Electromagnetics

<Chap. 8> Plane Electromagnetic waves

Section 8.1 ~ 8.4

(1st class of **week 3**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 8 | Contents for 1st class of week 3

Sec 1. Introduction

Sec 2. Plane waves in lossless media

- Uniform plane wave [Transverse Electromagnetic (TEM) wave]
- Polarization of plane wave
- Doppler Effect

Chap. 8 | Time-harmonic (sinusoidal) electromagnetics

*time-harmonic source ρ and **J*** ***with a given frequency ω***	*Maxwell's Equations* →	*time-harmonic **E** and **H** fields* ***with the same frequency ω***

$\boldsymbol{E}(\boldsymbol{R},t) = \mathrm{Re}\left[\boldsymbol{E}(\boldsymbol{R})e^{j\omega t}\right]$ where $\boldsymbol{E}(\boldsymbol{R})$ is a ***vector phasor*** with ***direction, magnitude, and phase*** information

Phasor notation

Differential & Integral of Time-varying vector	*Vector phasor*
$\dfrac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{R},t)$	$j\omega\boldsymbol{E}(\boldsymbol{R})$
$\int \boldsymbol{E}(\boldsymbol{R},t)\,dt$	$\dfrac{1}{j\omega}\boldsymbol{E}(\boldsymbol{R})$

Time-harmonic Maxwell's equations in phasor notation

$$\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t} \quad\longrightarrow\quad \nabla\times\boldsymbol{E} = -\mu(j\omega\boldsymbol{H})$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \varepsilon\frac{\partial\boldsymbol{E}}{\partial t} \quad\longrightarrow\quad \nabla\times\boldsymbol{H} = \boldsymbol{J} + \varepsilon(j\omega\boldsymbol{E})$$

$$\nabla\cdot\boldsymbol{H} = 0 \quad\longrightarrow\quad \nabla\cdot\boldsymbol{H} = 0$$

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon} \quad\longrightarrow\quad \nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon}$$

Chap. 8 | Wave equations in source-free, lossless media

Maxwell's Equations
($\rho = 0$, $\sigma = 0 \rightarrow \boldsymbol{J} = \sigma\boldsymbol{E} = 0$)

Homogeneous Helmholtz Equation

Time-varying vector fields

$$\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t}$$
$$\nabla\times\boldsymbol{H} = \cancel{\boldsymbol{J}} + \varepsilon\frac{\partial\boldsymbol{E}}{\partial t}$$
$$\nabla\cdot\boldsymbol{H} = 0$$
$$\nabla\cdot\boldsymbol{E} = \frac{\cancel{\rho}}{\varepsilon}$$

$\Rightarrow$

$$\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t}$$
$$\nabla\times\boldsymbol{H} = \varepsilon\frac{\partial\boldsymbol{E}}{\partial t}$$
$$\nabla\cdot\boldsymbol{H} = 0$$
$$\nabla\cdot\boldsymbol{E} = 0$$

$\Rightarrow$

$$\begin{cases}\nabla^2\boldsymbol{E} - \mu\varepsilon\dfrac{\partial^2\boldsymbol{E}}{\partial t^2} = 0\\[2ex] \nabla^2\boldsymbol{H} - \mu\varepsilon\dfrac{\partial^2\boldsymbol{H}}{\partial t^2} = 0\end{cases}$$

Phasor Notation

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H}$$
$$\nabla\times\boldsymbol{H} = \cancel{\boldsymbol{J}} + j\omega\varepsilon\boldsymbol{E}$$
$$\nabla\cdot\boldsymbol{H} = 0$$
$$\nabla\cdot\boldsymbol{E} = \frac{\cancel{\rho}}{\varepsilon}$$

$\Rightarrow$

$$\nabla\times\boldsymbol{E} = -j\omega\mu\boldsymbol{H}$$
$$\nabla\times\boldsymbol{H} = j\omega\varepsilon\boldsymbol{E}$$
$$\nabla\cdot\boldsymbol{H} = 0$$
$$\nabla\cdot\boldsymbol{E} = 0$$

$\Rightarrow$

$$\begin{cases}\nabla^2\boldsymbol{E} + \omega^2\mu\varepsilon\boldsymbol{E} = 0\\ \nabla^2\boldsymbol{H} + \omega^2\mu\varepsilon\boldsymbol{H} = 0\end{cases} \quad \left(\because \frac{\partial^2\boldsymbol{E}}{\partial t^2} \rightarrow (-j\omega)^2\boldsymbol{E}\right)$$

$\Downarrow$

$$\begin{cases}\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = 0\\ \nabla^2\boldsymbol{H} + k^2\boldsymbol{H} = 0\end{cases}$$ where $k = \omega\sqrt{\mu\varepsilon}$

Chap. 8 | Plane waves in free space (1/5)

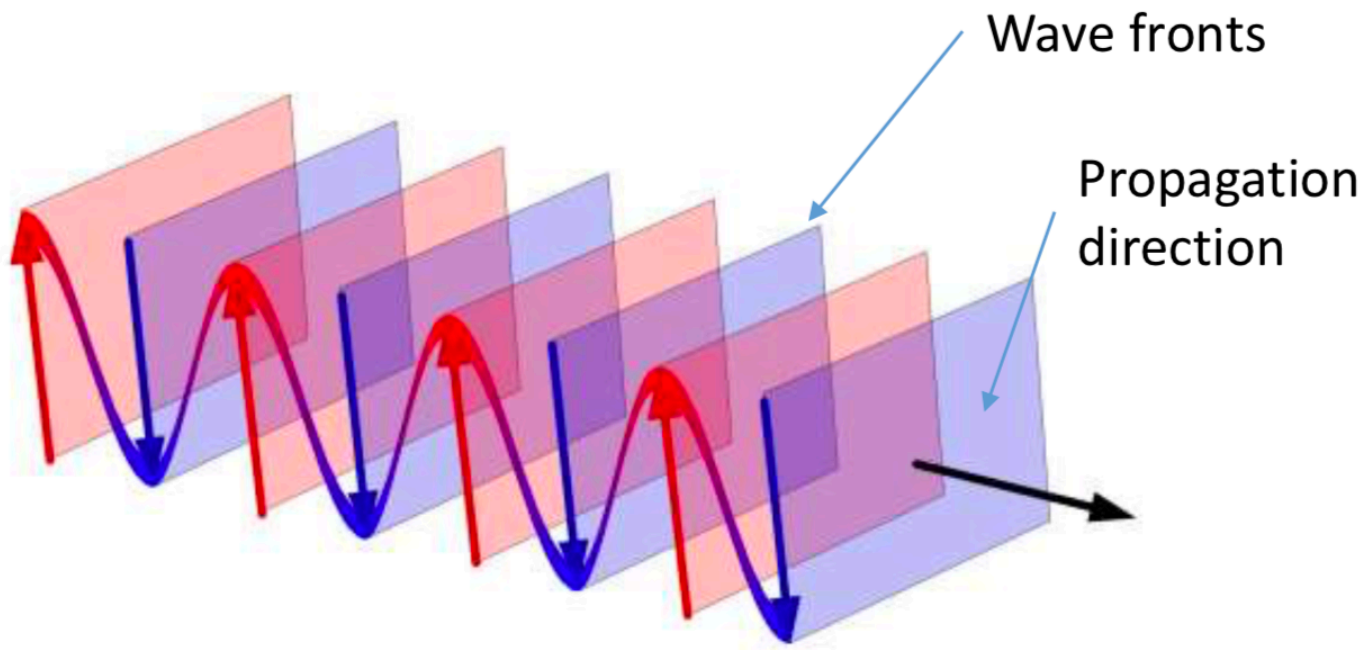
Homogeneous Helmholtz's equations for free space

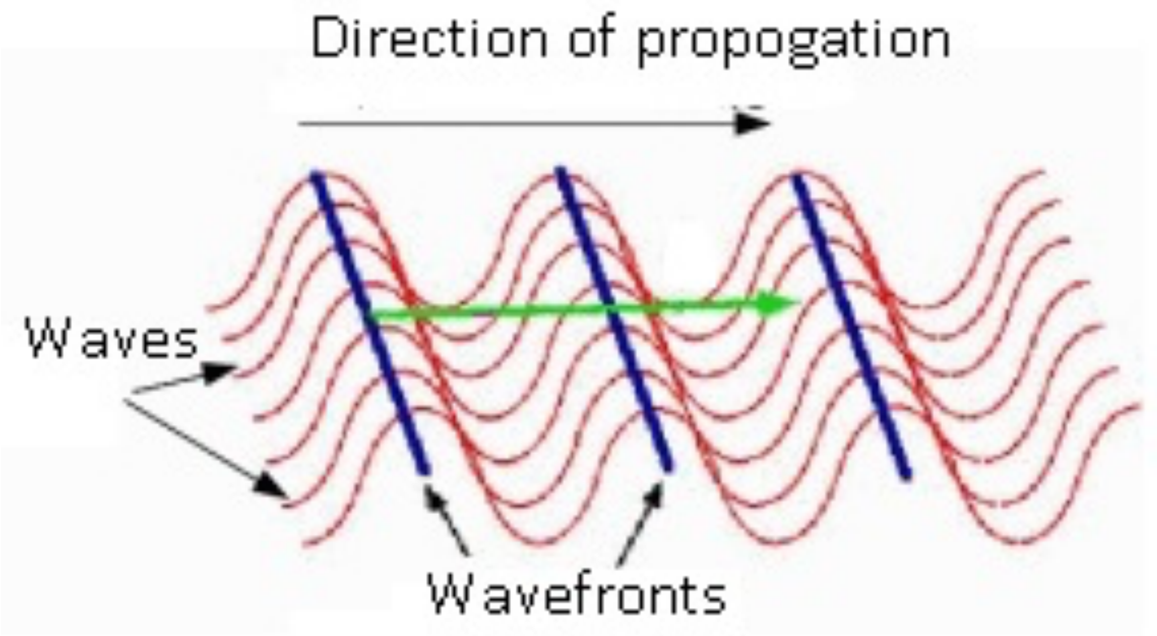
$$\nabla^2 \boldsymbol{E} + \omega^2 \mu_0 \varepsilon_0 \boldsymbol{E} = 0 \quad \rightarrow \quad \boxed{\nabla^2 \boldsymbol{E} + k_0^{\ 2} \boldsymbol{E} = 0}$$

where $k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \dfrac{\omega}{c}$ (rad/m) is free-space wavenumber

"Plane" wave

: a wave whose ***wavefronts*** (i.e. surfaces of constant phase) are *parallel planes* ***normal to propagation direction***





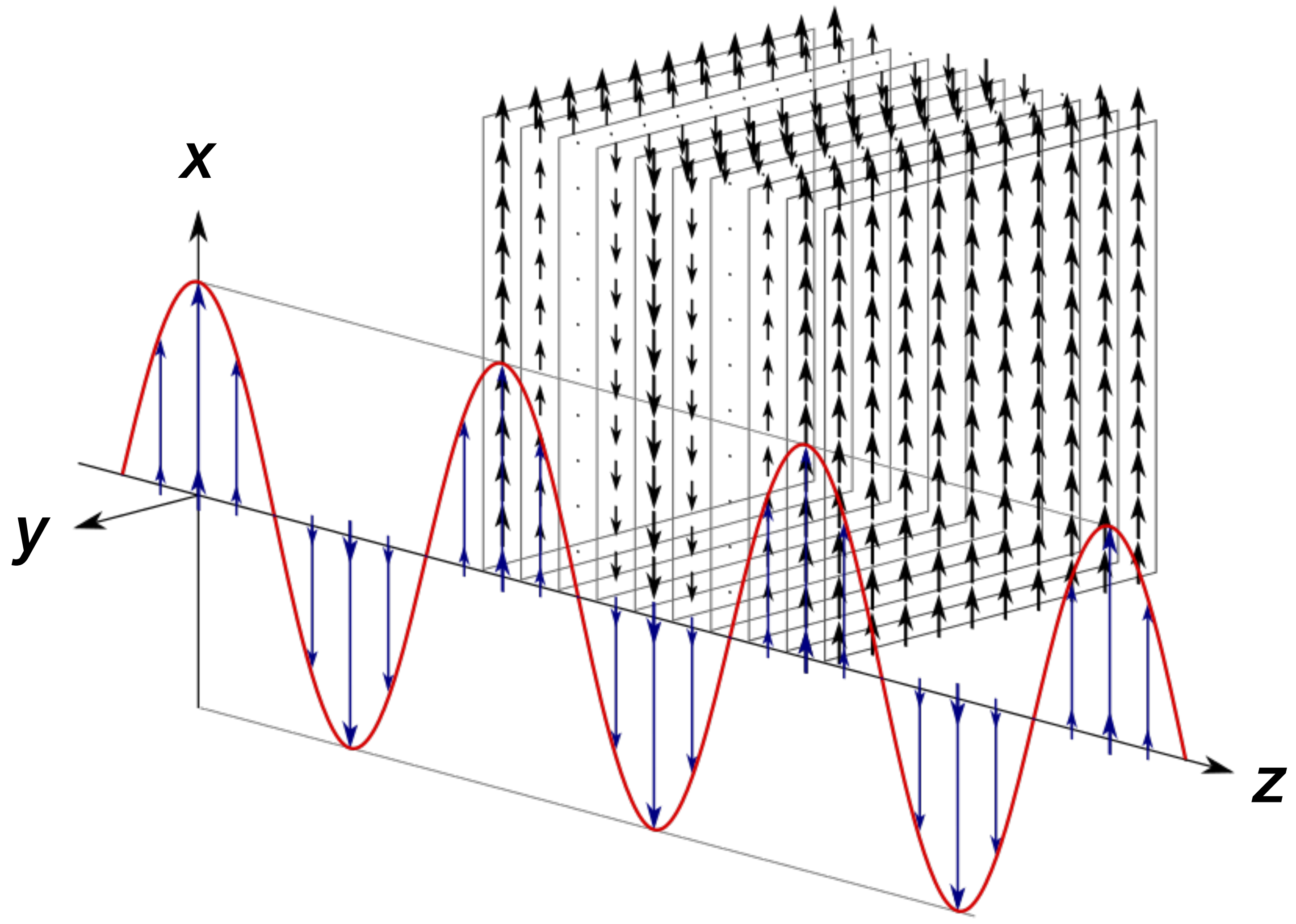
"Uniform" Plane wave

Plane wave characterized by "uniform" E over the wavefronts

: ***E*** has uniform magnitude and phase ***on the plane normal to z-axis (i.e. xy plane)***

$$\boldsymbol{E} = \boldsymbol{a}_x E_x \quad \text{where} \quad \frac{\partial^2 E_x}{\partial x^2} = 0 \ \text{ and } \ \frac{\partial^2 E_x}{\partial y^2} = 0$$

$$\nabla^2 \boldsymbol{E} + k_0^{\ 2} \boldsymbol{E} = 0 \ \rightarrow \ \boldsymbol{a}_x \left(\cancel{\frac{\partial^2}{\partial x^2}} + \cancel{\frac{\partial^2}{\partial y^2}} + \frac{\partial^2}{\partial z^2} + k_0^{\ 2} \right) E_x = 0 \ \rightarrow \ \boxed{\frac{d^2 E_x}{dz^2} + k_0^{\ 2} E_x = 0}$$



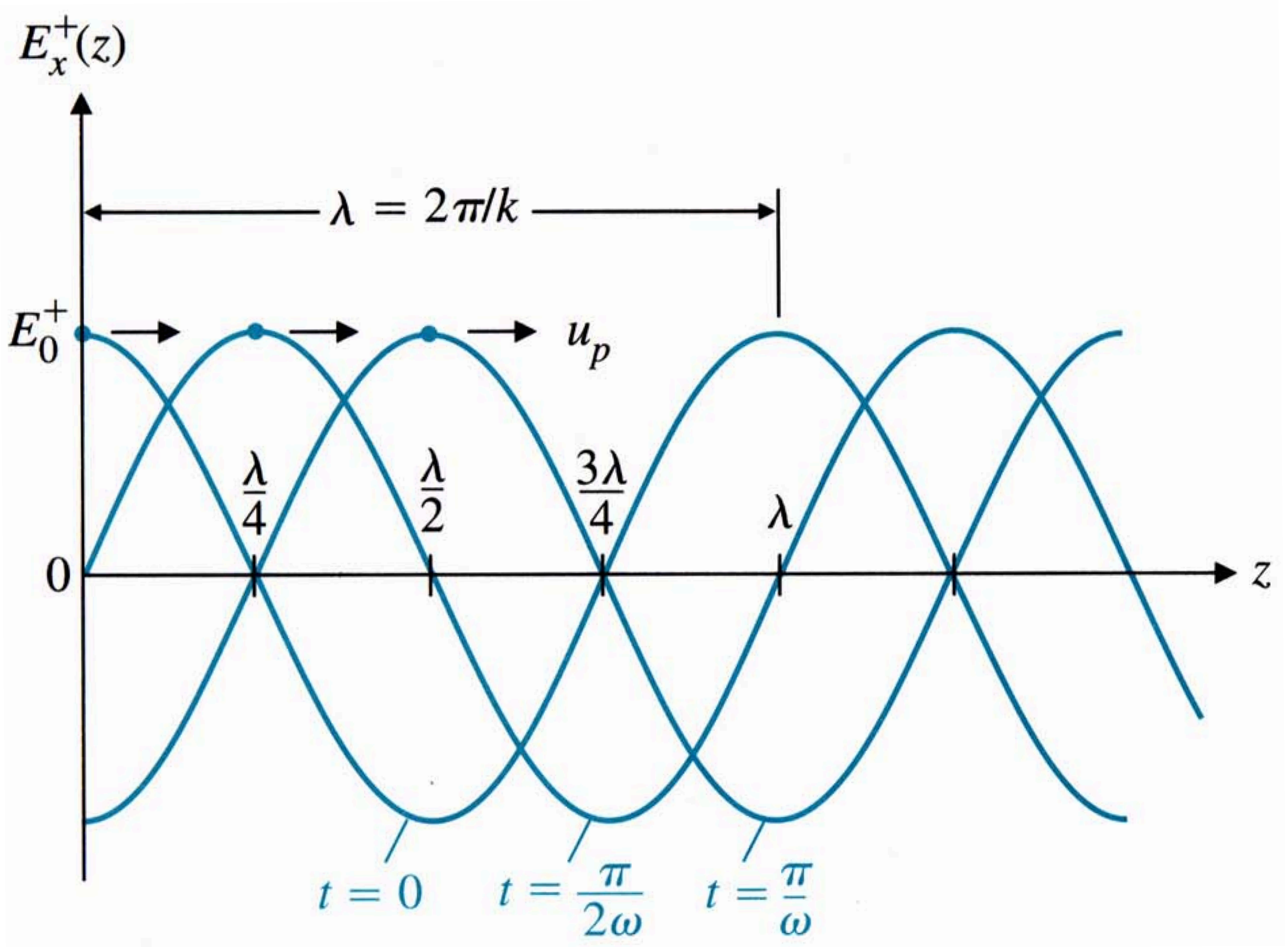
Img srcs: miniphysics.com, Wikipedia

 Plane waves in free space (2/5)

Uniform Plane wave propagating in z-direction

$$\frac{d^2 E_x}{dz^2} + k_0{}^2 E_x = 0$$: ODE because E_x is only a function of z

$$E_x(z) = E_x^+(z) + E_x^-(z) = \underbrace{E_0^+ e^{-jk_0 z}}_{\text{propagating in +z direction}} + \underbrace{E_0^- e^{jk_0 z}}_{\text{propagating in –z direction}}$$



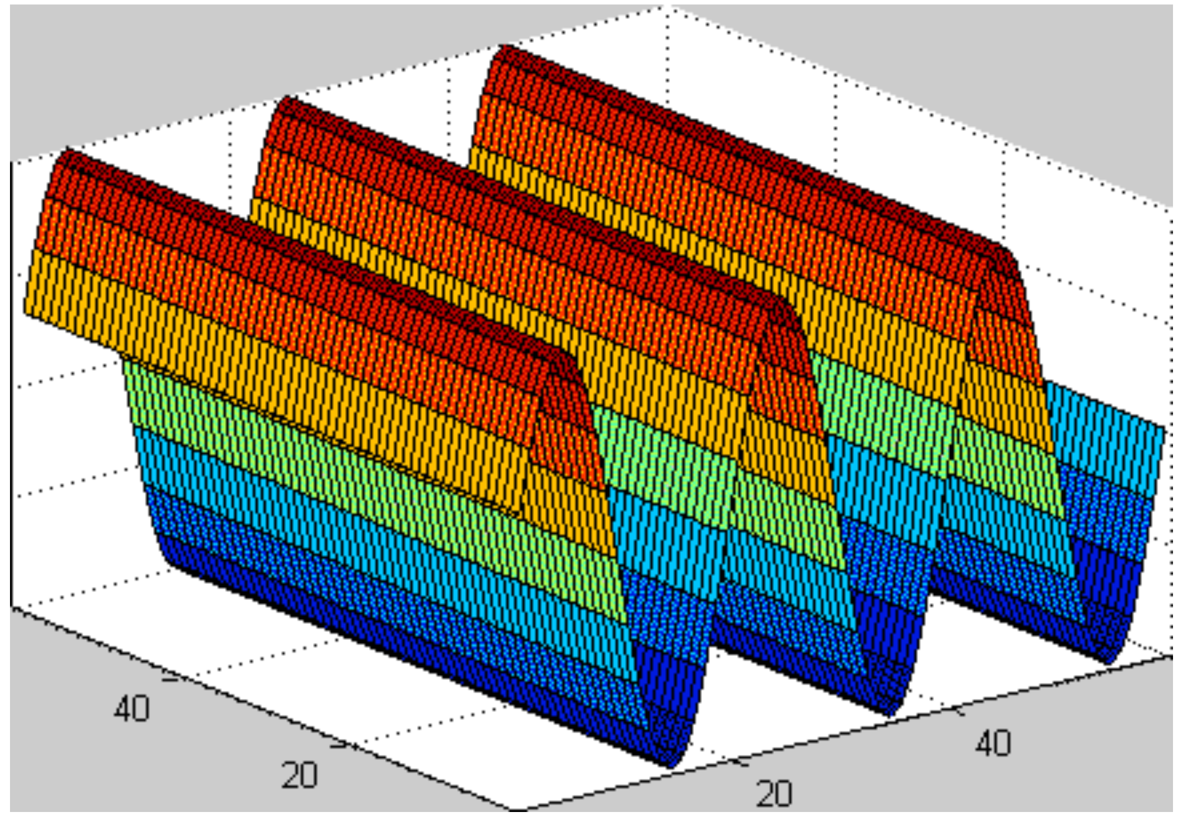
Uniform plane wave in real time (traveling wave)

$$E_x^+(z,t) = \mathrm{Re}\left[E_x^+(z)e^{j\omega t}\right] = \mathrm{Re}\left[E_0^+ e^{-jk_0 z} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[E_0^+ e^{j(\omega t - k_0 z)}\right] = E_0^+ \cos(\omega t - k_0 z)$$

(Let's omit "+" for simplicity)

$$\boxed{E_x(z,t) = E_0 \cos(\omega t - k_0 z)}$$ Traveling wave



At successive times, the curve travels in the positive z direction

gif src: Wikipedia

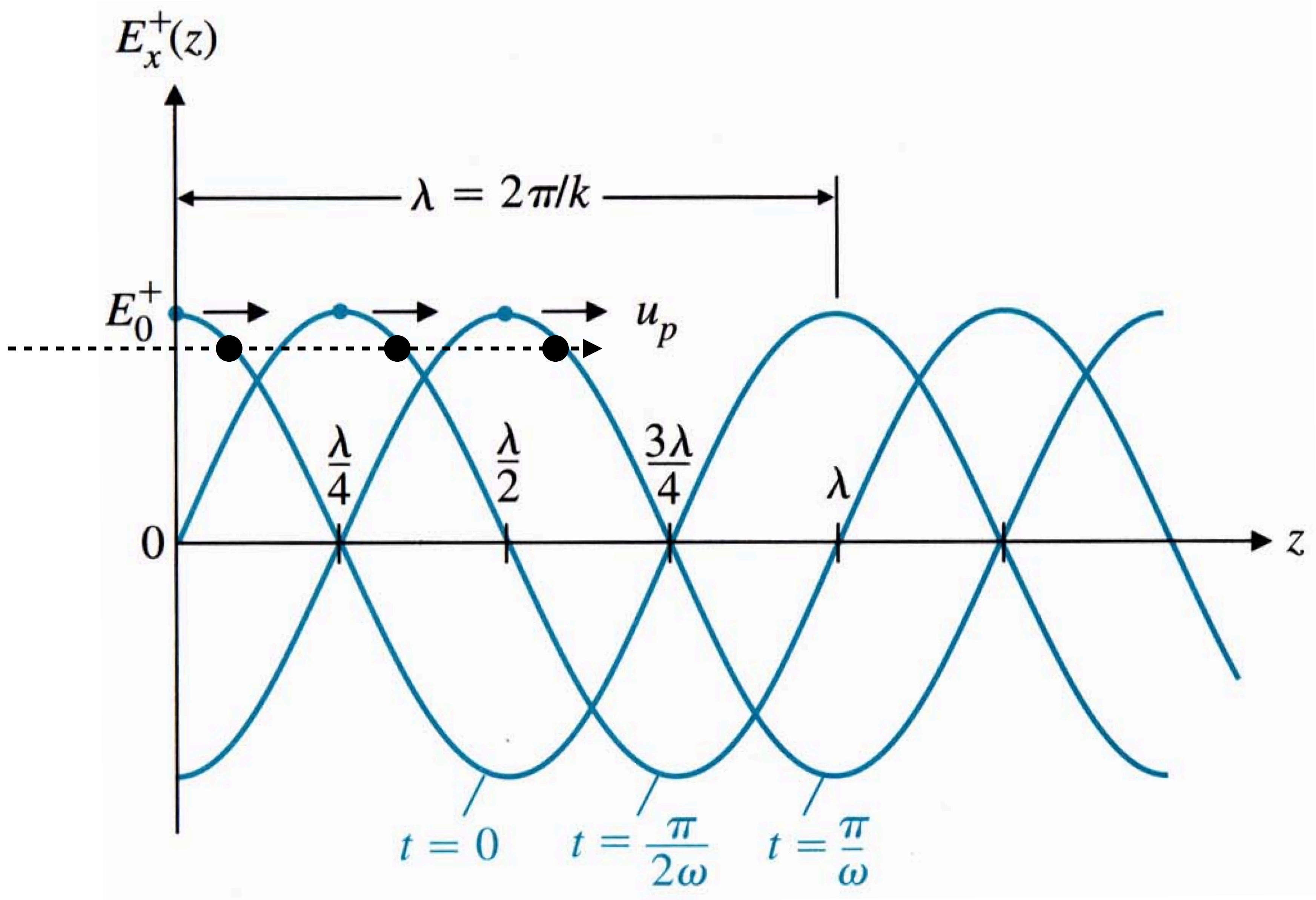
Uniform plane wave in real time (traveling wave)

- ***Phase velocity***

  : velocity of propagation of ***an equi-phase front***
  (= traveling speed of ***the point of particular phase***)

$$E_x(z,t) = E_0 \cos(\omega t - k_0 z)$$

$$\cos(\omega t - k_0 z) = \text{Constant}$$



$$\omega t - k_0 z = \text{Constant} \rightarrow \boxed{u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c}$$

- ***Wavenumber and wavelength***

$$k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0} \quad (\text{rad/m})$$

: the number of waves per unit distance
∴How strongly (many times) the wave oscillates (Strength of the oscillation)

$$\lambda_0 = \frac{c}{f} \quad (\text{m})$$

: How long the wave travels in one oscillation

Plane waves in free space (4/5)

Traveling wave in real time

- *Associated magnetic field*

Since $\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H}$, $\quad \nabla \times \boldsymbol{E} = \begin{vmatrix} \boldsymbol{a_x} & \boldsymbol{a_y} & \boldsymbol{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0\left(\boldsymbol{a_x}H_x + \boldsymbol{a_y}H_y + \boldsymbol{a_z}H_z\right)$

we get $H_x = 0, \quad H_y = \dfrac{1}{-j\omega\mu_0}\dfrac{\partial E_x(z)}{\partial z}, \quad H_z = 0.$

Here, $\quad H_y(z) = \dfrac{1}{-j\omega\mu_0}\dfrac{\partial E_x(z)}{\partial z} = \dfrac{1}{-j\omega\mu_0}\left(-jk_0 E_x(z)\right) \quad \left(\because \dfrac{\partial E_x(z)}{\partial z} = \dfrac{\partial}{\partial z}\left(E_0 e^{-jk_0 z}\right) = -jk_0 E_x(z)\right)$

$$\boxed{H_y(z) = \frac{k_0}{\omega\mu_0}E_x(z) = \frac{1}{\eta_0}E_x(z)}$$

where $\boxed{\eta_0 = \dfrac{\omega\mu_0}{k_0} = \dfrac{\omega\mu_0}{\omega\sqrt{\mu_0\varepsilon_0}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \cong 377\ (\Omega)}$ is *intrinsic impedance of free space.*

Instantaneous expression for magnetic field

$$\boxed{\boldsymbol{H}(z,t) = \boldsymbol{a_y}H_y(z,t) = \boldsymbol{a_y}\,\mathrm{Re}\left[H_y(z)e^{j\omega t}\right] = \boldsymbol{a_y}\frac{E_0}{\eta_0}\cos(\omega t - k_0 z)}$$

Chap. 8 | Plane waves in free space (5/5)

Characteristics of uniform plane wave

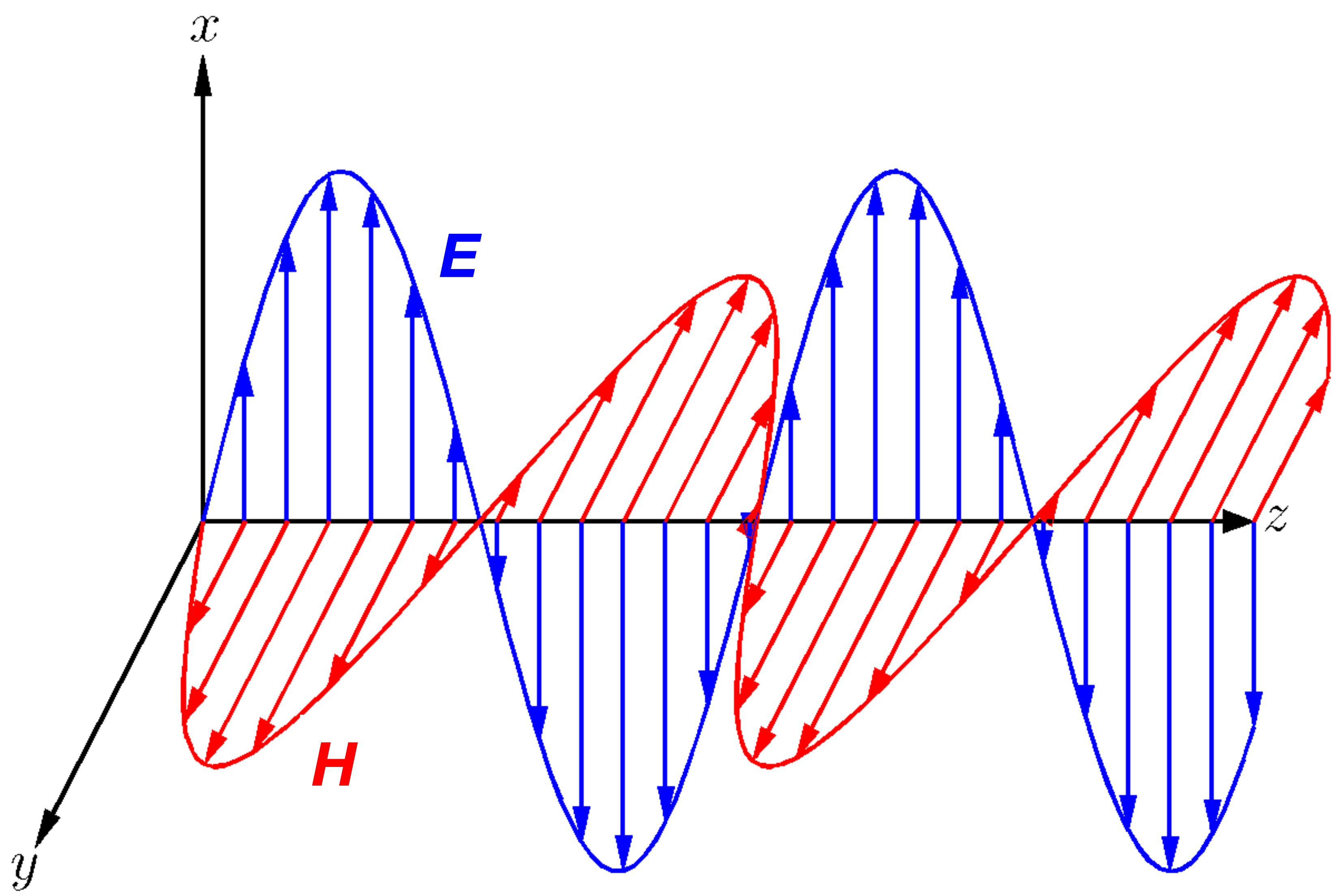
$$\begin{cases} \boldsymbol{E}(z,t) = \boldsymbol{a}_x \operatorname{Re}\left[E_x(z)e^{j\omega t}\right] = \boldsymbol{a}_x E_0 \cos(\omega t - k_0 z) \\ \boldsymbol{H}(z,t) = \boldsymbol{a}_y \operatorname{Re}\left[H_y(z)e^{j\omega t}\right] = \boldsymbol{a}_y \dfrac{E_0}{\eta_0} \cos(\omega t - k_0 z) \end{cases}$$

- ***E***(*z*,*t*) and ***H***(*z*,*t*) are in phase

- The ratio of magnitudes of ***E***- and ***H***-fields = ***intrinsic impedance*** of the medium

$$\frac{\boldsymbol{E}(z,t)}{\boldsymbol{H}(z,t)} \neq \eta_0 = \frac{E_x(z)}{H_x(z)}$$

Should be obtained from Phasors!



Transverse Electromagnetic (TEM) Waves

- Both ***E***(*z*,*t*) and ***H***(*z*,*t*) are ***transverse (or normal)*** to propagation direction (*z*)

- ***E***(*z*,*t*) and ***H***(*z*,*t*) are perpendicular to each other

Img src: inspirehep.net

Chap. 8 | Transverse Electromagnetic Waves (1/4)

Uniform plane wave propagating in "an arbitrary direction"

$$\boxed{\boldsymbol{E}(\boldsymbol{R}) = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{R}} = \boldsymbol{E}_0 e^{-jk\boldsymbol{a}_n\cdot\boldsymbol{R}}}$$

$\boldsymbol{E}(z) = E_0 e^{-jkz}$ *:Uniform plane wave propagating "in the +z-direction"*

where $\boldsymbol{k} = \boldsymbol{a}_x k_x + \boldsymbol{a}_y k_y + \boldsymbol{a}_z k_z = k\boldsymbol{a}_n$ is ***wavenumber vector*** where $k = \sqrt{k_x{}^2 + k_y{}^2 + k_z{}^2}$ and $\boldsymbol{a}_n$ denotes propagation direction.

Here, $\boldsymbol{R} = \boldsymbol{a}_x x + \boldsymbol{a}_y y + \boldsymbol{a}_z z$ is ***position vector***.

- ***E-field vs. propagation direction***

In a source-free region, $\nabla\cdot\boldsymbol{E} = 0$

$$\nabla\cdot\boldsymbol{E} = \nabla\cdot\left(\boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right) = \left(e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right)\cancel{\nabla\cdot\boldsymbol{E}_0} + \boldsymbol{E}_0\cdot\nabla\left(e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right)$$

$$\boxed{\nabla\cdot(f\boldsymbol{A}) = f\nabla\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\nabla f} \qquad \boxed{\nabla\left(e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right) = -j\boldsymbol{k}e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}}$$

$$\nabla\cdot\boldsymbol{E} = -j\left(\boldsymbol{E}_0\cdot\boldsymbol{k}\right)e^{-j\boldsymbol{k}\cdot\boldsymbol{R}} = 0$$

$$\boxed{\therefore \boldsymbol{E}_0\cdot\boldsymbol{k} = \boldsymbol{E}_0\cdot\boldsymbol{a}_n = 0}$$

E-field is transverse (normal) to propagation direction!

Uniform plane wave propagating in "an arbitrary direction"

- ***Associated Magnetic field***

Since $\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}$,

$$\boldsymbol{H}(\boldsymbol{R}) = -\frac{1}{j\omega\mu}\nabla \times \boldsymbol{E}(\boldsymbol{R}) = \frac{1}{\eta}\boldsymbol{a}_n \times \boldsymbol{E}(\boldsymbol{R})$$

$$= \frac{1}{\eta}(\boldsymbol{a}_n \times \boldsymbol{E}_0)e^{-j\boldsymbol{k}\cdot\boldsymbol{R}} = \boldsymbol{H}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}$$

$$\boxed{\nabla \times (f\boldsymbol{A}) = f\cdot(\nabla \times \boldsymbol{A}) + \nabla f \times \boldsymbol{A}}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{R}) = \nabla \times \left(\boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right) = e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\left(\cancel{\nabla \times \boldsymbol{E}_0}\right) - \nabla\left(e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right) \times \boldsymbol{E}_0$$

$$= -j\boldsymbol{k} \times \left(\boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{R}}\right)$$

$$= -jk\boldsymbol{a}_n \times \boldsymbol{E}(\boldsymbol{R}) \quad \text{where } k = \omega\sqrt{\mu\varepsilon}$$

where $\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$ (Ω) *: Intrinsic impedance of the medium*

$$\boxed{\therefore \boldsymbol{H}_0 \cdot \boldsymbol{k} = \boldsymbol{H}_0 \cdot \boldsymbol{a}_n = 0}$$

H-field is also transverse (normal) to propagation direction!

c.f.) $\boxed{\therefore \boldsymbol{E}_0 \cdot \boldsymbol{k} = \boldsymbol{E}_0 \cdot \boldsymbol{a}_n = 0}$

- *A uniform plane wave propagating in $\boldsymbol{a}_n$*

= *Transverse Electromagnetic (TEM) wave such that $\boldsymbol{E} \perp \boldsymbol{H}$ and both $\boldsymbol{E}$ & $\boldsymbol{H}$ are normal to $\boldsymbol{a}_n$*

- ***Relationship between H(R) & E(R)***

$$\boxed{\boldsymbol{E}(\boldsymbol{R}) = -\eta\boldsymbol{a}_n \times \boldsymbol{H}(\boldsymbol{R})}$$

Transverse Electromagnetic Waves (3/4)

Polarization of plane waves

: describing time-varying behavior of the ***E**-field* at a given point in space

e.g.) ***E**-field* of plane wave $\boldsymbol{E} = \boldsymbol{a}_x E_x$: *Linearly polarized* in *x*-direction

* ***H***-field does not need to be specified → ***H***-field can be determined by ***E***-field by $\boldsymbol{H}(\boldsymbol{R}) = \frac{1}{\eta}\boldsymbol{a}_n \times \boldsymbol{E}(\boldsymbol{R})$

Example: Circularly polarized wave

- *Superposition of two linearly-polarized waves*

$$\boldsymbol{E}(z) = \boldsymbol{a}_x E_1(z) + \boldsymbol{a}_y E_2(z)$$

$$= \underbrace{\boldsymbol{a}_x E_{10} e^{-jkz}}_{\text{Polarized in } x\text{-direction}} - \underbrace{\boldsymbol{a}_y jE_{20} e^{-jkz}}_{\text{Polarized in } y\text{-direction, but lagging in time phase by } 90^\circ\ (\pi/2)} \qquad \boxed{-j = e^{-j\frac{\pi}{2}}}$$

- *Instantaneous expression for **E***

$$\boldsymbol{E}(z,t) = \mathrm{Re}\left[\boldsymbol{E}(z)e^{j\omega t}\right] = \mathrm{Re}\left[\left\{\boldsymbol{a}_x E_1(z) + \boldsymbol{a}_y E_2(z)\right\}e^{j\omega t}\right]$$

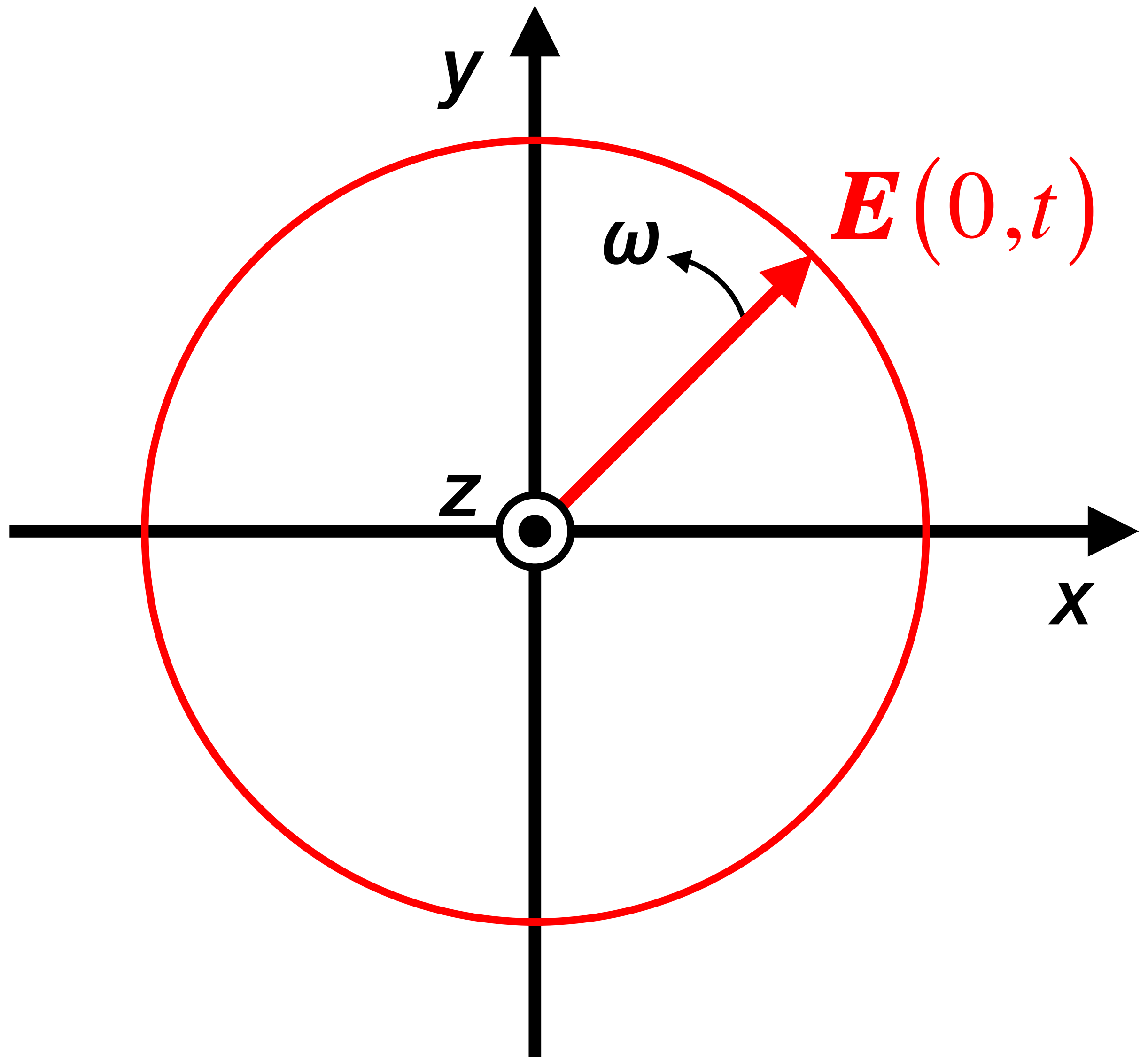
$$= \boldsymbol{a}_x E_{10}\cos(\omega t - kz) + \boldsymbol{a}_y E_{20}\cos\left(\omega t - kz - \frac{\pi}{2}\right)$$

Chap. 8 | Transverse Electromagnetic Waves (4/4)

Example: Circularly polarized wave

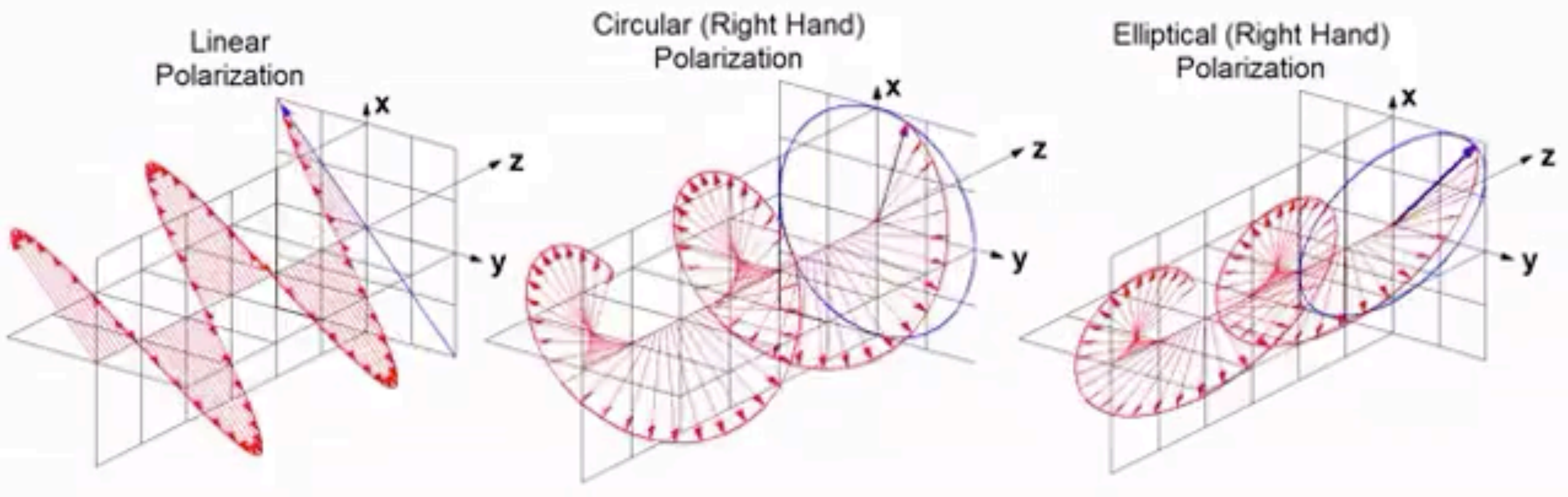
$$\boldsymbol{E}(z,t) = \boldsymbol{a}_x E_{10}\cos(\omega t - kz) + \boldsymbol{a}_y E_{20}\cos\left(\omega t - kz - \frac{\pi}{2}\right)$$

$$\boldsymbol{E}(0,t) = \boldsymbol{a}_x E_{10}\cos\omega t + \boldsymbol{a}_y E_{20}\sin\omega t$$



If $E_{10} = E_{20}$, wave is ***circularly polarized***

If $E_{10} \neq E_{20}$, wave is ***elliptically polarized***



- ***Propagation direction***
  - ***Right-hand*** circularly polarized wave
    - Thumb of the ***right*** hand: propagation direction
    - Fingers of the ***right*** hand: rotation of ***E***

$$\boldsymbol{E}(0,t) = \boldsymbol{a}_x E_{10}\cos\omega t + \boldsymbol{a}_y E_{20}\sin\omega t$$

  - ***left-hand*** circularly polarized wave
    - Thumb of the ***left*** hand: propagation direction
    - Fingers of the ***left*** hand: rotation of ***E***

$$\boldsymbol{E}(0,t) = \boldsymbol{a}_x E_{10}\cos\omega t - \boldsymbol{a}_y E_{20}\sin\omega t$$

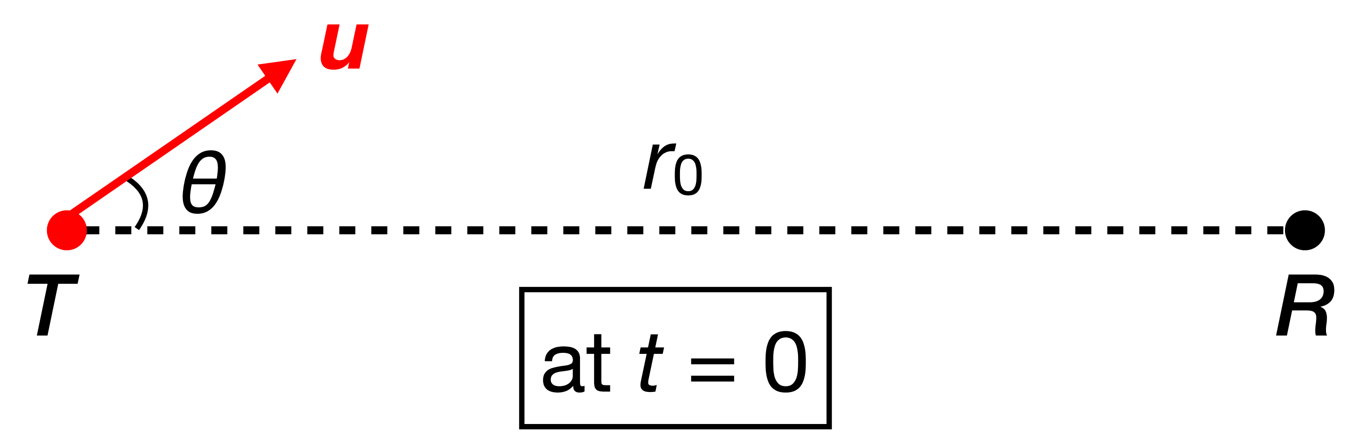
Circularly polarized wave =
*Sum of **TWO** linearly polarized waves in both space and time quadrature*

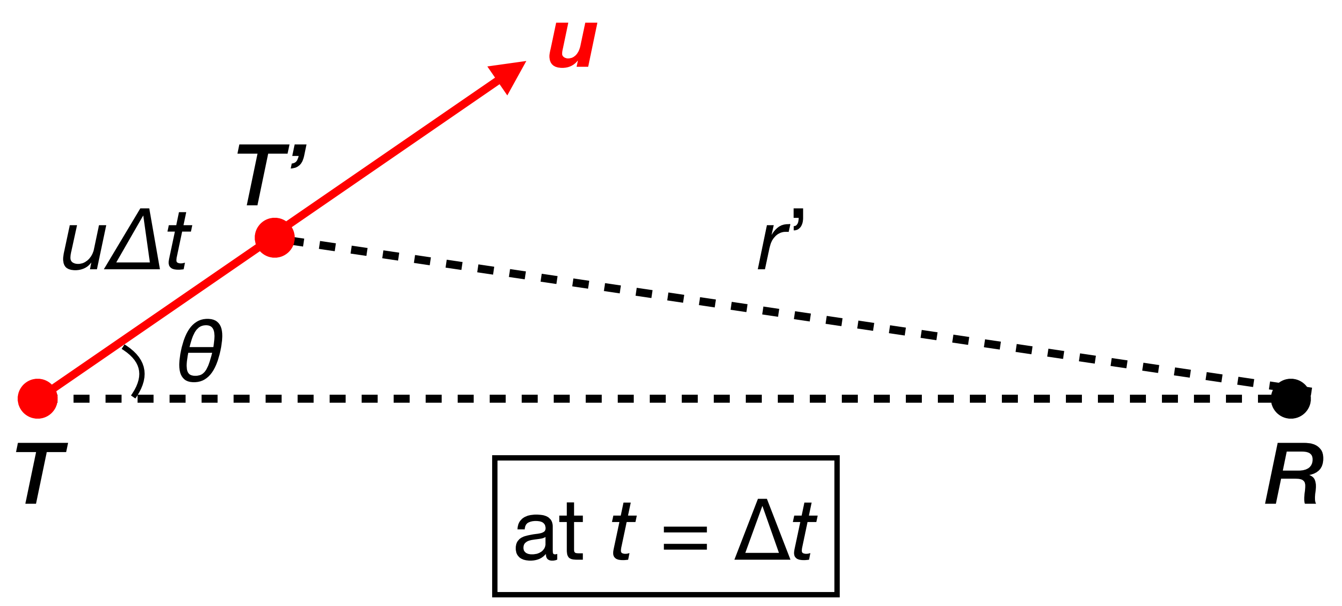
gif src: gfycat.com

Chap. 8 | Doppler Effect (1/2)

Doppler effect

- ***Frequency of the wave sensed by the receiver ≠ Frequency of the wave emitted by the source*** when there is ***relative motion between them***





Time (t_1) that EM wave emitted at t = 0 from ***T*** will reach at ***R***

$$t_1 = \frac{r_0}{c}$$

Time (t_2) that EM wave emitted at t = Δt from ***T'*** will reach at ***R***

$$t_2 = \Delta t + \frac{r'}{c} = \Delta t + \frac{1}{c}\sqrt{{r_0}^2 - 2r_0\left(u\Delta t\right)\cos\theta + \left(u\Delta t\right)^2}$$

$$\cong \Delta t + \frac{r_0}{c}\left(1 - \frac{u\Delta t}{r_0}\cos\theta\right) \qquad \text{if } \left(u\Delta t\right)^2 \ll {r_0}^2$$

- ***Elapsed time at R*** when the second wave arrives after the first wave arrived

$$t_2 - t_1 = \Delta t' = \Delta t\left(1 - \frac{u}{c}\cos\theta\right)$$

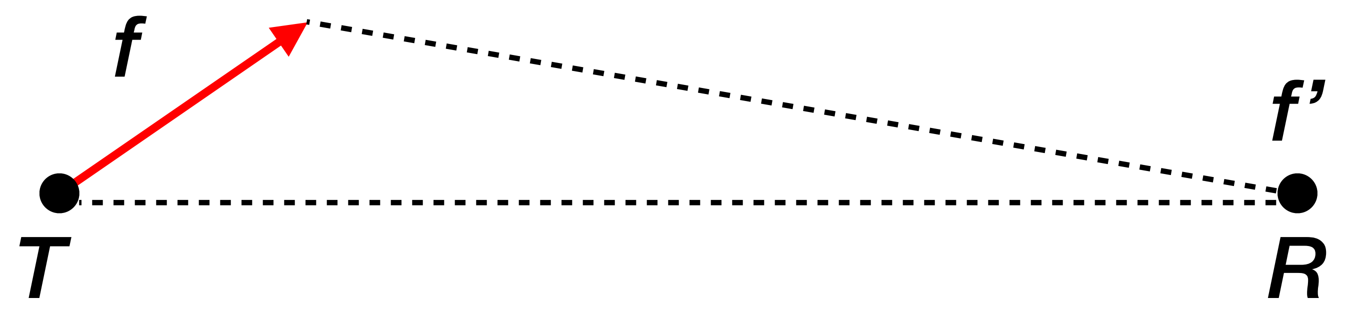
- ***If Δt = 1/f*: a period of the time-harmonic source,**

$$\boxed{f' = \frac{1}{\Delta t'} = \frac{f}{1 - \frac{u}{c}\cos\theta} \cong f\left(1 + \frac{u}{c}\cos\theta\right)} \qquad \text{if } \left(\frac{u}{c}\right)^2 \ll 1$$

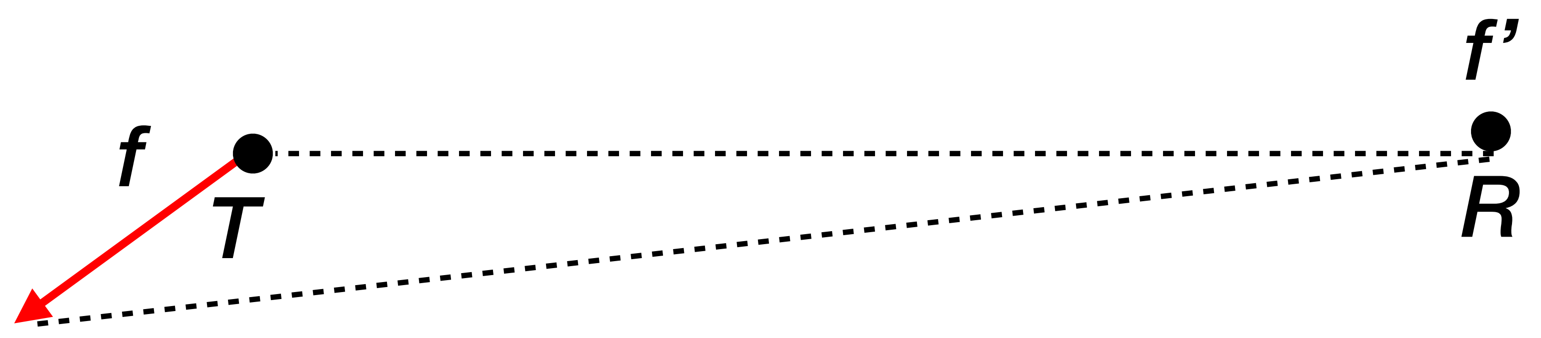
Doppler effect

$$f' = \frac{1}{\Delta t'} = \frac{f}{1-\frac{u}{c}\cos\theta} \cong f\left(1+\frac{u}{c}\cos\theta\right)$$

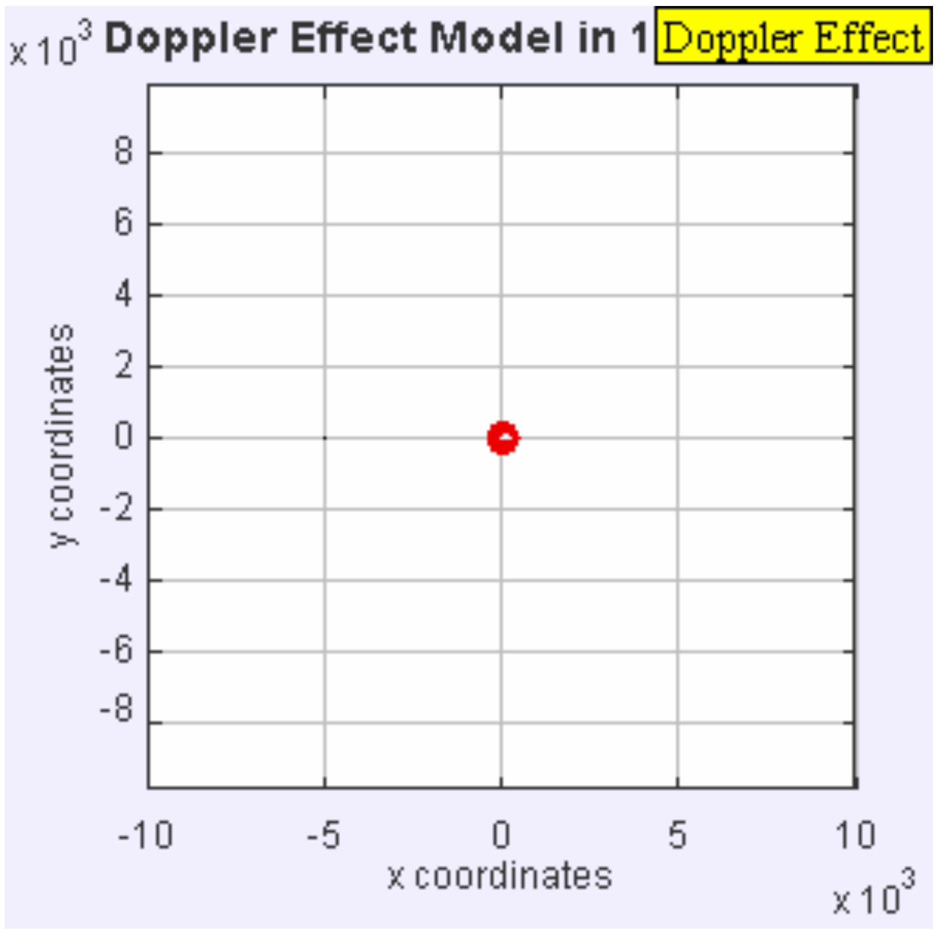
f**: frequency of the transmitted wave from **T
f'**: frequency of the received wave at **R



f' > f
When *T* moves ***toward*** *R*

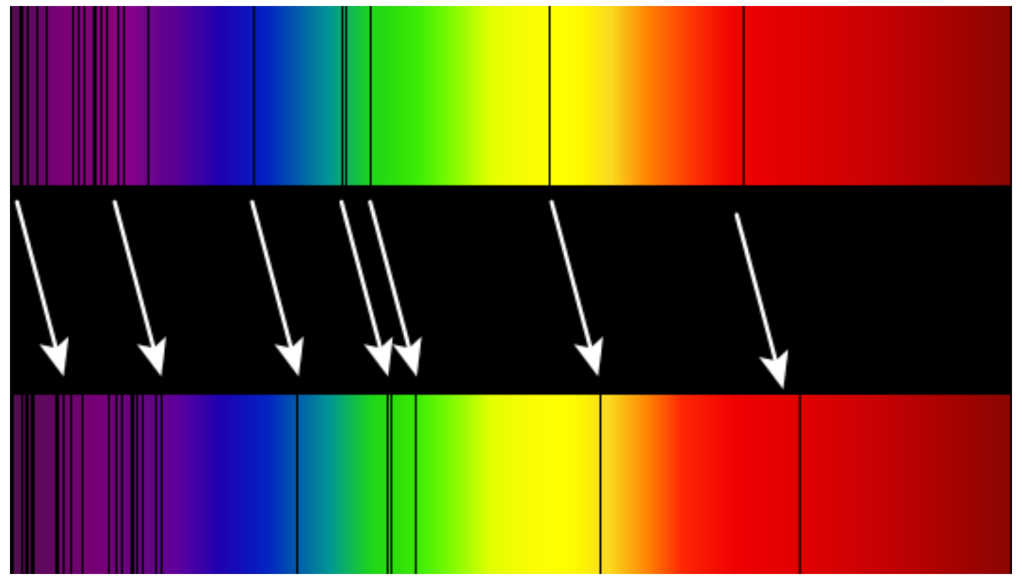


f' < f
When *T* moves ***away*** *R*



Doppler effect simulation

Doppler effect caused by
- ***motion of the source***
- ***motion of the observer***
- ***motion of the medium***



Redshift of spectral lines in the optical spectrum of a distant galaxy (bottom) vs. that of the sun (top)

Doppler effect example

- Police speed gun (HW!)
- Speed measurements for stars or galaxies
  - Approaching stars: blue shift
  - Receding stars: red shift

img, gif src: Wikipedia

Electromagnetics

<Chap. 8> Plane Electromagnetic waves

Section 8.1 ~ 8.4

(2nd class of **week 3**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Chap. 8 | Contents for 2nd class of week 3

Sec 3. Plane waves in “lossy” media

Sec 4. Group velocity

Chap. 8 | Plane waves in source-free "Lossy Media"

Wave equations

$$\nabla^2 \boldsymbol{E} + k_c^{\,2} \boldsymbol{E} = 0$$ where $k_c = \omega\sqrt{\mu\varepsilon_c}$ is a complex wavenumber

Propagation constant γ

$$\boxed{\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} \quad (\mathrm{m}^{-1})}$$ *Conventional notation used in transmission-line theory*

Since complex permittivity is given by $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega}$,

$$\gamma = j\omega\sqrt{\mu\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} = \alpha + j\beta$$

Plane wave in terms of γ

$$\nabla^2 \boldsymbol{E} - \gamma^2 \boldsymbol{E} = 0 \longrightarrow$$ ***Solution***: transverse electromagnetic (TEM) wave ***propagating in z-direction***

$$\boldsymbol{E} = \boldsymbol{a_x} E_x = \boldsymbol{a_x} E_0 e^{-\gamma z}$$ (wave is ***linearly polarized in the x-direction***)

$$E_x = E_0 e^{-\gamma z} = E_0 e^{-\alpha z} e^{-j\beta z}$$

- $e^{-\alpha z}$: Attenuation factor → ***α: Attenuation constant (Np/m)***
- $e^{-j\beta z}$: Phase factor → ***β: phase constant (rad/m)***

Complex permittivity (Review)

Complex permittivity

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \rightarrow \nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon \boldsymbol{E}$$

$$= (\sigma + j\omega\varepsilon)\boldsymbol{E} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\boldsymbol{E} = j\omega\varepsilon_c \boldsymbol{E}$$

where $\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega}$ (F/m) is *complex permittivity*

Physical origin of complex permittivity

- ***Out-of-phase polarization***
  - When external time-varying ***E***-field applied to material bodies → Slight displacements of bound charges (electric dipoles)
  - As frequency of time-varying ***E***-field increases
    - *Inertia of charged particles* resists *against **E**-field*
    - *Inertia of charged particles* prevents dipoles from being in phase with field change → ***Frictional damping***
- ***Ohmic loss***
  - if materials have sufficient amount of free charges

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$ *(representing damping and ohmic losses)*

Chap. 8 | Plane wave in "Low-loss" dielectrics (1/2)

Meaning of low-loss?

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\omega}\right)$$

$$= \varepsilon' - j\varepsilon'' = \varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)$$

σ is non-zero, but small → $\frac{\sigma}{\varepsilon\omega} \ll 1$ or $\frac{\varepsilon''}{\varepsilon'} \ll 1$

Propagation constant

$$\gamma \triangleq jk_c = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{\frac{1}{2}} \cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] = \alpha + j\beta$$

$\because$ *binomial expansion:* $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$

Attenuation constant $\alpha = -j\omega\sqrt{\mu\varepsilon'}\left(j\frac{\varepsilon''}{2\varepsilon'}\right) = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}$ (Np/m)

Phase constant $\beta = \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$ (rad/m)

Plane wave in "Low-loss" dielectrics (2/2)

Intrinsic impedance

$$\eta_c = \frac{E_x(z)}{H_x(z)} = \sqrt{\frac{\mu}{\varepsilon_c}}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-\frac{1}{2}} \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right) \quad (\Omega)$$

Complex Intrinsic Impedance →
Electric and Magnetic fields are NOT in time-phase
c.f.) They are in phase in a lossless medium (η_c is a real number)

Phase velocity

$$u_p = \frac{dz}{dt} = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \quad (\text{m/s}) \qquad \text{c.f.) } u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}$$

for plane wave in "lossless" medium

$$\boxed{\because \beta = \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]}$$

Chap. 8 | Plane wave in "good" conductors (1/2)

Meaning of "good" conductors?

$$\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \quad \sigma \text{ is large} \rightarrow \boxed{\frac{\sigma}{\varepsilon\omega} \gg 1} \longrightarrow \varepsilon_c \cong \varepsilon\left(\frac{\sigma}{j\omega\varepsilon}\right) = \frac{\sigma}{j\omega}$$

Propagation constant

$$\gamma \triangleq jk_c = j\omega\sqrt{\mu\varepsilon_c} \cong j\omega\sqrt{\mu\left(\frac{\sigma}{j\omega}\right)} = \sqrt{j}\sqrt{\omega\mu\sigma} \qquad \boxed{\because \sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}}}$$

$$= \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta$$

$$\boxed{\omega = 2\pi f}$$

$$\boxed{\therefore \alpha = \beta = \sqrt{\pi f\mu\sigma}}$$

Intrinsic impedance

$$\eta \triangleq \sqrt{\frac{\mu}{\varepsilon_c}} \cong \sqrt{\frac{\mu}{\left(\sigma/j\omega\right)}} = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} \quad (\Omega)$$

$$\boxed{\therefore \eta = (1+j)\frac{\alpha}{\sigma} \quad (\Omega)}$$

Phase angle of 45º
(Magnetic field lags behind Electric field by 45º)

Chap. 8 | Plane wave in “good” conductors (2/2)

Phase velocity

$$u_p = \frac{\omega}{\beta} \cong \frac{\omega}{\sqrt{\pi f \mu \sigma}} = 2\sqrt{\frac{\pi f}{\mu \sigma}} \quad \text{(m/s)}$$

Wavelength of a plane wave

$$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f} \cong 2\sqrt{\frac{\pi}{f \mu \sigma}} \quad \text{(m)}$$

Skin depth (Depth of penetration)

$$e^{-\alpha\delta} = e^{-1} \sim 0.368 \qquad \delta = \frac{1}{\alpha}$$

: Distance through which amplitude of wave is attenuated by a factor of e^{-1}

e.g.) For copper where σ = 5.8 x 10^7 (S/m) and μ = 4π x 10^{-7} (H/m),

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}} = 0.038 \ \text{(mm)} \quad \textit{at } f = 3 \text{ (MHz)}$$

$$= 0.66 \ (\mu\text{m}) \quad \textit{at } f = 10 \text{ (GHz)}$$

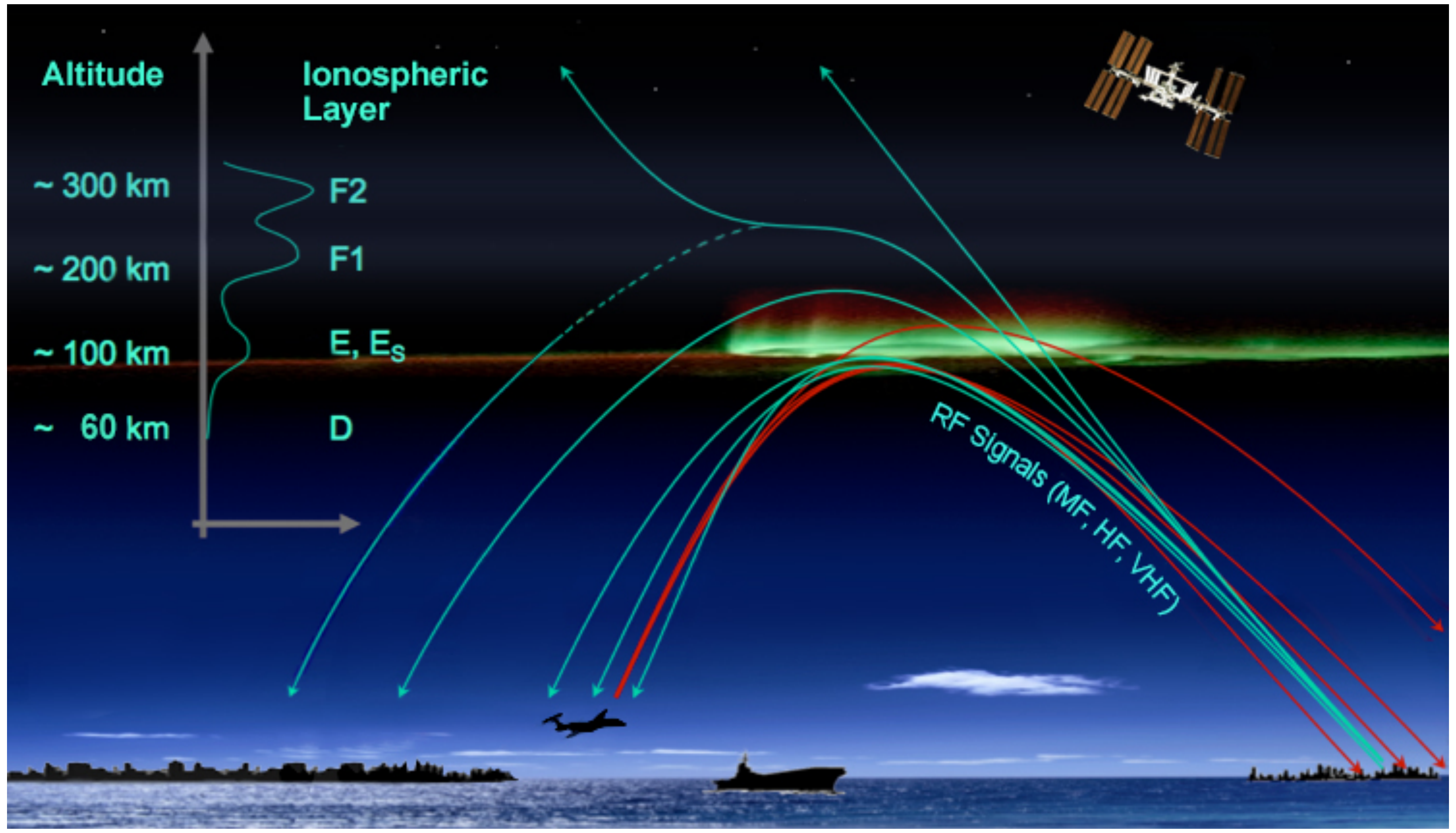
∴ ***At high frequency***, EM wave is ***attenuated very rapidly*** in a good conductor

→ Fields and currents are ***confined in a very thin layer of the conductor surface***

Chap. 8 | Plane wave in ionized gases (1/3)

Ionosphere

Img src: Nasa



Img src: astrosurf.com

- Ionosphere ranges from 60 km (37 mi) ~ 1,000 km (620 mi) altitude
- Ionosphere = free electrons + positive ions (Ionized by solar radiation or cosmic rays)
- Such ***ionized gases*** with ***equal number*** of electrons and ions: ***Plasma***
- Used for long-distance ***radio communication***

Plane wave in ionized gases (2/3)

Simplified model

- Due to lighter mass of electrons, they are more accelerated by E-field than positive ions
- Ionized gases ~ free electron gas, and motion of ions neglected

- An electron (-e) in a time-harmonic electric field (with angular frequency ω)

$$-e\mathbf{E} = m\frac{d^2\mathbf{x}}{dt^2} = -m\omega^2\mathbf{x} \quad \rightarrow \quad \mathbf{x} = \frac{e}{m\omega^2}\mathbf{E}$$

where $\mathbf{x}$ and $\mathbf{E}$ are ***phasors***, and x is ***displacement distance*** from positive ion

- ***Polarization***

$$\mathbf{p} = -e\mathbf{x} \quad \rightarrow \quad \mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$$

: Volume density of electric dipole moment (or polarization vector)
where N is the number of electrons per unit volume

- ***Plasma oscillation frequency***

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\left(1 - \frac{Ne^2}{m\omega^2\varepsilon_0}\right)\mathbf{E} = \varepsilon_0\left(1 - \frac{\omega_p^{\ 2}}{\omega^2}\right)\mathbf{E}$$

where $\boxed{\omega_p = \sqrt{\dfrac{Ne^2}{m\varepsilon_0}} \quad \text{(rad/s)}}$ ***: Plasma angular frequency***

Corresponding plasma frequency

$$f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m\varepsilon_0}} \quad \text{(Hz)}$$

Chap. 8 | Plane wave in ionized gases (3/3)

Plasma oscillation

$$\boldsymbol{D} = \varepsilon_0 \underline{\left(1 - \frac{\omega_p^{\ 2}}{\omega^2}\right)} \boldsymbol{E} = \varepsilon_p \boldsymbol{E} \quad \text{where} \quad \boxed{\varepsilon_p = \varepsilon_0 \left(1 - \frac{\omega_p^{\ 2}}{\omega^2}\right) = \varepsilon_0 \left(1 - \frac{f_p^{\ 2}}{f^2}\right) \quad (\text{F/m})}$$

: *Permittivity of ionosphere (or plasma)*

"Effective" relative permittivity ε_r

- When $f = f_p \quad \rightarrow \quad \varepsilon_p = 0 \quad \rightarrow \quad \boldsymbol{D} = 0,$ although ***E*** still exists.

  Note that ***D*** depends ***only on free charges*** $\left(\because \nabla \cdot \boldsymbol{D} = \rho\right)$ *c.f.)* ***E*** depends on both free charges and polarization charges

$\therefore$ ***At f = f_p, an oscillating E-field exist in the plasma in the absence of free charges → "Plasma oscillation"***

Wave propagation

$$\gamma = j\omega\sqrt{\mu_0 \varepsilon_p} = j\omega\sqrt{\mu_0 \varepsilon_0}\sqrt{1 - \left(\frac{f_p}{f}\right)^2}$$

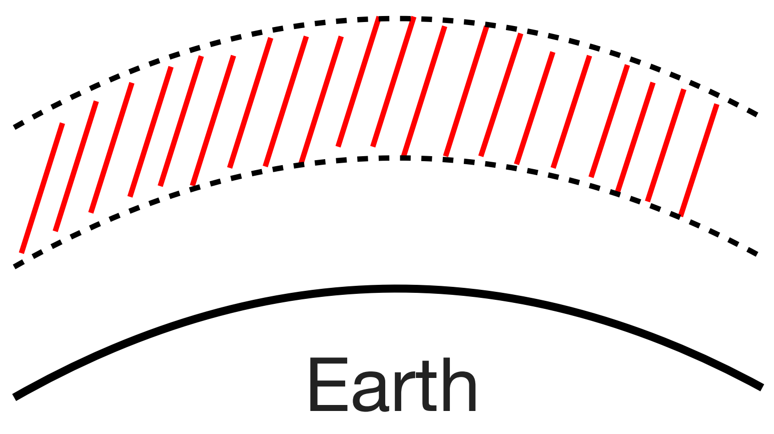
$f < f_p$: γ purely real → A reactive load with ***NO transmission of power***

$\boxed{f_p}$ **: *"Cut-off" frequency***

$f > f_p$: γ purely imaginary → EM wave propagating ***without attenuation***

- Radio communication in ionosphere

$$f_p = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m\varepsilon_0}} \sim 9\sqrt{N} \quad (\text{Hz})$$



N = 10^{12} /m³ → f_p ~ 9 MHz

N = 10^{10} /m³ → f_p ~ 0.9 MHz

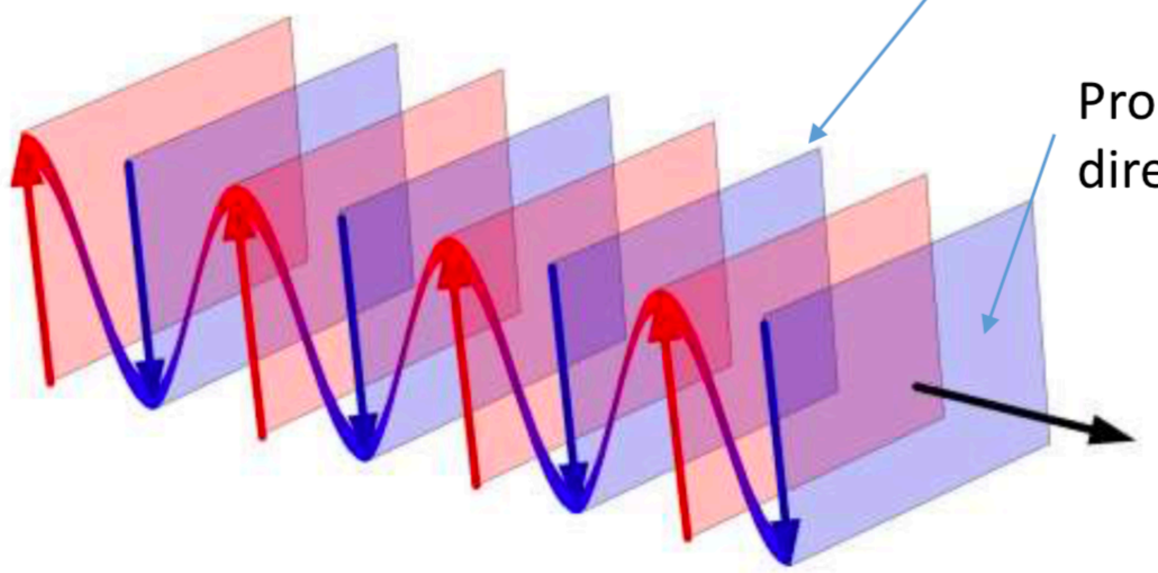
* N at a given altitude vs. time of the day, season, and other factors

* signal should be sent ***at a frequency larger than 9 (MHz)***

Chap. 8 | Group Velocity (1/3)

Phase velocity vs. frequency

$$u_p = \frac{\omega}{\beta} \quad \text{(m/s)}$$: Velocity of propagation of an equi-phase front



<img src: Wikipedia>

- For plane waves in a lossless medium

$$\beta = \omega\sqrt{\mu\varepsilon} \quad \rightarrow \quad u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$ ***Independent of frequency***

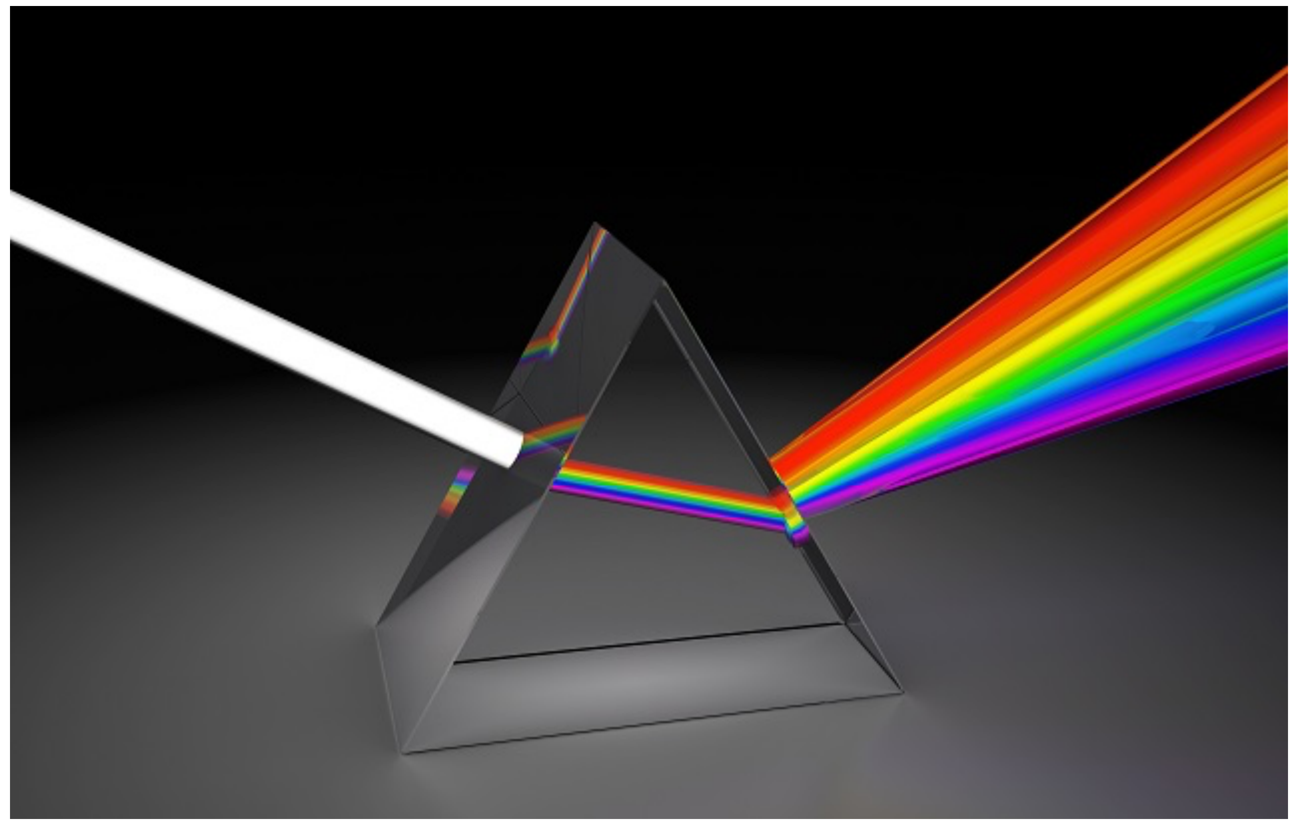
However, for plane waves in a "lossy dielectric", along a "transmission line", or in a "waveguide",

- Phase constant (β) is ***NOT a linear function*** of frequency (ω)
- Waves with ***different ω*** propagate with ***different u_p*** → ***"Distortion"*** of the signal

Dispersion

- The phenomenon of signal distortion caused by ***dependence of u_p vs. ω***
- ***Lossy dielectric = dispersive medium***

- e.g.) Dielectric prism = dispersive medium
  - *Colors are dispersed at the front face
  - *Colors are refracted at different angles
  - *Refractive index different for different colors

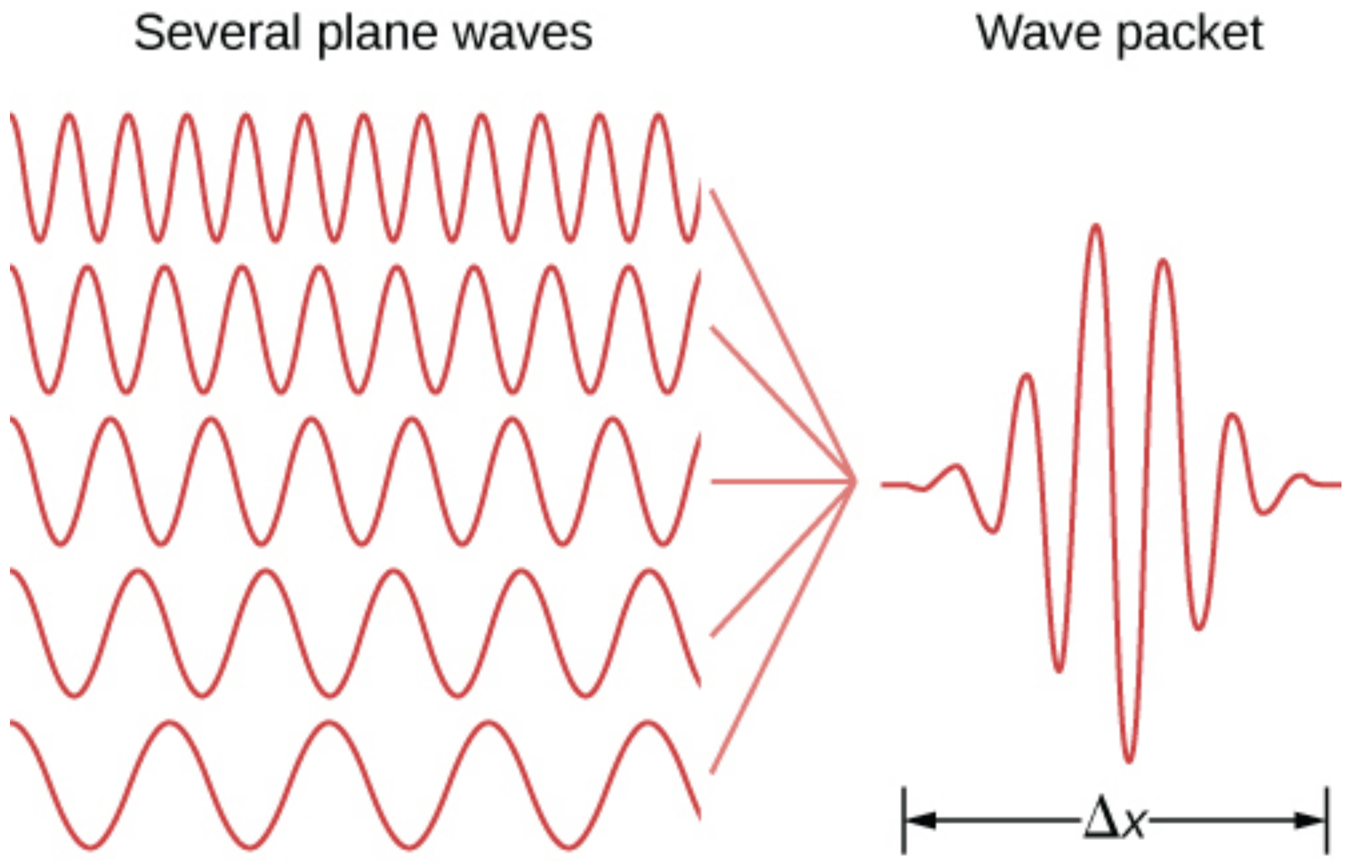

<Img source: AZoOptics>

Chap. 8 | Group Velocity (2/3)

Group velocity

- An information-bearing signal consists of a ***"group of frequencies"***
- There is ***a small spread of frequencies*** ($\Delta\omega$) around the central carrier frequency (ω_0)
- Such a group of frequencies forms a ***"wave packet"***

*Group velocity = velocity of propagation of the **wave packet envelop***



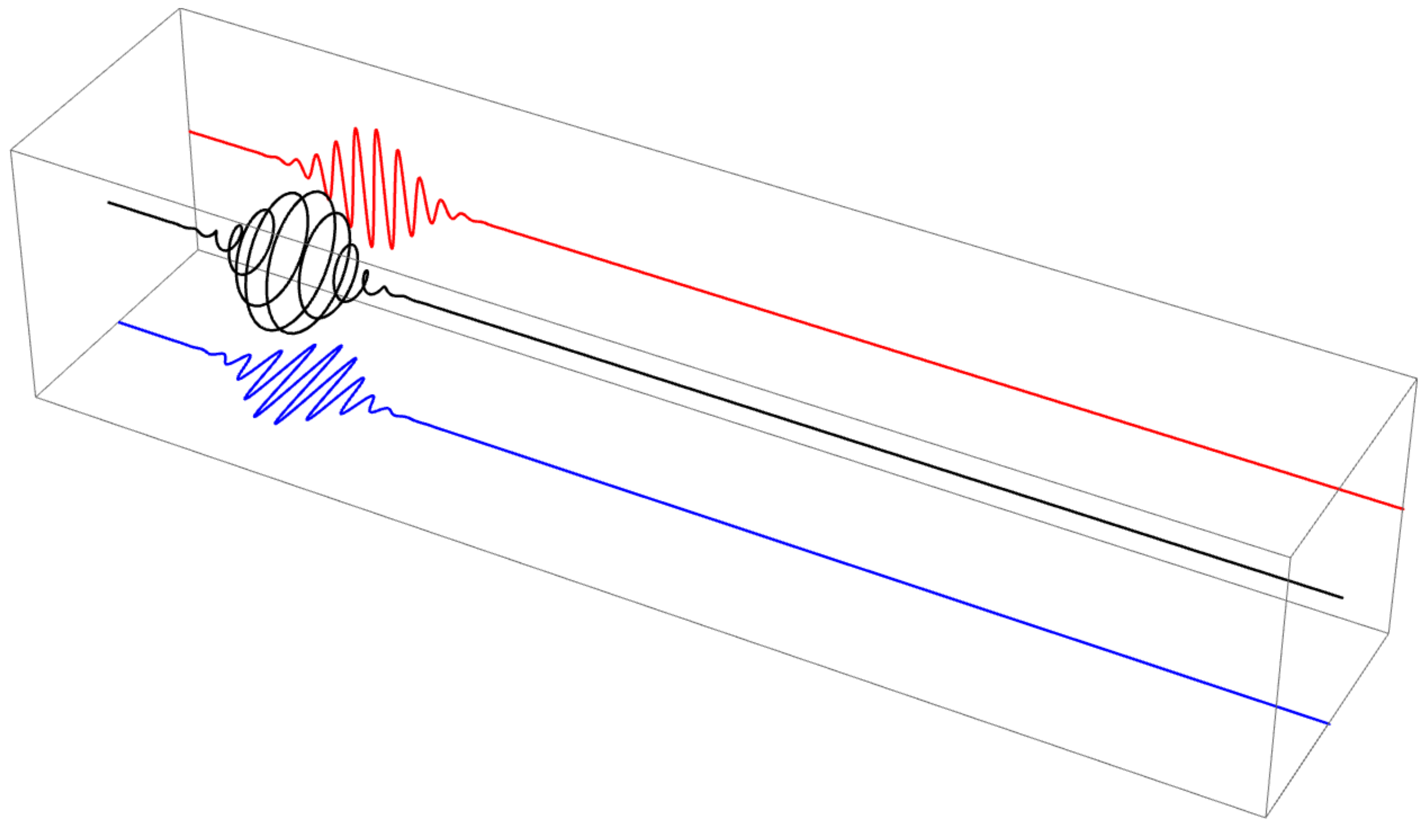
<img source: Physics Libretexts>

Simple example

- Two traveling waves with
  - Equal amplitude (E_0)
  - Slightly different angular frequencies ($\omega_0 \pm \Delta\omega$)
  - Slightly different phase constants ($\beta_0 \pm \Delta\beta$)

$$E(z,t) = E_0 \cos\left[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z\right] + E_0 \cos\left[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z\right]$$

$$= 2E_0 \underbrace{\cos(\Delta\omega t - \Delta\beta z)}_{\text{Wave Packet Envelop}} \cdot \underbrace{\cos(\omega_0 t - \beta_0 z)}_{\text{Waves}}$$



<gif source: GIPHY>

- Phase velocity

$$\omega_0 t - \beta_0 z = \text{Constant} \quad \rightarrow \quad u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}$$

- Group velocity

$$\Delta\omega t - \Delta\beta z = \text{Constant} \quad \rightarrow \quad u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta}$$

in dispersive medium,

$$\boxed{\therefore u_g = \frac{1}{d\beta/d\omega} \quad (\text{m/s})}$$

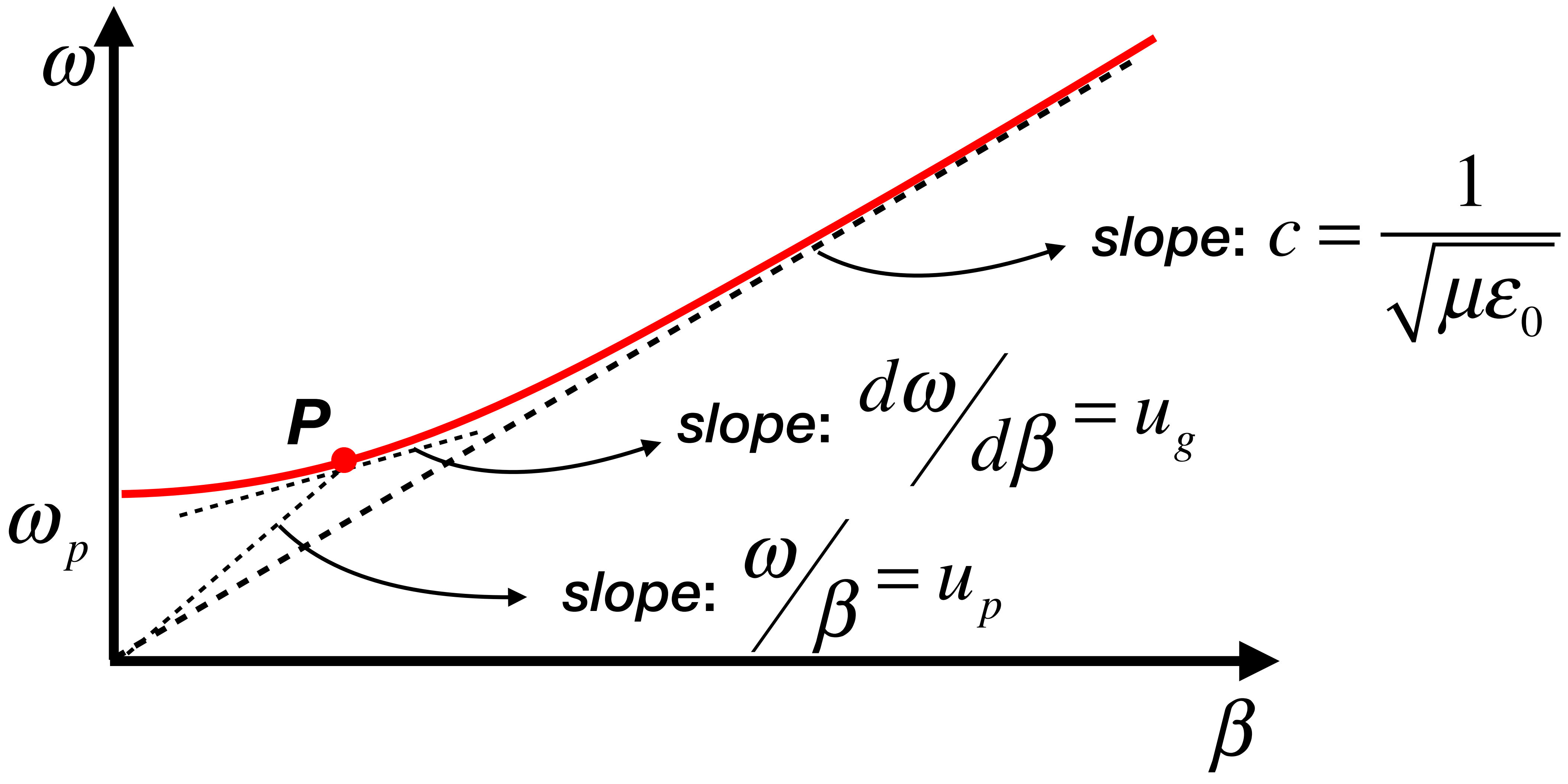
Chap. 8 | Group Velocity (3/3)

β vs. ω relationship (dispersion relationship)

- For an ionized medium (e.g. ionosphere)

$$\gamma \triangleq jk_c = j\omega\sqrt{\mu_0\varepsilon_p} = \alpha + j\beta \quad \rightarrow \quad \beta = \omega\sqrt{\mu_0\varepsilon_p} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

- For $\omega > \omega_p$ (γ purely imaginary = propagating without attenuation)



$$\boxed{\text{Phase velocity: } u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}} \qquad \text{Group velocity: } u_g = \frac{d\omega}{d\beta} = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$$

- $u_p \geq c$, $u_g \leq c$ and $u_p u_g = c^2$ in an ionized medium

Relationship between u_p and u_g

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{u_p}\right) = \frac{1}{u_p} - \frac{\omega}{u_p^{\,2}}\frac{du_p}{d\omega} \quad \rightarrow \quad \boxed{u_g = \frac{1}{d\beta/d\omega} = \frac{u_p}{1 - \frac{\omega}{u_p}\frac{du_p}{d\omega}}}$$

- *No dispersion* $\quad \frac{du_p}{d\omega} = 0 \quad \rightarrow \quad u_p = u_g$
- *Normal dispersion* $\quad \frac{du_p}{d\omega} < 0 \quad \rightarrow \quad u_p > u_g$