Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.1 ~ 10.2

(1st of **week 6**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 10 | Contents for 1st class of week 6

Sec 1. Introduction

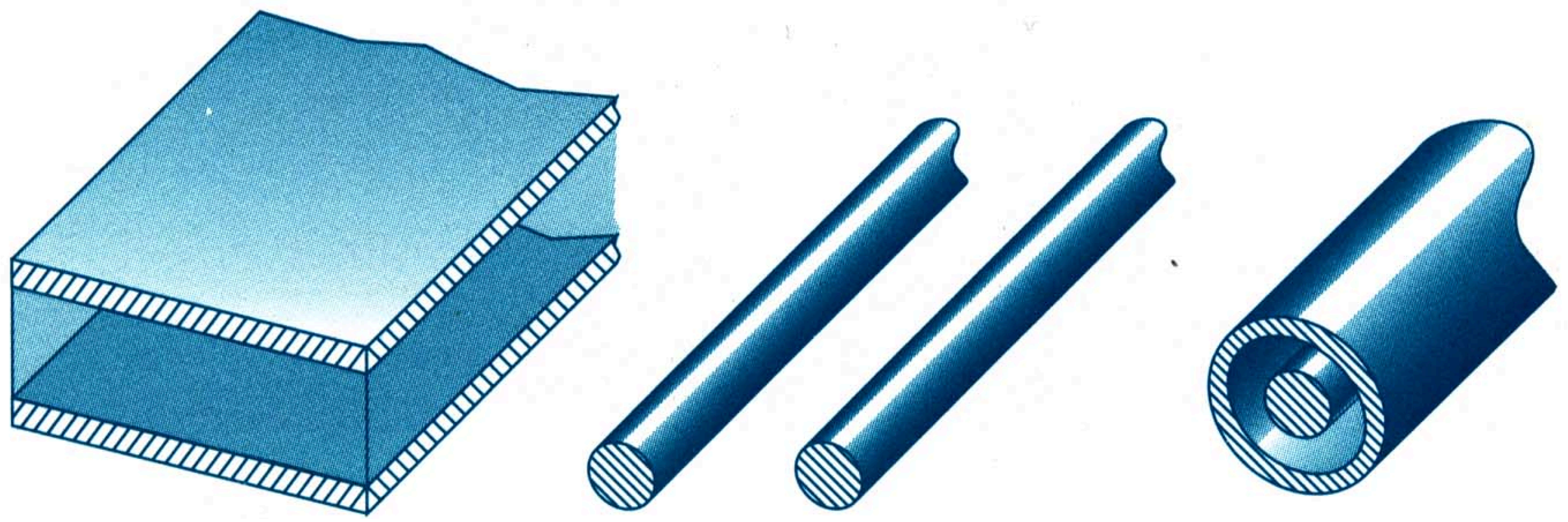
- Difference between Chap. 9 (Transmission lines) and Chap. 10 (Waveguides)
- General wave behaviors (TEM, TE, TM) in the guiding structures
- Various waveguides
  - Parallel-plate, rectangular, circular and dielectric-slab waveguides
- Cavity resonators

Sec 2. General wave behaviors along a uniform guiding structures

- For TEM and TM waves

Chap. 10 | Difference between Chap. 9 and Chap. 10

- In *<Chapter 9> Theory and Applications of Transmission Lines*,
  - Propagation of ***TEM wave*** in parallel-plate, two-wire, and coaxial transmission lines



Parallel plate *Two-wire* *Coaxial*

1. These are ***NOT THE ONLY*** wave-guiding structures (=waveguides)
2. TEM is ***NOT THE ONLY*** mode that these structures can support

- In *<Chapter 10> Waveguides and Cavity resonators,*
  - Propagation of all ***TEM, TM,TE wave*** not only in parallel plate, but also in other waveguides

- ***Attenuation coefficient*** for a general transmission line
  - Low-loss line ($R << \omega L$, $G << \omega L$)

$$\alpha \cong \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

  - Distortionless line ($R/L = G/C$)

$$\alpha = R\sqrt{\frac{C}{L}}$$

$\rightarrow$ $\alpha \propto R \propto \sqrt{f}$

where R results from finite conductivity of the lines

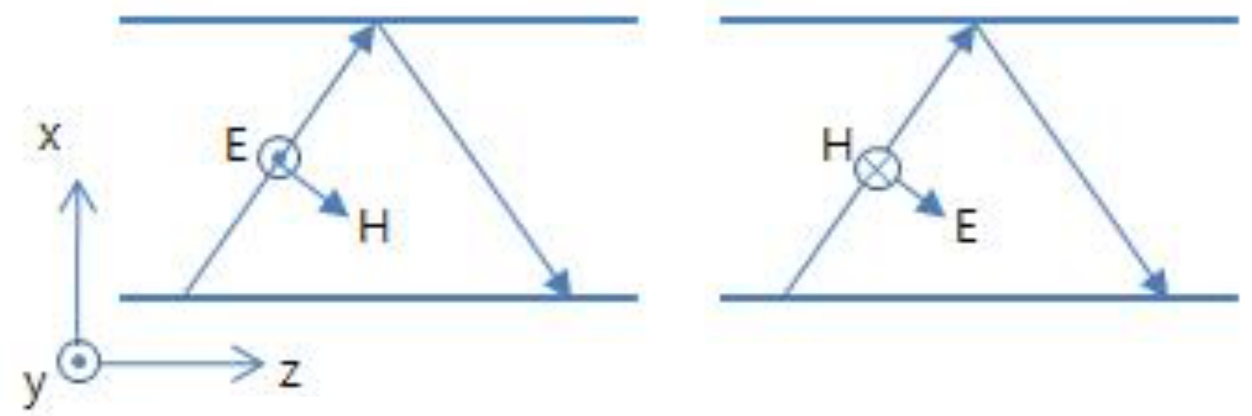
→ ***TEM not supported at microwave range!***

TR Lines	*Parallel plate*	*Two-wire*	*Coaxial*
$R\ (\Omega/m)$	$\frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}}$	$\frac{R_s}{\pi a}$	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$

where $R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$

Chap. 10 | Overview (1/2)

- Section 10.2 – Wave definition and general behavior
  - *TEM* wave: No field components in the propagation direction (Both $\mathbf{E} \perp \mathbf{k}$ and $\mathbf{H} \perp \mathbf{k}$)
  - *TE* wave: Having a longitudinal $\mathbf{H}$-field (Only $\mathbf{E} \perp \mathbf{k}$)
  - *TM* wave: Having a longitudinal $\mathbf{E}$-field (Only $\mathbf{H} \perp \mathbf{k}$)



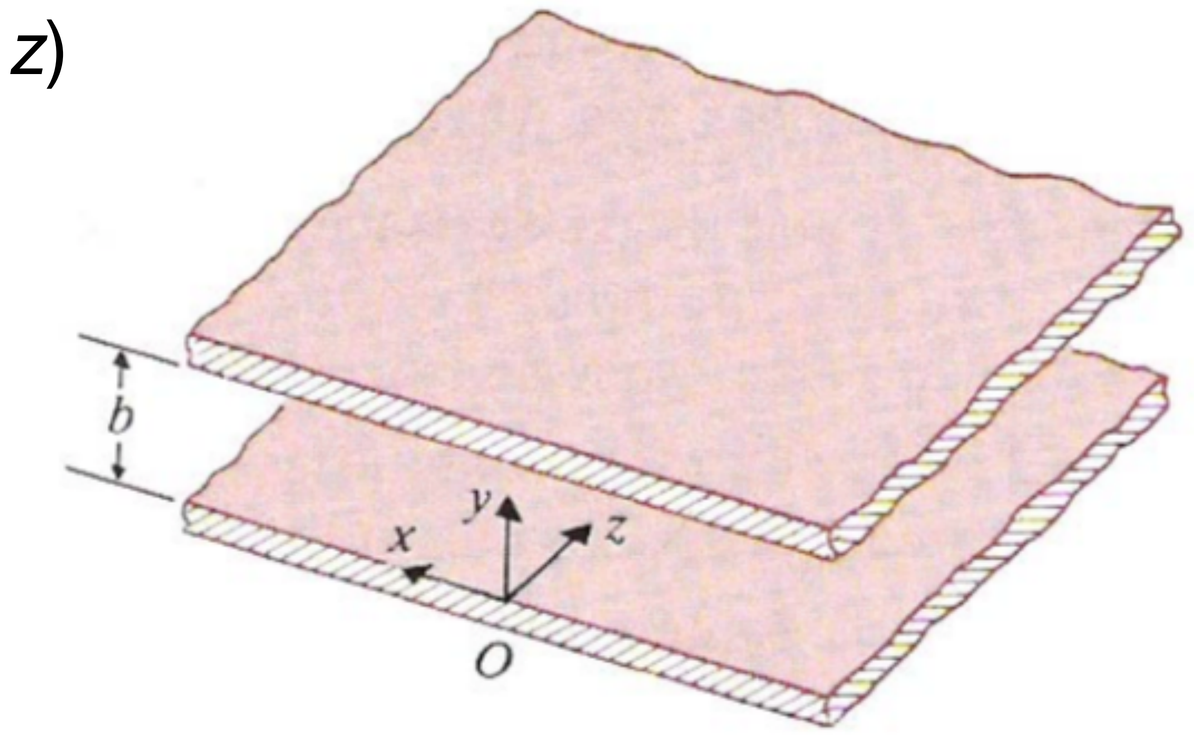
Parallel-plate transmission line

- **Characteristics of TE and TM waves**
  - Have a cut-off frequency (f_c)
  - Power & Signal transmission only possible when $f > f_c$ (→ High-pass filters)

- Section 10.3 – Parallel-plate waveguides with TE & TM modes
  - ***Transverse components*** (i.e., x and y) of the fields expressed in terms ***Longitudinal components*** (i.e., z)
    - For TM modes: $E_x, E_y, H_x, H_y = f(E_z)$ (E_z: longitudinal E-field, $H_z = 0$)
    - For TE modes: $E_x, E_y, H_x, H_y = f(H_z)$ (H_z: longitudinal H-field, $E_z = 0$)

  - ***Attenuation coefficient α due to imperfect conducting walls***
    - Depends on the mode of propagating wave and frequency
    - For TM modes: $f\uparrow \rightarrow \alpha\uparrow$
    - For TE modes: $f\uparrow \rightarrow \alpha\downarrow$

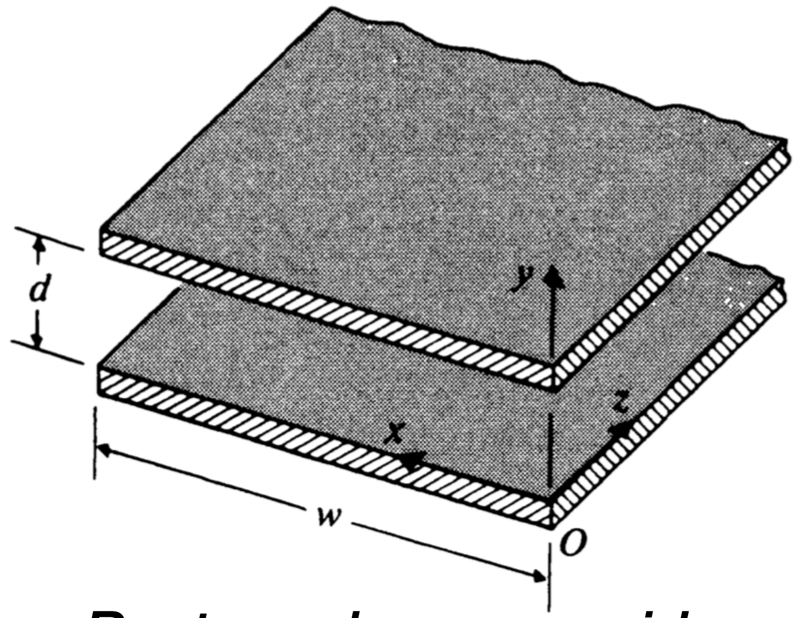
  * *Mode: A wave propagating in the structure with a particular frequency and energy*



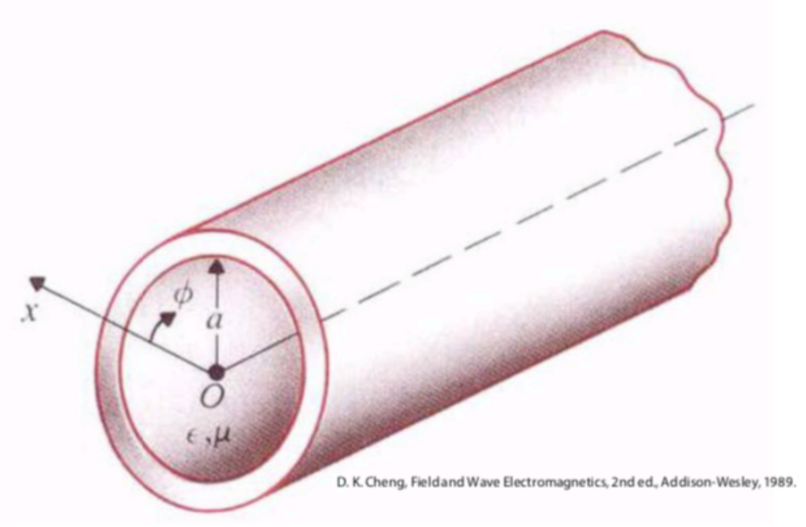
Parallel-plate waveguides

Chap. 10 | Overview (2/2)

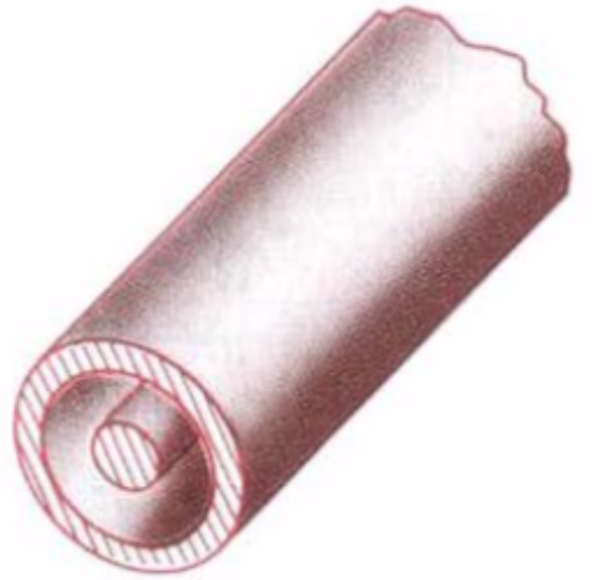
- **Section 10.4 & 10.5 – Hollow metal-pipe of an arbitrary cross-sections**
  - *TEM* CANNOT BE supported in such waveguides!
  - Only *TM* & *TE* waves are possible to pass



Rectangular waveguide
(Only TE and TM)

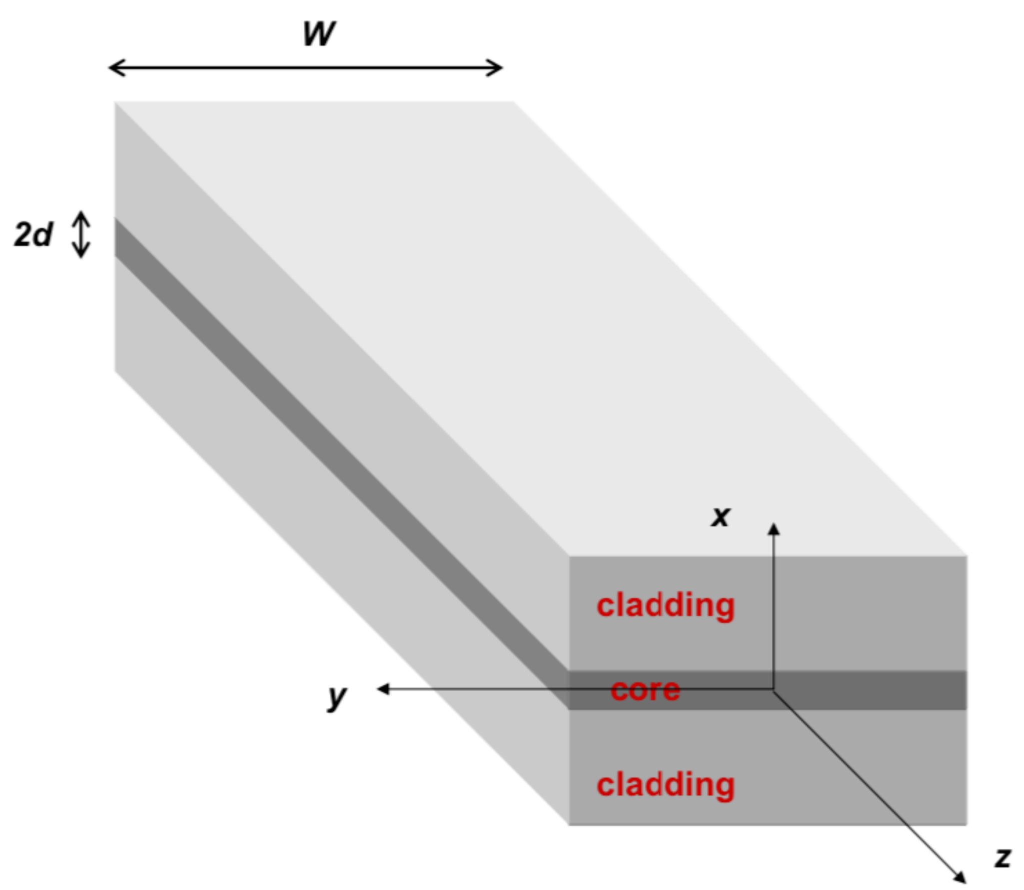


Circular waveguide
(Only TE and TM)

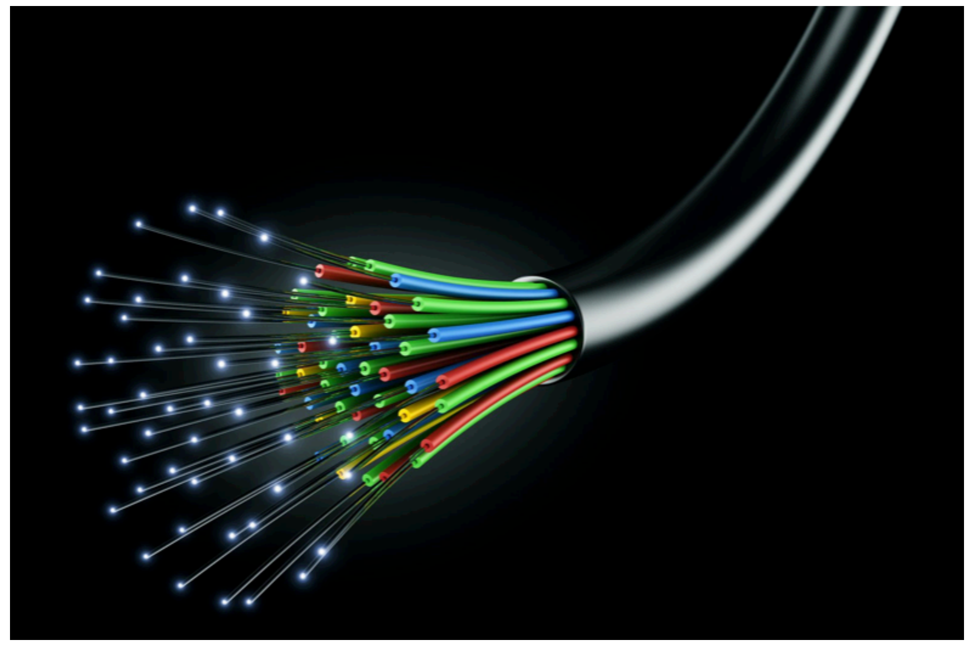


Coaxial transmission line
(All TE, TM, and TEM)

- **Section 10.6: Dielectric slab waveguide**
  - Fields are ***confined within the dielectric*** (core)
  - Fields ***decay rapidly away*** from the slab surface in the transverse plane (cladding)
  - TE & TM waves in dielectric slab wave-guide = "***Surface waves***"

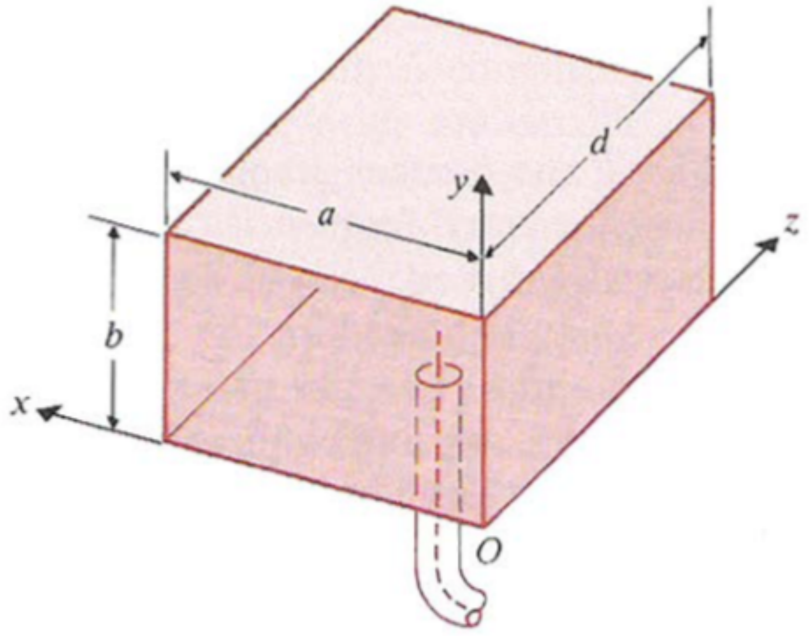


Dielectric slab waveguide
(Img src: Aalto Univ., ELEC-E3240)



Optical fiber
(Img src: tiaonline)

- **Section 10.7: Cavity resonator**
  - Resonator: Device or system exhibiting ***resonance*** (Waves oscillating at ***SOME frequency*** with ***greater amplitude*** than others)
  - A hollow metal box with proper dimension → "Resonant device"
  - Box walls providing large areas for current flow with extremely small losses → Resonance with very high Q-factor
    - e.g. 1) Acoustic resonator in musical instruments
    - e.g. 2) Quartz crystals in radio transmitter
    - e.g. 3) Quartz watches for oscillation of precise frequencies



Metal box resonator

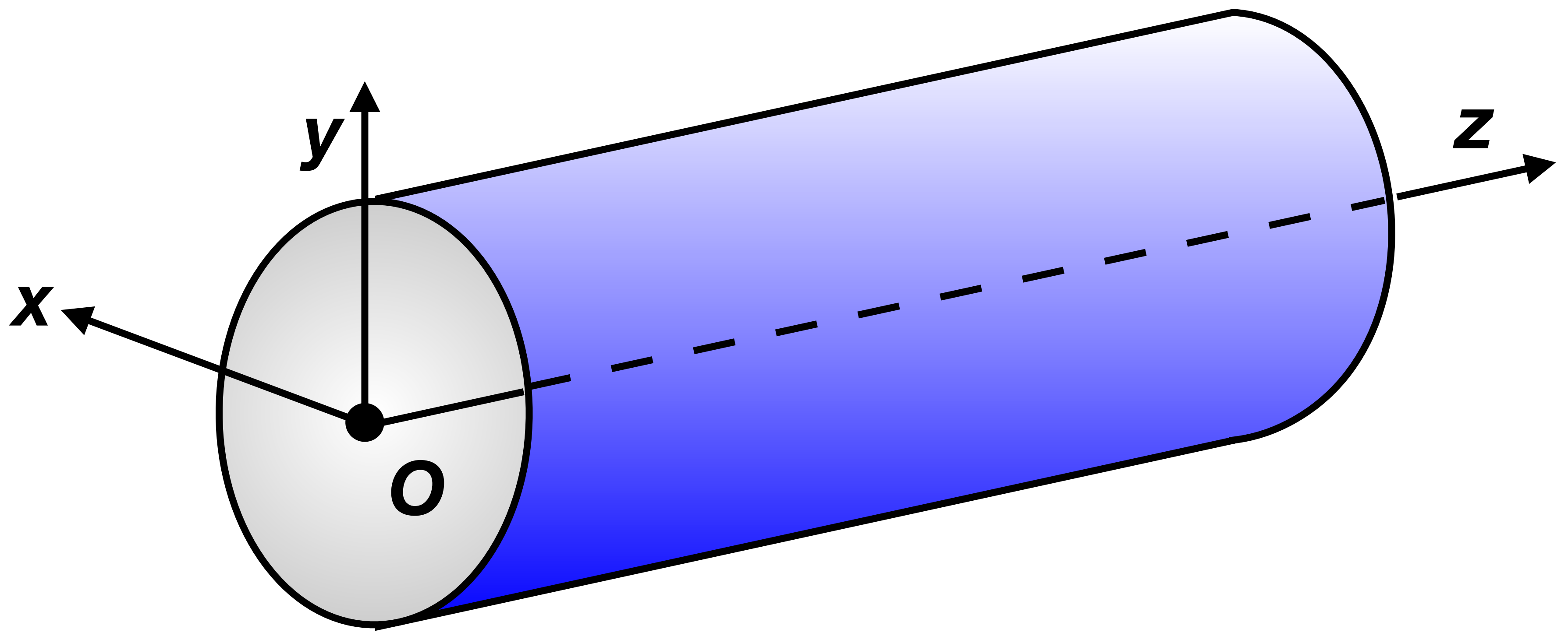
Chap. 10 | Waves within a uniform waveguide

- ***Characteristics for waves*** within a ***uniform dielectric waveguide***

  - Time-harmonic electromagnetic wave

$$\boldsymbol{E}(x,y,z,t)=\mathrm{Re}\left[\boldsymbol{E}(x,y,z)e^{j\omega t}\right] \quad \text{where} \quad \boxed{\boldsymbol{E}(x,y,z)=\boldsymbol{E}^0(x,y)e^{-\gamma z}}$$

Here, $\gamma=\alpha+j\beta$ is a *propagation constant*



<Uniform dielectric waveguide>

- *transverse plane: xy plane*
- *Propagation direction: z*

  - ***Homogenous wave eqns*** in "***charge-free dielectric region***"

$$\boxed{\begin{cases}\nabla^2\boldsymbol{E}+k^2\boldsymbol{E}=0\\ \nabla^2\boldsymbol{H}+k^2\boldsymbol{H}=0\end{cases}}$$

where $k=\omega\sqrt{\mu\varepsilon}$ is the wavenumber

where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$ is Laplacian operator

- Decomposition of Laplacian

$$\nabla^2=\nabla^2_{xy}+\nabla^2_{z}$$

∇^2_z → "Longitudinal" coordinate

∇^2_{xy} → "Cross-sectional" coordinate

$$\rightarrow \nabla^2\boldsymbol{E}=\nabla^2_{xy}\boldsymbol{E}+\nabla^2_{z}\boldsymbol{E}=\nabla^2_{xy}\boldsymbol{E}+\frac{\partial^2\boldsymbol{E}}{\partial z^2}$$

$$=\nabla^2_{xy}\boldsymbol{E}+\gamma^2\boldsymbol{E}$$

- New form of Homogenous wave eqns.

$$\boxed{\begin{cases}\nabla^2_{xy}\boldsymbol{E}+\left(\gamma^2+k^2\right)\boldsymbol{E}=0\\ \nabla^2_{xy}\boldsymbol{H}+\left(\gamma^2+k^2\right)\boldsymbol{H}=0\end{cases}}$$

* Still ***6 equations*** for E_x, E_y, E_z, H_x, H_y, H_z

* but in different notation

: Solution depends on

(i) cross sectional geometry

(ii) cladding-dielectric boundary condition

Transverse & longitudinal fields

- *Inter-relationship among six components*
  - ***E*** and ***H*** components are partly dependent and no need to solve all 6 equations!

- *Assumptions*
  - All the field quantities in the phasor ***depend only on x, y***
  - Only propagation factor $e^{-\gamma z}$ ***depends on z***

$$\begin{cases} \mathbf{E}(x,y,z) = \mathbf{E}^0(x,y)e^{-\gamma z} = \left(\mathbf{a}_x E_x^0 + \mathbf{a}_y E_y^0 + \mathbf{a}_z E_z^0\right)e^{-\gamma z} \\ \mathbf{H}(x,y,z) = \mathbf{H}^0(x,y)e^{-\gamma z} = \left(\mathbf{a}_x H_x^0 + \mathbf{a}_y H_y^0 + \mathbf{a}_z H_z^0\right)e^{-\gamma z} \end{cases}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \longleftarrow \boxed{\textit{Maxwell's Equations}} \longrightarrow \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x^0 e^{-\gamma z} & E_y^0 e^{-\gamma z} & E_z^0 e^{-\gamma z} \end{vmatrix} = -j\omega\mu\left(\mathbf{a}_x H_x^0 + \mathbf{a}_y H_y^0 + \mathbf{a}_z H_z^0\right)e^{-\gamma z}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x^0 e^{-\gamma z} & H_y^0 e^{-\gamma z} & H_z^0 e^{-\gamma z} \end{vmatrix} = j\omega\varepsilon\left(\mathbf{a}_x E_x^0 + \mathbf{a}_y E_y^0 + \mathbf{a}_z E_z^0\right)e^{-\gamma z}$$

$$\begin{cases} \dfrac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0 & \cdots(\mathrm{a}) \\ -\dfrac{\partial E_z^0}{\partial x} - \gamma E_x^0 = -j\omega\mu H_y^0 & \cdots(\mathrm{b}) \\ \dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 & \cdots(\mathrm{c}) \end{cases}$$

Curl Equations

Transverse components (E_x, E_y, H_x, H_y) can be expressed in terms of *longitudinal* components! (E_z, H_z)

$$\begin{cases} \dfrac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\varepsilon E_x^0 & \cdots(\mathrm{e}) \\ -\dfrac{\partial H_z^0}{\partial x} - \gamma H_x^0 = j\omega\varepsilon E_y^0 & \cdots(\mathrm{f}) \\ \dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0 & \cdots(\mathrm{g}) \end{cases}$$

Chap. 10 | General behavior of wave within a guide

• *Transverse (E_x, E_y, H_x, H_y) components in terms of longitudinal (E_z, H_z) components*

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial x} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \end{cases} \quad \begin{cases} H_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

where $h^2 = \gamma^2 + k^2$

i.e., (E_x, E_y, H_x, H_y) are functions of (E_z, H_z)!!

• *Procedures to determine EM wave within a waveguide*

1. Solve the wave eqns for E_z, H_z with given boundary conditions

$$\begin{cases} \nabla_{xy}^2 \boldsymbol{E} + \left(\gamma^2 + k^2\right)\boldsymbol{E} = 0 \\ \nabla_{xy}^2 \boldsymbol{H} + \left(\gamma^2 + k^2\right)\boldsymbol{H} = 0 \end{cases}$$

$\Rightarrow$

2. Substitute E_z, H_z into above eqns to obtain E_x, E_y, H_x, H_y

$$\begin{cases} E_x^0 = \Im\left(E_z^0, H_z^0\right) \\ E_y^0 = \Im\left(E_z^0, H_z^0\right) \end{cases}, \quad \begin{cases} H_x^0 = \Im\left(E_z^0, H_z^0\right) \\ H_y^0 = \Im\left(E_z^0, H_z^0\right) \end{cases}$$

• *Classification of EM wave in terms of (E_z, H_z)*

- *TEM wave*: No longitudinal components, H_z, E_z = 0 (wave in unbounded medium [Chap. 8], wave in transmission lines [Chap. 9])
- *Transverse Magnetic (TM) wave*: H_z = 0, but nonzero E_z
- *Transverse Electric (TE) wave*: E_z = 0, but nonzero H_z

TEM wave within a guide (1/3)

- ***TEM wave within a guide***

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial E_z^0}{\partial x} + j\omega\mu\dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial E_z^0}{\partial y} - j\omega\mu\dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \end{cases} \quad \begin{cases} H_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon\dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon\dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

Since $E_z^0, H_z^0 = 0$ for TEM,

1. Trivial case: All (E_x^0, E_y^0, H_x^0, H_y^0) are zero (meaningless!)
2. Nontrivial case: $\boxed{h^2 = \gamma^2 + k^2 = 0}$

- ***Characteristics of TEM waves within a guide***

- Propagation constant γ_{TEM}

$$h^2 = \gamma_{TEM}{}^2 + k^2 = 0 \quad \rightarrow \quad \boxed{\gamma_{TEM} = jk = j\omega\sqrt{\mu\varepsilon}}$$

* Exactly same as γ for ***uniform plane wave*** in an unbounded medium or in ***lossless transmission lines***!

- Velocity of propagation (phase velocity)

$$\boxed{u_{p(TEM)} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(m/s)}}$$

- Wave impedance

$$\boxed{Z_{TEM} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} = \frac{\gamma_{TEM}}{j\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \quad (\Omega)}$$

In the Curl eqns (b) and (e), set $E_z^0 = 0$ and $H_z^0 = 0$

$$\begin{cases} \dfrac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0 & \cdots(a) \\ -\cancel{\dfrac{\partial E_z^0}{\partial x}} - \gamma E_x^0 = -j\omega\mu H_y^0 & \cdots(b) \\ \dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 & \cdots(c) \end{cases} \quad \begin{cases} \cancel{\dfrac{\partial H_z^0}{\partial y}} + \gamma H_y^0 = j\omega\varepsilon E_x^0 & \cdots(e) \\ -\dfrac{\partial H_z^0}{\partial x} - \gamma H_x^0 = j\omega\varepsilon E_y^0 & \cdots(f) \\ \dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0 & \cdots(g) \end{cases}$$

$$\boxed{\gamma_{TEM} E_x^0 = j\omega\mu H_y^0} \qquad \boxed{\gamma_{TEM} H_y^0 = j\omega\varepsilon E_x^0}$$

Chap. 10 | TEM wave within a guide (2/3)

- ***Characteristics of TEM wave within a guide***

$$\boxed{u_{p(TEM)} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(m/s)} \qquad Z_{TEM} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \quad (\Omega)}$$

Both phase velocity & wave impedance
independent of frequency!

- ***Relationship between E and H via Z_TEM***

$$\begin{cases} \cancel{\dfrac{\partial E_z^0}{\partial y}} + \gamma E_y^0 = -j\omega\mu H_x^0 & \cdots\text{(a)} \\ -\cancel{\dfrac{\partial E_z^0}{\partial x}} - \gamma E_x^0 = -j\omega\mu H_y^0 & \cdots\text{(b)} \\ \dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 & \cdots\text{(c)} \end{cases} \qquad \begin{cases} \cancel{\dfrac{\partial H_z^0}{\partial y}} + \gamma H_y^0 = j\omega\varepsilon E_x^0 & \cdots\text{(e)} \\ -\cancel{\dfrac{\partial H_z^0}{\partial x}} - \gamma H_x^0 = j\omega\varepsilon E_y^0 & \cdots\text{(f)} \\ \dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0 & \cdots\text{(g)} \end{cases}$$

by setting $E_z^0 = 0$ and $H_z^0 = 0$ in the Curl equations (b) and (e),

$$Z_{TEM} \triangleq \frac{E_x^0}{H_y^0} = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega) \;\rightarrow\; H_y^0 = \frac{1}{Z_{TEM}} E_x^0 \quad \cdots(1)$$

by setting $E_z^0 = 0$ and $H_z^0 = 0$ in the Curl equations (a) and (f),

$$\frac{E_y^0}{H_x^0} = -\sqrt{\frac{\mu}{\varepsilon}} = -Z_{TEM} \quad (\Omega) \;\rightarrow\; H_x^0 = -\frac{1}{Z_{TEM}} E_y^0 \quad \cdots(2)$$

by combining (1) and (2), we get

$$\boxed{\boldsymbol{H} = \frac{1}{Z_{TEM}} \boldsymbol{a}_z \times \boldsymbol{E} \quad \text{(A/m)}}$$

c.f.) ***E*** and ***H*** in an unbounded medium

$$\boldsymbol{H} = \frac{1}{\eta} \boldsymbol{a}_z \times \boldsymbol{E} \quad \text{(A/m)}$$

Chap. 10 | TEM wave within a guide (3/3)

- ***“single conductor” waveguide cannot support TEM wave!***
  (see <Fig.1>, <Fig.2>)

- ***Proof of the statement***
  - Suppose that TEM wave exists in such a guide
  - Its ***B*** and ***H*** should form *a closed loop in a transverse plane (xy)*
    $\because \nabla \cdot \boldsymbol{B} = 0$ (*Magnetic flux lines close upon themselves*)

  - According to *Ampere’s circuital law (see <Fig. 4> and <Fig.5>),*

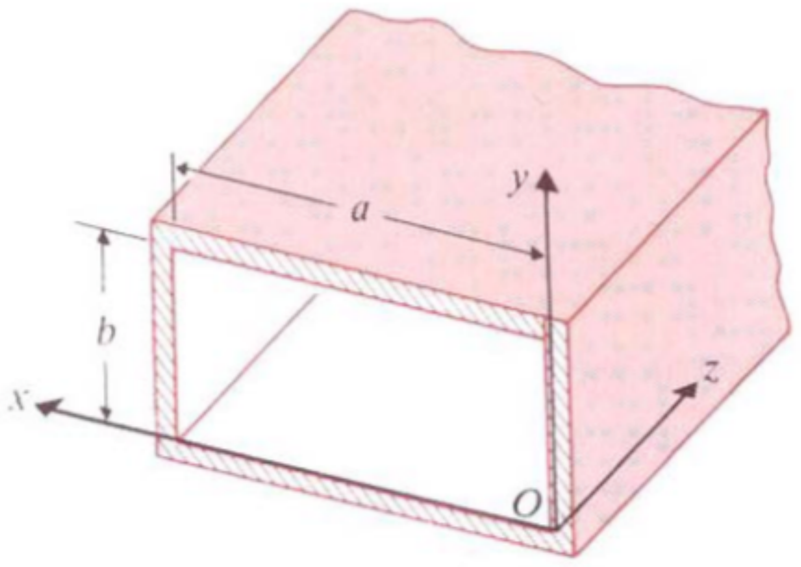
$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$$

*Line integral of **H** around any closed loop C in a transverse plane*

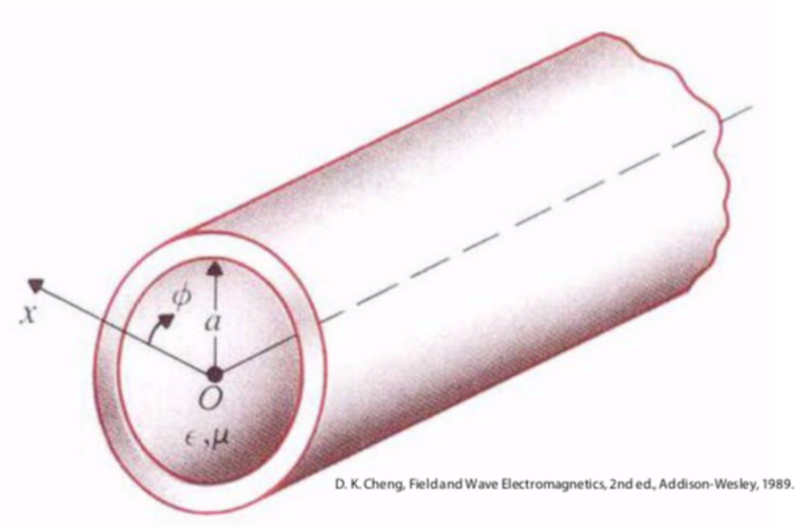
=

Longitudinal conduction & displacement currents through the loop C

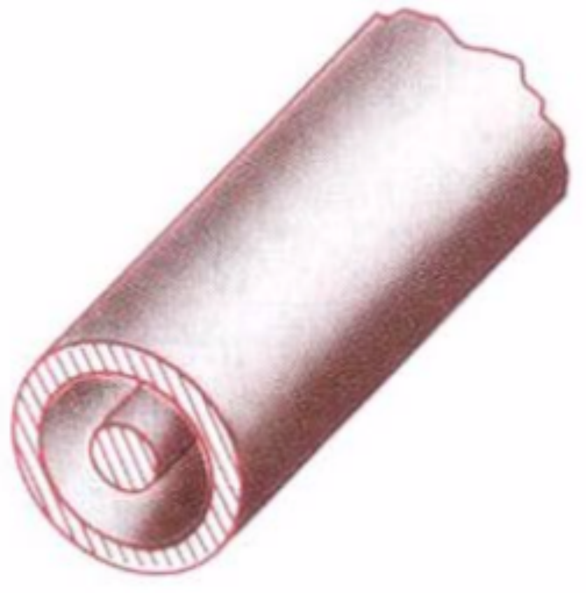
  - By definition, TEM wave does not have longitudinal E_z
    → No longitudinal current (***J*** and *δ**D**/δt*) can flow
    → Thus, ***B*** and ***H*** *do not exist* in a transverse plane
    → Thus, ***E*** and ***D*** also *do not exist* in a transverse plane
    ∴ TEM cannot exist in a single conductor waveguide!



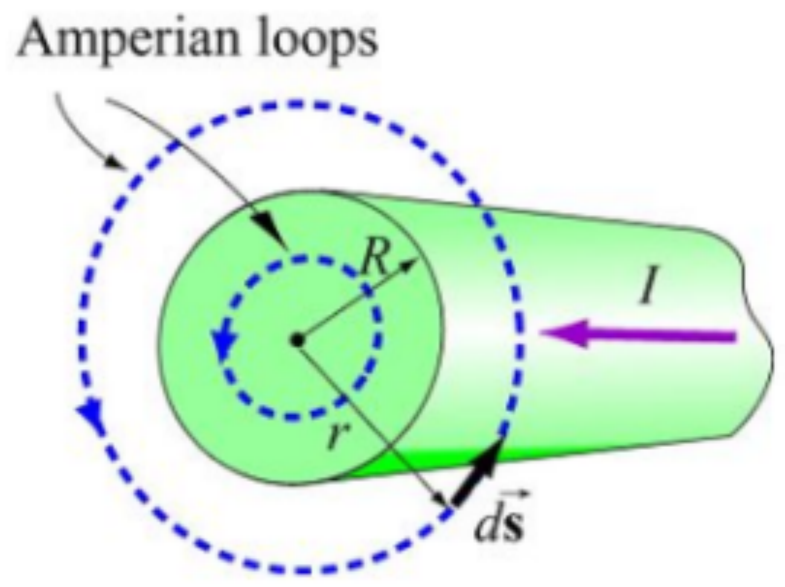
Rectangular waveguide
(Only TE and TM)
<Fig. 1>



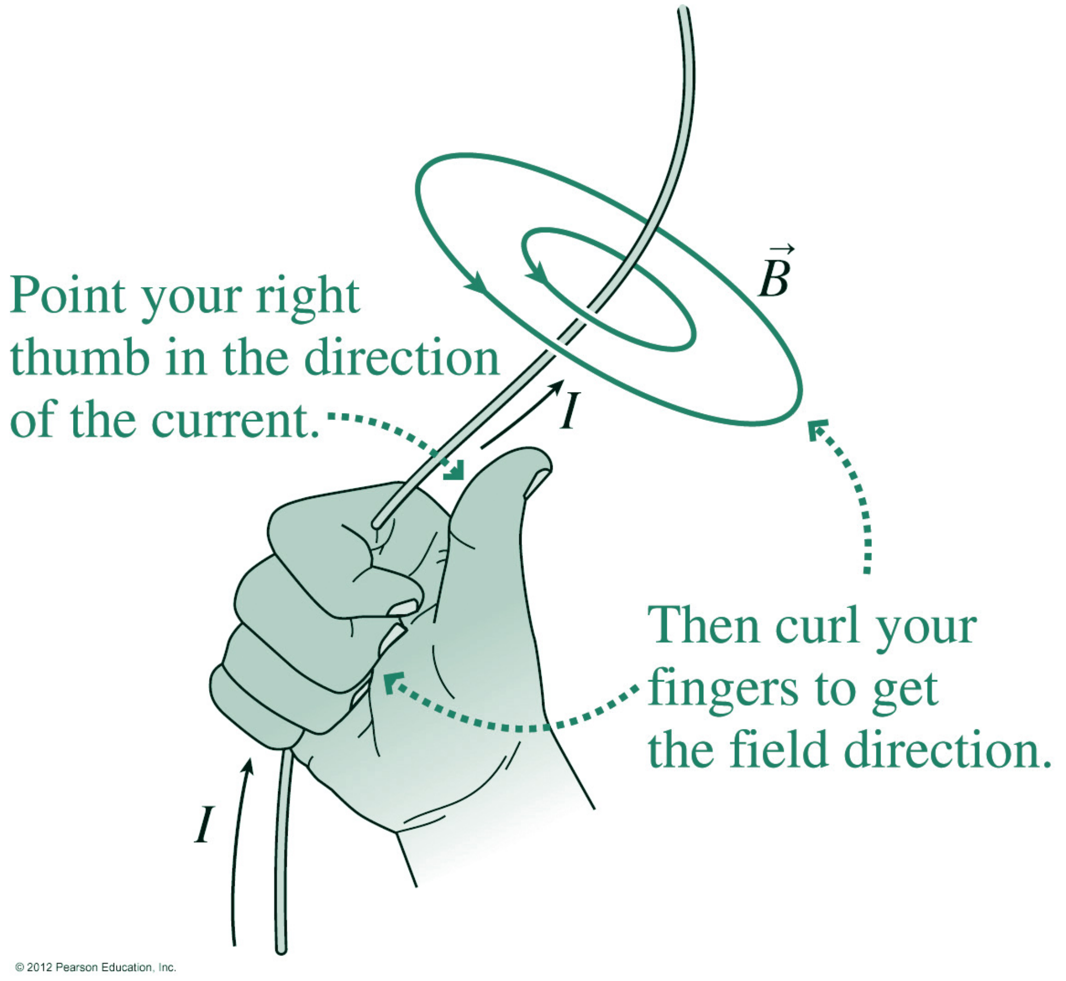
Circular waveguide
(Only TE and TM)
<Fig. 2>



Coaxial transmission line
(All TE, TM, and TEM)
<Fig. 3>



Ampere’s circuital law
<Fig. 4>
(Img src: Toppr)



Ampere’s circuital law
<Fig. 5>
(img src: Pearson Education)

- ***However, if there is an inner conductor as in <Fig. 3>, TEM can be supported!***

Chap. 10 | TM wave within a guide (1/5)

- ***Transverse Magnetic (TM) waves within a uniform guide***
  - ***H***-field = 0 in the propagation direction → $H_z = 0$
  - ***E***-field has a non-zero longitudinal component → $E_z \neq 0$

- ***Procedure to determine the actual field components***
  - Solve the homogeneous equation for longitudinal E_z^0 [***Eqn. (A)***] *with given B.C.*
  - Plug E_z into ***Eqns. (B)*** to find transverse E_x^0, E_y^0, H_x^0, H_y^0
    - Since $H_z = 0$, ***Eqns. (B)*** reduce to ***Eqns. (B')*** as

$$(\boldsymbol{A}) - \left\{ \nabla_{xy}^2 E_z^0 + \left(\gamma^2 + k^2\right) E_z^0 = 0 \right.$$

$$(\boldsymbol{B}) - \begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial x} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

$$(\boldsymbol{B'}) - \begin{cases} E_x^0 = -\dfrac{\gamma}{h^2}\dfrac{\partial E_z^0}{\partial x} & \cdots(1)' \\ E_y^0 = -\dfrac{\gamma}{h^2}\dfrac{\partial E_z^0}{\partial y} & \cdots(2)' \\ H_x^0 = \dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z^0}{\partial y} & \cdots(3)' \\ H_y^0 = -\dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z^0}{\partial x} & \cdots(4)' \end{cases}$$

→ By combining (1)' and (2)', we get

$$\left[\boldsymbol{a_x}E_x^0 + \boldsymbol{a_y}E_y^0 \triangleq \boldsymbol{E}_{TM}^0\right] = \left[-\frac{\gamma}{h^2}\left(\boldsymbol{a_x}\frac{\partial}{\partial x} + \boldsymbol{a_y}\frac{\partial}{\partial y}\right)E_z^0 \triangleq -\frac{\gamma}{h^2}\nabla_T E_z^0\right]$$

$$\boxed{\therefore \boldsymbol{E}_{TM}^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0}$$

where $\boldsymbol{E}_{TM}^0$ is transverse electric field

where $\nabla_T E_z^0$ is the gradient of E_z^0 in the transverse plane

formula for finding E_x^0 and E_y^0 from E_z^0

TM wave within a guide (2/5)

- ***Wave impedance***
  - By combining equations (1)' and (4)', or (2)' and (3)',

$$\boxed{Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma}{j\omega\varepsilon} \quad (\Omega)} \longrightarrow \boxed{\boldsymbol{H} = \frac{1}{Z_{TM}}\left(\boldsymbol{a}_z \times \boldsymbol{E}\right) \quad (\text{A/m})}$$

$$(\boldsymbol{B'}) - \begin{cases} E_x^0 = -\dfrac{\gamma}{h^2}\dfrac{\partial E_z^0}{\partial x} & \cdots(1)' \\ E_y^0 = -\dfrac{\gamma}{h^2}\dfrac{\partial E_z^0}{\partial y} & \cdots(2)' \\ H_x^0 = \dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z^0}{\partial y} & \cdots(3)' \\ H_y^0 = -\dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_z^0}{\partial x} & \cdots(4)' \end{cases}$$

- ***Solutions to homogeneous wave eqn.* (A)**
  - Subject to B.C. of a given waveguide

$$(\boldsymbol{A}) - \left\{ \nabla_{xy}^2 E_z^0 + \left(\gamma^2 + k^2\right) E_z^0 = 0 \right.$$

  - Solution only possible for ***discrete values of h!*** ⟶ ***Characteristic values*** or ***eigenvalues*** of the boundary-value problem
  - ***Each eigenvalue*** determines the ***characteristics of a particular mode*** of the ***given waveguide***
  - Eigenvalues in many cases are ***real numbers***

Will be covered in greater detail with a particular waveguide! (next class)

Chap. 10 | TM wave within a guide (3/5)

• ***Frequency-dependence of TM waves***

- f where [propagation constant (γ) = 0] is given by

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{h^2 - \omega^2 \mu\varepsilon} = 0 \quad \rightarrow \quad \omega_c{}^2 \mu\varepsilon = h^2 \quad \rightarrow \quad \boxed{f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}}$$

: cut-off frequency

(depending on the eigenvalue)

$$\boxed{\therefore \gamma = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}}$$

• ***When f > f_c → γ: purely imaginary***

$$\gamma = j\beta = jk\sqrt{1 - \left(\frac{h}{k}\right)^2} = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

→ propagating mode with $\beta = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$ (rad/m)

- Corresponding wavelength *in the guide*

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - \left(f_c/f\right)^2}} > \lambda,$$ where $\lambda = \frac{2\pi}{k} = \frac{1}{f\sqrt{\mu\varepsilon}} = \frac{u}{f}$ is a wavelength of a plane wave *in an unbounded dielectric medium (μ, ε)*

where u is the velocity of light *in that medium*

Chap. 10 | TM wave within a guide (4/5)

- ***When f > f_c → γ: purely imaginary***

  - Phase velocity of the propagating wave in the guide

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} > u$$

→ (1) u_p ***within a waveguide*** is always higher than *u* ***in an unbounded medium***

(2) u_p is ***frequency-dependent***

→ Waveguide for TM = *dispersive system*

  - Group velocity

$$u_g = \frac{1}{d\beta/d\omega} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2} < u$$

$$\boxed{\therefore u^2 = u_p u_g}$$

Group velocity in a lossless medium

= ***Velocity of signal propagation*** (or ***energy transport***) [will be discussed in next class]

  - Wave impedance Z_{TM}

By plugging $\gamma = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}$ into $Z_{TM} = \frac{\gamma}{j\omega\varepsilon}$,

$$\boxed{\therefore Z_{TM} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

(1) Purely ***resistive***

(2) Always smaller than intrinsic impedance of the dielectric

Chap. 10 | TM wave within a guide (5/5)

- ***When $f < f_c$ → γ: real***

  - Propagation constant (γ)

$$\gamma = \alpha = h\sqrt{1-\left(\frac{f}{f_c}\right)^2}$$ ***: Attenuation constant*** → Wave propagating with $e^{-\gamma z} = e^{-\alpha z}$ (Rapidly decaying with z → *Evanescent mode*)

  - Wave with $f < f_c$ attenuates
  - Wave with $f > f_c$ propagates with β

→ a waveguide for TM wave acts like a ***high-pass filter***!

  - Wave impedance Z_{TM}

By plugging $\gamma = h\sqrt{1-\left(\frac{f_c}{f}\right)^2}$ into $Z_{TM} = \frac{\gamma}{j\omega\varepsilon}$,

$$\boxed{\therefore Z_{TM} = -j\frac{h}{\omega\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Purely reactive below cutoff frequency

→ ***E*** and ***H*** in a phase quadrature

→ No power flow associated with such an evanescent wave because

$$P_{av} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}\times\boldsymbol{H}^*\right) = 0$$

Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.1 ~ 10.2

(1st of **week 6**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

Chap. 10 | Contents for 2nd class of week 6

Review of the last class

- General wave behavior along a uniform dielectric guide

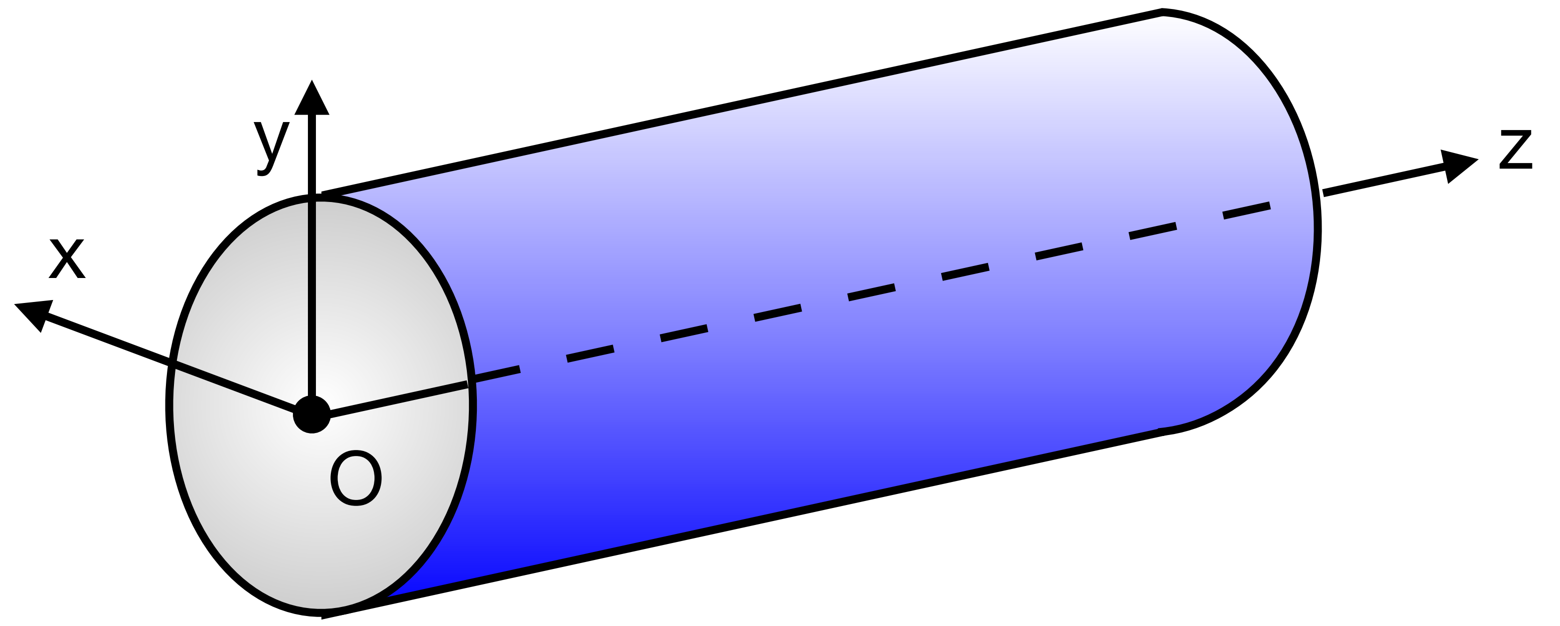
Sec 2. General wave behaviors along a uniform guiding structures (Cont'd.)

- TE and TM wave characteristics, commonality and difference

Chap. 10 | General wave behavior within a uniform dielectric guide (1/2)

- ***Propagating waves*** along a ***uniform dielectric waveguide***

$$\begin{cases} \mathbf{E}(x,y,z) = \mathbf{E}^0(x,y)e^{-\gamma z} = \left[\mathbf{a}_x E_x^0(x,y) + \mathbf{a}_y E_y^0(x,y) + \mathbf{a}_z E_z^0(x,y)\right]e^{-\gamma z} \\ \mathbf{H}(x,y,z) = \mathbf{H}^0(x,y)e^{-\gamma z} = \left[\mathbf{a}_x H_x^0(x,y) + \mathbf{a}_y H_y^0(x,y) + \mathbf{a}_z H_z^0(x,y)\right]e^{-\gamma z} \end{cases}$$



<Uniform dielectric waveguide>

- *Uniform cross-section & composition*
- *transverse plane: xy plane*
- *Propagation direction: z*

$$\boxed{\begin{cases} \nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \\ \nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0 \end{cases}} \Leftrightarrow \boxed{\begin{cases} \nabla_{xy}^2 \mathbf{E} + (\gamma^2 + k^2)\mathbf{E} = 0 \\ \nabla_{xy}^2 \mathbf{H} + (\gamma^2 + k^2)\mathbf{H} = 0 \end{cases}}$$

We have to solve "source-free" Non-homogeneous wave equations!

Maxwell's Equations

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}$$

$$\begin{cases} \dfrac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0 & \cdots(\text{a}) \\ -\dfrac{\partial E_z^0}{\partial x} - \gamma E_x^0 = -j\omega\mu H_y^0 & \cdots(\text{b}) \\ \dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0 & \cdots(\text{c}) \end{cases}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E}$$

$$\begin{cases} \dfrac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\varepsilon E_x^0 & \cdots(\text{e}) \\ -\dfrac{\partial H_z^0}{\partial x} - \gamma H_x^0 = j\omega\varepsilon E_y^0 & \cdots(\text{f}) \\ \dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\varepsilon E_z^0 & \cdots(\text{g}) \end{cases}$$

Transverse components
(E_x, E_y, H_x, H_y)
can be expressed in terms of
longitudinal components!
(E_z, H_z)

 General wave behavior within a uniform dielectric guide (2/2)

- *Transverse (E_x, E_y, H_x, H_y) components in terms of* ***longitudinal*** *(E_z, H_z) components*

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial E_z^0}{\partial x} + j\omega\mu\dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial E_z^0}{\partial y} - j\omega\mu\dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \end{cases} \quad \begin{cases} H_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon\dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon\dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

where $h^2 = \gamma^2 + k^2$

(E_x, E_y, H_x, H_y) are functions of (E_z, H_z)!!

- *Procedure to determine EM wave within a waveguide*

1. Solve the wave eqns for E_z, H_z with given boundary conditions

$$\begin{cases} \nabla_{xy}^2 \mathbf{E} + \left(\gamma^2 + k^2\right)\mathbf{E} = 0 \\ \nabla_{xy}^2 \mathbf{H} + \left(\gamma^2 + k^2\right)\mathbf{H} = 0 \end{cases}$$

2. Substitute E_z, H_z into above eqns to obtain E_x, E_y, H_x, H_y

$$\begin{cases} E_x^0 = \Im\left(E_z^0, H_z^0\right) \\ E_y^0 = \Im\left(E_z^0, H_z^0\right) \end{cases}, \quad \begin{cases} H_x^0 = \Im\left(E_z^0, H_z^0\right) \\ H_y^0 = \Im\left(E_z^0, H_z^0\right) \end{cases}$$

- *Classification of EM wave in terms of (E_z, H_z)*
  - *TEM wave*: No longitudinal components → $\mathbf{H}, \mathbf{E} \perp \mathbf{k}$ → H_z, $E_z = 0$
  - *Transverse Magnetic (TM) wave*: $\mathbf{H} \perp \mathbf{k}$ → $H_z = 0$, but nonzero E_z
  - *Transverse Electric (TE) wave*: $\mathbf{E} \perp \mathbf{k}$ → $E_z = 0$, but nonzero H_z

Chap. 10 | TE wave within a guide (1/2)

- ***Transverse Electric (TE) waves***
  - $\boldsymbol{E} \perp \boldsymbol{k} \rightarrow E_z = 0$ (Longitudinal component of ***E***-field = 0)
  - Nonzero H_z

- ***Characterization of TE waves***
  - i.e. How to obtain E_x^0, E_y^0, $\cancel{E_z^0}$, H_x^0, H_y^0, H_z^0?

STEP 1

$$\nabla_{xy}^2 H_z^0 + \left(\gamma^2 + k^2\right) H_z^0 = 0$$

Solve non-homogenous equation for H_z^0 with ***given B.C.*** of a guide
(*Will be dealt in greater details in section 10.2~7*)

STEP 2

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu \dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial y}}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) & \cdots(4) \end{cases}$$

Use H_z^0 from STEP 1 and $E_z^0 = 0$ to determine ***transverse H-field components***

If we combine eqns. (3) and (4),

$$\left[\left(\boldsymbol{H}_T^0\right)_{TE} \triangleq \boldsymbol{a}_x H_x^0 + \boldsymbol{a}_y H_y^0\right] = \left[-\frac{\gamma}{h^2}\left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y}\right) H_z^0 \triangleq -\frac{\gamma}{h^2} \nabla_T H_z^0\right]$$

$$\boxed{\therefore \left(\boldsymbol{H}_T^0\right)_{TE} = -\frac{\gamma}{h^2} \nabla_T H_z^0}$$

Gradient of H_z^0 in a transverse plane

: Transverse *H*-fields (H_x^0 and H_y^0) can be obtained by longitudinal *H*-field (H_z^0)!

Chap. 10 | TE wave within a guide (2/2)

- ***Characterization of TE waves***
  - i.e. How to obtain E_x^0, E_y^0, E_z^0, H_x^0, H_y^0, H_z^0?

$\boxed{\text{STEP 3}}$

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu\dfrac{\partial H_z^0}{\partial y}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu\dfrac{\partial H_z^0}{\partial x}\right) & \cdots(2) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial x} - \cancel{j\omega\varepsilon\dfrac{\partial E_z^0}{\partial y}}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial y} + \cancel{j\omega\varepsilon\dfrac{\partial E_z^0}{\partial x}}\right) & \cdots(4) \end{cases}$$

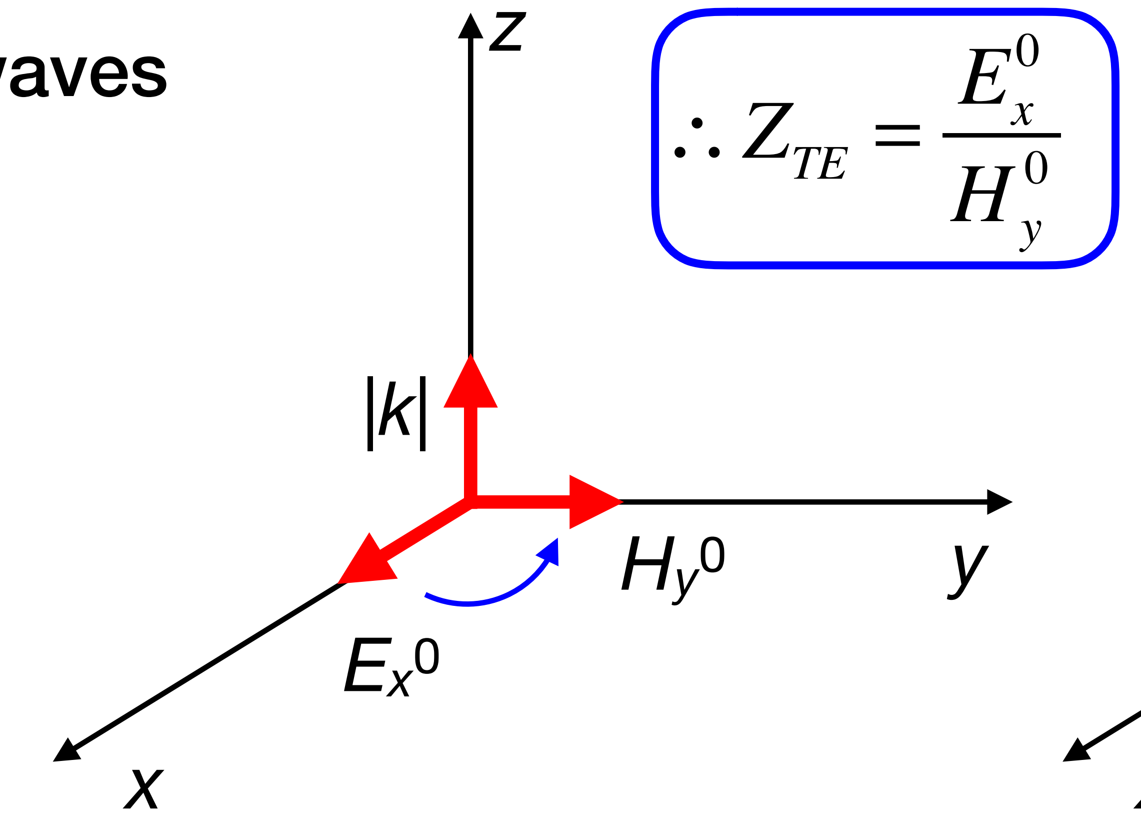
By using *a wave impedance*, obtain *transverse E-fields* from *transverse H-fields*

Wave impedance

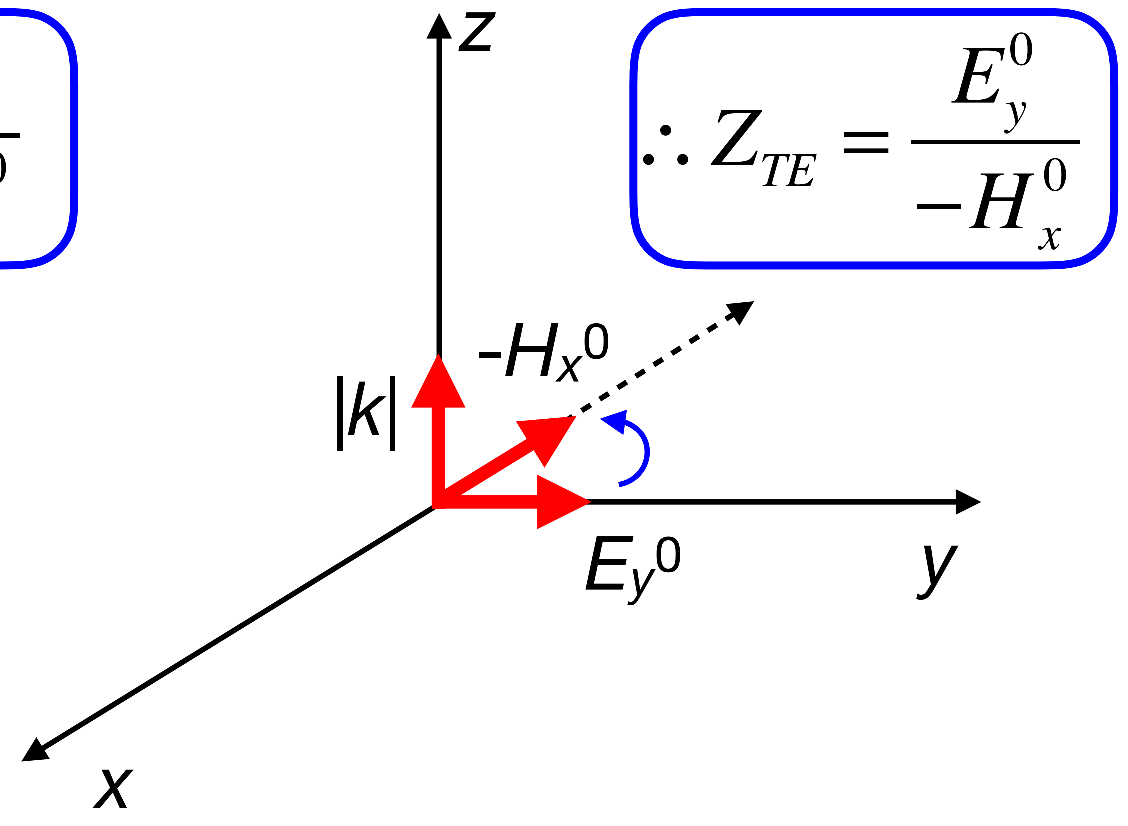
- A ratio of transverse components of the ***E*** and ***H***-fields
- By using eqns. (1)/(4) or eqns. (2)/(3),

$$\boxed{Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma} \quad (\Omega)} \rightarrow \boxed{\therefore \boldsymbol{E}_T = -Z_{TE}\left(\boldsymbol{a}_z \times \boldsymbol{H}_T\right)}$$

- Wave impedance is defined according to "right-hand" rule for propagating waves
- "Right-hand" rule
  - Determining the directions of ***E***, ***H***, and ***k*** associated with a EM wave
  - Steps
    - Point the fingers of your right hand in ***E***-field direction
    - Bend them in the ***H***-field direction
    - Then, your thumb points in the ***k*** direction



$$\boxed{\therefore Z_{TE} = \frac{E_x^0}{H_y^0}}$$



$$\boxed{\therefore Z_{TE} = \frac{E_y^0}{-H_x^0}}$$

Chap. 10 | TM wave within a guide (1/2)

- ***Transverse Magnetic (TM) waves***
  - $\boldsymbol{H} \perp \boldsymbol{k} \rightarrow H_z = 0$ (Longitudinal component of ***H***-field = 0)
  - Nonzero E_z

- ***Characterization of TM waves***
  - i.e. How to obtain E_x^0, E_y^0, E_z^0, H_x^0, H_y^0, $\cancel{H_z^0}$?

STEP 1 $\quad \nabla_{xy}^2 E_z^0 + \left(\gamma^2 + k^2\right) E_z^0 = 0$

Solve non-homogenous equation for E_z^0 with ***given B.C.*** within a guide

STEP 2

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial x} + j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial y}}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) & \cdots(2) \\ H_x^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma \dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma \dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

Use E_z^0 from STEP 1 and $H_z^0 = 0$ to obtain ***transverse E-field components***

If we combine eqns. (1) and (2),

$$\left[\left(\boldsymbol{E}_T^0\right)_{TM} \triangleq \boldsymbol{a}_x E_x^0 + \boldsymbol{a}_y E_y^0\right] = \left[-\frac{\gamma}{h^2}\left(\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y}\right) E_z^0 \triangleq -\frac{\gamma}{h^2} \nabla_T E_z^0\right]$$

$$\boxed{\therefore \left(\boldsymbol{E}_T^0\right)_{TE} = -\frac{\gamma}{h^2} \nabla_T E_z^0}$$

Gradient of E_z^0 in a transverse plane

: Transverse *E*-fields (E_x^0 and E_y^0) can be obtained by longitudinal *E*-field (E_z^0)!

TM wave within a guide (2/2)

- ***Characterization of TM waves***
  - i.e. How to obtain E_x^0, E_y^0, E_z^0, H_x^0, H_y^0, H_z^0?

STEP 3

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial x} + j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial y}}\right) & \cdots(1) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) & \cdots(2) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) & \cdots(3) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \dfrac{\partial E_z^0}{\partial x}\right) & \cdots(4) \end{cases}$$

By using ***a wave impedance***, obtain ***transverse H-fields*** from ***transverse E-fields***

Wave impedance

- By having eqns. (1)/(4) or (2)/(3),

$$\boxed{Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma_{TM}}{j\omega\varepsilon} \quad (\Omega)} \rightarrow \boxed{\therefore (\boldsymbol{H}_T)_{TM} = \frac{1}{Z_{TM}}\left(\boldsymbol{a}_z \times (\boldsymbol{E}_T)_{TM}\right)}$$

- c.f.) For TE and TEM,

$$\boxed{Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma_{TE}} \quad (\Omega)} \rightarrow \boxed{(\boldsymbol{E}_T)_{TE} = -Z_{TE}\left(\boldsymbol{a}_z \times (\boldsymbol{H}_T)_{TE}\right)}$$

$$\boxed{Z_{TEM} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta \quad (\Omega)} \rightarrow \boxed{(\boldsymbol{H}_T)_{TEM} = \frac{1}{\eta}\left(\boldsymbol{a}_z \times (\boldsymbol{E}_T)_{TEM}\right)}$$

- **Wave impedance for TEM vs. TE / TM**
  - Z_{TEM} is ***independent of frequency*** $\leftarrow \gamma_{TEM} = jk = j\omega\sqrt{\mu\varepsilon} \leftarrow h^2 = \gamma_{TEM}^2 + k^2 = 0$
  - Z_{TE} and Z_{TM} are ***frequency-dependent*** $\leftarrow \gamma_{TE}$ and $\gamma_{TM} \neq jk$

Chap. 10 | TE & TM waves within a guide (Commonality, 1/2)

- *Cut-off frequency*
  - *frequency* at which [$\gamma = 0$] is given by

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{h^2 - \omega^2 \mu\varepsilon} = 0 \quad \rightarrow \quad \omega_c^{\ 2} \mu\varepsilon = h^2$$

"cut-off frequency"

$$\boxed{\therefore f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}} \longrightarrow \boxed{\gamma = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}}$$

$$\begin{cases} \nabla_{xy}^2 \boldsymbol{E} + \left(\underline{\gamma^2 + k^2}\right)\boldsymbol{E} = 0 \\ \nabla_{xy}^2 \boldsymbol{H} + \left(\underline{\gamma^2 + k^2}\right)\boldsymbol{H} = 0 \end{cases} \qquad \gamma^2 + k^2 = h^2$$

- h: *Characteristic values* or *eigenvalues* determined by boundary condition
  - Only ***discrete & real*** values are allowed!
  - ***Each eigenvalue*** determines the ***characteristics of a particular mode*** of the ***given waveguide***

- ***When $f > f_c$ → γ: purely imaginary***
  - *Phase constant (β)*

$$\gamma = \cancel{\alpha} + j\beta = jk\sqrt{1 - \left(\frac{h}{k}\right)^2} = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \longrightarrow$$

propagating mode with $\boxed{\beta = k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad (\mathrm{rad/m})}$

  - Corresponding wavelength

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} > \lambda, \ \text{where } \lambda = \frac{2\pi}{k} = \frac{1}{f\sqrt{\mu\varepsilon}} = \frac{u}{f}$$

λ: wavelength of a plane wave *in an unbounded dielectric medium (μ, ε)*
u: velocity of light *in that medium*

Chap. 10 | TE & TM waves within a guide (Commonality, 2/2)

- ***When f > f_c → γ: purely imaginary (cont'd)***
  - *Phase velocity*

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} > u$$

→ (1) ***u_p within a waveguide*** is "always faster" than ***u in an unbounded medium***

(2) u_p is ***frequency-dependent***

→ Waveguides for TE/TM = *"dispersive systems"*

$$\boxed{\beta = k\sqrt{1-\left(\frac{f_c}{f}\right)^2} \quad (\text{when } f > f_c)}$$

  - Group velocity

$$u_g = \frac{1}{d\beta/d\omega} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2} < u$$

$$\boxed{\therefore u^2 = u_p u_g}$$

Group velocity in a lossless medium

= ***Velocity of signal propagation*** (or ***energy transport***) [will be discussed later]

- ***When f < f_c → γ: real***
  - *Waves become **"attenuating"** or **"evanescent"** modes*
  - *Waveguides for TE/TM = "high-pass" filters (f > f_c propagating, f < f_c attenuated)*

$$\boxed{\gamma = h\sqrt{1-\left(\frac{f}{f_c}\right)^2}}$$

$$\gamma = \alpha = h\sqrt{1-\left(\frac{f}{f_c}\right)^2} \quad \longrightarrow \quad \begin{cases} Z_{TM} = \dfrac{\gamma_{TM}}{j\omega\varepsilon} \\ Z_{TE} = \dfrac{j\omega\mu}{\gamma_{TE}} \end{cases}$$

Wave impedance for evanescent TM and TE modes

- Purely *Imaginary* → Purely ***"reactive"***

$$Z = j|Z| = |Z|e^{j\pi/2} = \frac{\boldsymbol{E}_T}{\boldsymbol{H}_T}$$

- ***E_T*** and ***H_T*** are in phase-quadrature

∴ No power flow for evanescent waves ← $\left(\because P_{av} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E}_T \times \boldsymbol{H}_T^*\right) = 0\right)$

TE & TM waves within a guide (Difference)

- *Wave impedance at f > f_c*

Since $\gamma = j\beta = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}$,

"Purely imaginary"
→ Propagating modes

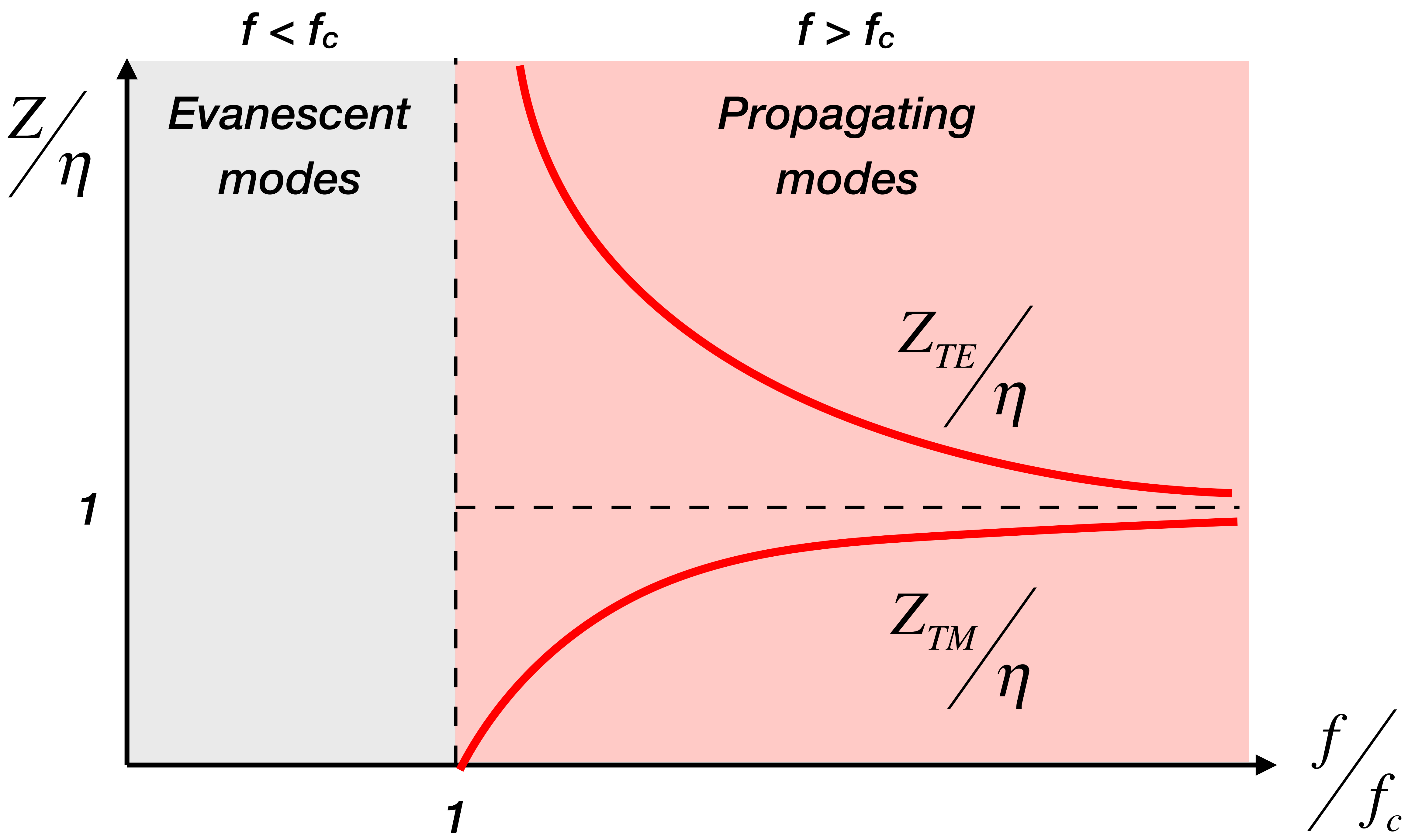
$$Z_{TM} = \frac{\gamma_{TM}}{j\omega\varepsilon} = \frac{jk}{j\omega\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \boxed{\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$Z_{TE} = \frac{j\omega\mu}{\gamma_{TE}} = \frac{j\omega\mu}{jk}\frac{1}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \boxed{\frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

(a) Wave impedances for both TE and TM are real & purely resistive!

(b) $Z_{TE} > Z_{TM}$ for all $f > f_c$

(c) At very high f, both asymptotically converge to η



Chap. 10 | Propagating TE & TM waves within a guide

- *Summary for propagating mode*

Mode	*Wave impedance*	*Waveguide*	*Phase velocity*	*Group velocity*
TEM	$\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$	$\lambda_g = \dfrac{2\pi}{k} = \lambda$	$u = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\varepsilon}}$	$u = \dfrac{1}{d\beta/d\omega} = \dfrac{1}{\sqrt{\mu\varepsilon}}$
TM	$\eta\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}$	$\lambda_g = \dfrac{\lambda}{\sqrt{1-(f_c/f)^2}} > \lambda$	$u_p = \dfrac{u}{\sqrt{1-(f_c/f)^2}} > u$	$u_g = u\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} < u$
TE	$\dfrac{\eta}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}}$			

$\gamma = j\beta$ *where*

$$\beta = k\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

(when $f > f_c$)

- *ω-β relationship (Frequency dependence of β)*

  - Determining characteristics of propagating waves along a waveguide

TEM $\quad \beta = \omega\sqrt{\mu\varepsilon}$

TE&TM $\quad \beta = k\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\dfrac{\omega_c}{\omega}\right)^2}$ *where* $\omega_c = \dfrac{h}{\sqrt{\mu\varepsilon}}$

h (eigenvalue) depends on a particular TE or TM mode in a waveguide of given cross-section

$$\left(\frac{\omega}{\beta}\right)_{TEM} = u_{TEM} = \frac{1}{\sqrt{\mu\varepsilon}}$$