Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.3 ~ 10.4

(1st of **week 7**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

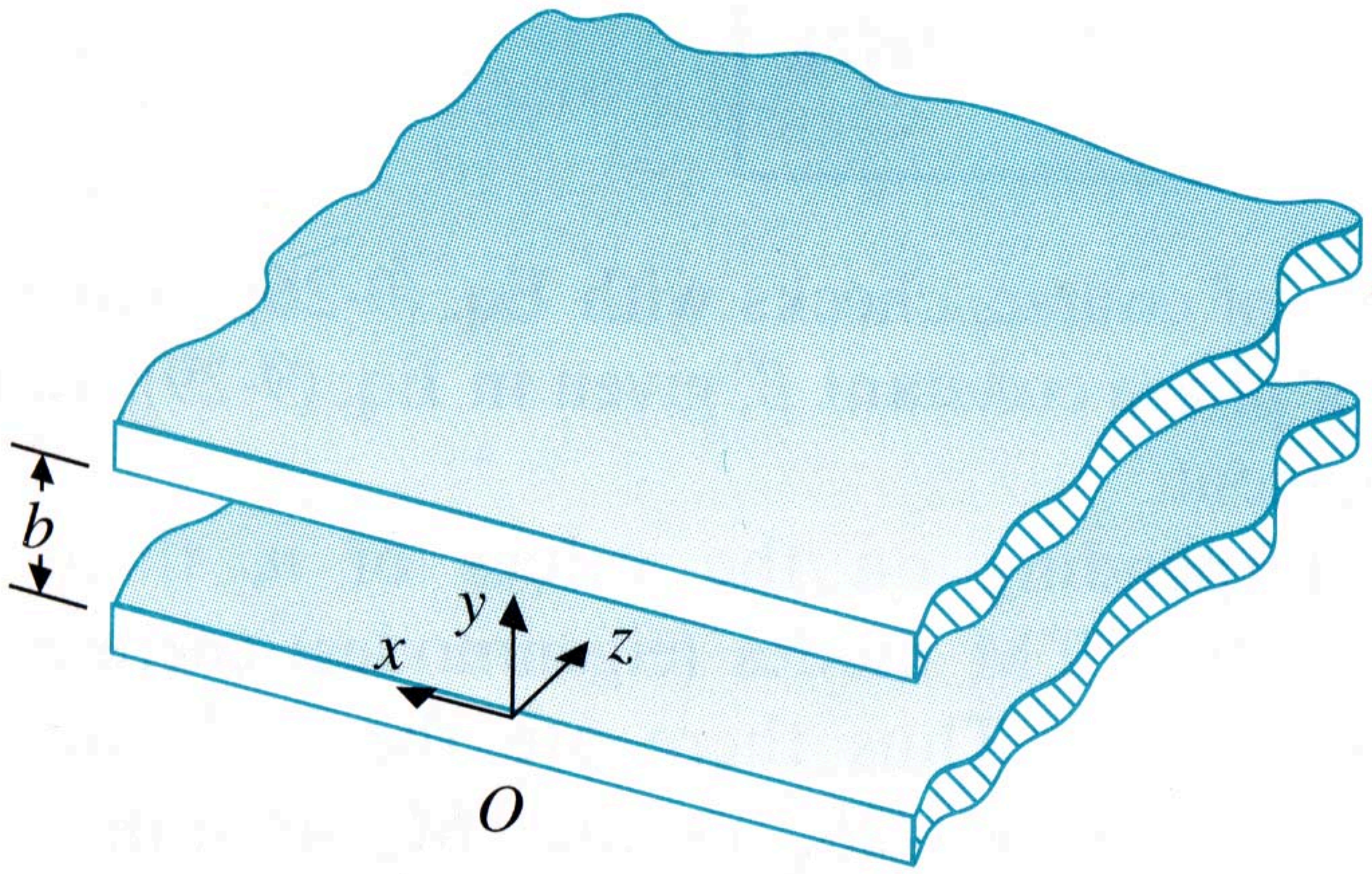
Chap. 10 | Contents for 1st class of week 7

Sec 3. Parallel-plate waveguide

- Characteristics of TE and TM wave propagation
- Energy transport velocity
- Attenuation in the waveguide

Chap. 10 | Parallel-plate waveguide and "TM" waves (1/6)

- ***Infinite parallel plate waveguide***
  - Two ***"perfectly conducting"*** plates separated by a ***"dielectric"*** medium (μ, ε)
  - All ***TEM, TM, TE*** waves can be supported
  - ***"Infinite in extent"*** in *x*-direction
    - Fields do not vary in x-direction → $\frac{\partial \boldsymbol{E}}{\partial x} = 0,\ \frac{\partial \boldsymbol{H}}{\partial x} = 0\ \left(\boldsymbol{E} \neq 0,\ \boldsymbol{H} \neq 0\right)$
    - Edge effects negligible



- ***TM waves between parallel plates***
  - Longitudinal components

$$H_z^0 = 0,\ E_z^0 \neq 0$$

  - Phasor notation for longitudinal *E*-field

$$E_z(y,z) = E_z^0(y)e^{-\gamma z}$$

(no dependence of x!)

  - Wave equation

z-dependence taken care by γ²

$$h^2 \triangleq \gamma^2 + k^2$$

$$\left(\nabla_{xy}^2 + \nabla_z^2\right)E_z(y,z) + k^2 E_z(y,z) = 0 \quad \rightarrow \quad \nabla_{xy}^2 E_z^0(y) + \left(\gamma^2 + k^2\right)E_z^0(y) = 0 \quad \rightarrow \quad \frac{d^2 E_z^0(y)}{dy^2} + h^2 E_z^0(y) = 0$$

  - Boundary condition

$$E_z^0(y) = 0 \ \text{ at } \ y = 0 \text{ and } y = b$$

(∵E-field vanishes at the conducting interface!)

Chap. 10 | Parallel-plate waveguide and “TM” waves (2/6)

- ***TM waves between parallel plates***

  - Solution to the wave equation

$$\frac{d^2E_z^0(y)}{dy^2}+h^2E_z^0(y)=0 \quad \text{with boundary condition} \quad E_z^0(y)=0 \quad \text{at} \quad y=0 \text{ and } y=b$$

$$\boxed{\therefore E_z^0(y)=A_n\sin(hy)=A_n\sin\left(\frac{n\pi y}{b}\right),} \quad n=0,1,2\cdots$$

where A_n ~ strength of a particular TM mode (***not our interest here***)

Longitudinal E-field for TM mode

$$\boxed{h=\frac{n\pi}{b}, \quad n=0,1,2\cdots}$$

Eigenvalues
(Depending on geometry!)

- ***Transverse E and H-field components***

  Transverse components
  in terms of longitudinal components

$$\left\{\begin{aligned}
E_x^0&=-\frac{1}{h^2}\left(\gamma\cancel{\frac{\partial E_z^0}{\partial x}}+j\omega\mu\cancel{\frac{\partial H_z^0}{\partial y}}\right)\\
E_y^0&=-\frac{1}{h^2}\left(\gamma\frac{\partial E_z^0}{\partial y}-j\omega\mu\cancel{\frac{\partial H_z^0}{\partial x}}\right)\\
H_x^0&=-\frac{1}{h^2}\left(\gamma\cancel{\frac{\partial H_z^0}{\partial x}}-j\omega\varepsilon\frac{\partial E_z^0}{\partial y}\right)\\
H_y^0&=-\frac{1}{h^2}\left(\gamma\cancel{\frac{\partial H_z^0}{\partial y}}+j\omega\varepsilon\cancel{\frac{\partial E_z^0}{\partial x}}\right)
\end{aligned}\right.$$

$$\left\{\begin{aligned}
H_z^0&=0\\
\frac{\partial E_z^0}{\partial x}&=0
\end{aligned}\right. \qquad \Longrightarrow$$

$$\left\{\begin{aligned}
E_x^0&=0\\
E_y^0&=-\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial y}\\
H_x^0&=\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z^0}{\partial y}\\
H_y^0&=0
\end{aligned}\right. \qquad \xrightarrow{E_z^0(y)=A_n\sin\left(\frac{n\pi y}{b}\right)}$$

$$\boxed{\left\{\begin{aligned}
E_x^0(y)&=0\\
E_y^0(y)&=-\frac{\gamma}{h^2}A_n\cos\left(\frac{n\pi y}{b}\right)\\
H_x^0(y)&=\frac{j\omega\varepsilon}{h}A_n\cos\left(\frac{n\pi y}{b}\right)\\
H_y^0(y)&=0
\end{aligned}\right.}$$

Chap. 10 | Parallel-plate waveguide and “TM” waves (3/6)

- ***TM waves between parallel plates***

  - Propagation constant

$$\gamma = \sqrt{h^2 - k^2} = j\sqrt{k^2 - h^2} = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2} \quad \rightarrow \quad \boxed{\therefore \beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2}}$$

  - Cut-off frequency ($\gamma = 0$)

$$\boxed{f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2b\sqrt{\mu\varepsilon}} \quad (\text{Hz})}$$

f > *f*c: Propagate with a phase constant β

f < *f*c: Evanescent wave

**** Cut-off frequency is determined by geometry & material composition of a waveguide!***

(Typically, microwave (0.3~300GHz) used in waveguide for communications. Why?)

- ***Possible TM wave (= eigenmode, TM$_n$)***

  - Characterized by eigenvalue $h = \dfrac{n\pi}{b}, \quad n = 0,1,2\cdots$

- ***TM$_0$ mode*** **(*n*=0)**

Longitudinal

$$\begin{cases} H_z^0(y) = 0 \\ E_z^0(y) = A_n \sin\left(\frac{n\pi y}{b}\right) = 0 \end{cases}$$

Transverse

$$\begin{cases} E_x^0(y) = 0 \\ E_y^0(y) = -\frac{\gamma}{h^2} A_n \cos\left(\frac{n\pi y}{b}\right) = -\frac{\gamma}{h^2} A_0 \\ H_x^0(y) = \frac{j\omega\varepsilon}{h} A_n \cos\left(\frac{n\pi y}{b}\right) = \frac{j\omega\varepsilon}{h} A_0 \\ H_y^0(y) = 0 \end{cases}$$

Zero longitudinal E and H-fields

Non-zero transverse E and H-fields

→ ***TEM mode!***

→ ***f*c = 0 (No cutoff frequency)**

Chap. 10 | Parallel-plate waveguide and "TM" waves (4/6)

- ***TM₀ mode*** **(*n* = *0*)**
  - ***TM₀*** = ***TEM*** with ***f*$_c$** **= *0***
  - The mode with ***lowest cutoff frequency*** = ***"Dominant mode"*** of the waveguide (→ *lowest attenuation, why?*)

    ∴ Dominant mode for parallel-plate waveguides = TM₀ mode (TEM mode)

- ***TMₙ mode*** **(*n* > *0*)**
  - Cut-off frequency $f_c = \dfrac{h}{2\pi\sqrt{\mu\varepsilon}} = \dfrac{n}{2b\sqrt{\mu\varepsilon}}$ (Hz)
  - Each mode (*n*) has its own λ_g, u_p, u_g, and Z_{TMn}

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1-\left(f_c/f\right)^2}} \quad \text{where} \quad \lambda = \frac{2\pi}{k} = \frac{1}{f\sqrt{\mu\varepsilon}}$$

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-\left(f_c/f\right)^2}}$$

$$u_p = \frac{1}{d\beta/d\omega} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2} \qquad Z_{TM_n} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

β in terms of fc

$$\beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{n\pi}{b\omega\sqrt{\mu\varepsilon}}\right)^2}$$

$$= k\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2} = k\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Conclusion:

∴Each TMₙ mode has its own propagating characteristics

with distinct f$_c$, λ$_g$, u$_p$, u$_g$, and Z$_{TMn}$!

Chap. 10 | Parallel-plate waveguide and “TM” waves (5/6)

Example 10-4

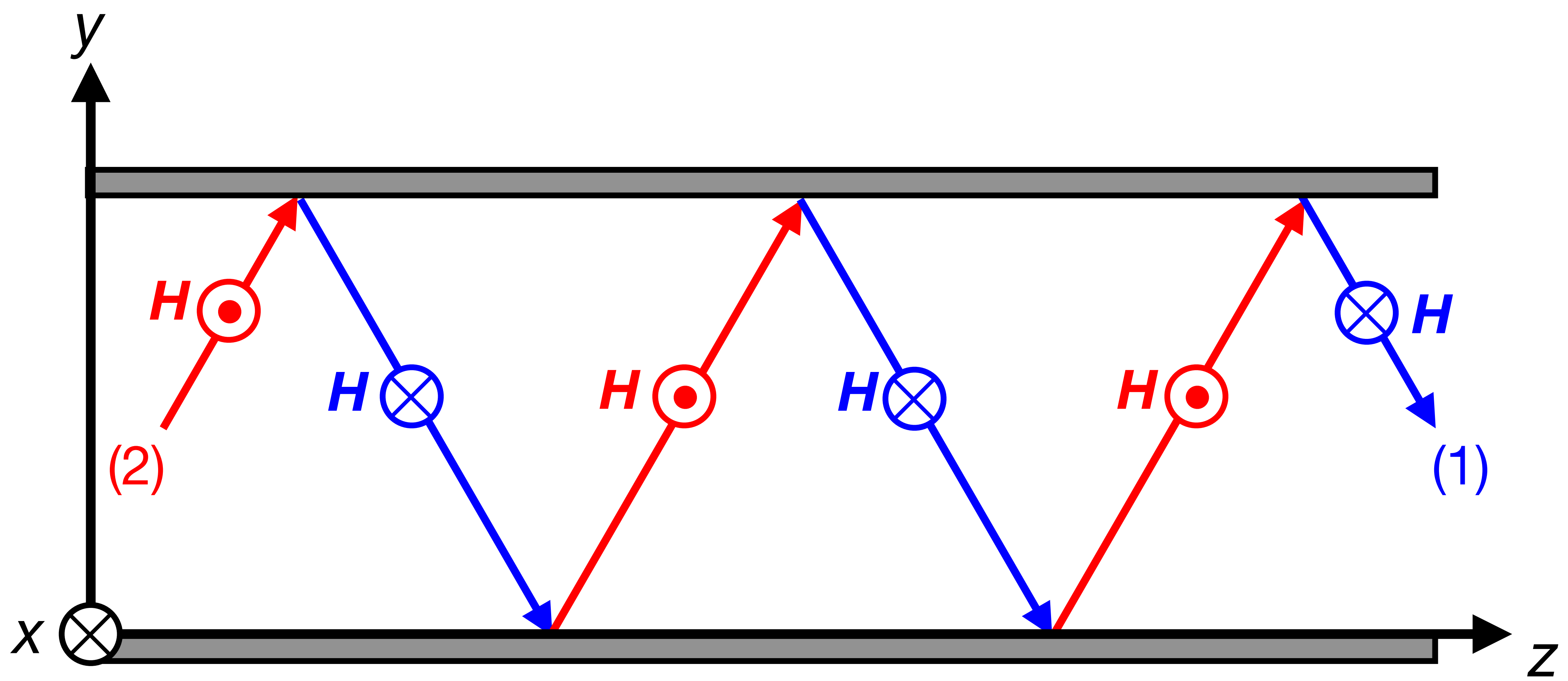
Propagating TM_1 wave in a parallel plate waveguide =
Superposition of two plane waves bouncing back and forth obliquely between two plates

- ***Longitudinal electric field for TM_1 mode***

$$E_z(y,z) = E_z^0(y)e^{-\gamma z} = A_1 \sin\left(\frac{\pi y}{b}\right)e^{-j\beta z} = \frac{A_1}{2j}\left(e^{j\frac{\pi y}{b}} - e^{-j\frac{\pi y}{b}}\right)e^{-j\beta z} = \frac{A_1}{2j}\left(\underbrace{e^{-j\left(\beta z-\frac{\pi y}{b}\right)}}_{(1)} - \underbrace{e^{-j\left(\beta z+\frac{\pi y}{b}\right)}}_{(2)}\right)$$

(1) Plane wave propagating obliquely ***in z and -y direction***

(2) Plane wave propagating obliquely ***in z and +y direction***



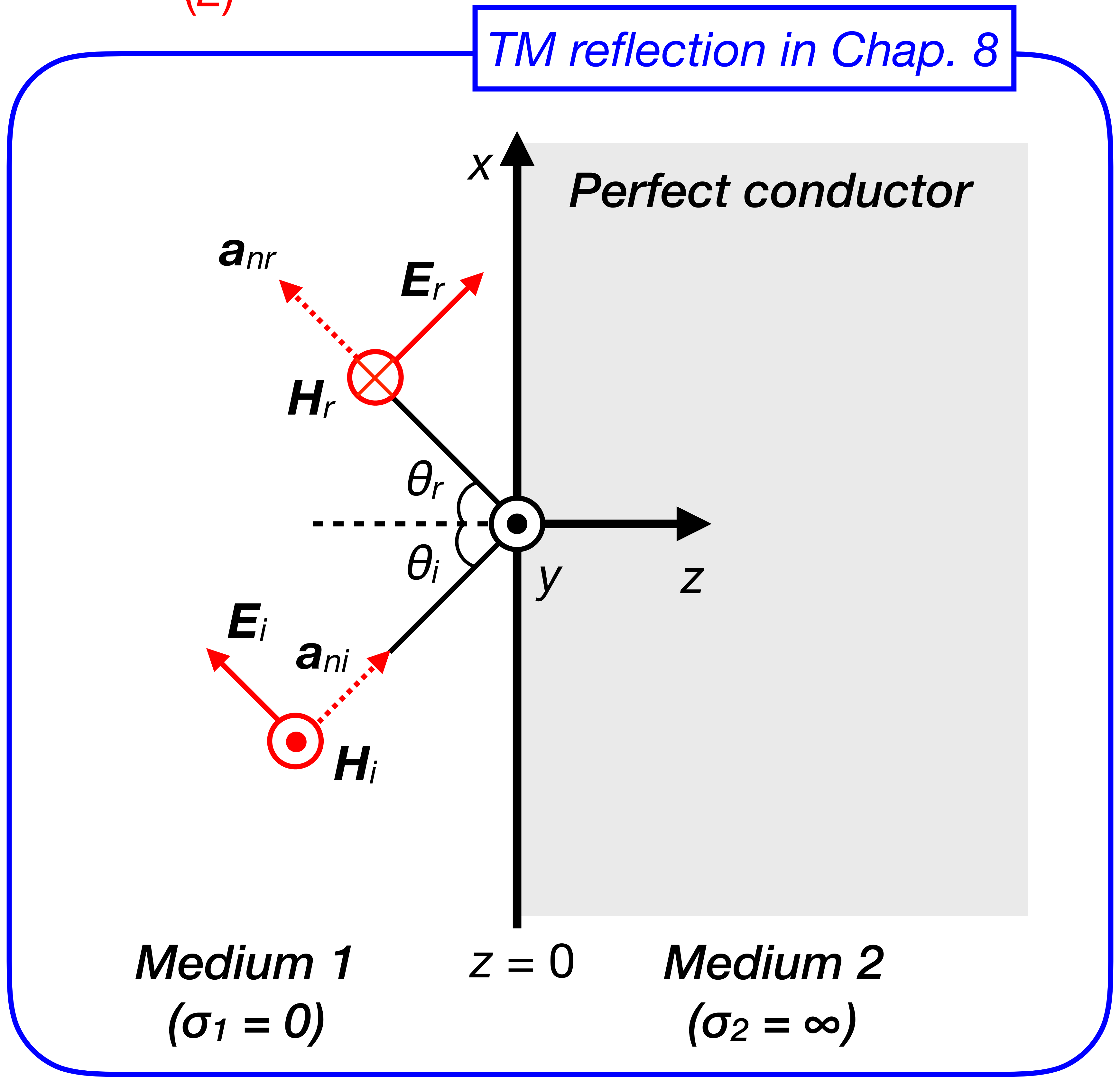
Parallel plates lying along xz plane

Plane of incidence: yz plane

TM_1 wave propagating in z direction

Replacing
x with z
&
z with -y

TM reflection in Chap. 8



Chap. 10 | Parallel-plate waveguide and “TM” waves (6/6)

Example 10-4 Propagating *TM*₁ wave in a parallel plate waveguide =

Superposition of two plane waves bouncing back and forth obliquely between two plates

- ***Total E-field for TM wave from Chap. 8-7.2***

$$\boldsymbol{E}_1(x,z) = -2E_{i0}\left[\boldsymbol{a}_x j\cos\theta_i\sin(\beta_1 z\cos\theta_i) + \boldsymbol{a}_z\sin\theta_i\cos(\beta_1 z\cos\theta_i)\right]e^{-j\beta_1 z\sin\theta_i}$$

$$= \boldsymbol{a}_x E_x(x,z) + \boldsymbol{a}_z E_z(x,z) \quad \longrightarrow \quad E_x(x,z) = -j2E_{i0}\cos\theta_i\sin(\beta_1 z\cos\theta_i)e^{-j\beta_1 z\sin\theta_i}$$

From previous page

(notation used in ***Chap. 10***)

by replacing *x* with *z* and *z* with -*y*, $E_x(x,z)$ becomes

$$E_x(x,z) \quad \rightarrow \quad E_z(z,y) = -j2E_{i0}\cos\theta_i\sin(\beta_1 y\cos\theta_i)e^{-j\beta_1 z\sin\theta_i} \quad \Longleftrightarrow \quad \boxed{E_z(y,z) = A_1\sin\left(\frac{\pi y}{b}\right)e^{-j\beta z}}$$

Solutions:

$$\boxed{\begin{cases}\beta_1\sin\theta_i = \beta \\ \beta_1\cos\theta_i = \dfrac{\pi}{b}\end{cases}} \quad \longrightarrow \quad \boxed{\begin{gathered}\beta = \sqrt{\beta_1^2 - \left(\frac{\pi}{b}\right)^2} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{\pi}{b}\right)^2} \\ \cos\theta_i = \frac{\pi}{b\beta_1} = \frac{\lambda}{2b} \quad \text{where} \quad \beta_1 = \frac{2\pi}{\lambda}\end{gathered}} \quad \longrightarrow$$

(Here, λ: wavelength in an unbounded dielectric medium)

***** Solution condition*****

- θ_i exists only if $\lambda/2b \le 1$
- at $\lambda/2b = 1$

$\rightarrow f = \dfrac{u}{\lambda} = \dfrac{1}{2b\sqrt{\mu\varepsilon}} = f_c$: Cut-off frequency for *n* = 1

($\cos\theta_i = 1$, $\sin\theta_i = 0$)

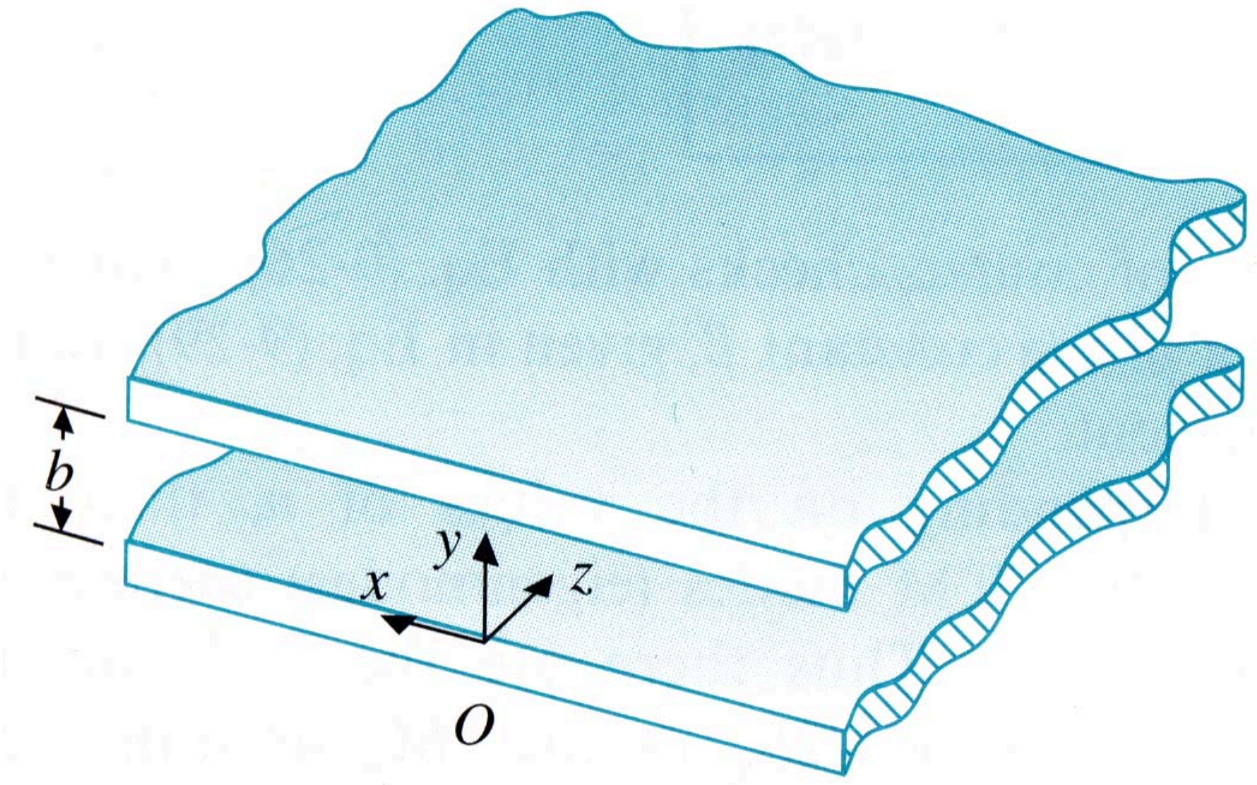
→ Waves bounding back & forth in *y* direction

→ No propagation in *z* direction!

∴ Propagation only possible when $\lambda < 2b = \lambda_c$ or $f > f_c$

Chap. 10 | Parallel-plate waveguide and "TE" waves (1/2)

- *TE waves between parallel plates*



- Longitudinal components

$$E_z^0 = 0,\ H_z^0 \neq 0$$

- Phasor notation for longitudinal *H*-field

$$H_z(y,z) = H_z^0(y)e^{-\gamma z}$$ *(no dependence of x!)* ⟵ $\because \dfrac{\partial H_z^0(y)}{\partial x} = 0$

- Wave equation

$$\left(\nabla_{xy}^2 + \nabla_z^2\right)H_z(y,z) + k^2 H_z(y,z) = 0 \;\rightarrow\; \nabla_{xy}^2 H_z^0(y) + \left(\gamma^2 + k^2\right)H_z^0(y) = 0 \;\rightarrow\; \boxed{\frac{d^2 H_z^0(y)}{dy^2} + h^2 H_z^0(y) = 0}$$

- Boundary condition

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

$$\begin{cases} E_z^0 = 0 \\ \dfrac{\partial H_z^0}{\partial x} = 0 \end{cases}$$

$$\longrightarrow \begin{cases} E_x^0(y) = -\dfrac{j\omega\mu}{h^2}\dfrac{dH_z^0(y)}{dy} \\ E_y^0(y) = 0 \\ H_x^0(y) = 0 \\ H_y^0(y) = -\dfrac{\gamma}{h^2}\dfrac{dH_z^0(y)}{dy} \end{cases}$$

$$E_x^0(y) = -\frac{j\omega\mu}{h^2}\frac{dH_z^0(y)}{dy} = 0\Bigg|_{y=0 \text{ and } y=b}$$

(at the surface of conducting plates)

⬇

$$\boxed{\therefore \frac{dH_z^0(y)}{dy} = 0 \quad \text{at } y = 0 \text{ and } y = b}$$

Chap. 10 | Parallel-plate waveguide and "TE" waves (2/2)

- ***TE waves between parallel plates***

- Solution to the wave equation

$$\frac{d^2H_z^0(y)}{dy^2}+h^2H_z^0(y)=0 \text{ with boundary condition } \frac{dH_z^0(y)}{dy}=0 \text{ at } y=0 \text{ and } y=b$$

$$\boxed{\therefore H_z^0(y)=B_n\cos(hy)=B_n\cos\left(\frac{n\pi y}{b}\right),}\quad n=0,1,2\cdots \text{ where } B_n \sim \text{strength of a particular TE mode (\textit{not our interest here})}$$

Longitudinal H-field for TE mode

- Transverse E and H-field components

$$\begin{cases} E_x^0(y)=-\dfrac{j\omega\mu}{h^2}\dfrac{dH_z^0(y)}{dy}=\dfrac{j\omega\mu}{h}B_n\sin\left(\dfrac{n\pi y}{b}\right) \\ H_y^0(y)=-\dfrac{\gamma}{h^2}\dfrac{dH_z^0(y)}{dy}=\dfrac{\gamma}{h}B_n\sin\left(\dfrac{n\pi y}{b}\right) \end{cases}$$

Here, $\gamma=\sqrt{h^2-k^2}=j\sqrt{k^2-h^2}=j\sqrt{\omega^2\mu\varepsilon-\left(\frac{n\pi}{b}\right)^2}$

(Same as that for TM modes!)

- Cut-off frequency (γ = 0)

$$f_c=\frac{h}{2\pi\sqrt{\mu\varepsilon}}=\frac{n}{2b\sqrt{\mu\varepsilon}} \text{ (Hz)}$$ *(Same as that for TM modes!)*

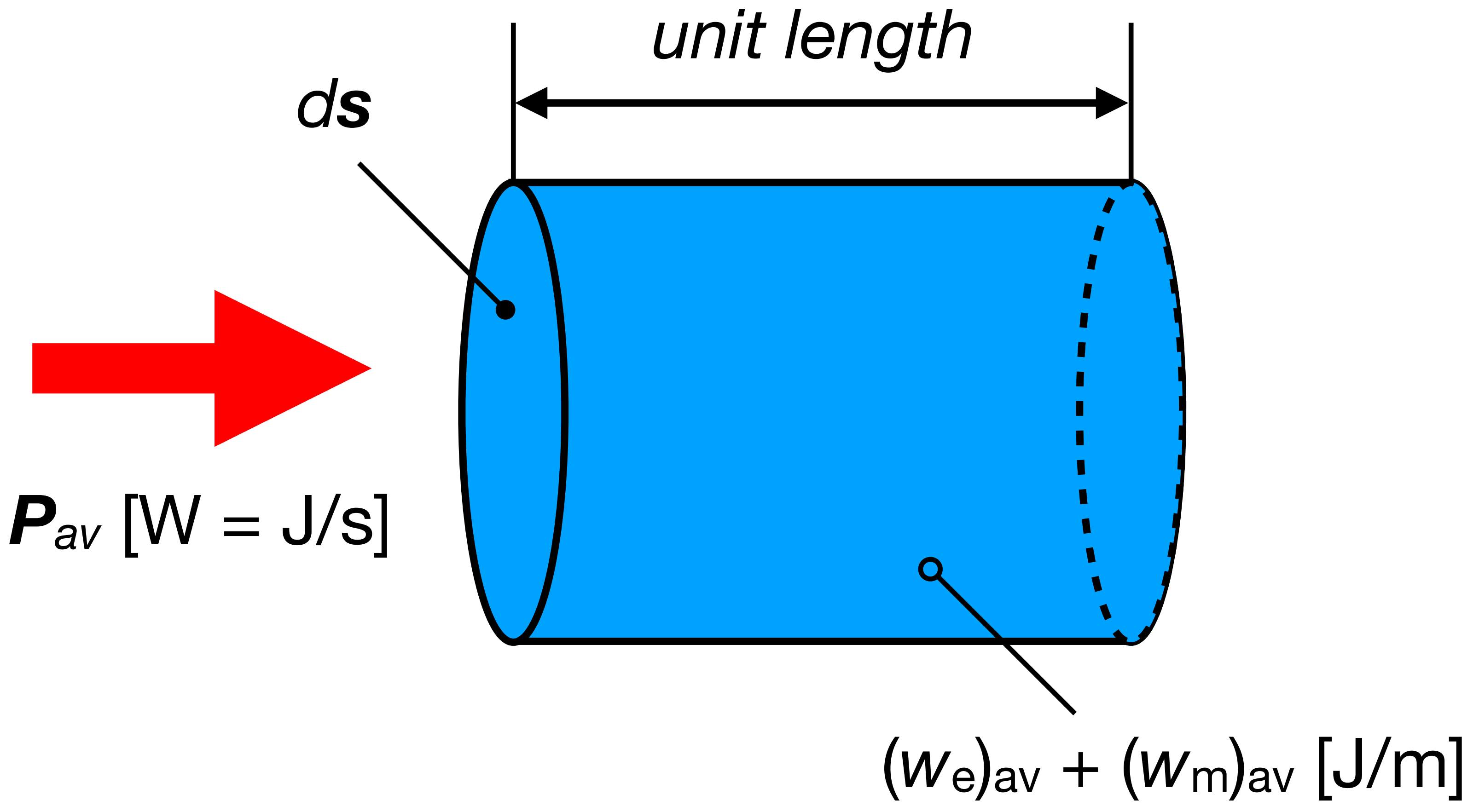
- Dominant mode?

- n = 0 → All transverse fields vanish!

  No TE$_0$ exists in a parallel-plate waveguide

- n = 1 → However, ***TE$_1$ ≠ dominant mode! (Why?)***

Chap. 10 | Energy transport velocity (1/2)



- ***Energy transport velocity***
  - Velocity at which energy propagates along a waveguide
    - ***Energy transport velocity (u_{en}) = group velocity (u_g) in "lossless medium"***
    - ✳In lossy media, group velocity loses its physical meaning (beyond our scope)
  - Definition

$$u_{en} \triangleq \frac{(P_z)_{av}}{W'_{av}} = \frac{\int_S \boldsymbol{P}_{av} \cdot d\boldsymbol{s}}{\int_V \left[(w_e)_{av} + (w_m)_{av}\right] dv}$$

Ratio of *time-average propagated power* across the guide (*S*) to *time-average stored energy* within the volume of unit length (*l* = 1)

Example 10-6 u_{ev} for TM_n mode in a lossless parallel-plate waveguide

- ***Time-average Poynting vector***

$$\boldsymbol{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) \quad \text{where} \quad \begin{cases} \boldsymbol{E} = \boldsymbol{a}_y E_y^0(y) + \boldsymbol{a}_z E_z^0(y) \\ \boldsymbol{H} = \boldsymbol{a}_x H_x^0(y) \end{cases}$$

$$= \frac{1}{2}\mathrm{Re}\left(-\boldsymbol{a}_z E_y^0 H_x^{0*} + \boldsymbol{a}_y E_z^0 H_x^{0*}\right)$$

- ***Integration of P_{av} across the cross-section of a unit width (w = 1)***

$$\int_S \boldsymbol{P}_{av} \cdot d\boldsymbol{s} = w \int_0^b \left(\boldsymbol{P}_{av} \cdot \boldsymbol{a}_z\right) dy = -\frac{1}{2}\int_0^b E_y^0 H_x^{0*}\, dy$$

$$\text{where} \quad \begin{cases} E_y^0(y) = -\dfrac{\gamma}{h^2} A_n \cos\left(\dfrac{n\pi y}{b}\right) \\ H_x^0(y) = \dfrac{j\omega\varepsilon}{h} A_n \cos\left(\dfrac{n\pi y}{b}\right) \end{cases}$$

Chap. 10 | Energy transport velocity (2/2)

$$u_{en} \triangleq \frac{(P_z)_{av}}{W'_{av}} = \frac{\int_S \boldsymbol{P}_{av} \cdot d\boldsymbol{s}}{\int_S \left[(w_e)_{av} + (w_m)_{av} \right] ds}$$

Example 10-6 u_{ev} for TM_n mode in a lossless parallel-plate waveguide

$$\begin{cases} \boldsymbol{E} = \boldsymbol{a}_y E_y^0(y) + \boldsymbol{a}_z E_z^0(y) \\ \boldsymbol{H} = \boldsymbol{a}_x H_x^0(y) \end{cases}$$

$$\begin{cases} E_z^0(y) = A_n \sin\left(\frac{n\pi y}{b}\right) \\ E_y^0(y) = -\frac{\gamma}{h^2} A_n \cos\left(\frac{n\pi y}{b}\right) \\ H_x^0(y) = \frac{j\omega\varepsilon}{h} A_n \cos\left(\frac{n\pi y}{b}\right) \end{cases}$$

- *Integration of P_{av} across the cross-section of a unit width* (w = 1)

$$\int_S \boldsymbol{P}_{av} \cdot d\boldsymbol{s} = w \int_0^b \left(\boldsymbol{P}_{av} \cdot \boldsymbol{a}_z \right) dy = -\frac{1}{2} \int_0^b E_y^0 H_x^{0*} \, dy$$

$$= -\frac{1}{2} \int_0^b \mathrm{Re}\left[\frac{j\omega\varepsilon\gamma}{h^2} A_n^2 \cos^2\left(\frac{n\pi y}{b}\right) \right] dy = \frac{\omega\varepsilon\beta}{2h^2} A_n^2 \int_0^b \cos^2\left(\frac{n\pi y}{b}\right) dy = \frac{\omega\varepsilon\beta b}{4h^2} A_n^2 \quad \cdots (1)$$

- *Time-average stored E and H-field energies within a volume of unit length* (l = 1)

$$w_e = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \boldsymbol{E} \cdot \boldsymbol{E}^* \quad \rightarrow \quad (w_e)_{av} = \frac{\varepsilon}{4} \mathrm{Re}\left(\boldsymbol{E} \cdot \boldsymbol{E}^* \right) = \frac{\varepsilon}{4} A_n^2 \left[\sin^2\left(\frac{n\pi y}{b}\right) + \frac{\beta^2}{h^2} \cos^2\left(\frac{n\pi y}{b}\right) \right]$$

$$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu \boldsymbol{H} \cdot \boldsymbol{H}^* \quad \rightarrow \quad (w_m)_{av} = \frac{\mu}{4} \mathrm{Re}\left(\boldsymbol{H} \cdot \boldsymbol{H}^* \right) = \frac{\mu}{4} \left(\frac{\omega^2\varepsilon^2}{h^2} \right) A_n^2 \cos^2\left(\frac{n\pi y}{b}\right)$$

$$\int_V \left[(w_e)_{av} + (w_m)_{av} \right] dv = lw \int_0^b \left[(w_e)_{av} + (w_m)_{av} \right] dy = 2 \times \frac{\varepsilon b}{8h^2} k^2 A_n^2 \quad \cdots (2)$$

$$\uparrow$$

$$dv = lwdy$$

$$\therefore u_{en} \triangleq \frac{(1)}{(2)} = \frac{\frac{\omega\varepsilon\beta b}{4h^2} A_n^2}{\frac{\varepsilon b}{4h^2} k^2 A_n^2} = \frac{\omega\beta}{k^2} = \left(\frac{\omega}{k}\right)\left(\frac{\beta}{k}\right)$$

$$= u\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = u_g$$

Chap. 10 | Attenuation in parallel-plate waveguides (1/4)

- ***Attenuation in any waveguide caused by…***
  - (1) lossy dielectric
  - (2) Imperfectly conducting walls
  - Assumed ***E*** and ***H***-fields are not altered by such losses

$\alpha = \alpha_c + \alpha_d$ where α_d: Losses in the dielectric

α_c: Ohmic losses in the imperfectly conducting walls

- ***TEM modes (Mostly from Chap. 9)***

$$\alpha = \frac{1}{2R_0}\left(R + G\left|Z_0\right|^2\right) \cong \frac{R}{2R_0} + \frac{GR_0}{2} = \alpha_c + \alpha_d$$

- Attenuation in dielectric (α_d)

$$\alpha_d = \frac{GR_0}{2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\sigma}{2}\eta \quad \text{for a low-loss dielectric}$$

- Attenuation in parallel plates (α_c) *[Frequency-dependent]*

$$\alpha_c = \frac{R}{2R_0} = \frac{2}{b}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \cdot \frac{1}{2\eta} = \frac{1}{b}\sqrt{\frac{\pi f \varepsilon}{\sigma_c}} \quad (\because \text{non-magnetic media})$$

$$\therefore \alpha = \alpha_d + \alpha_c = \frac{\sigma}{2}\eta + \frac{1}{b}\sqrt{\frac{\pi f \varepsilon}{\sigma_c}}$$

$$\begin{cases} \alpha_d \to 0 \ \text{ as } \ \sigma \to 0 \\ \alpha_c \to 0 \ \text{ as } \ \sigma_c \to \infty \end{cases}$$

Transmission line modeling in Chap. 9

for a low-loss line ($R << \omega L$, $G << \omega C$)

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left[1+\frac{1}{2j\omega}\left(\frac{R}{L}-\frac{G}{C}\right)\right] \cong \sqrt{\frac{L}{C}} = \sqrt{\frac{\eta}{\varepsilon}} = R_0$$

$$R = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}},\ L = \mu\frac{d}{w},\ G = \sigma\frac{w}{d},\ C = \varepsilon\frac{w}{d}$$

***** *At high frequency (i.e. microwave), α_c dominates and TEM cannot be supported in a parallel-plate waveguide!***

Chap. 10 | Attenuation in parallel-plate waveguides (2/4)

- *TM modes*

  - attenuation constant (α_d) due to losses in dielectric at $f > f_c$

$$\gamma = \sqrt{h^2 - k^2} = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2} \leftarrow \boxed{\varepsilon_d = \varepsilon + \frac{\sigma}{j\omega}}$$

$$= j\sqrt{\omega^2\mu\varepsilon\left(1 - \frac{j\sigma}{\omega\varepsilon}\right) - \left(\frac{n\pi}{b}\right)^2} = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2}\sqrt{1 - j\omega\mu\sigma\left[\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2\right]^{-1}}$$

$$\cong j\sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2}\left(1 - \frac{1}{2}j\omega\mu\sigma\left[\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2\right]^{-1}\right)$$

  - Let's express above in terms of cut-off frequency (f_c)

$$f_c = \frac{n}{2b\sqrt{\mu\varepsilon}} \quad \rightarrow \quad \frac{n\pi}{b} = \omega_c\sqrt{\mu\varepsilon} \quad \rightarrow \quad \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = 2\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\gamma \cong \underbrace{\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1-(f_c/f)^2}}}_{\triangleq\, \alpha_d} + \underbrace{j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}_{\triangleq\, \beta}$$

$$\boxed{\therefore \begin{cases} \alpha_d = \dfrac{\sigma}{2}\sqrt{\dfrac{\mu}{\varepsilon}}\dfrac{1}{\sqrt{1-(f_c/f)^2}} \\[2ex] \beta = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2} \end{cases}} \rightarrow$$

f↑ → α_d↓ for TM!

Chap. 10 | Attenuation in parallel-plate waveguides (3/4)

- *TM modes*
  - attenuation constant (α_c) due to imperfectly conducting walls

$$\alpha_c = \frac{P_L(z)}{2P(z)}$$ ***(from Law of conservation)***

  - $P(z)$: Time-average *power flowing through cross-section* of width w
  - $P_L(z)$: Time-average *power lost in two plates* per unit length

  - *From Example 10-6,*

$$P(z) = \int_S \mathbf{P}_{av}\cdot d\mathbf{s} = w\int_0^b \left(\mathbf{P}_{av}\cdot \mathbf{a}_z\right)dy = -\frac{w}{2}\int_0^b E_y^0 H_x^{0*}\,dy$$

$$= \frac{w\omega\varepsilon\beta b}{4h^2}A_n^2 = w\omega\varepsilon\beta b\left(\frac{bA_n}{2n\pi}\right)^2 \qquad \cdots(1)$$

  - Surface current densities on two plates (of same magnitude!)

$$\left|J_{Sz}^0\right| = \left|H_x^0\left(y=0\right)\right| = \frac{\omega\varepsilon A_n}{h} = \frac{\omega\varepsilon b A_n}{n\pi}$$

  - Total power loss per unit length in two plates of width w

$$P_L(z) = 2w\left(\frac{1}{2}\left|J_{Sz}^0\right|^2 R_s\right) = w\left(\frac{\omega\varepsilon b A_n}{n\pi}\right)^2 R_s \qquad \cdots(2)$$

Chap. 9-3

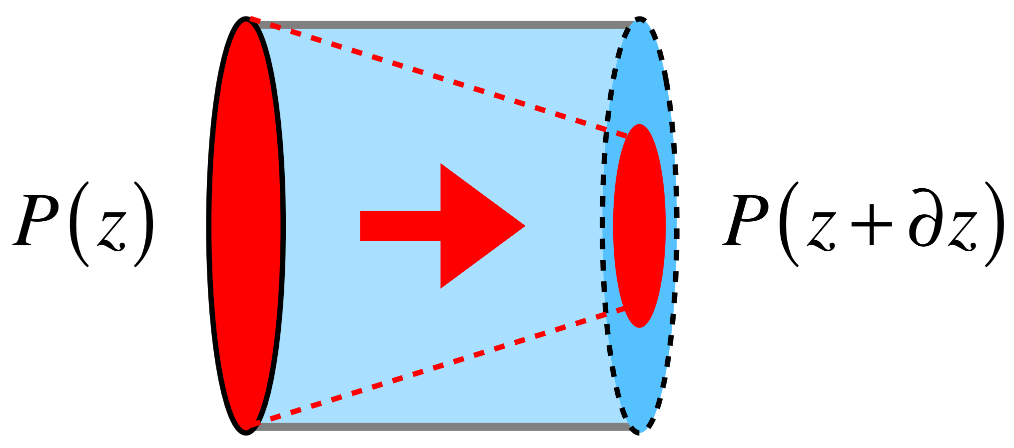
$$P(z) = \frac{1}{2}\mathrm{Re}\left[V(z)I^*(z)\right] = \frac{V_0^2}{2\left|Z_0\right|^2}R_0 e^{-2\alpha z} \qquad \begin{cases} V(z) = V_0 e^{-(\alpha+j\beta)z} \\ I(z) = I_0 e^{-(\alpha+j\beta)z} \\ Z_0 = R_0 + jX_0 \end{cases}$$

Law of conservation

$$-\frac{\partial P(z)}{\partial z} = P_L(z) = 2\alpha P(z)$$ ***: Rate of decrease of P(z) with distance along the line = time-average power loss per unit length***

TM in parallel-plate

$$\begin{cases} E_z^0(y) = A_n \sin\left(\dfrac{n\pi y}{b}\right) \\ E_y^0(y) = -\dfrac{\gamma}{h^2}A_n \cos\left(\dfrac{n\pi y}{b}\right) \\ H_x^0(y) = \dfrac{j\omega\varepsilon}{h}A_n \cos\left(\dfrac{n\pi y}{b}\right) \end{cases}$$



Chap. 10 | Attenuation in parallel-plate waveguides (4/4)

- *TM modes*
  - By having Eqns. (1) and (2) into α_c,

$$\alpha_c = \frac{P_L(z)}{2P(z)} = \frac{2\omega\varepsilon R_s}{\beta b} = \frac{2R_s}{\eta b\sqrt{1-(f_c/f)^2}} = \boxed{\frac{2}{\eta b}\sqrt{\frac{\pi\mu_c}{\sigma_c}}\sqrt{\frac{f}{1-(f_c/f)^2}}}$$

TE in parallel-plate

$$\begin{cases} H_z^0(y) = B_n\cos\left(\frac{n\pi y}{b}\right) \\ E_x^0(y) = \frac{j\omega\mu}{h}B_n\sin\left(\frac{n\pi y}{b}\right) \\ H_y^0(y) = \frac{\gamma}{h}B_n\sin\left(\frac{n\pi y}{b}\right) \end{cases}$$

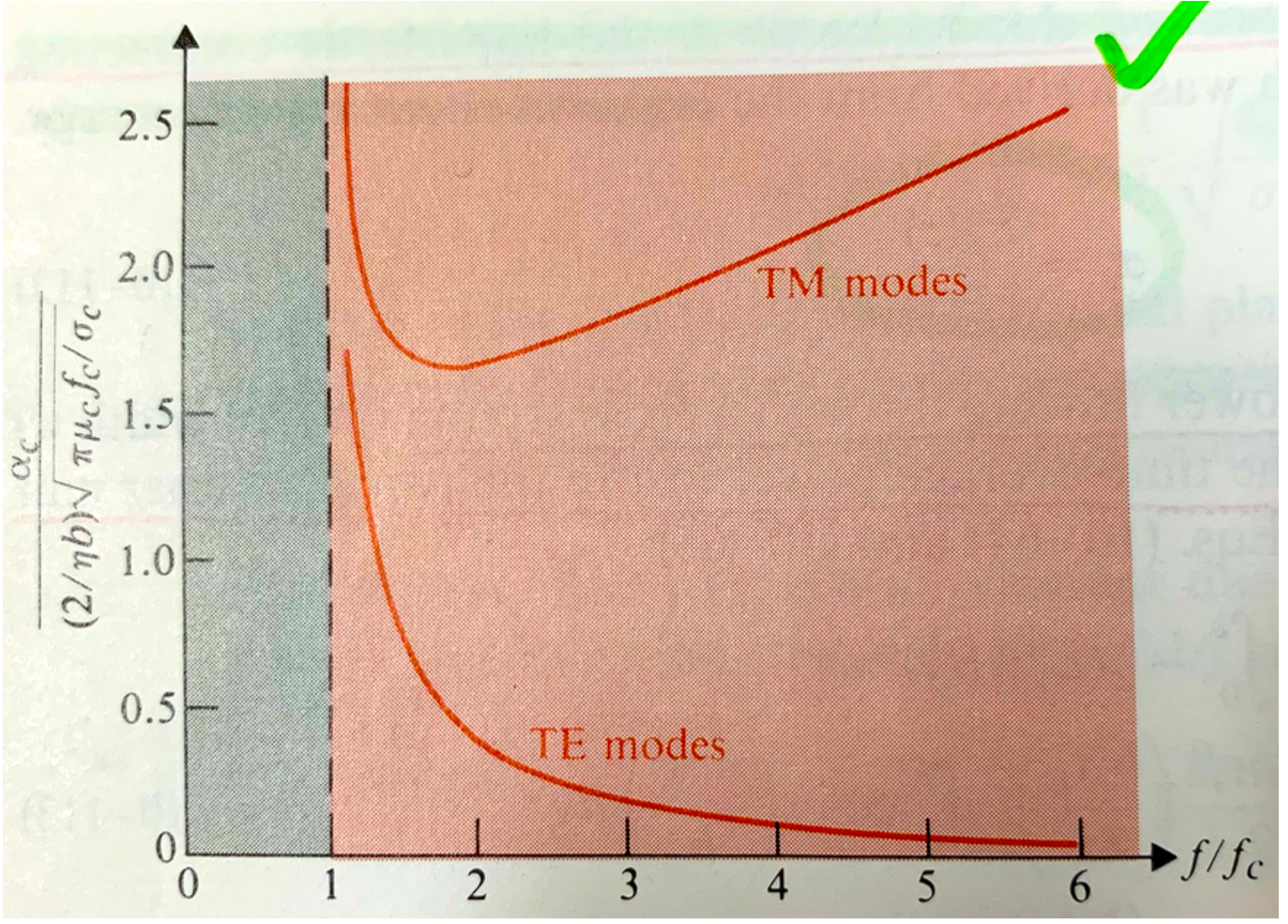
$$R_s = \sqrt{\frac{\pi f\mu_c}{\sigma_c}}\quad(\Omega)$$ *Eq. (9-26b)*

- *TE modes*
  - Since $\gamma_{TE} = \gamma_{TM} \rightarrow (\alpha_d)_{TE} = (\alpha_d)_{TM} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\sqrt{1-(f_c/f)^2}}$ ***(Due to dielectric loss)***
  - α_c *due to imperfectly conducting walls*

$$P(z) = \int_S \boldsymbol{P}_{av}\cdot d\boldsymbol{s} = w\int_0^b(\boldsymbol{P}_{av}\cdot\boldsymbol{a_z})dy = \frac{w}{2}\int_0^b E_x^0 H_y^{0*}\,dy = \frac{w\omega\mu\beta}{2}\left(\frac{bB_n}{n\pi}\right)^2\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy = w\omega\mu\beta b\left(\frac{bB_n}{2n\pi}\right)^2$$

$$P_L(z) = 2w\left(\frac{1}{2}\left|J_{Sx}^0\right|^2 R_s\right) = w\left|H_z^0(y=0)\right|^2 R_s = wB_n^2 R_s$$

$$\therefore \alpha_c = \frac{P_L(z)}{2P(z)} = \frac{2R_s}{\omega\mu\beta b}\left(\frac{n\pi}{b}\right)^2 = \frac{2R_s f_c^2}{\eta b f^2\sqrt{1-(f_c/f)^2}} = \boxed{\frac{2}{\eta b}\sqrt{\frac{\pi\mu_c}{\sigma_c}}\frac{f_c^2}{f^{3/2}\sqrt{1-(f_c/f)^2}}}$$



Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.3 ~ 10.4

(1st of **week 7**)

Jaesang Lee
Dept. of Electrical and Computer Engineering
Seoul National University
(email: jsanglee@snu.ac.kr)

Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

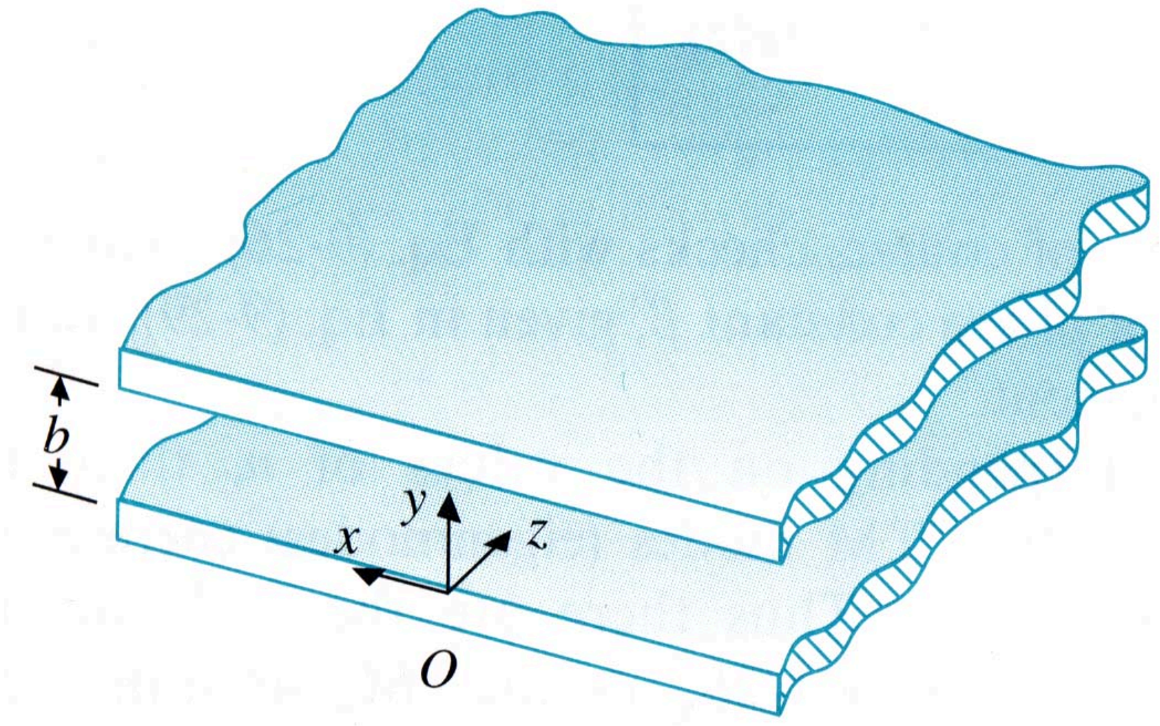
Chap. 10 | Contents for 2nd class of week 7

Sec 4. Rectangular waveguide

- Characteristics of TE and TM wave propagation

- Attenuation in the waveguide

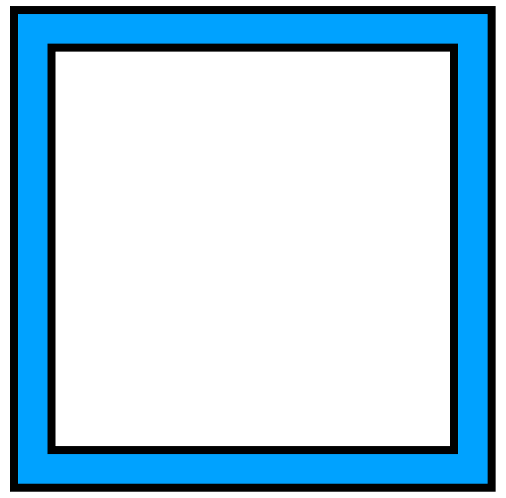
Chap. 10 | Introduction



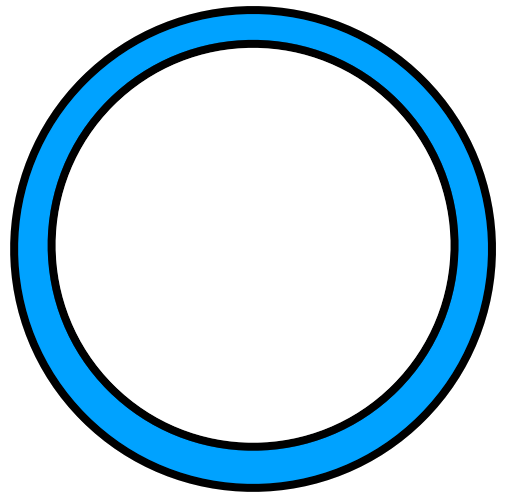
<Parallel-plate waveguide>

- ***Previously in a parallel-plate waveguide…***
  - Two assumptions
    - ***Infinite in extent*** in *x* direction → Fields do not vary in x-direction → $\frac{\partial \boldsymbol{E}}{\partial x} = 0, \ \frac{\partial \boldsymbol{H}}{\partial x} = 0 \ (\boldsymbol{E} \neq 0, \boldsymbol{H} \neq 0)$
    - Edge effects negligible
  - In practical cases
    - Dimensions are always ***finite*** (i.e. finite width)
    - ***Fringing fields*** exist → i) EM leak through the sides of a guide, ii) Undesirable coupling to other circuits and systems
    - Practical waveguide: A uniform dielectric ***enclosed*** by metallic skin

- ***Simplest structures***



<Rectangular waveguide>



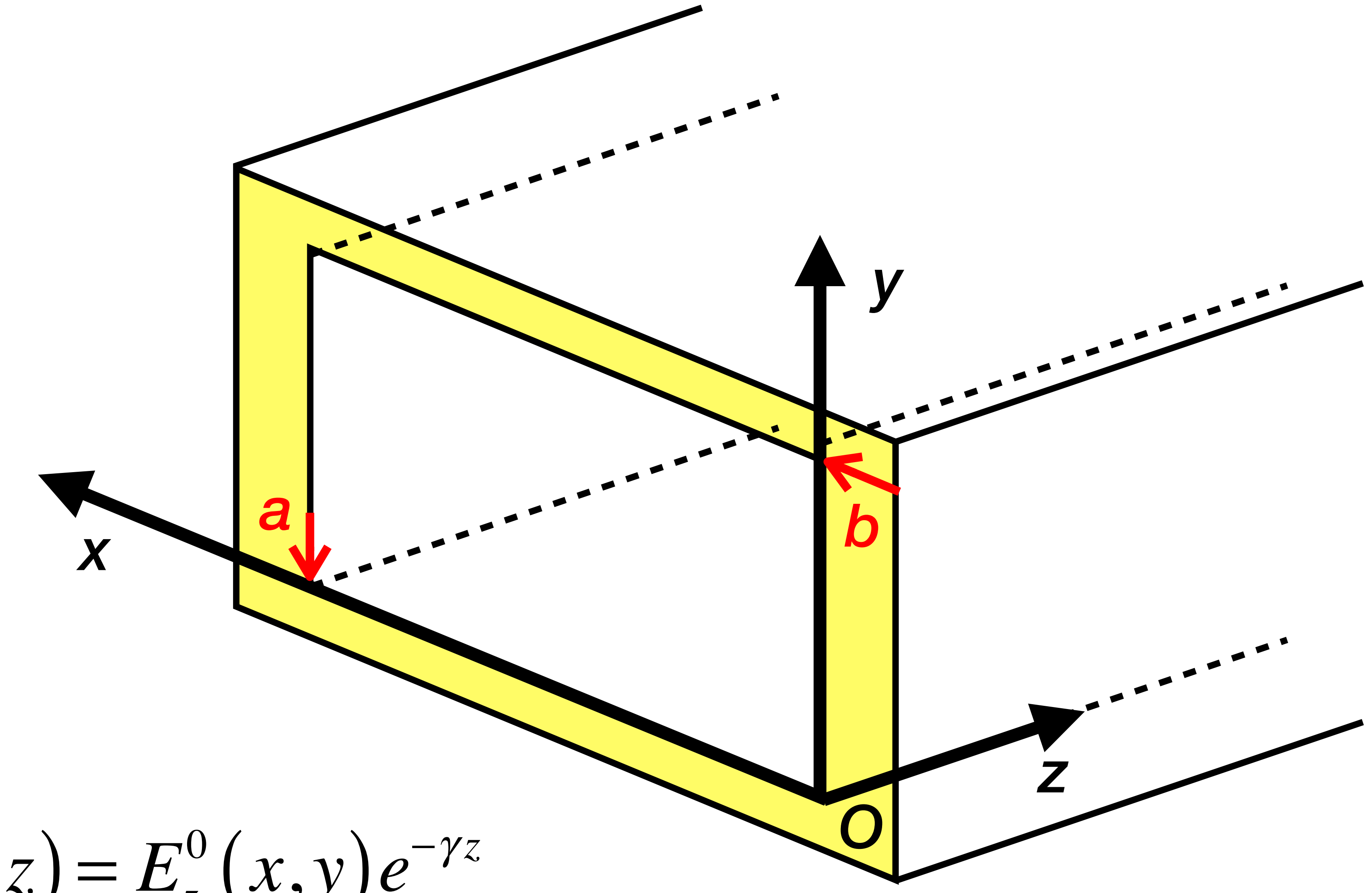
<Circular waveguide>

- Wave behavior in such waveguides
  - TM & TE modes are supported
  - ***TEM mode CANNOT be supported (why?)***
- Rectangular waveguide ***more commonly used in RF/microwave*** than circular waveguide
  - It is desirable to operate waveguides with ***only one allowed mode (i.e. a dominant mode)***
  - Rectangular has a ***larger bandwidth*** than circular for a single mode

Img src: Fairview Microwave

Chap. 10 | TM waves in rectangular waveguide (1/4)

- ***Rectangular waveguide***
  - A waveguide of ***rectangular cross-section*** of widths *a* and *b*
  - Dielectric (*μ* and *ε*) enclosed by metallic skin



- ***Longitudinal field components***
  - $H_z^0 = 0$ (***By definition***)
  - Wave equation for E_z^0

$$\nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0 \quad \rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right) E_z^0 = 0 \quad \cdots(1)$$

where $E_z^0(x,y,z) = E_z^0(x,y)e^{-\gamma z}$

c.f.) $E_z^0(y,z) = E_z^0(y)e^{-\gamma z}$ for a parallel plate waveguide

  - Separation of variables

$$E_z^0(x,y) = X(x)Y(y) \quad \cdots(2)$$

  - By substituting (2) into (1), we have

$$Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)h^2 = 0 \quad \rightarrow \quad \underbrace{-\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\triangleq k_x^2} = \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + h^2}_{\triangleq -k_y^2}$$

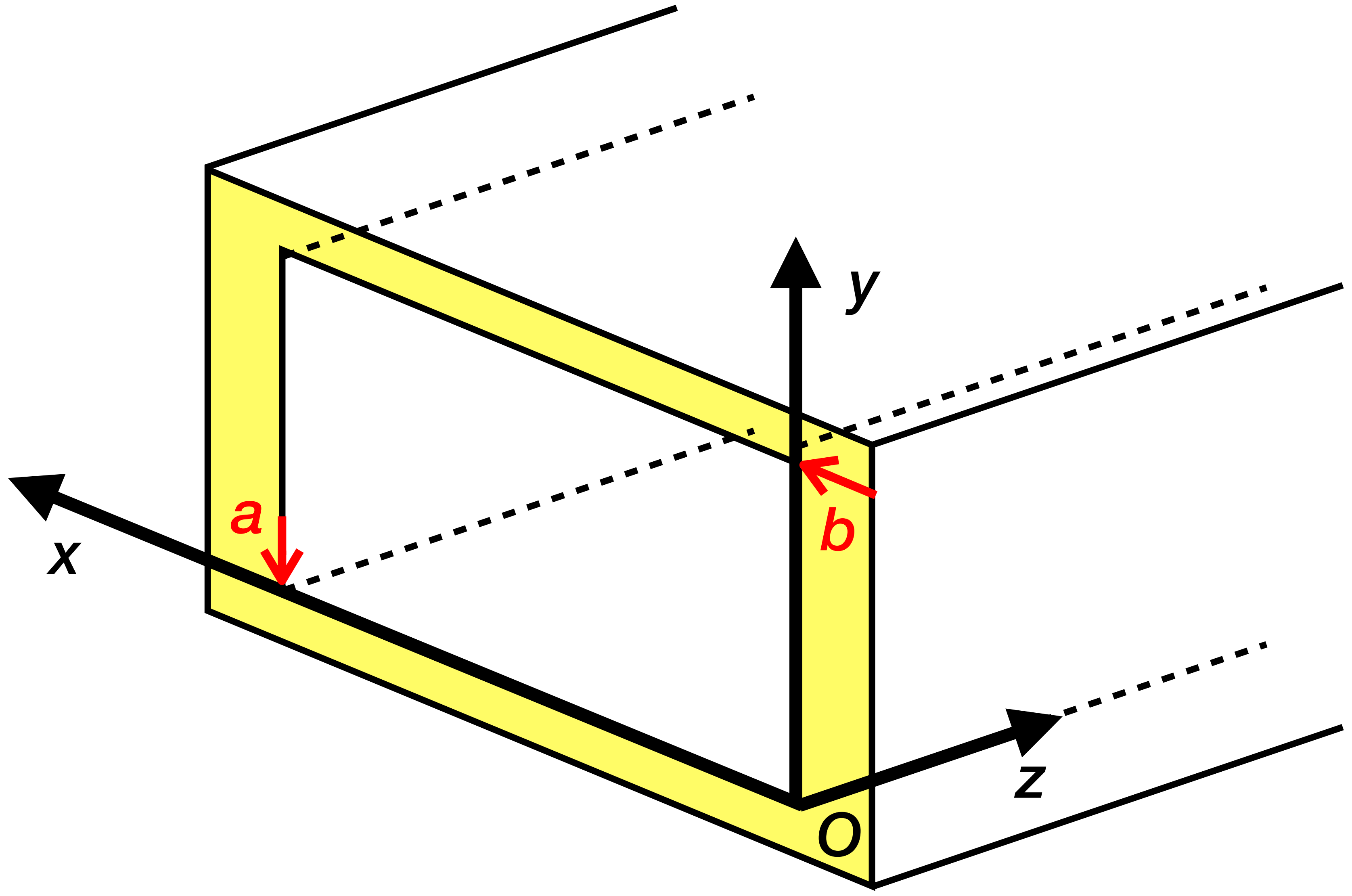
Both sides are equal to constants to hold for all x, y!

Chap. 10 | TM waves in rectangular waveguide (2/4)

- ***Longitudinal field components***

  - Two separable ODEs

$$E_z^0(x,y)=X(x)Y(y) \quad \rightarrow \quad \begin{cases} \dfrac{d^2X(x)}{dx^2}+k_x^2X(x)=0 \\ \dfrac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0 \end{cases} \quad \text{where} \quad h^2=k_x^2+k_y^2$$

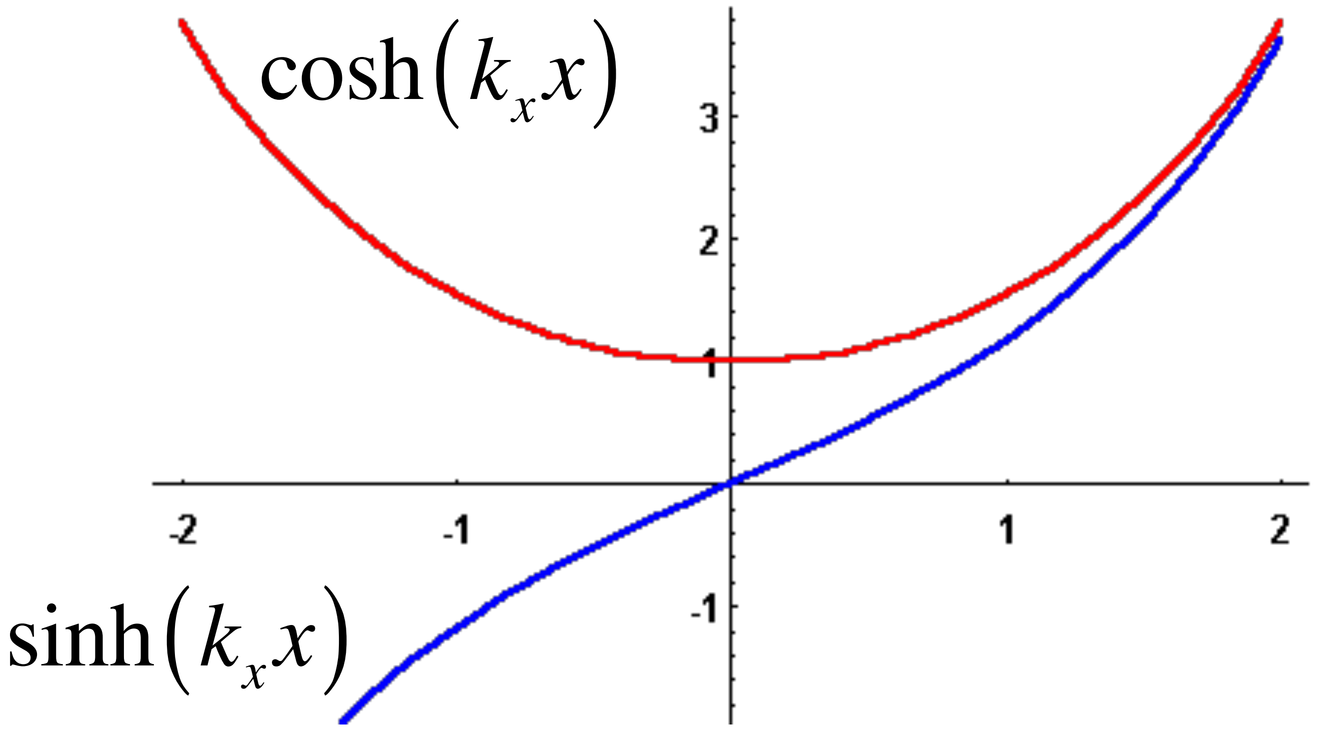


  - Form of solution determined by ***boundary condition***

At the lateral walls, (*x*-direction) $\left[\begin{array}{l} E_z^0(0,y)=0 \\ E_z^0(a,y)=0 \end{array}\right.$ At the vertical walls, (*y*-direction) $\left[\begin{array}{l} E_z^0(x,0)=0 \\ E_z^0(x,b)=0 \end{array}\right.$

→ *X*(*x*) and *Y*(*y*) should be in ***sinusoidal forms***, because *E*-fields vanish at both ends!

→ Other forms, sinh(*kx*) and cosh(*kx*) do not vanish, except at *x* = 0



<Hyperbolic functions>

Possible solution forms of $\dfrac{d^2X(x)}{dx^2}+k_x^2X(x)=0$

Condition	k_x	$X(x)$	*Exponential form*
$k_x^2=0$	0	A_0x+B_0	
$k_x^2>0$	k	$A_1\sin kx+B_1\cos kx$	$C_1e^{jkx}+D_1e^{-jkx}$
$k_x^2<0$	jk	$A_2\sinh kx+B_2\cosh kx$	$C_2e^{kx}+D_2e^{-kx}$

Chap. 10 | TM waves in rectangular waveguide (3/4)

- ***Longitudinal field components***

- *X*(*x*) and *Y*(*y*)

$$\begin{cases} X(x) \rightarrow \sin k_x x = \sin\left(\dfrac{m\pi}{a}x\right), \quad m = 1,2,3,\cdots \\ Y(y) \rightarrow \sin k_y y = \sin\left(\dfrac{n\pi}{b}y\right), \quad n = 1,2,3,\cdots \end{cases}$$

$$\leftarrow \left[\begin{aligned} E_z^0(0,y) &= 0 \\ E_z^0(a,y) &= 0 \end{aligned}\right. \qquad \leftarrow \left[\begin{aligned} E_z^0(x,0) &= 0 \\ E_z^0(x,b) &= 0 \end{aligned}\right.$$

$$\boxed{\begin{aligned} &\therefore E_z^0(x,y) = X(x)Y(y) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \\ &\text{where} \quad \begin{aligned} m &= 1,2,3,\cdots \\ n &= 1,2,3,\cdots \end{aligned} \end{aligned}}$$

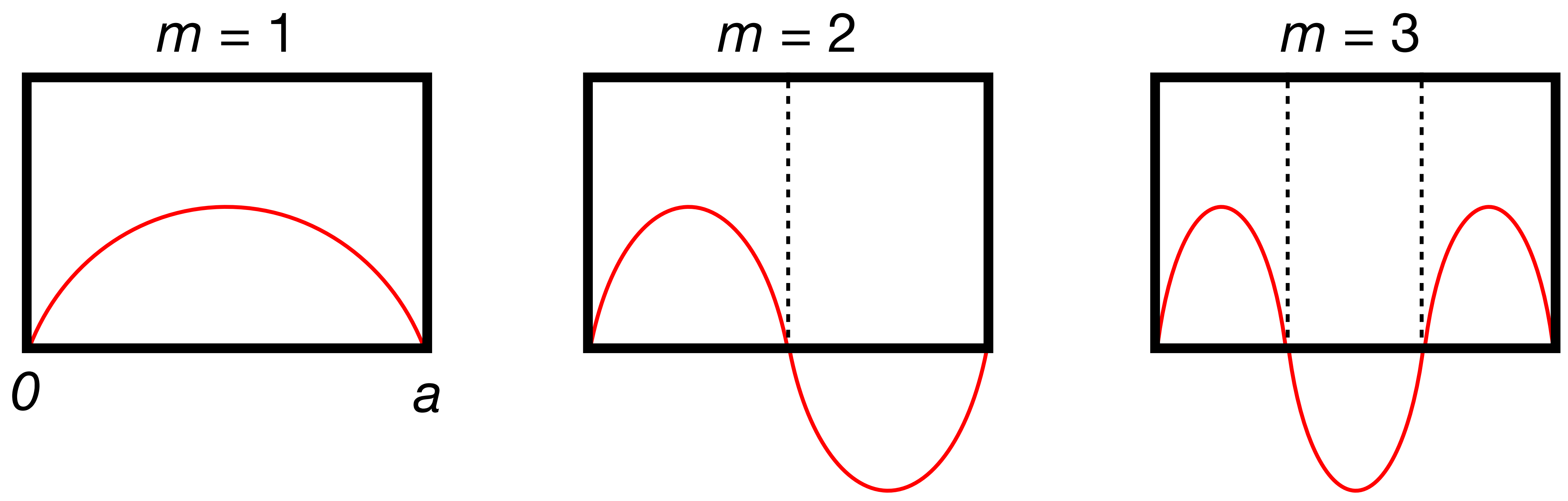
- Eigenvalues

$$\boxed{h^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Depending on the geometry of a waveguide!

- Meaning of integer *m*, *n*

$$\sin(k_x x) = \sin\left(\frac{m\pi}{a}x\right) \rightarrow \lambda_x = \frac{2\pi}{k_x} = \frac{2a}{m}$$



- *m* and *n*: ***Number of half-cycle variations*** of the fields in *x*, *y* directions
- A combination of *m* and *n* determines TM$_{mn}$ mode characteristics!

Chap. 10 | TM waves in rectangular waveguide (4/4)

- *Transverse field components*

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial x} + j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial y}}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \dfrac{\partial E_z^0}{\partial x}\right) \end{cases}$$

$$H_z^0(x,y) = 0$$

$$E_z^0(x,y) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$\longrightarrow$$

where $\gamma = j\beta = j\sqrt{k^2 - h^2}$

$$= j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\boxed{\begin{cases} E_x^0(x,y) = -\dfrac{\gamma}{h^2}\left(\dfrac{m\pi}{a}\right)E_0 \cos\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right) \\ E_y^0(x,y) = -\dfrac{\gamma}{h^2}\left(\dfrac{n\pi}{b}\right)E_0 \sin\left(\dfrac{m\pi}{a}x\right)\cos\left(\dfrac{n\pi}{b}y\right) \\ H_x^0(x,y) = \dfrac{j\omega\varepsilon}{h^2}\left(\dfrac{n\pi}{b}\right)E_0 \sin\left(\dfrac{m\pi}{a}x\right)\cos\left(\dfrac{n\pi}{b}y\right) \\ H_y^0(x,y) = -\dfrac{j\omega\varepsilon}{h^2}\left(\dfrac{m\pi}{a}\right)E_0 \cos\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right) \end{cases}}$$

- *Cutoff frequency (γ = 0)*

$$(f_c)_{mn} = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- *Cutoff wavelength*

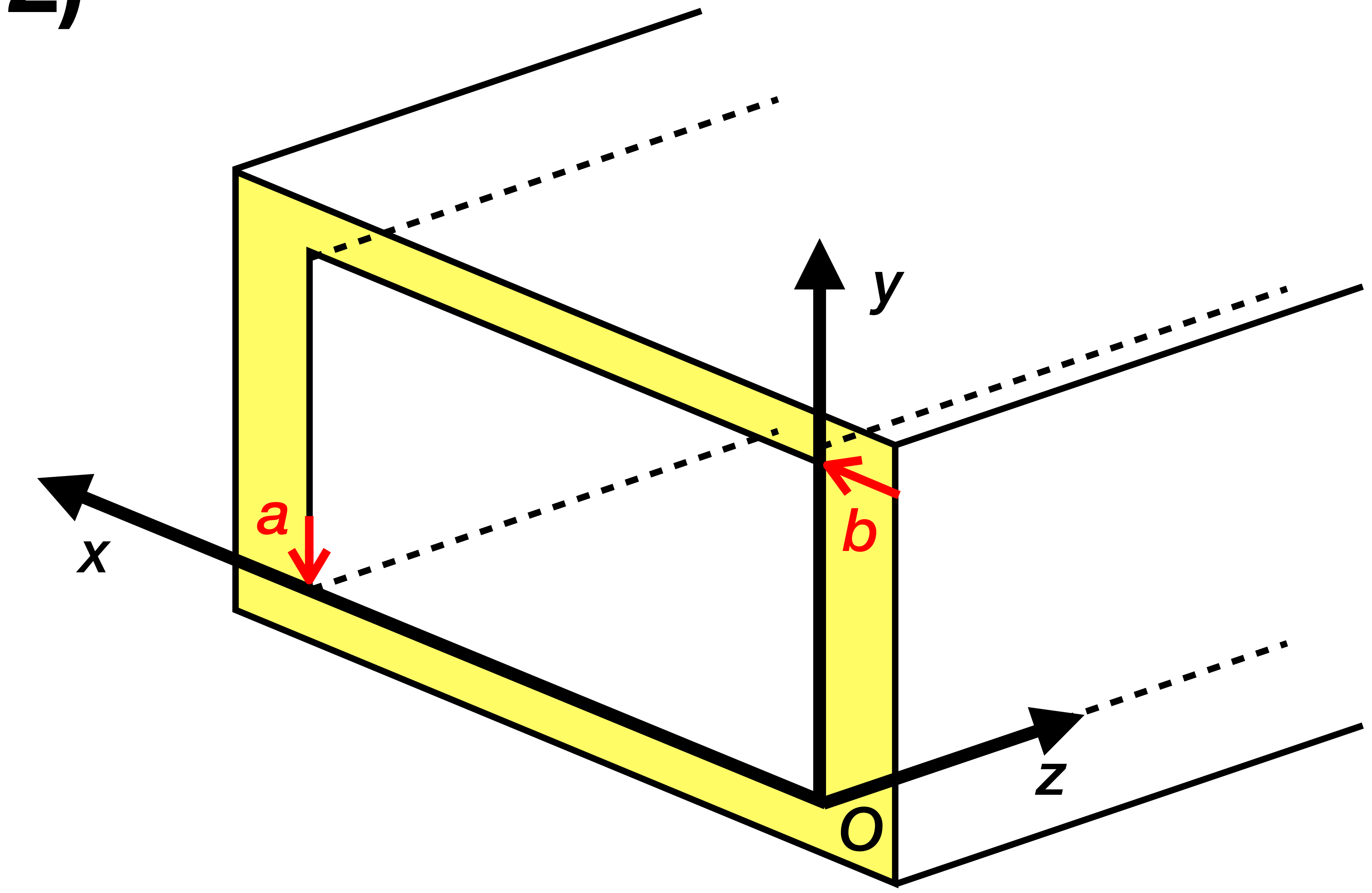
$$(\lambda_c)_{mn} = \frac{u}{f} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

- ***Lowest cutoff frequency***
  - If either *m* = 0 or *n* = 0, $E_z^0 = 0$ → TM_{00}, TM_{01}, TM_{10} = TEM
  - *TEM mode CANNOT be supported by a single-conductor waveguide!*

∴ TM_{11} mode = lowest cutoff frequency *"among TM modes"*

→ *Is it a dominant mode or not? (→ cannot know yet)*

Chap. 10 | TE waves in rectangular waveguide (1/2)



- ***Longitudinal components***
  - $E_z^0 = 0$ (***By definition***)
  - Wave equation for H_z^0

$$\nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0 \quad \rightarrow \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+h^2\right)H_z^0 = 0 \quad \cdots(1)$$

where $H_z^0(x,y,z) = H_z^0(x,y)e^{-\gamma z}$

- ***B.C.*** provided by transverse fields components

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma\cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu\dfrac{\partial H_z^0}{\partial y}\right) & \cdots(2) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma\cancel{\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu\dfrac{\partial H_z^0}{\partial x}\right) & \cdots(1) \end{cases}$$

$$\begin{cases} E_x^0(x,y) = -\dfrac{\gamma}{h^2}\left(\dfrac{m\pi}{a}\right)E_0\cos\left(\dfrac{m\pi}{a}x\right)\underline{\sin\left(\dfrac{n\pi}{b}y\right)} \\ E_y^0(x,y) = -\dfrac{\gamma}{h^2}\left(\dfrac{n\pi}{b}\right)E_0\underline{\sin\left(\dfrac{m\pi}{a}x\right)}\cos\left(\dfrac{n\pi}{b}y\right) \end{cases}$$

- At the lateral walls (x = 0 and x = a)

$$(1)\begin{cases} \left.\dfrac{\partial H_z^0}{\partial x}\right|_{(0,y)} = \dfrac{h^2}{j\omega\mu}E_y^0(0,y) = 0 \\ \left.\dfrac{\partial H_z^0}{\partial x}\right|_{(a,y)} = \dfrac{h^2}{j\omega\mu}E_y^0(a,y) = 0 \end{cases}$$

$$\frac{\partial H_z^0}{\partial x} \quad \rightarrow \quad \sin k_x x = \sin\left(\frac{m\pi}{a}x\right)$$

- At the vertical walls (y = 0 and y = b)

$$(2)\begin{cases} \left.\dfrac{\partial H_z^0}{\partial y}\right|_{(x,0)} = -\dfrac{h^2}{j\omega\mu}E_x^0(x,0) = 0 \\ \left.\dfrac{\partial H_z^0}{\partial y}\right|_{(x,b)} = -\dfrac{h^2}{j\omega\mu}E_x^0(x,b) = 0 \end{cases}$$

$$\frac{\partial H_z^0}{\partial y} \quad \rightarrow \quad \sin k_y y = \sin\left(\frac{n\pi}{b}y\right)$$

$$\boxed{\therefore H_z^0(x,y) = H_0\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)}$$

Chap. 10 | TE waves in rectangular waveguide (2/2)

- *Transverse components*

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu\dfrac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\cancel{\gamma\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu\dfrac{\partial H_z^0}{\partial x}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial x} - j\omega\varepsilon\cancel{\dfrac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma\dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon\cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

$$E_z^0(x,y) = 0$$

$$H_z^0(x,y) = H_0\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$\longrightarrow$$

where $\gamma = j\beta = j\sqrt{k^2 - h^2}$

$$= j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\boxed{\begin{cases} E_x^0(x,y) = \dfrac{j\omega\mu}{h^2}\left(\dfrac{n\pi}{b}\right)H_0\cos\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right) \\ E_y^0(x,y) = -\dfrac{j\omega\mu}{h^2}\left(\dfrac{m\pi}{a}\right)H_0\sin\left(\dfrac{m\pi}{a}x\right)\cos\left(\dfrac{n\pi}{b}y\right) \\ H_x^0(x,y) = \dfrac{\gamma}{h^2}\left(\dfrac{m\pi}{a}\right)H_0\sin\left(\dfrac{m\pi}{a}x\right)\cos\left(\dfrac{n\pi}{b}y\right) \\ H_x^0(x,y) = \dfrac{\gamma}{h^2}\left(\dfrac{n\pi}{b}y\right)H_0\cos\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right) \end{cases}}$$

- *Cutoff frequency*
  - Either *m* or *n* can be zero (Not both! → Why?)
  - Lowest cutoff frequency: If *a* > *b*, ***TE₁₀ mode*** has the lowest *f*c

$$(f_c)_{mn} = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

- ***Is* TE₁₀ *mode a dominant mode?***
  - ***Yes! (Although it was not shown why yet)***
  - TE₁₀ has the ***lowest attenuation coefficient*** (*Shown later*)

$$\boxed{\therefore (f_c)_{10} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{u}{2a} \quad (\text{Hz})} \quad \leftrightarrow \quad \boxed{(\lambda_c)_{10} = 2a \quad (\text{m})}$$

Longest wavelength that can propagate!

Chap. 10 | Surface current for TE mode

Example 10-8 For TE$_{01}$ mode, obtain the surface current on the guide walls at t = 0

- Surface current provided by boundary condition for the H-field, $\boldsymbol{J}_s = \boldsymbol{a_n} \times \boldsymbol{H}$
- Expressions for ***instantaneous field components*** are given by

$$\begin{cases} E_x^0(x,y,z,t)=0 \\ E_y^0(x,y,z,t)=\dfrac{\omega\mu}{h^2}\left(\dfrac{\pi}{a}\right)H_0\sin\left(\dfrac{\pi}{a}x\right)\sin(\omega t-\beta z) \\ E_z^0(x,y,z,t)=0 \longleftarrow \textit{By definition} \end{cases}$$

$$\begin{cases} H_x^0(x,y,z,t)=-\dfrac{\beta}{h^2}\left(\dfrac{\pi}{a}\right)H_0\sin\left(\dfrac{\pi}{a}x\right)\sin(\omega t-\beta z) \\ H_y^0(x,y,z,t)=0 \\ H_z^0(x,y,z,t)=H_0\cos\left(\dfrac{\pi}{a}x\right)\cos(\omega t-\beta z) \end{cases}$$

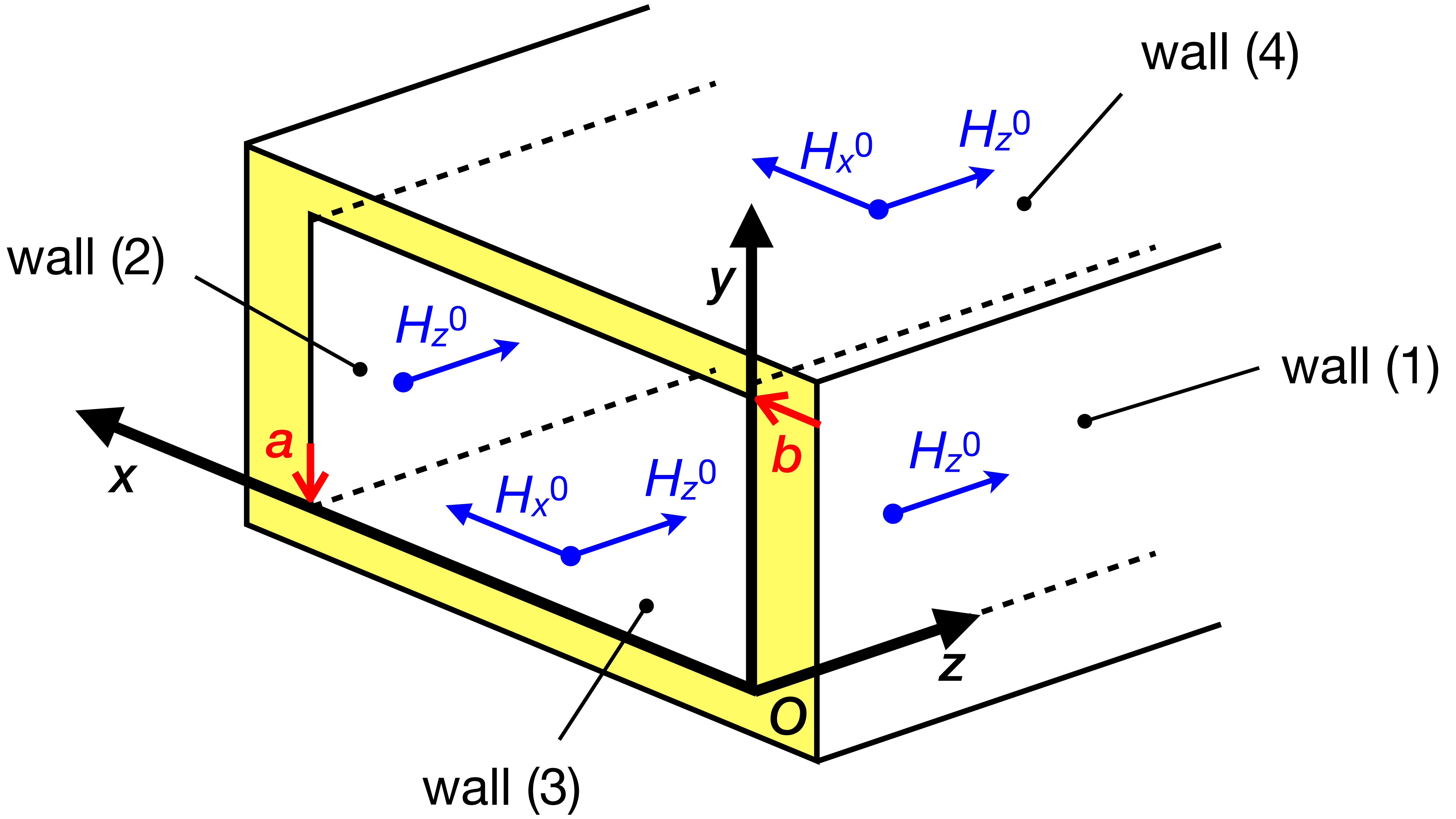
$$\text{(1) } \boldsymbol{J}_s(x=0)=\boldsymbol{a_x}\times\boldsymbol{a_z}H_z^0(0,y,z,0)=-\boldsymbol{a_y}\cos\beta z$$

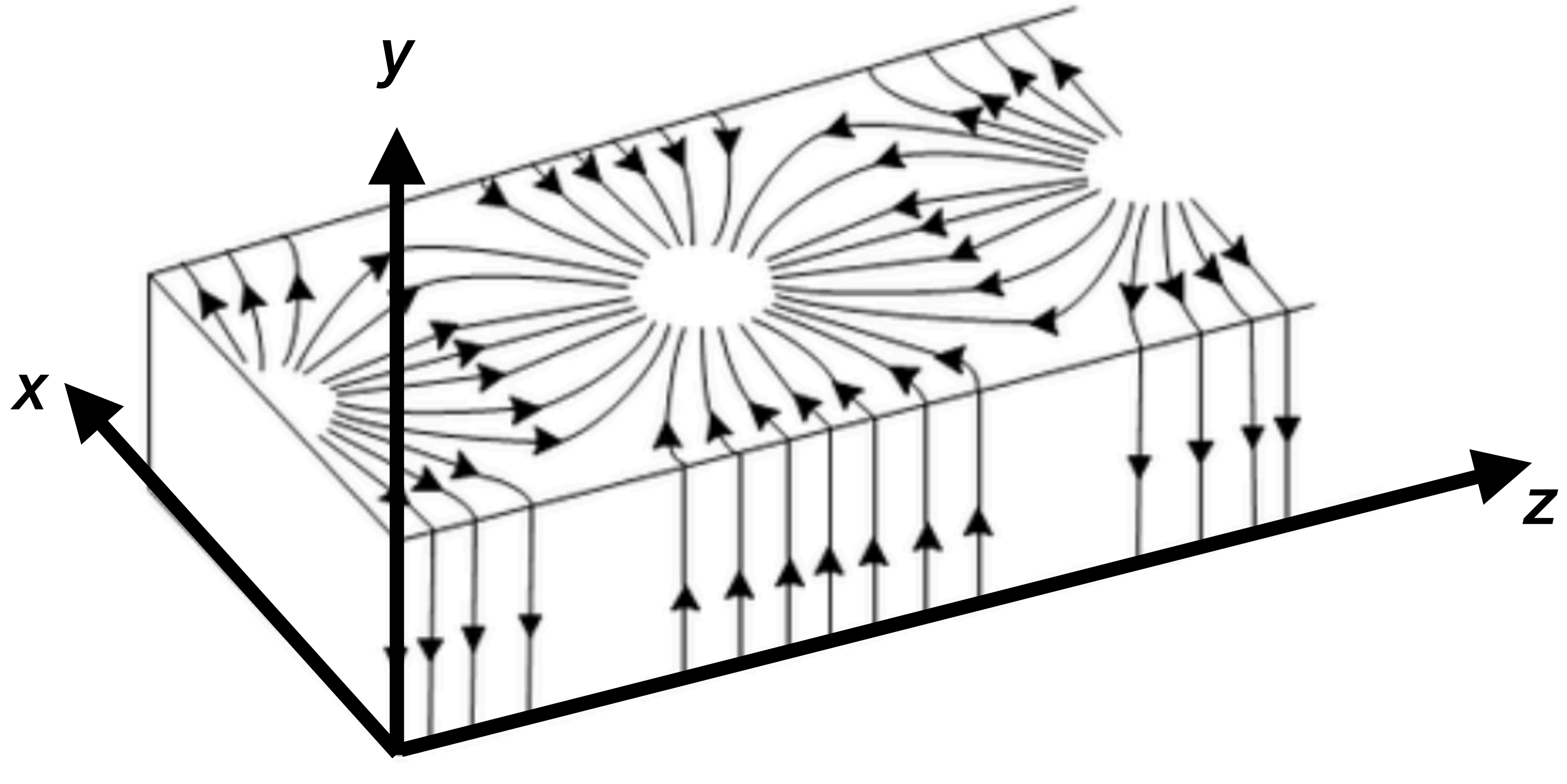
$$\text{(2) } \boldsymbol{J}_s(x=a)=-\boldsymbol{a_x}\times\boldsymbol{a_z}H_z^0(a,y,z,0)=-\boldsymbol{a_y}\cos\beta z$$

$$\text{(3) } \boldsymbol{J}_s(y=0)=\boldsymbol{a_y}\times\boldsymbol{a_z}H_z^0(x,0,z,0)+\boldsymbol{a_y}\times\boldsymbol{a_x}H_x^0(x,0,z,0)$$

$$=\boldsymbol{a_x}H_0\cos\left(\frac{\pi}{a}x\right)\cos\beta z-\boldsymbol{a_z}\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_0\sin\left(\frac{\pi}{a}x\right)\sin\beta z$$

$$\text{(4) } \boldsymbol{J}_s(y=b)=-\boldsymbol{J}_s(y=0)$$





<Surface current for TE$_{10}$ mode>

Chap. 10 | Operating frequency range for TE$_{01}$ mode

Example 10-9 Obtain the range of operating frequency for a standard air-filled waveguide for radar bands “X”

WG-16 for X-band, a certain type of the rectangular waveguide, has widths of ***a* = 2.29 (cm), *b* = 1.02 (cm).**

WG-16 has to ***operate only in the dominant TE$_{10}$ mode*** & its frequency should be $1.25(f_c)_{TE_{10}} \le f \le 0.95(f_c)_{TE_{mn}}$

Here, TE$_{mn}$ is the mode of the next higher cutoff frequency.

$$(f_c)_{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

where *c* is the speed of light

mn	$(f_c)_{mn}$
10	**6.55 GHz**
01	14.7 GHz
11	16.1 GHz
20	**13.1 GHz**
02	29.4 GHz

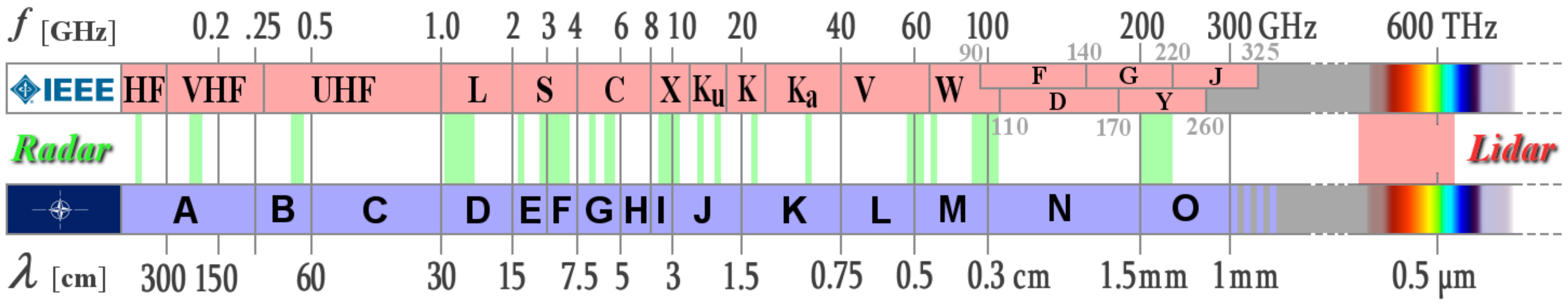
← ***Lowest: “Dominant” mode*** (10)

← ***Next lowest*** (20)

$$1.25(f_c)_{TE_{10}} \le f \le 0.95(f_c)_{TE_{mn}}$$

$$\rightarrow 1.25 \times 6.55\ (GHz) \le f \le 0.95 \times 13.1\ (GHz)$$

$$\therefore 8.19\ (GHz) \le f \le 12.5\ (GHz)$$



<Microwave frequency range>

X band (8.0~12.0GHz) is used for radar, satellite communication, and wireless computer networks.

Img src: radartutorial.eu

Chap. 10 | Attenuation in rectangular waveguides (1/4)

- *Attenuation for propagating modes*
  - Loss in *dielectric*
  - Loss in *imperfectly conducting wall*

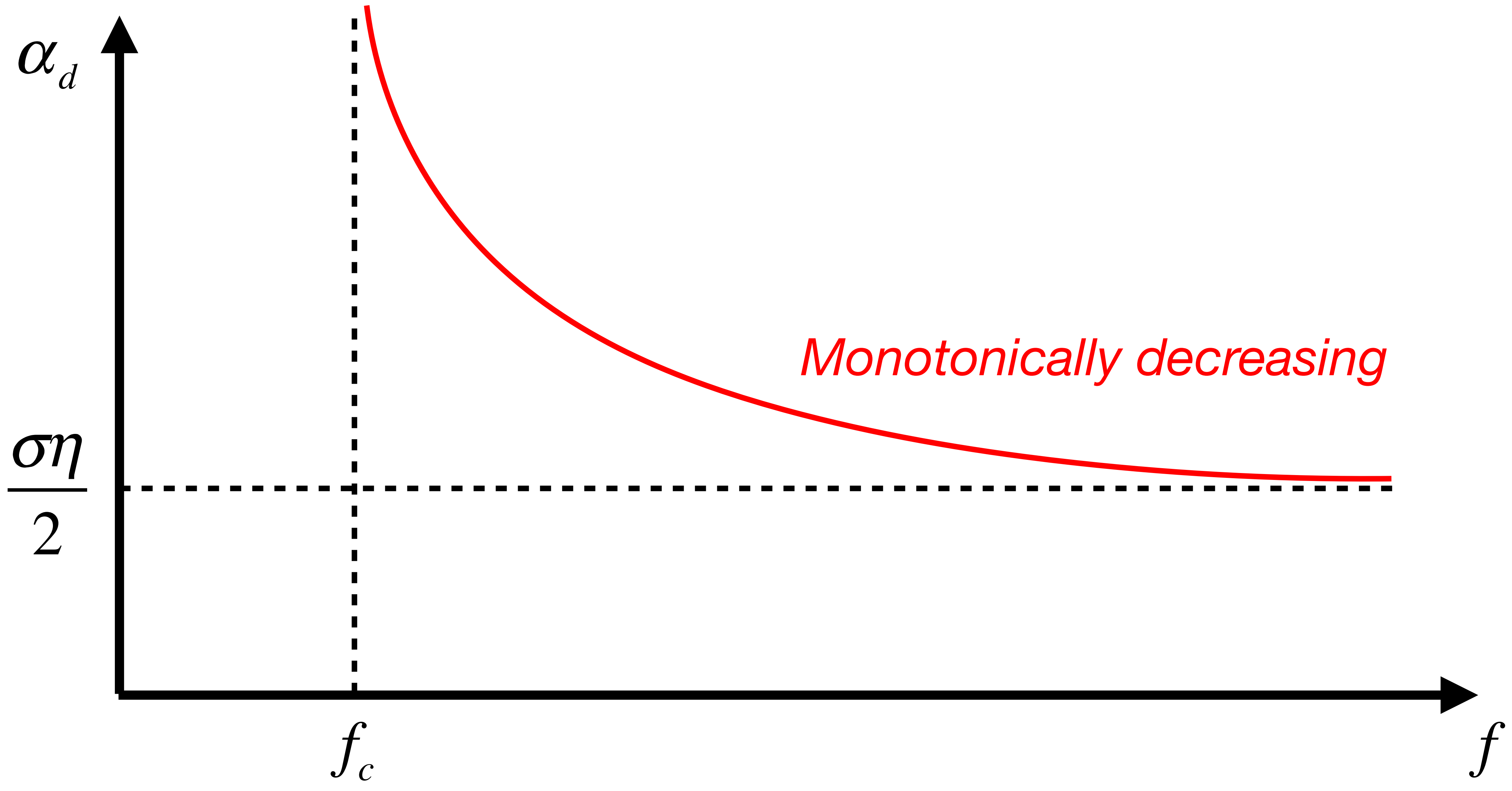
- ***Loss in "dielectric"***

$$\varepsilon \;\rightarrow\; \varepsilon_d = \varepsilon + \frac{\sigma}{j\omega} \;\Rightarrow\; \gamma_d = j\beta_d = j\sqrt{\omega^2\mu\varepsilon_d - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \alpha_d + j\beta$$

For derivation, please refer to last lecture note

$$\boxed{\therefore \alpha_d = \frac{\sigma\eta}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

where σ: conductivity, η: intrinsic impedance of a dielectric



Chap. 10 | Attenuation in rectangular waveguides (2/4)

- *Loss due to imperfectly conducting wall*

  - Attenuation coefficient is given by

$$\alpha_c = \frac{P_L(z)}{2P(z)}$$

(∵Law of conservation)

- P(z): Time-average power flow through cross section

- $P_L(z)$: Time-average power lost in the walls per unit length

- α_c for TM_{mn} is very complicated and not useful
- α_c for TE_{10}, the dominant mode, is more important!

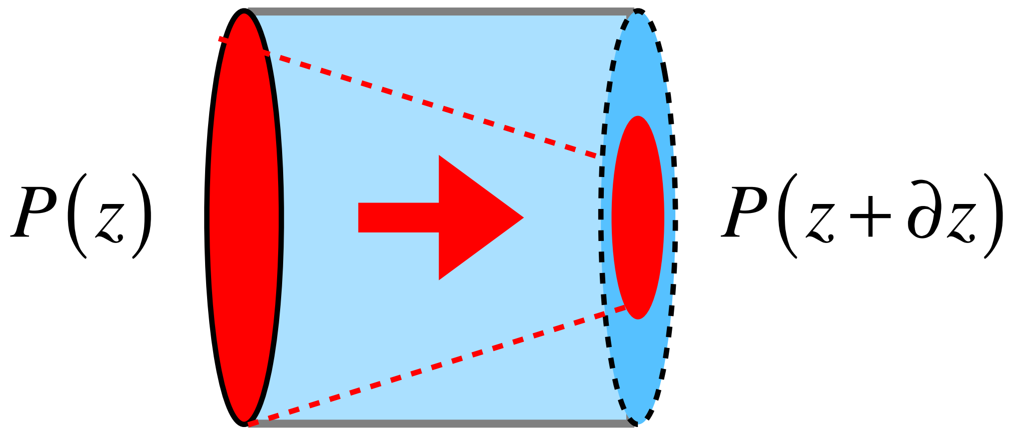
- *P(z)*

$$P(z) = \int_S \boldsymbol{P}_{av} \cdot d\boldsymbol{s} = \int_S \boldsymbol{P}_{av} \cdot \boldsymbol{a_z}\, ds = \int_0^b \int_0^a \frac{1}{2} \mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) \cdot \boldsymbol{a_z}\, dx\, dy$$

$$\frac{1}{2} \mathrm{Re}\left(\boldsymbol{E} \times \boldsymbol{H}^*\right) = \left[\boldsymbol{a_y} C_y \sin\left(\frac{\pi}{a}x\right) \times \left\{\boldsymbol{a_x} C_x \sin\left(\frac{\pi}{a}x\right) + \cancel{\boldsymbol{a_z} C_z^* \cos\left(\frac{\pi}{a}x\right)}\right\}\right]$$

$$= \boldsymbol{a_z} C_x C_y \sin^2\left(\frac{\pi}{a}x\right)$$

$$P(z) = C_x C_y \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx\, dy = \omega\mu\beta ab \left(\frac{aH_0}{2\pi}\right)^2 \quad \cdots(1)$$



Chap. 9-3 Law of conservation

$$P(z) = \frac{1}{2} \mathrm{Re}\left[V(z) I^*(z)\right] = \frac{V_0^{\,2}}{2|Z_0|^2} R_0 e^{-2\alpha z}$$

$$\begin{cases} V(z) = V_0 e^{-(\alpha+j\beta)z} \\ I(z) = I_0 e^{-(\alpha+j\beta)z} \\ Z_0 = R_0 + jX_0 \end{cases}$$

Law of conservation

$$-\frac{\partial P(z)}{\partial z} = P_L(z) = 2\alpha P(z)$$

: Rate of decrease of P(z) with distance along the line = time-average power loss per unit length

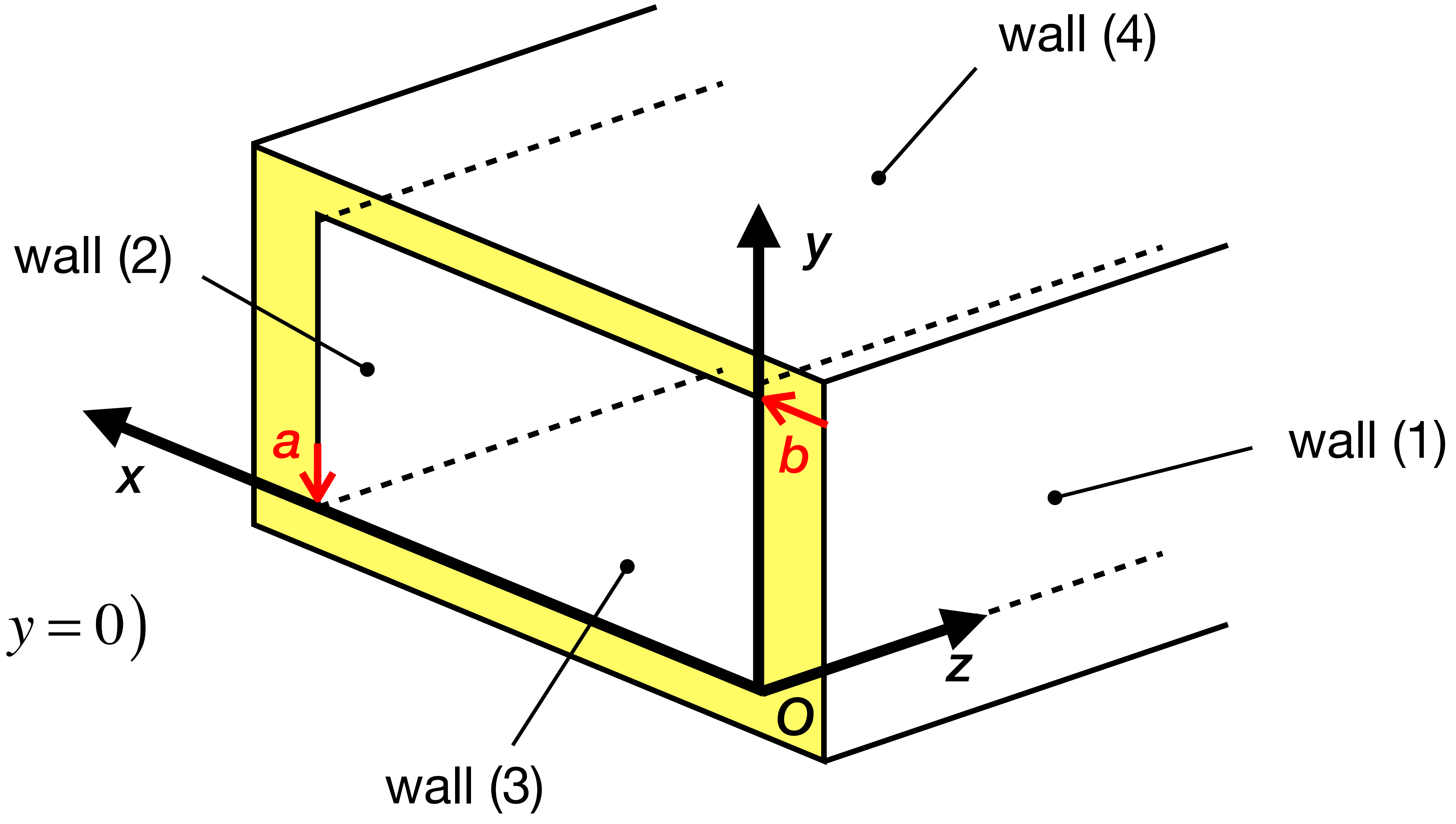
Field for TE_{10} mode

$$\begin{cases} E_x^0(x,y) = 0 \\ E_y^0(x,y) = -\dfrac{j\omega\mu}{h^2}\left(\dfrac{\pi}{a}\right) H_0 \sin\left(\dfrac{\pi}{a}x\right) \\ E_z^0(x,y) = 0 \end{cases}$$

$$\begin{cases} H_x^0(x,y) = -\dfrac{j\beta}{h^2}\left(\dfrac{\pi}{a}\right) H_0 \sin\left(\dfrac{\pi}{a}x\right) \\ H_y^0(x,y) = 0 \\ H_z^0(x,y) = H_0 \cos\left(\dfrac{\pi}{a}x\right) \end{cases}$$

Chap. 10 | Attenuation in rectangular waveguides (3/4)

- *$P_L(z)$: Time-average power loss in the walls per unit length*
  - Consider surface current flowing in all four walls $\boldsymbol{J}_s = \boldsymbol{a_n}\times\boldsymbol{H}$



$$(1)\ \boldsymbol{J}_s(x=0) = \boldsymbol{a_x}\times\boldsymbol{a_z}H_z^0(0,y) = -\boldsymbol{a_y}H_0 \quad (\mathrm{A/m})$$

$$(2)\ \boldsymbol{J}_s(x=a) = -\boldsymbol{a_x}\times\boldsymbol{a_z}H_z^0(a,y) = -\boldsymbol{a_y}H_0$$

$$(3)\ \boldsymbol{J}_s(y=0) = \boldsymbol{a_y}\times\left[\boldsymbol{a_z}H_z^0(x,0) + \boldsymbol{a_x}H_x(x,0)\right]$$

$$= \boldsymbol{a_x}\underline{H_0\cos\left(\frac{\pi}{a}x\right)} - \boldsymbol{a_z}\underline{\frac{j\beta a}{\pi}H_0\sin\left(\frac{\pi}{a}x\right)} = \boldsymbol{a_x}J_{sx}(y=0) - \boldsymbol{a_z}J_{sz}(y=0)$$

$$(4)\ \boldsymbol{J}_s(y=b) = -\boldsymbol{J}_s(y=0)$$

- Total power losses in the walls

$$P_L(z) = 2\left[P_L(z)\right]_{x=0} + 2\left[P_L(z)\right]_{y=0}$$

where $\left[P_L(z)\right]_{x=0} = \int_0^b \frac{1}{2}\left|\boldsymbol{J}_s(x=0)\right|^2 R_s\,dy = \frac{b}{2}H_0^2R_s$

$$\left[P_L(z)\right]_{y=0} = \int_0^a \frac{1}{2}\left|J_{sx}(y=0)\right|^2 R_s + \frac{1}{2}\left|J_{sz}(y=0)\right|^2 R_s\,dy$$

$$= \frac{a}{4}\left[1+\left(\frac{\beta a}{\pi}\right)^2\right]H_0^2R_s \quad \cdots(2)$$

Field for TE

$$\begin{cases} E_x^0(x,y) = 0 \\ E_y^0(x,y) = -\dfrac{j\omega\mu}{h^2}\left(\dfrac{\pi}{a}\right)H_0\sin\left(\dfrac{\pi}{a}x\right) \\ E_z^0(x,y) = 0 \end{cases}$$

$$\begin{cases} H_x^0(x,y) = -\dfrac{j\beta}{h^2}\left(\dfrac{\pi}{a}\right)H_0\sin\left(\dfrac{\pi}{a}x\right) \\ H_y^0(x,y) = 0 \\ H_z^0(x,y) = H_0\cos\left(\dfrac{\pi}{a}x\right) \end{cases}$$

Chap. 10 | Attenuation in rectangular waveguides (4/4)

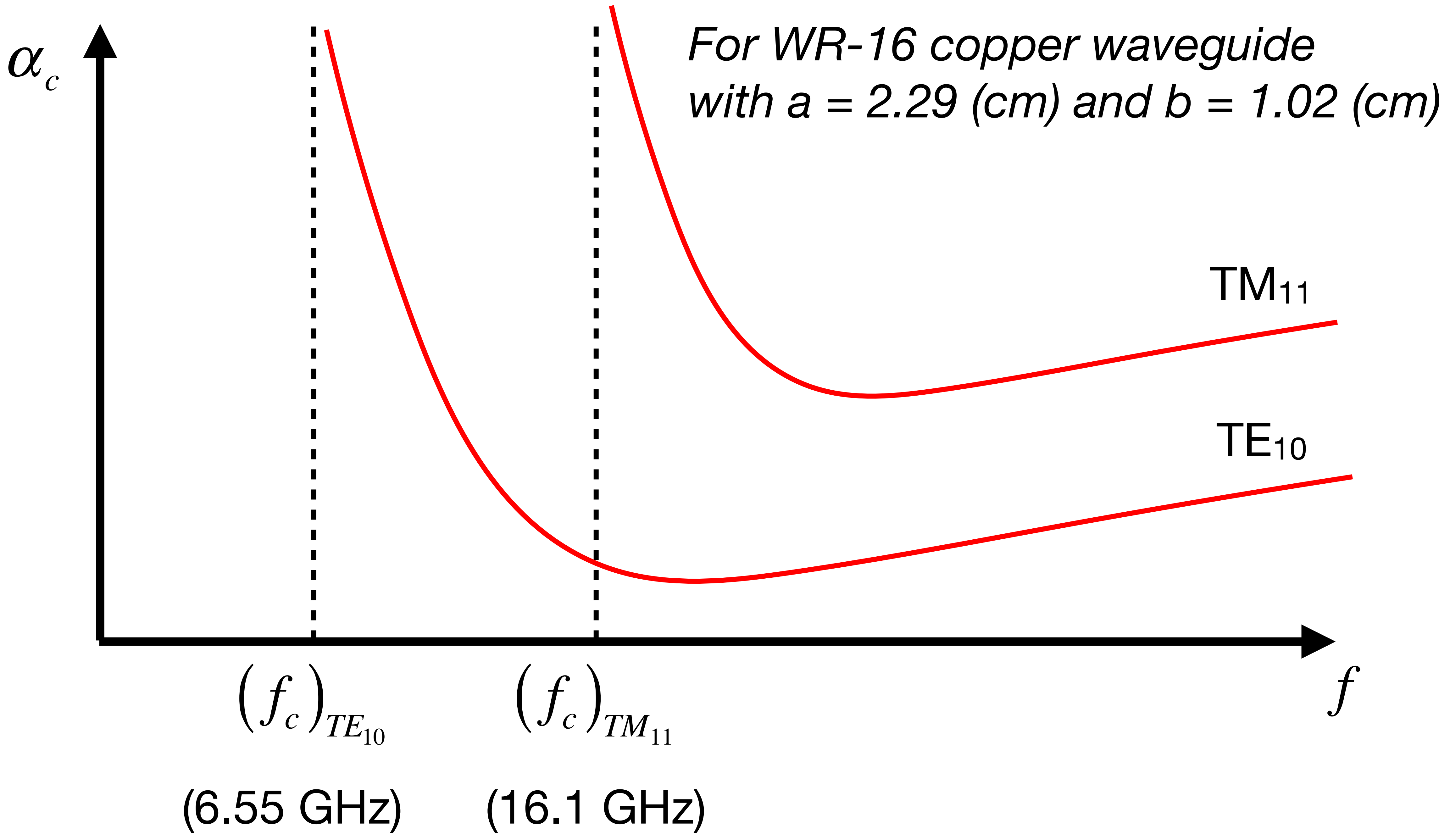
- Attenuation coefficient α_c for TE_{01} mode

$$\boxed{\therefore (\alpha_c)_{TE_{10}} = \frac{P_L(z)}{2P(z)} = \frac{R_s\left[1+\left(\frac{2b}{a}\right)\left(\frac{f_c}{f}\right)^2\right]}{\eta b\sqrt{1-\left(\frac{f_c}{f}\right)^2}}}$$

where $R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$ (Ω)

c.f.) $$(\alpha_c)_{TM_{11}} = \frac{2R_s\left[\frac{b}{a^2}+\frac{a}{b^2}\right]}{\eta ab\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Please try it at home!



For WR-16 copper waveguide with a = 2.29 (cm) and b = 1.02 (cm)

- Analysis on α_c

  - *b*↓ → α_c↓ with a given width of *a* (***Good!***)
  - *But, b*↓ → f_c↓ of next higher order mode (TM_{11} or TE_{20}) (***Bad!***)

    →Available ***bandwidth for*** TE_{10}, a dominant mode, reduced

    (Bandwidth: frequency range where only TE_{10} *mode allowed)*

    ∴Compromise made at *b/a* ≥ 1/2

  - **$(\alpha_c)_{TE10} < (\alpha_c)_{TM11}$ for all *frequency***

    → Reason why the ***dominant mode is used as an operating mode*** over others