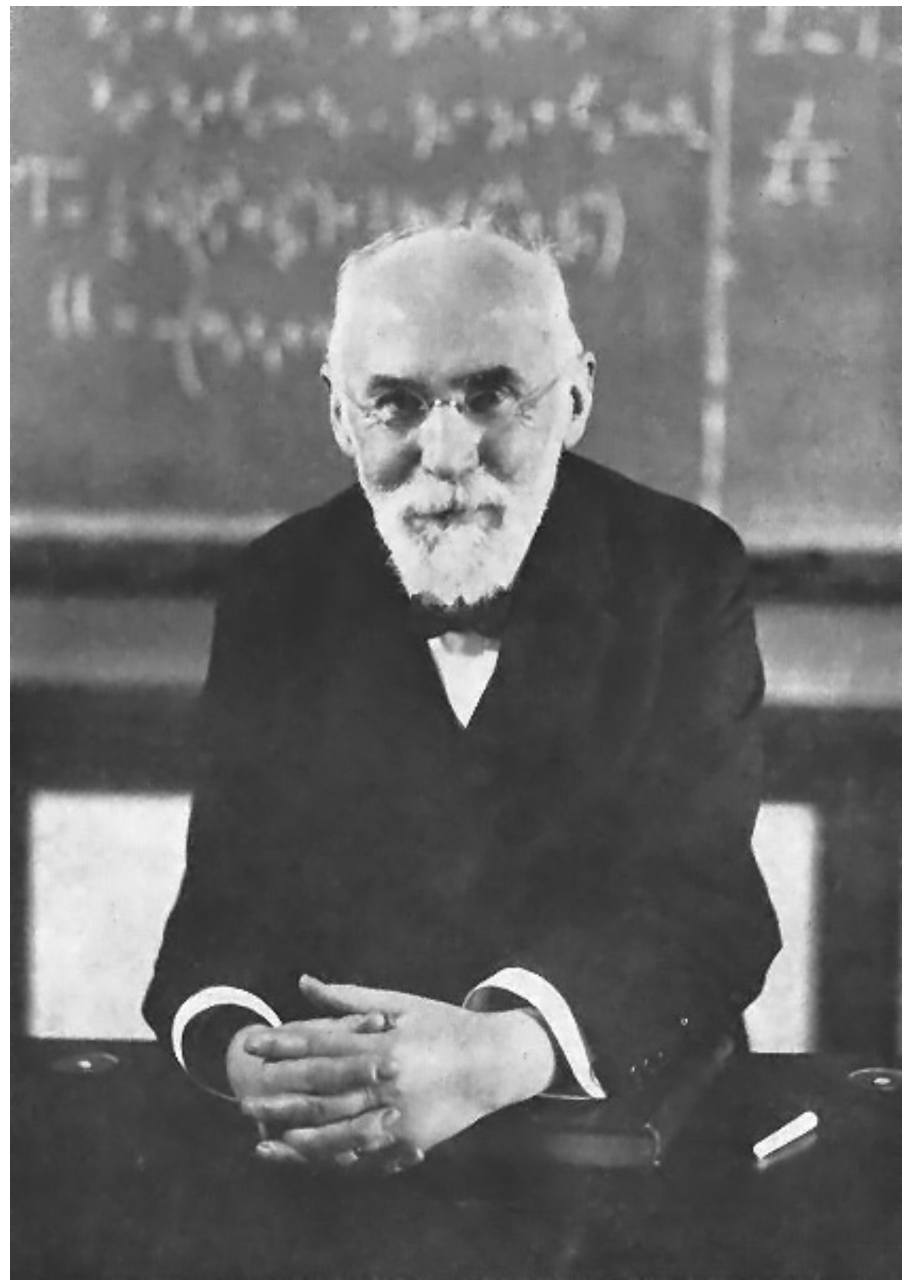
$$\boxed{\therefore \oint_C \boldsymbol{E}' \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} + \oint_C \left(\boldsymbol{u} \times \boldsymbol{B}\right) \cdot d\boldsymbol{l}}$$

General form of Faraday's law

$\oint_C \boldsymbol{E}' \cdot d\boldsymbol{l} \quad (\mathrm{V})$: ***induced emf*** in the "*moving frame of reference*"

$-\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s} \quad (\mathrm{V})$: ***transformer emf*** caused by time-varying ***B***

$\oint_C \left(\boldsymbol{u} \times \boldsymbol{B}\right) \cdot d\boldsymbol{l} \quad (\mathrm{V})$: ***motional emf*** due to the motion of circuit in ***B***



Hendrik Lorentz
(1853~1928)

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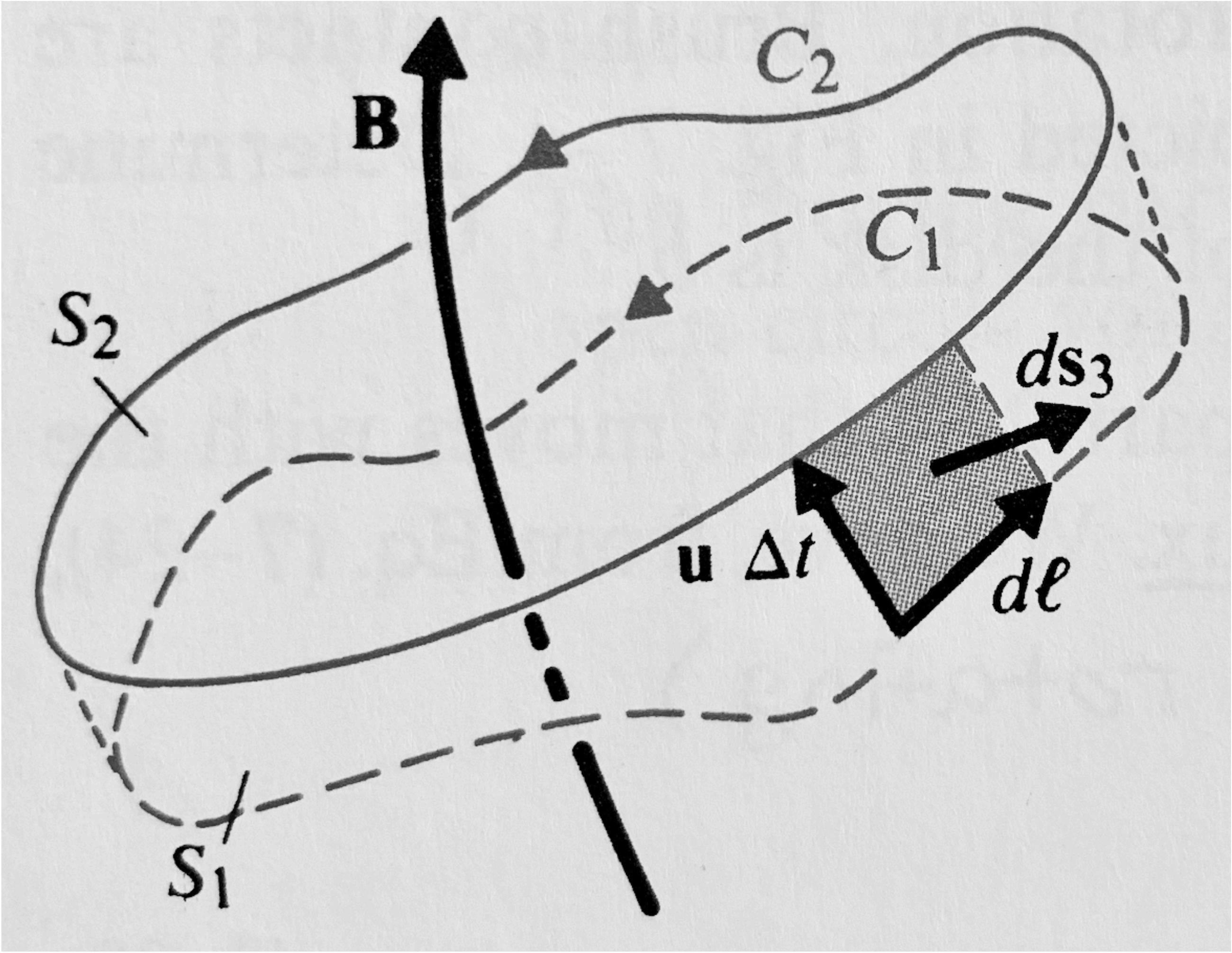
Chap. 7 | Moving circuit in a time-varying *B* (2/3)



Derivation of Faraday's law for a moving circuit

- Time rate of change of magnetic flux Φ through the contour C is

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_S \boldsymbol{B}\cdot d\boldsymbol{s}$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\boldsymbol{B}(t+\Delta t)\cdot d\boldsymbol{s}_2 - \int_{S_1}\boldsymbol{B}(t)\cdot d\boldsymbol{s}_1\right] \quad \cdots(1)$$

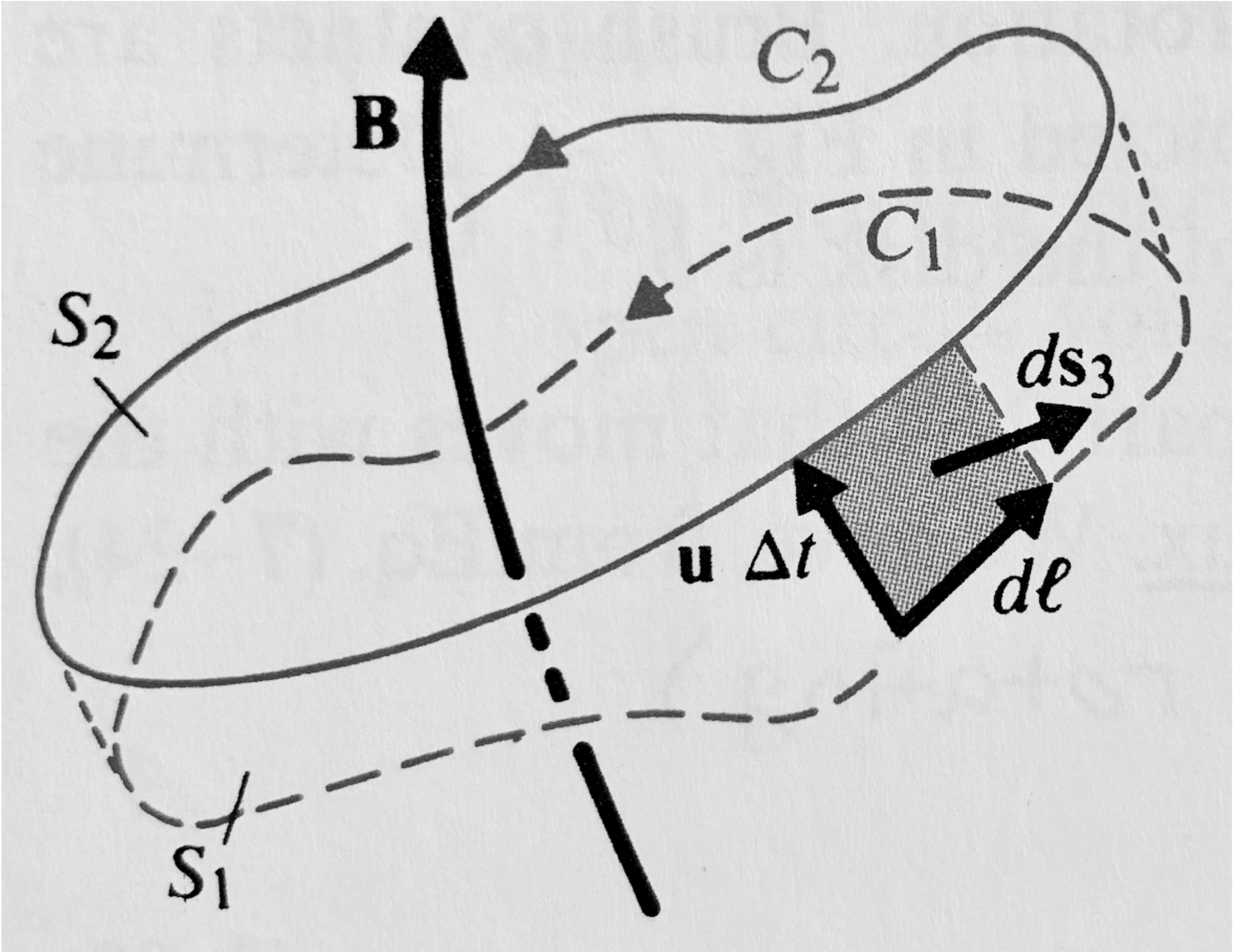
- Here, $\boldsymbol{B}(t+\Delta t)$ can be expanded as Taylor's series:

$$\boldsymbol{B}(t+\Delta t) = B(t) + \frac{\partial \boldsymbol{B}(t)}{\partial t}\Delta t + H.O.T \quad \cdots(2)$$

- If we plug (2) into (1), we get

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot ds + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\boldsymbol{B}\cdot d\boldsymbol{s}_2 - \int_{S_1}\boldsymbol{B}(t)\cdot d\boldsymbol{s}_1 + H.O.T\right] \quad \cdots(3)$$

Moving circuit in a time-varying *B* (3/3)



- Now, Let's take volume integral for divergence of ***B***

$$\int_V \nabla\cdot\boldsymbol{B}\,dv = \int_{S_2}\boldsymbol{B}\cdot d\boldsymbol{s}_2 - \int_{S_1}\boldsymbol{B}\cdot d\boldsymbol{s}_1 + \int_{S_3}\boldsymbol{B}\cdot d\boldsymbol{s}_3 = 0 \quad (\because \nabla\cdot\boldsymbol{B}=0)$$

where $d\boldsymbol{s}_3 = d\boldsymbol{l}\times\boldsymbol{u}\Delta t$

$$\rightarrow \int_{S_2}\boldsymbol{B}\cdot d\boldsymbol{s}_2 - \int_{S_1}\boldsymbol{B}\cdot d\boldsymbol{s}_1 = -\Delta t\oint_C(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{l} \quad \cdots(4)$$

- By plugging (4) into (3), we get

$$\frac{d\Phi}{dt} = \int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot ds + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\boldsymbol{B}\cdot d\boldsymbol{s}_2 - \int_{S_1}\boldsymbol{B}(t)\cdot d\boldsymbol{s}_1 + \cancel{H.O.T}\right] \quad \cdots(3)$$

$$= \int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot ds - \oint_C(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{l}$$

$$\boxed{\because \oint_C \boldsymbol{E}'\cdot d\boldsymbol{l} = -\int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{s} + \oint_C(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{l}}$$

General form of Faraday's law

- According to a general form of Faraday's law,

$$\frac{d\Phi}{dt} = -\left(-\int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{s} + \oint_C(\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{l}\right) = -\left(\oint_C \boldsymbol{E}'\cdot d\boldsymbol{l}\right) \triangleq -v'$$

where v' is induced emf in circuit C *"measured in the moving frame"*

$$\boxed{\therefore v' = -\frac{d\Phi}{dt} \quad \text{(V)}}$$

- If a circuit does not move, v' reduces to v
- *Faraday's law* applies to both moving and stationary circuits

Maxwell's Equations (1/3)

Curl postulate for *H*

- Previously,

Under time-varying condition

$$\nabla\times\boldsymbol{H}=\boldsymbol{J} \;\rightarrow\; \nabla\cdot(\nabla\times\boldsymbol{H})=0\neq\nabla\cdot\boldsymbol{J} \qquad \because \nabla\cdot\boldsymbol{J}=-\frac{\partial\rho}{\partial t}$$

Null Identity

: Equation of continuity that must hold **at all times** under **time-varying condition**

- Thus, under time-varying condition,

$$\nabla\cdot(\nabla\times\boldsymbol{H})=0=\nabla\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}$$

$$\rightarrow\; \nabla\cdot(\nabla\times\boldsymbol{H})=\nabla\cdot\left(\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}\right)$$

Displacement current density

$$\therefore \nabla\times\boldsymbol{H}=\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}$$

:Major contributions of James Clerk Maxwell

Maxwell's Equations

James Clerk Maxwell (1831~1879)

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}+\frac{\partial\boldsymbol{D}}{\partial t}$$

$$\nabla\cdot\boldsymbol{B}=0$$

$$\nabla\cdot\boldsymbol{D}=\rho$$

where $\boldsymbol{J}=\underline{\rho\boldsymbol{u}}$ or $\boldsymbol{J}=\underline{\sigma\boldsymbol{E}}$

Convection current (ρ**u**) due to motion of free charge distribution

Conduction current (σ**E**) caused by presence of E-field in conducting medium

Img src: Wikipedia

Maxwell's Equations

$$\begin{cases} \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{D} = \rho \end{cases} \quad \begin{cases} \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

+

Equation of Continuity

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}$$

+

Lorentz Force Equation

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)$$

=

"Foundation of Electromagnetic Theory"

: Explain and predict all Macroscopic Electromagnetic Phenomena

When solving electromagnetic problems,

- Need to understand inter-dependence of Maxwell's Equations such that

Two divergence equations $\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$ can be derived from two curl equations $\begin{cases} \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \\ \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \end{cases}$ **(Proof, HW!)**

- To determine **12 unknown** variables for ***E***, ***B***, ***D*** and ***H***

6 Equations from Maxwell's Equations *(Not 12, because of inter-dependence)* **+** **6 Equations** from *Constitutive Relation* **=** **Sufficient to solve EM problems!**

$$\begin{cases} \boldsymbol{D} = \varepsilon \boldsymbol{E}, \\ \boldsymbol{H} = \dfrac{1}{\mu} \boldsymbol{B} \end{cases}$$

Chap. 7 | Maxwell's Equations (3/3)

Differential form	Integral form	
$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \dfrac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s}$	*Faraday's law of electromagnetic induction*
$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t}$	$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$	*Ampere's circuital law*
$\nabla \cdot \boldsymbol{B} = 0$	$\int_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$	*Law of conservation of magnetic flux*
$\nabla \cdot \boldsymbol{D} = \rho$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \int_V \rho \, dv = Q$	*Gauss's law*

In explaining electromagnetic phenomena in a physical environment,
integral forms are more useful *in applying to the* ***finite objects of specified shapes and boundary condition.***

André-Marie Ampère (1775-1836)

Carl F. Gauss (1777~1855)

Michael Faraday (1791~1867)

James C. Maxwell (1831~1879)

Img src: Wikipedia

Vector magnetic potential *A*

- Starting from the curl equation for ***H***

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (\mathrm{T}) \quad (\because \nabla \cdot \boldsymbol{B} = 0)$$

- By substituting above equation into $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$, we get

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{A}) \quad \text{or} \quad \nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0$$

- Thus, we get

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V$$

$$\boxed{\therefore \boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \quad (\mathrm{V/m})}$$

Electric field
- spatial distribution of charges $-\nabla V$
- Time-varying magnetic field $-\frac{\partial \boldsymbol{A}}{\partial t}$

Non-homogeneous wave equations for *A*

- Starting from the curl equation for ***H***

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}.$$ Since $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} = \frac{1}{\mu}\nabla \times \boldsymbol{A}$ and $\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right)$, we get

$$\nabla \times \nabla \times \boldsymbol{A} = \mu \boldsymbol{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right)$$

$$\rightarrow \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu \boldsymbol{J} - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right) - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

$$\rightarrow \nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} + \nabla\left(\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

$$\boxed{\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0}$$

Lorentz Condition for potentials

- Consistent with Equation of Continuity (***HW!***)
- Reduces to $\nabla \cdot \boldsymbol{A} = 0$ for static fields

- If we apply Lorentz condition to above equation, we get

$$\boxed{\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}}$$

Nonhomogeneous equation for magnetic potential A

Potential functions (3/3)

Non-homogeneous wave equations for *V*

• Starting from the divergence equation for ***D***

$$\nabla \cdot \boldsymbol{D} = \rho. \quad \text{Since} \quad \boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right), \quad \text{we get}$$

$$-\nabla \cdot \varepsilon\left(\nabla V + \frac{\partial \boldsymbol{A}}{\partial t}\right) = \rho$$

$$\rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon}$$

• By plugging **Lorentz Condition** $\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$ into above equation, we get

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

Nonhomogeneous equation for electric potential V

c.f.) $$\boxed{\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}}$$

*Nonhomogeneous equation for magnetic potential **A***

∴ *Lorentz condition **decouples the Maxwell's equations** for V and for **A***

Chap. 7 | Electromagnetic Boundary Condition (1/2)

How to obtain Boundary Condition (B.C.) for time-varying electromagnetic fields?

• B.C. obtained by applying the ***integral form of Maxwell's equations*** to a small region at an interface between two media

- Application of integral form of a ***"curl"*** equation to ***Scheme 1*** → B.C. for ***tangential*** components
- Application of integral form of a ***"divergence"*** equation to ***Scheme 2*** → B.C. for ***normal*** components

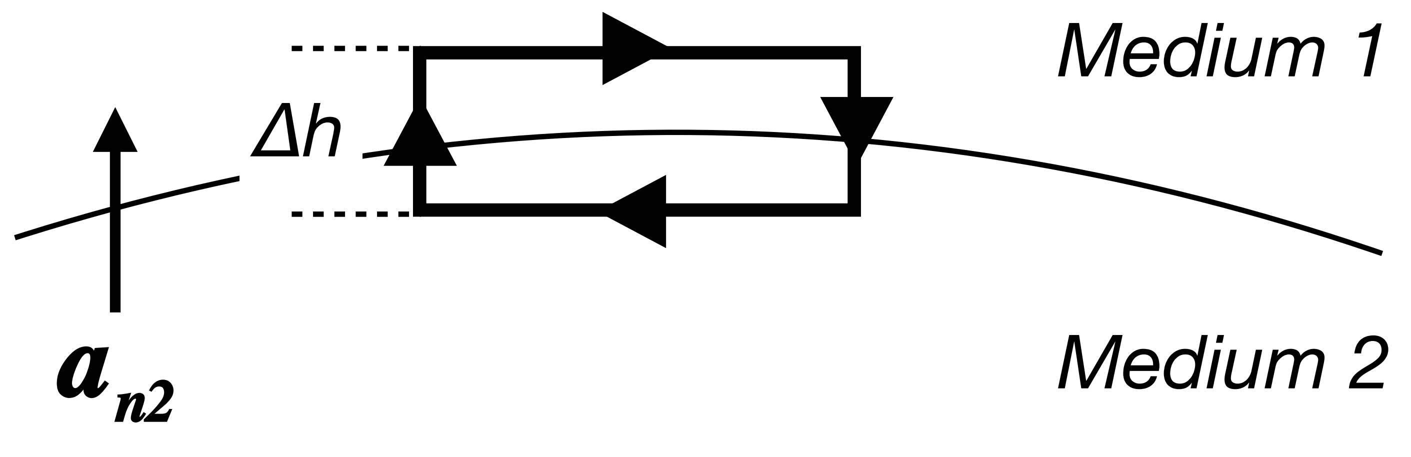
(HW!)

B.C. for tangential components of *E* and *H*

$$\oint_C \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_S \cancel{\frac{\partial \boldsymbol{B}}{\partial t}}\cdot d\boldsymbol{s} \rightarrow \boxed{E_{1t} = E_{2t} \quad (\text{V/m})}$$

$$\oint_C \boldsymbol{H}\cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}}\right)\cdot d\boldsymbol{s} \rightarrow \boxed{\boldsymbol{a}_{n2}\times\left(\boldsymbol{H}_1 - \boldsymbol{H}_2\right) = \boldsymbol{J}_S \quad (\text{A/m})}$$

Time-varying terms vanishes as S goes to 0!

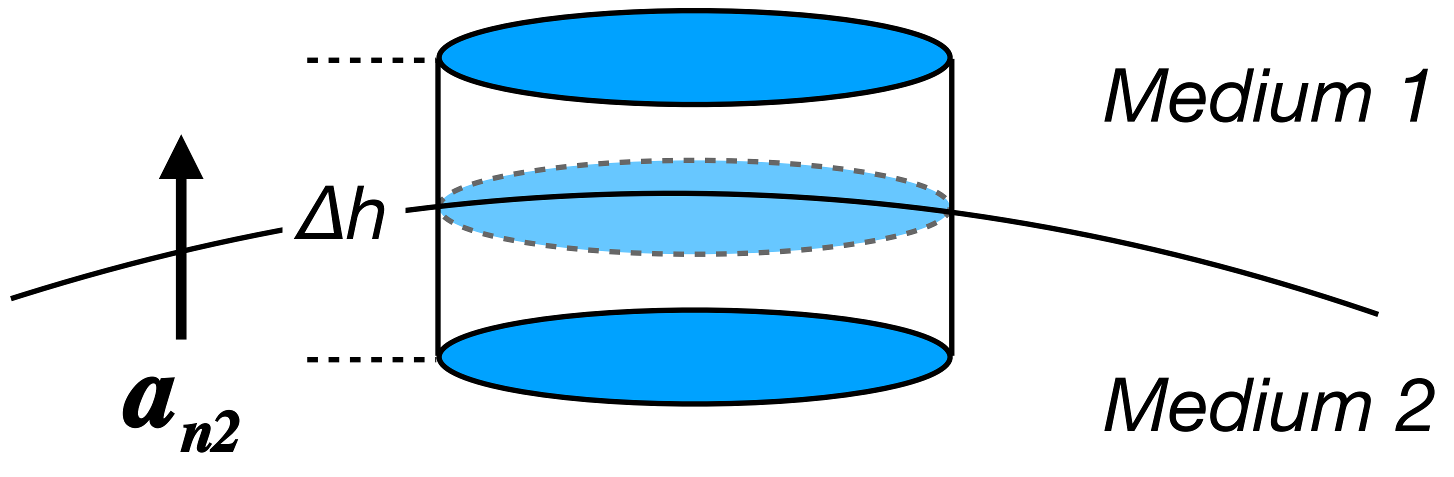


Scheme 1

B.C. for normal components of *D* and *B*

$$\nabla\cdot\boldsymbol{D} = \rho \rightarrow \boxed{\boldsymbol{a}_{n2}\cdot\left(\boldsymbol{D}_1 - \boldsymbol{D}_2\right) = \rho_S \quad (\text{C/m}^2)}$$

$$\nabla\cdot\boldsymbol{B} = 0 \rightarrow \boxed{B_{1n} = B_{2n} \quad (\text{T})}$$



Scheme 2

Chap. 7 | Electromagnetic Boundary Condition (2/2)

Lossless non-conductive media

- $\sigma = 0 \longrightarrow$ No free charges and no free surface currents at the interface $\begin{cases} \rho_S = 0 \\ \boldsymbol{J_S} = 0 \end{cases}$

$$\left(\because P = \int_V \boldsymbol{E} \cdot \boldsymbol{J}\, dv = \int_V \sigma E^2\, dv\right)$$ *Joule's law*

General B.C.

$$E_{1t} = E_{2t}$$

$$\boldsymbol{a_{n2}} \times \left(\boldsymbol{H_1} - \boldsymbol{H_2}\right) = \cancel{\boldsymbol{J_S}}$$

$$\boldsymbol{a_{n2}} \cdot \left(\boldsymbol{D_1} - \boldsymbol{D_2}\right) = \cancel{\rho_S}$$

$$B_{1n} = B_{2n}$$

$\longrightarrow$

B.C. between two lossless media

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

$$D_{1n} = D_{2n} \rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

dielectric / Perfect conductor

- For perfect conductor, $\sigma \sim \infty$
  - ***E*** = 0 in the *interior of a perfect conductor*
  - ***B***, ***H*** = 0 because they are interrelated through Maxwell's equations

General B.C.

$$E_{1t} = E_{2t}$$

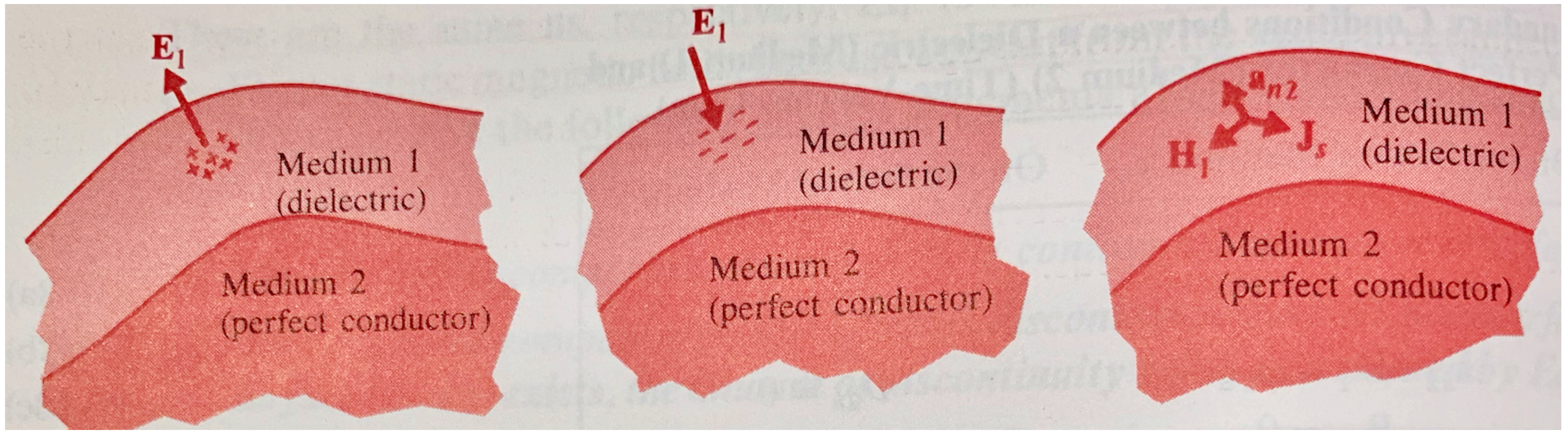
$$\boldsymbol{a_{n2}} \times \left(\boldsymbol{H_1} - \boldsymbol{H_2}\right) = \boldsymbol{J_S}$$

$$\boldsymbol{a_{n2}} \cdot \left(\boldsymbol{D_1} - \boldsymbol{D_2}\right) = \rho_S$$

$$B_{1n} = B_{2n}$$

$\longrightarrow$

Medium 1 (dielectric)	*Medium 2* (Perfect conductor)
$E_{1t} = 0$	$E_{2t} = 0$
$\boldsymbol{a_{n2}} \times \boldsymbol{H_1} = \boldsymbol{J_S}$	$H_{2t} = 0$
$\boldsymbol{a_{n2}} \cdot \boldsymbol{D_1} = \rho_S$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$



Electromagnetics

<Chap. 7> Time-varying fields and Maxwell's Equations

Section 7.1 ~ 7.7

(2nd class of **week 2**)

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Chap. 7 | Contents for 2nd class of week 2

Sec 6. Electromagnetic wave equations and their solutions

Sec 7. Time-harmonic electromagnetic fields

Time-varying electric & magnetic fields

Differential form	*Integral form*
$\nabla \times \boldsymbol{E} = \boxed{-\frac{\partial \boldsymbol{B}}{\partial t}}$	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{s}$
$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \boxed{\frac{\partial \boldsymbol{D}}{\partial t}}$	$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$
$\nabla \cdot \boldsymbol{B} = 0$	$\int_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$
$\nabla \cdot \boldsymbol{D} = \rho$	$\oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = \int_V \rho\, dv = Q$

Constitutive relation

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{H} = \frac{1}{\mu} \boldsymbol{B}$$

Maxwell's equations

- Explain and predict all electric and magnetic phenomena under static or time-varying condition
- New curl postulates with time-dependent term introduced
  - Curl of ***E***: consistent with Faraday's law
  - Curl of ***H***: consistent with Equation of continuity
  - Mutual dependence between ***E*** and ***B***
- Two divergence equations can be derived from two curl equations (**HW!**)
- Along with constitutive relation, Maxwell's equations are sufficient to solve all electromagnetic problems with given boundary conditions
- A particular solution to Maxwell's equations: Electromagnetic wave propagating with speed of light

 Potential functions and Non-homogeneous equations

Potential functions

$$\begin{cases} \boldsymbol{B} = \nabla \times \boldsymbol{A} & (\because \nabla \cdot \boldsymbol{B} = 0) \\ \boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t} & \left(\because \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}\right) \end{cases}$$

A goal of solving EM problems

"Obtain *E* and *B*" ← Non-homogeneous Equations for *A* and *V* ← *Maxwell's Equations*

This slide is about…

$$\boxed{\begin{aligned} \boldsymbol{H} &= \frac{1}{\mu}\boldsymbol{B} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \\ \boldsymbol{D} &= \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right) \end{aligned}} \rightarrow \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho \leftarrow \boxed{\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right)}$$

$$\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} + \nabla\left(\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon}$$

$$\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$$

Lorenz condition

$$\boxed{\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}}$$

Non-homogeneous equation for A

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

Non-homogeneous equation for V

Chap. 7 | Retarded Potential functions

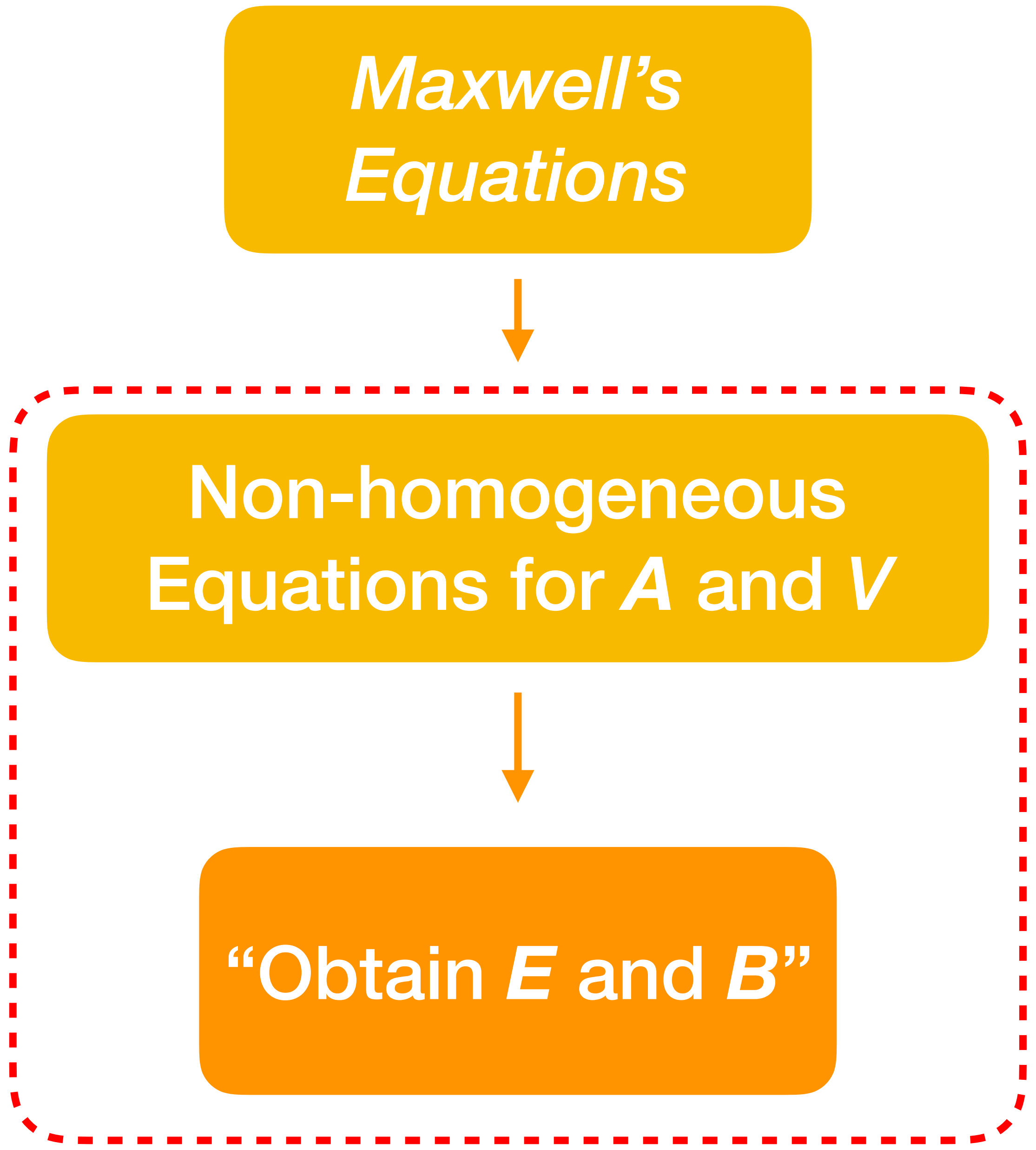
Procedures to obtain E and B from Non-homogeneous eqns

(1) we solve non-homogeneous equations for V and $\boldsymbol{A}$ with given ρ and $\boldsymbol{J}$

$$\begin{cases} \nabla^2 V - \mu\varepsilon \dfrac{\partial^2 V}{\partial t^2} = -\dfrac{\rho}{\varepsilon} \\ \nabla^2 \boldsymbol{A} - \mu\varepsilon \dfrac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} \end{cases}$$

(2) With determined V and $\boldsymbol{A}$, apply $\begin{cases} \boldsymbol{B} = \nabla \times \boldsymbol{A} \\ \boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t} \end{cases}$ to obtain $\boldsymbol{B}$ and $\boldsymbol{E}$.

This slide is about…



Solutions to non-homogeneous equations

delayed time

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} \quad \rightarrow \quad V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho(t - R/u)}{R} dv' \quad \text{(V)}$$

$$\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} \quad \rightarrow \quad \boldsymbol{A}(R,t) = \frac{\mu}{4\pi}\int_{V'} \frac{\boldsymbol{J}(t - R/u)}{R} dv' \quad \text{(Wb/m)}$$

where $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$ (m/s) is a ***velocity of propagation***

Retarded potential V

- Potential measured at ***R*** created by a charge distribution ρ in a volume V' at origin
- Takes time (R/u) for the effect of ρ to be sensed at ***R*** → It is called "*Retarded potential*"
- Equivalently, it takes time for EM waves to travel and for the effects of time-varying charges (ρ) to be sensed at a distant point (***R***).

Chap. 7 | Source-free wave equations

EM waves in a source-free region (ρ = 0 and $\boldsymbol{J}$ = 0)

- We are more interested in how EM waves are propagated than how they are originated (generated)
- If the wave is in a simple (linear, isotropic, and homogeneous) non-conducting medium, Maxwell's equations read

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla\times\boldsymbol{H} = \cancel{\boldsymbol{J}} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\cdot\boldsymbol{D} = \cancel{\rho}$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu},\ \boldsymbol{D} = \varepsilon\boldsymbol{E}$$

$$\nabla\times\boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla\times\boldsymbol{H} = \varepsilon\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla\cdot\boldsymbol{H} = 0$$

$$\nabla\cdot\boldsymbol{E} = 0$$

- Let's take a curl to each side of $\nabla\times\boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t}$

$$\rightarrow \nabla\times\nabla\times\boldsymbol{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\boldsymbol{H})$$

$$\text{(l.h.s)}\ \nabla\times\nabla\times\boldsymbol{E} = \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E} = -\nabla^2\boldsymbol{E}$$

$$\text{(r.h.s)}\ -\mu\frac{\partial}{\partial t}(\nabla\times\boldsymbol{H}) = -\mu\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}\right) = -\mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\text{(l.h.s)} = \text{(r.h.s)} \rightarrow -\nabla^2\boldsymbol{E} = -\mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\boxed{\therefore \nabla^2\boldsymbol{E} - \frac{1}{u^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0}$$ *Homogenous vector wave equation*

Similarly, $\boxed{\therefore \nabla^2\boldsymbol{H} - \frac{1}{u^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0}$ where $u = \frac{1}{\sqrt{\mu\varepsilon}}$ is a ***velocity of propagation***

Chap. 7 | Time-harmonic electromagnetic fields

Source — $\rho, \boldsymbol{J}$ → *Field quantities* — $\boldsymbol{E}, \boldsymbol{H}$

Maxwell's Equations

- Function forms of fields (***E***, ***H***) = Function forms of source (ρ, ***J***)
- Arbitrary *periodic* time functions can be expanded into ***Fourier series of harmonic sinusoidal components***

$$x(t) = a_0 + \sum_{k=-\infty}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$$

- Non-periodic time functions can be expressed as Fourier integrals

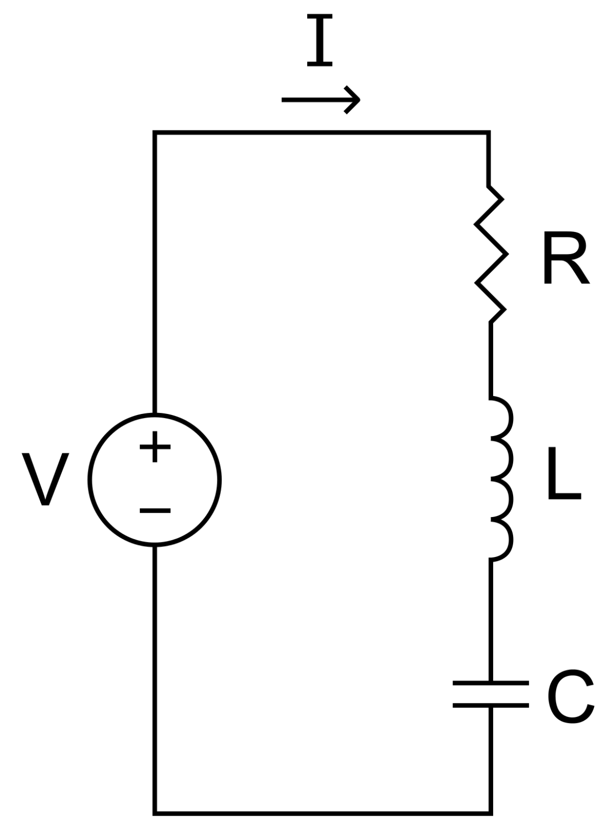
Sinusoidal (time-harmonic) source
ρ and J with a given frequency ω

$\xrightarrow[\text{(linear PDE)}]{\text{Maxwell's Equations}}$

Sinusoidal (time-harmonic) fields
E and H with the same frequency ω

Chap. 7 | Phasor

Example: Loop equation for a series RLC circuit



$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)=V\cos\omega t$$

We want to get $i(t)=I\cos(\omega t+\phi)$ where

I: Amplitude
ω: angular frequency (*ω*=2π*f* [rad/s])
ϕ: phase

- By using exponential functions for convenience, we get

$$v(t)=V\cos\omega t=\mathrm{Re}\left[\left(Ve^{j0}\right)e^{j\omega t}\right]=\mathrm{Re}\left(V_S e^{j\omega t}\right)$$

$$i(t)=I\cos(\omega t+\phi)=\mathrm{Re}\left[\left(Ie^{j\phi}\right)e^{j\omega t}\right]=\mathrm{Re}\left[I_S e^{j\omega t}\right]$$

Here,

$$V_S=Ve^{j0}=V$$
$$I_S=Ie^{j\phi}$$

Phasor

- Containing "***Amplitude***" and "***phase***" info
- ***Independent of time***

Since

$$\frac{di(t)}{dt}=\mathrm{Re}\left(j\omega I_S e^{j\omega t}\right)$$

$$\int i(t)dt=\mathrm{Re}\left(\frac{1}{j\omega}I_S e^{j\omega t}\right),$$

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)dt=v(t)=V\cos\omega t$$

$$\downarrow$$

$$\mathrm{Re}\left[\left(R+j\omega L+\frac{C}{j\omega}\right)I_S e^{j\omega t}\right]=\mathrm{Re}\left[E_S e^{j\omega t}\right]$$

$$\downarrow$$

Time-independent!

$$\therefore\left(R+j\omega L+\frac{C}{j\omega}\right)I_S=V_S$$

$$\therefore i(t)=\mathrm{Re}\left[I_S e^{j\omega t}\right]$$

Chap. 7 | Time-harmonic electromagnetics

Time-harmonic (sinusoidal) E-field

$$\boldsymbol{E}(x,y,z,t) = \mathrm{Re}\left[\boldsymbol{E}(x,y,z)e^{j\omega t}\right]$$

*It is customary to use **cos(ωt) as a reference** instead of sin(ωt)!*

where $\boldsymbol{E}(x,y,z)$ is a ***vector phasor*** with ***direction, magnitude, and phase*** information

Differential & Integral of Time-varying vector	*Vector phasor*
$\boldsymbol{E}(x,y,z,t)$	$\boldsymbol{E}(x,y,z)$
$\frac{\partial}{\partial t}\boldsymbol{E}(x,y,z,t)$	$j\omega\boldsymbol{E}(x,y,z)$
$\int \boldsymbol{E}(x,y,z,t)dt$	$\frac{1}{j\omega}\boldsymbol{E}(x,y,z)$

Time-harmonic Maxwell's equations

$$\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t} \rightarrow \nabla\times\boldsymbol{E} = -\mu(j\omega\boldsymbol{H})$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \varepsilon\frac{\partial\boldsymbol{E}}{\partial t} \rightarrow \nabla\times\boldsymbol{H} = \boldsymbol{J} + \varepsilon(j\omega\boldsymbol{E})$$

$$\nabla\cdot\boldsymbol{H} = 0 \rightarrow \nabla\cdot\boldsymbol{H} = 0$$

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon} \rightarrow \nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon}$$

where ***E***, ***H*** are vector field phasors and *ρ* and ***J*** are source phasors
Once again, phasors are ***NOT a function of time (t)***.

Chap. 7 | Time-harmonic wave equations and potential functions

Time-harmonic wave equations

$$\nabla^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} \quad \rightarrow \quad \nabla^2 V + \mu\varepsilon\omega^2 V = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 \boldsymbol{A} - \mu\varepsilon\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J} \quad \rightarrow \quad \nabla^2 \boldsymbol{A} + \mu\varepsilon\omega^2 \boldsymbol{A} = -\mu\boldsymbol{J}$$

Non-homogeneous Helmholtz's Equations

$$\boxed{\begin{aligned} \nabla^2 V + k^2 V &= -\frac{\rho}{\varepsilon} \\ \nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} &= -\mu\boldsymbol{J} \end{aligned}}$$

where ***A***, *V* are phasors, and

$$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{\lambda}$$

λ wavelength

is called ***wavenumber***.

Lorentz Condition for potentials

$$\nabla\cdot\boldsymbol{A} + \mu\varepsilon\frac{\partial V}{\partial t} = 0 \quad \rightarrow \quad \nabla\cdot\boldsymbol{A} + j\omega\mu\varepsilon V = 0$$

Solutions to Non-homogeneous Equations (retarded potential)

$$V(R,t) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(t-R/u)}{R}dv' \quad \rightarrow \quad V(R) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho e^{-jkR}}{R}dv'$$

$$\boldsymbol{A}(R,t) = \frac{\mu}{4\pi}\int_{V'}\frac{\boldsymbol{J}(t-R/u)}{R}dv' \quad \rightarrow \quad \boldsymbol{A}(R) = \frac{\mu}{4\pi}\int_{V'}\frac{\boldsymbol{J}e^{-jkR}}{R}dv'$$

Relation to the static case

$$e^{-jkR} = 1 - \cancel{jkR + \frac{k^2R^2}{2} + \cdots}$$

If $kR = 2\pi R/\lambda << 1$, Helmholtz solutions reduce to those for *quasi-static* fields.

FROM NOW ON,
WE WILL ***NEARLY EXCLUSIVELY USE "PHASORS"***
SINCE WE WILL ONLY DEAL WITH ***TIME-HARMONIC ELECTROMAGNETICS***
IN THE REST OF THE COURSE.

*ALTHOUGH NOT SPECIFIED VECTOR FIELD QUANTITIES (**E**, **D**, **B**, **H**) THAT WE WILL USE ARE GOING TO BE* ***PHASORS****, AND THEY ARE ONLY FUNCTIONS OF SPACE AND* ***NOT A FUNCTION OF TIME.***

Chap. 7 | "Source-free" EM fields in simple media

Time-harmonic Maxwell's Equations
in a simple, nonconducting, source-free media
(ρ = 0, $\boldsymbol{J}$ = 0, σ = 0)

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E} \\ \nabla \cdot \boldsymbol{H} &= 0 \\ \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon} \end{aligned} \Rightarrow \begin{aligned} \nabla \times \boldsymbol{E} &= -j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= j\omega\varepsilon\boldsymbol{E} \\ \nabla \cdot \boldsymbol{H} &= 0 \\ \nabla \cdot \boldsymbol{E} &= 0 \end{aligned}$$

Homogeneous Vector Helmholtz's equations

$$\begin{aligned} \nabla\boldsymbol{E}^2 - \frac{1}{u^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} &= 0 \\ \boldsymbol{H}^2 - \frac{1}{u^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} &= 0 \end{aligned} \Rightarrow \begin{aligned} \nabla\boldsymbol{E}^2 + k^2\boldsymbol{E} &= 0 \\ \nabla\boldsymbol{H}^2 + k^2\boldsymbol{H} &= 0 \end{aligned}$$

Principle of Duality

- If (***E***, ***H***) are solutions of source-free Maxwell's equations in a simple medium, then so are (***E'***, ***H'***) where

$$\boldsymbol{E}' = \eta\boldsymbol{H}$$

$$\boldsymbol{H}' = -\frac{1}{\eta}\boldsymbol{E}$$

(Ch 7-7.3 for proof. Fairly simple!)

where η is called ***intrinsic impedance*** of the medium.

Chap. 7 | Simple conducting, lossy medium (1/2)

Complex permittivity (σ ≠ 0)

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \rightarrow \nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon \boldsymbol{E}$$

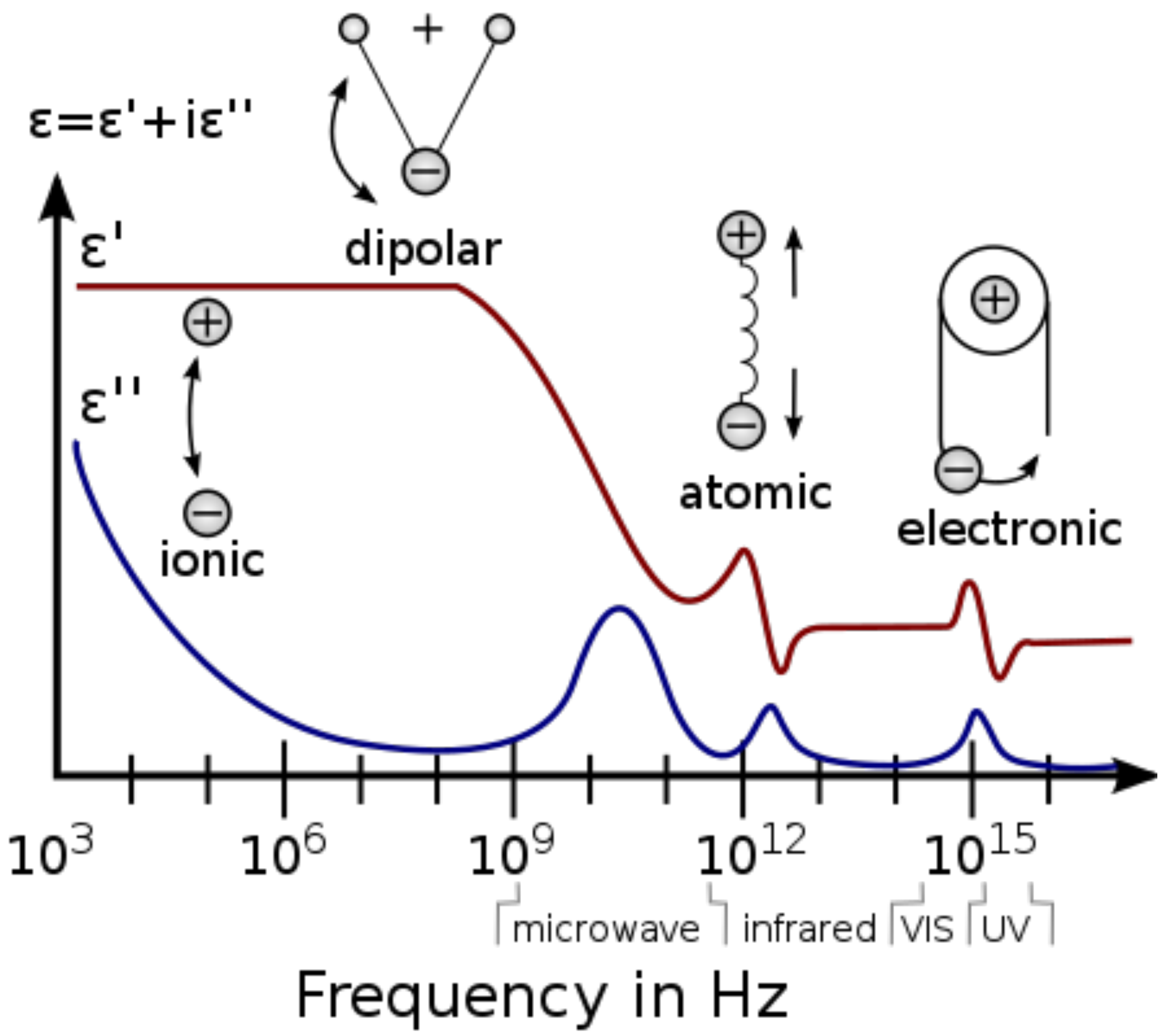
$$= (\sigma + j\omega\varepsilon)\boldsymbol{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\boldsymbol{E} = j\omega\varepsilon_c \boldsymbol{E}$$

where $\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$ (F/m) is ***complex permittivity***

Physical meaning of complex permittivity

- Indicates that ***materials polarization does not change instantaneously*** when E-field is applied (i.e. out-of-phase polarization)
- When external time-varying ***E***-field applied to material bodies → Slight displacements of bound charges → a volume polarization density
- As frequency of time-varying ***E***-field increases
  - ***Inertia of charged particles*** resist ***against E-field*** and prevent from being in phase with field change → ***Frictional damping***
- ***Ohmic loss*** if materials have sufficient amount of free charges

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$$ ***(including damping and ohmic losses)***



Simple conducting, lossy medium (2/2)

Complex permittivity

- due to out-of-phase polarization

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$$

Complex permeability

- due to out-of-phase magnetization

$\mu_c = \mu' - j\mu''$ (but, for ferromagnetic materials, μ’ is dominant and μ” is negligible)

Wavenumber in Helmholtz’s equations for a lossy medium

$$\boxed{\begin{aligned} \nabla^2 V + k^2 V &= -\frac{\rho}{\varepsilon} \\ \nabla^2 \mathbf{A} + k^2 \mathbf{A} &= -\mu \mathbf{J} \end{aligned}}$$

$$k = \omega\sqrt{\mu\varepsilon} \Rightarrow \boxed{k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')}}$$

A measure of the power loss

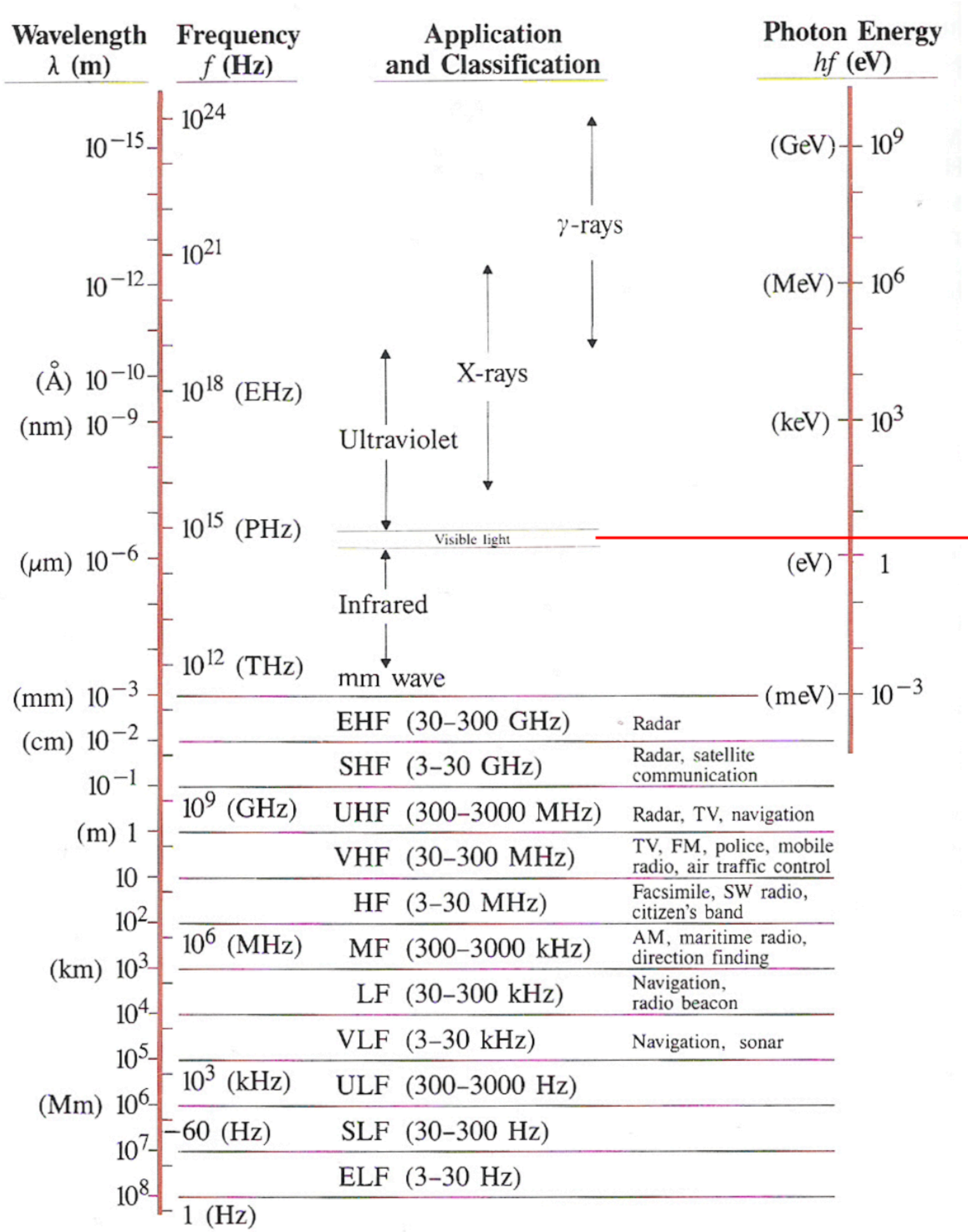
$$\varepsilon_c = \left|\frac{\mathbf{D}}{\mathbf{E}}\right| e^{j\delta} = \left|\frac{\mathbf{D}}{\mathbf{E}}\right| (\cos\delta + j\sin\delta) = \left|\frac{\mathbf{D}}{\mathbf{E}}\right| \cos\delta(1 + j\tan\delta) \Leftrightarrow \varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right) = \varepsilon\left(1 - j\frac{\sigma}{\omega}\right)$$

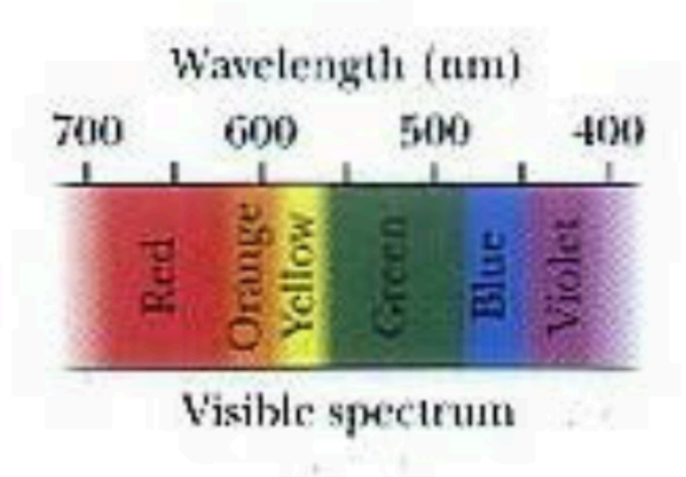
$$\boxed{|\tan\delta| \triangleq \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}}$$

Loss tangent

If $\sigma >> \omega\varepsilon$: Good conductor
If $\sigma << \omega\varepsilon$: Good insulator

Chap. 7 | Electromagnetic wave vs. frequency





Visible light (400 nm ~ 700 nm)

***All EM waves in ANY frequency range** propagate in a medium*

***with the same velocity**,* $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$ $\left(c \cong 3\times10^{8}\,(\mathrm{m/s})\text{ in air}\right)$