Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.5 ~ 10.6

(1st of **week 8**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

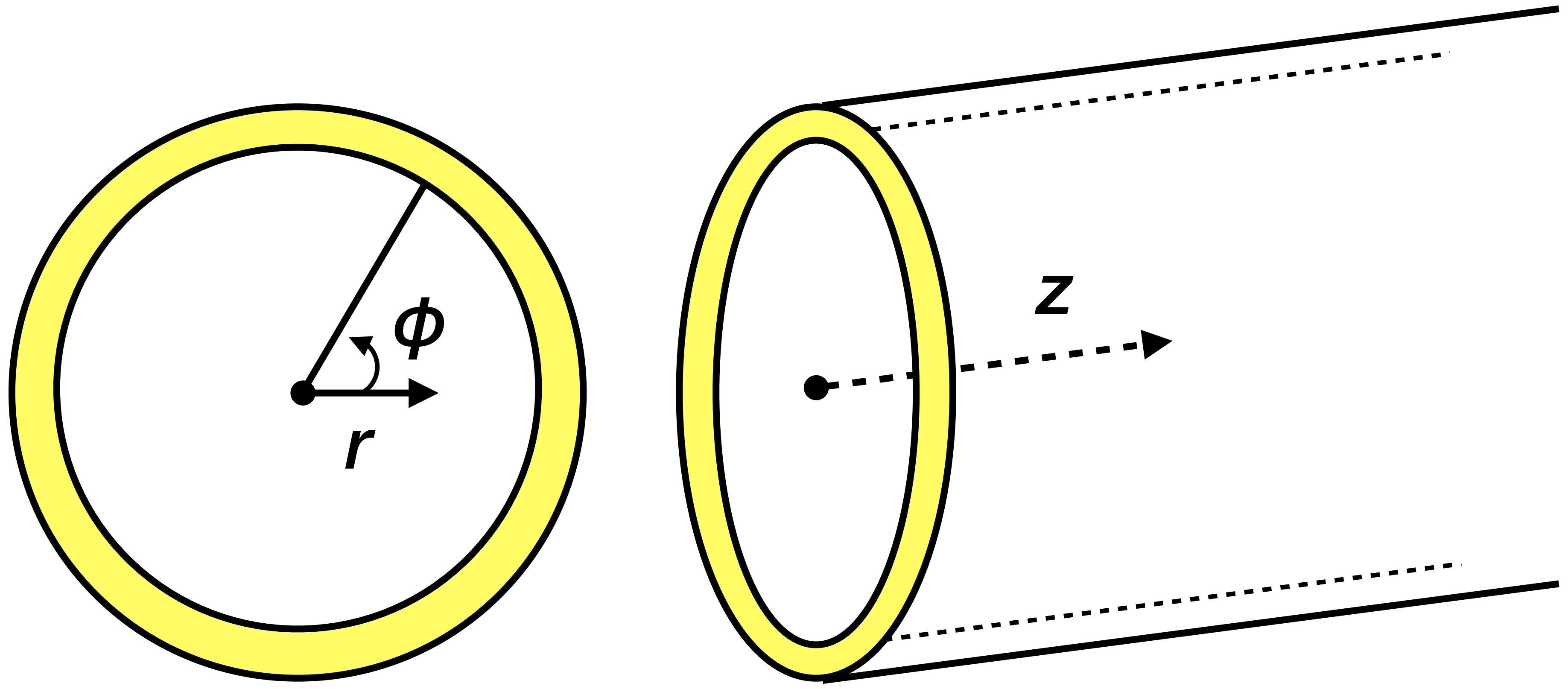
Chap. 10 | Contents for 1st class of week 8

Sec 5. Circular waveguides

- Bessel’s differential equation and Bessel function

- Characteristics of TE and TM wave propagation

Introduction: Circular waveguide



- *Circular waveguide*
  - Round metal pipe having a uniform circular cross-section
  - Inside filled with a dielectric (μ and ε)

- *Wave equations for EM waves in a circular waveguide*

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \end{cases} \rightarrow \begin{cases} \left(\nabla^2_{r\phi} + \nabla^2_z\right)\boldsymbol{E} + k^2 \boldsymbol{E} = 0 \\ \left(\nabla^2_{r\phi} + \nabla^2_z\right)\boldsymbol{H} + k^2 \boldsymbol{H} = 0 \end{cases}$$

where $\begin{cases} \nabla^2_{r\phi} : \text{Laplacian for a transverse polar plane } (r \text{ and } \phi) \\ \nabla^2_z : \text{Laplacian for a longitudinal axis } (z) \end{cases}$

Here, $\begin{cases} \boldsymbol{E} = \boldsymbol{E}_T + \boldsymbol{a_z} E_z \\ \boldsymbol{H} = \boldsymbol{H}_T + \boldsymbol{a_z} H_z \end{cases}$ where $\begin{cases} E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma z} \\ H_z(r,\phi,z) = H_z^0(r,\phi)e^{-\gamma z} \end{cases}$

- *Longitudinal field components*
  - For TM mode

$$\begin{cases} H_z = 0 \quad \text{(By definition)} \\ \left(\nabla^2_{r\phi} + \nabla^2_z\right)E_z + k^2 E_z = 0 \end{cases}$$

$$\rightarrow \nabla^2_{r\phi} E_z^0 + \left(\gamma^2 + k^2\right)E_z^0 = 0$$

$$\rightarrow \nabla^2_{r\phi} E_z^0 + h^2 E_z^0 = 0$$

  - For TE mode

$$\begin{cases} E_z = 0 \quad \text{(By definition)} \\ \left(\nabla^2_{r\phi} + \nabla^2_z\right)H_z + k^2 H_z = 0 \end{cases}$$

$$\rightarrow \nabla^2_{r\phi} H_z^0 + \left(\gamma^2 + k^2\right)H_z^0 = 0$$

$$\rightarrow \nabla^2_{r\phi} H_z^0 + h^2 H_z^0 = 0$$

<Example of circular waveguide>

Chap. 10 | Bessel's differential equations and Bessel functions (1/3)

- ***Wave equation***
  - In cylindrical coordinate

$$\nabla^2_{r\phi} E_z^0 + h^2 E_z^0 = 0 \quad \rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z^0}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2} + h^2 E_z^0 = 0 \quad \cdots(1)$$ *(HW!)*

  - Separation of variables

$$E_z^0(r,\phi) = R(r)\Phi(\phi) \quad \cdots(2)$$

  - By substituting (2) into (1),

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial R(r)}{\partial r} \right)\Phi(\phi) + \frac{1}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} R(r) + h^2 R(r)\Phi(\phi) = 0 \right] \longleftarrow \times \frac{r^2}{R(r)\Phi(\phi)}$$

$$\rightarrow \boxed{\frac{r}{R(r)}\frac{d}{dr}\left(r \frac{dR(r)}{dr} \right) + h^2 r^2} = \boxed{-\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d^2\phi}} = n^2$$ *: Both sides equal to* ***the constant*** *to be satisfied for* ***all r and φ!***

only a function of r! *only a function of φ!*

  - Two ODEs

$$\boxed{\begin{cases} \dfrac{d^2\Phi(\phi)}{d^2\phi} + n^2\Phi(\phi) = 0 \\ \dfrac{r}{R(r)}\dfrac{d}{dr}\left(r \dfrac{dR(r)}{dr} \right) + h^2 r^2 = n^2 \end{cases}}$$

Friedrich Wilhelm Bessel
(Prussian (German))
(1784-1846)

Bessel's Differential Equation

$$\boxed{\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(h^2 - \frac{n^2}{r^2} \right) R(r) = 0}$$

Img src: Wikipedia

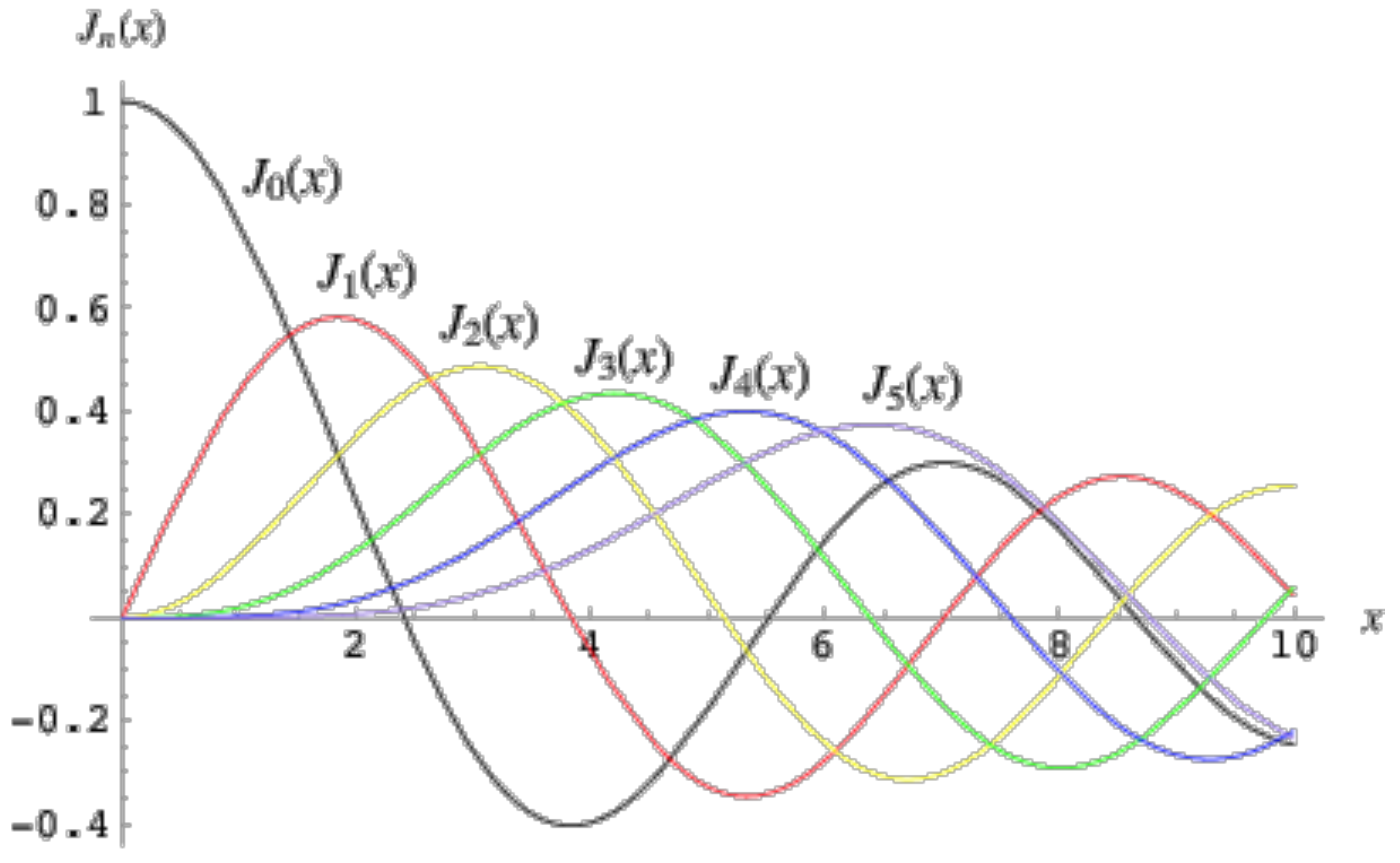
Chap. 10 | Bessel's differential equations and Bessel functions (2/3)

- ***Bessel's differential equation***

  - Second-order equation → *Two linearly independent solutions* exist!

$$\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}+\left(h^2-\frac{n^2}{r^2}\right)R(r)=0$$

Refer to Engineering Mathematics
For derivation!

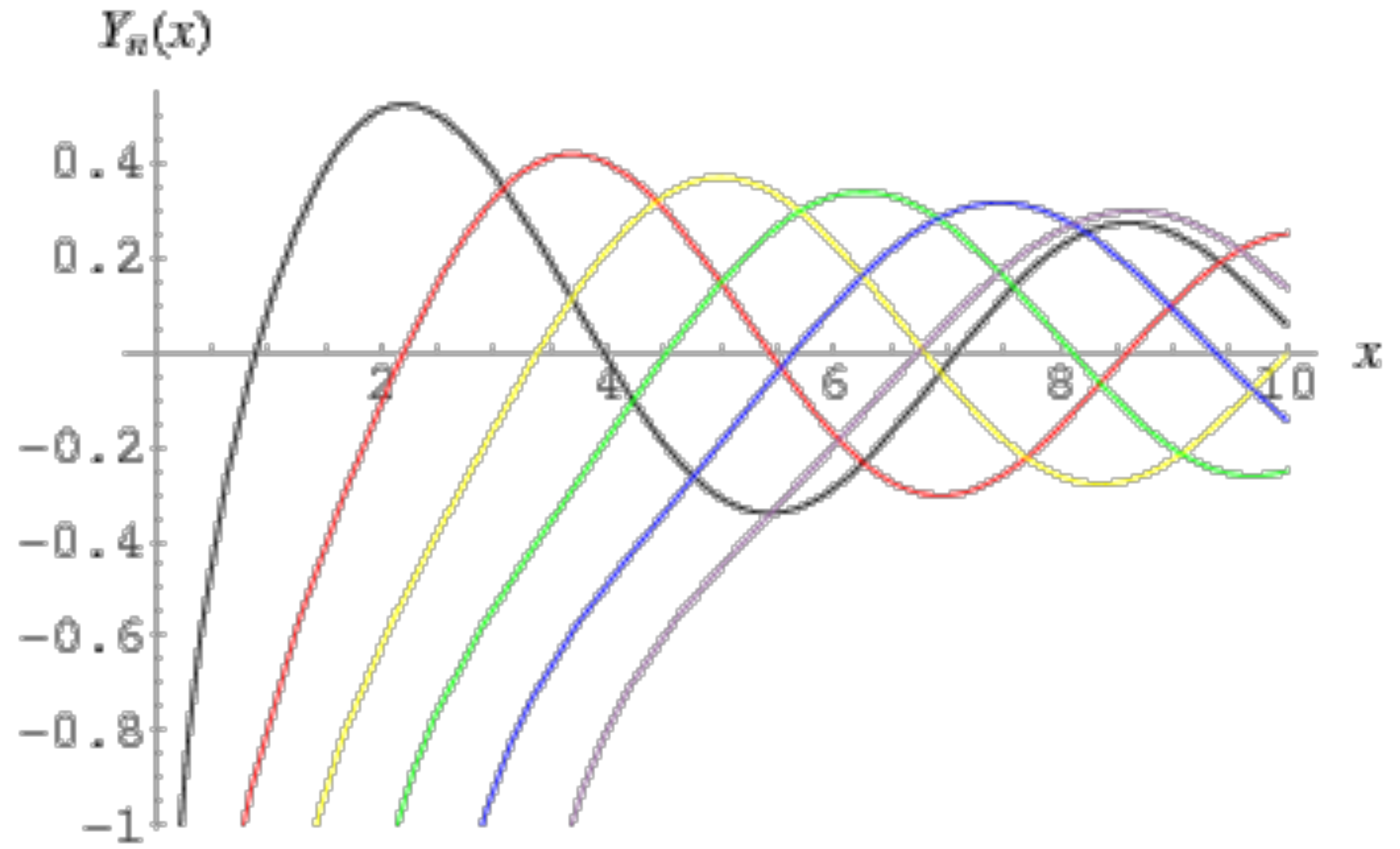


<Bessel function of the 1st kind>

- ***Solution 1: Bessel function of the 1st kind (of nth order)***

$$J_n(hr)=\sum_{m=0}^{\infty}\frac{(-1)^m(hr)^{n+2m}}{m!(n+m)!2^{n+2m}}$$ where n is ***an integer value***

  - $J_n(0) = 0$ for all n, except for $J_0(0) = 1$
  - $J_n(x)$: ***i)*** Alternating functions of decreasing amplitudes that ***ii)*** cross the zero level at ***iii)*** progressively shorter intervals. ***iv)*** As x becomes large, $J_n(x)$ approach a sinusoidal form



<Bessel function of the 2nd kind>

- ***Solution 2: Bessel function of the 2nd kind (of nth order)***

$$Y_n(hr)=\frac{(\cos n\pi)J_n(hr)-J_{-n}(hr)}{\sin n\pi}$$ where n is ***an integer value***

- ***General solution***

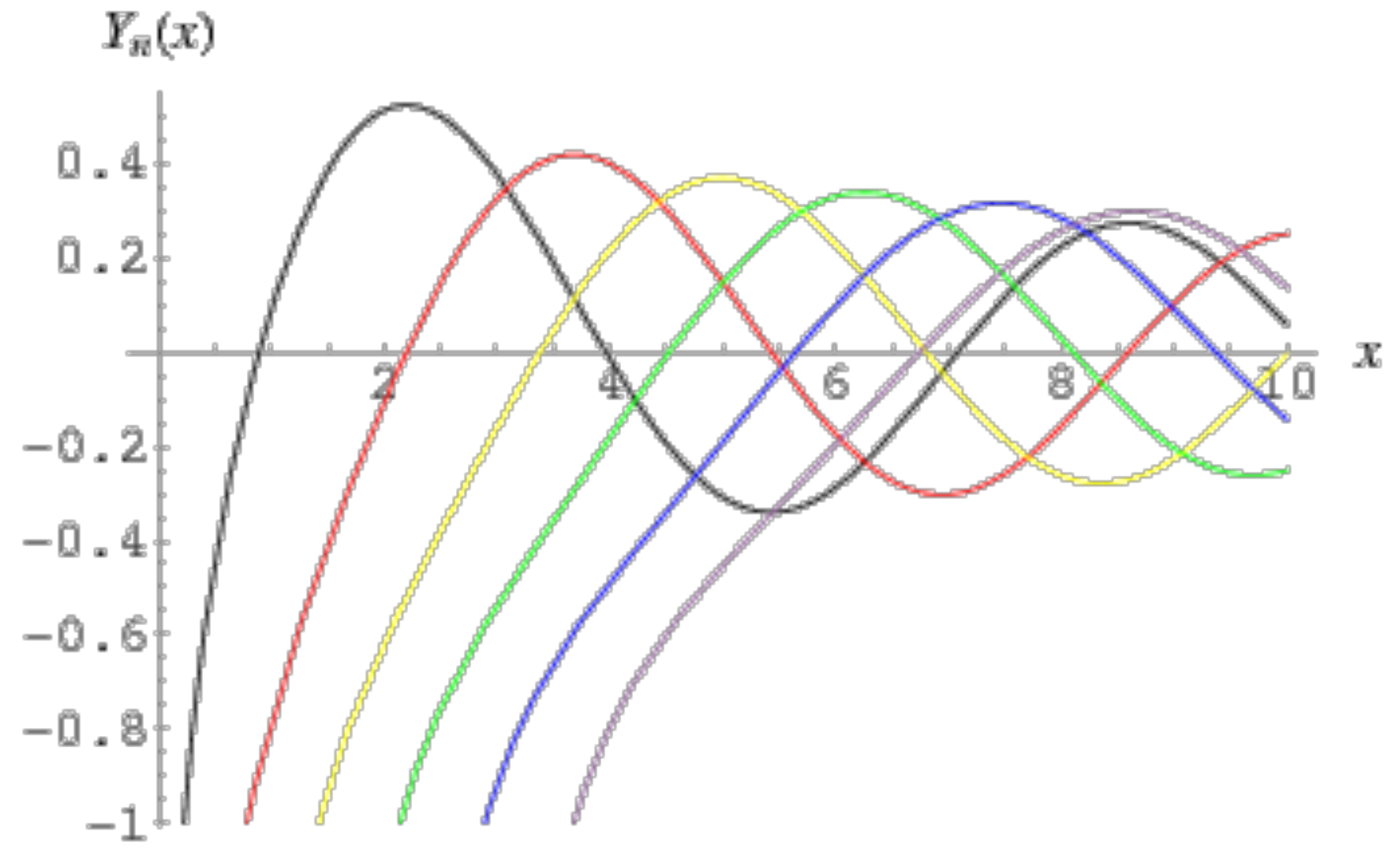
$$R(r)=C_nJ_n(hr)+D_nY_n(hr)$$

- ***Bessel solution for a circular waveguide***
  - Characteristics of Bessel function of the 2nd kind

    $$\text{If } hr \to 0,\ Y_n(hr) \to \infty$$

  - However, our region of interest should include the axis where $r = 0$
  - $\therefore$ A solution $R(r)$ CANNOT have $Y_n(hr)$ that leads to unphysical situation!

$$\boxed{\therefore R(r) = C_n J_n(hr)}$$



<Bessel function of the 2nd kind>

- ***"Zeros" of Bessel function of the 1st kind***

$$J_n(hr) = \sum_{m=0}^{\infty} \frac{(-1)^m (hr)^{n+2m}}{m!(n+m)!2^{n+2m}} = 0$$

There are several hr values (zeros) that make $J_n(hr) = 0$!

<Table 1>
Zeros of $J_n(x) = x_{np}$

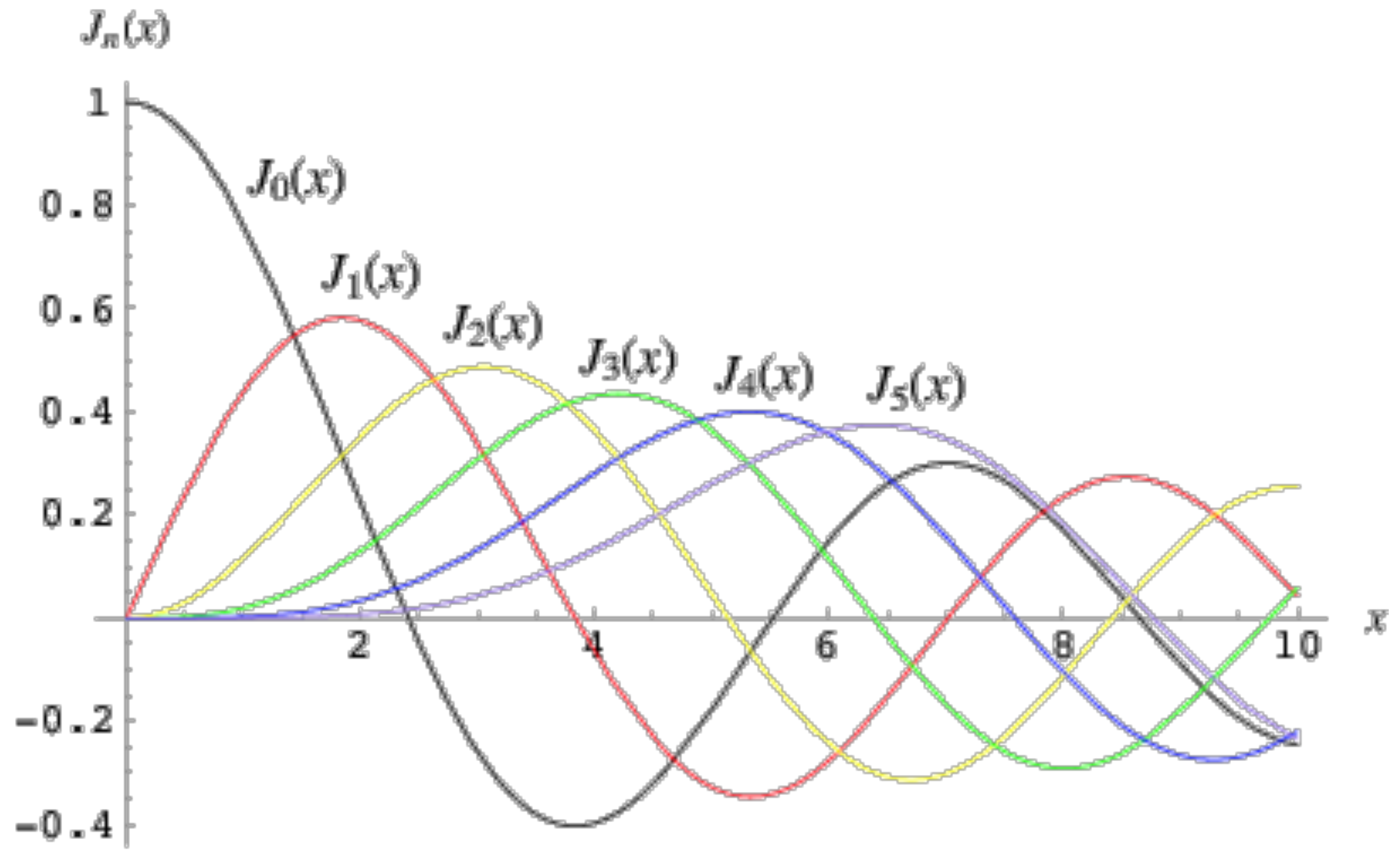
p \ n	0	1	2	…
1	2.405	3.832	5.136	…
2	5.520	7.016	8.417	…
…	…	…	…	…

→Determine eigenvalues for ***TM mode!***

<Table 2>
Zeros of $J_n'(x) = x'_{np}$

p \ n	0	1	2	…
1	3.832	1.841	3.054	…
2	7.016	5.331	6.706	…
…	…	…	…	…

→Determine eigenvalues for ***TE mode!***



<Bessel function of the 1st kind>

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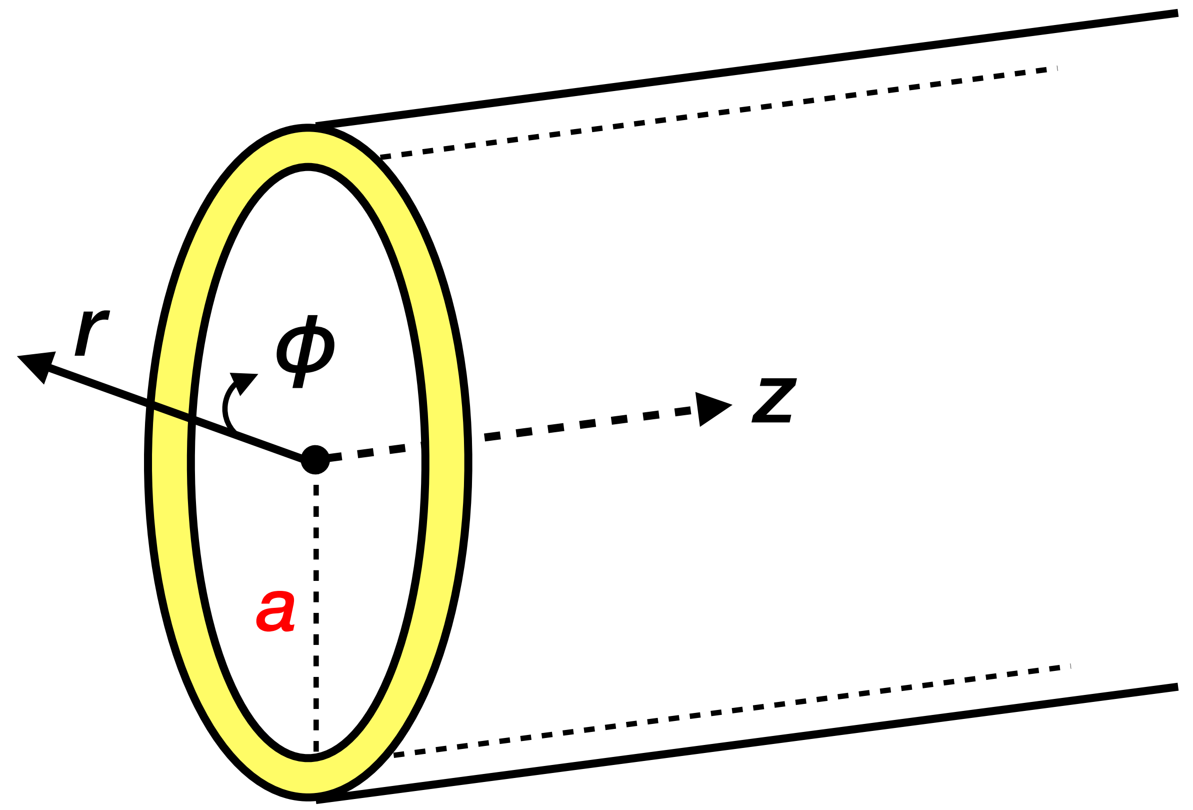
TM waves in circular waveguide (1/5)



- ***Circular waveguide***

  - Circular waveguide of radius "*a*"
  - Dielectric medium (μ and ε) enclosed by metallic skin

- ***Longitudinal field components***

$$\begin{cases} H_z = 0 & \text{(By definition)} \\ E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma z} & \text{where } \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0 \text{ and } E_z^0(r,\phi) = R(r)\Phi(\phi) \end{cases}$$

  - Solution components

$$\begin{cases} R(r) = C_n J_n(hr) \\ \Phi(\phi) \longleftarrow \text{Solution of } \dfrac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0 \end{cases}$$

* All the field components are ***periodic with respect to ϕ*** (period = 2π)

* $\Phi(\phi)$ should be ***in a sinusoidal form***!

* Because of the periodicity, n should be ***integer values***

$$\boxed{\therefore E_z^0(r,\phi) = C_n J_n(hr)\cos n\phi \quad \text{(TM modes)}}$$

* sin($n\phi$) and cos($n\phi$) does not matter!
(only reference changes)

Chap. 10 | TM waves in circular waveguide (2/5)

- ***Transverse field components*** *(Recall 10.2: General wave behaviors along uniform guides)*
  - ***Transverse E-field components*** expressed in terms of ***longitudinal E-field*** for ***TM modes***

Cartesian: $$\left(\boldsymbol{E}_T^0\right)_{TM} = \boldsymbol{a}_x E_x^0 + \boldsymbol{a}_y E_y^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0 \quad \text{where } \nabla_T = \boldsymbol{a}_x\frac{\partial}{\partial x} + \boldsymbol{a}_y\frac{\partial}{\partial y}$$ *(Gradient in transverse plane)*

Cylindrical: $$\left(\boldsymbol{E}_T^0\right)_{TM} = \boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0 \quad \text{where } \nabla_T = \boldsymbol{a}_r\frac{\partial}{\partial r} + \boldsymbol{a}_\phi\frac{\partial}{r\,\partial \phi}$$

$$\boxed{\rightarrow \left(\boldsymbol{E}_T^0\right)_{TM} = \boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0 = \boldsymbol{a}_r\left(-\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial r}\right) + \boldsymbol{a}_\phi\left(-\frac{\gamma}{h^2 r}\frac{\partial E_z^0}{\partial \phi}\right) \quad \cdots(1)}$$

- ***Transverse H-fields*** related to ***transverse E-fields*** via impedance Z_{TM}

$$\left(\boldsymbol{H}_T\right)_{TM} = \frac{1}{Z_{TM}}\left[\boldsymbol{a}_z \times \left(\boldsymbol{E}_T\right)_{TM}\right] \quad \text{where } Z_{TM} = \frac{\gamma}{j\omega\varepsilon} \quad (\Omega)$$

$$\left(\boldsymbol{H}_T\right)_{TM} = \boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0 = \frac{j\omega\varepsilon}{\gamma}\boldsymbol{a}_z \times \left(\boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0\right)$$

$$\boxed{= \boldsymbol{a}_r\left(-\frac{j\omega\varepsilon}{\gamma}E_\phi^0\right) + \boldsymbol{a}_\phi\left(\frac{j\omega\varepsilon}{\gamma}E_r^0\right) \quad \cdots(2)}$$

Right-hand rule

$$\boldsymbol{a}_r \times \boldsymbol{a}_\phi = \boldsymbol{a}_z$$
$$\boldsymbol{a}_\phi \times \boldsymbol{a}_z = \boldsymbol{a}_r$$
$$\boldsymbol{a}_z \times \boldsymbol{a}_r = \boldsymbol{a}_\phi$$

Chap. 10 | TM waves in circular waveguide (3/5)

- *Transverse field components*

  - From equations (1), (2), and (3)

$$\boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0 = \boldsymbol{a}_r\left(-\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial r}\right) + \boldsymbol{a}_\phi\left(-\frac{\gamma}{h^2 r}\frac{\partial E_z^0}{\partial \phi}\right) \quad \cdots(1)$$

$$E_z^0(r,\phi) = C_n J_n(hr)\cos n\phi \quad \cdots(3)$$

$$\boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0 = \boldsymbol{a}_r\left(-\frac{j\omega\varepsilon}{\gamma}E_\phi^0\right) + \boldsymbol{a}_\phi\left(\frac{j\omega\varepsilon}{\gamma}E_r^0\right) \quad \cdots(2)$$

  - *We can obtain transverse E and H-fields!*

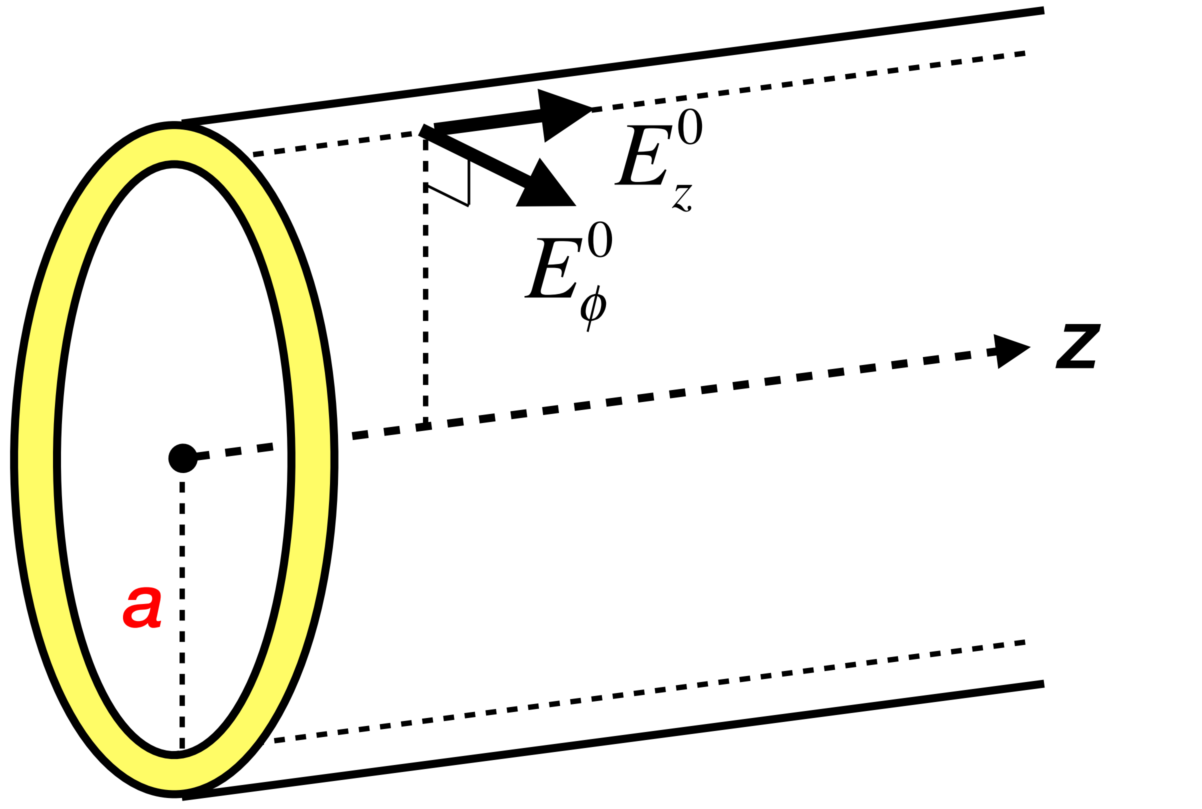
$$\begin{cases} E_r^0 = -\dfrac{j\beta}{h^2}\dfrac{\partial E_z^0}{\partial r} = -\dfrac{j\beta}{h} C_n J_n'(hr)\cos n\phi \\[2ex] E_\phi^0 = -\dfrac{j\beta}{h^2 r}\dfrac{\partial E_z^0}{\partial \phi} = \dfrac{j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi \\[2ex] H_r^0 = -\dfrac{\omega\varepsilon}{\beta}E_\phi^0 = -\dfrac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr)\sin n\phi \\[2ex] H_\phi^0 = \dfrac{\omega\varepsilon}{\beta}E_r^0 = -\dfrac{j\omega\varepsilon}{h} C_n J_n'(hr)\cos n\phi \end{cases}$$

- ***Eigenvalues h***

  - Eigenvalues provided by B.C. where ***tangential E-fields = 0 at r = a***

Medium 1 (dielectric)	Medium 2 (Conductor)
$E_{1t} = 0$	$E_{2t} = 0$
$\boldsymbol{a}_{n2} \times \boldsymbol{H}_1 = \boldsymbol{J}_S$	$H_{2t} = 0$
$\boldsymbol{a}_{n2} \cdot \boldsymbol{D}_1 = \rho_S$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

From Chap. 7-5.1



$E_\phi^0 = E_z^0 = 0$ for all ϕ at $r = a$

$$\boxed{\therefore J_n(ha) = 0}$$

TM waves in circular waveguide (4/5)

$$\boxed{J_n(ha)=0}$$

- ***Lowest cutoff frequency for TM modes***
  - From <*Table 1*>, the lowest zero of $J_n(x)$ is x_{01} = 2.405
  - Thus, the smallest *ha for* $J_0(hr) = 0 \rightarrow x_{01}$

$$ha = 2.405 \quad \rightarrow \quad h_{TM01} = \frac{2.405}{a}$$

  - Since cutoff frequency for TM modes is given by

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} \quad \rightarrow \quad \boxed{\therefore (f_c)_{TM01} = \frac{h_{TM01}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.383}{a\sqrt{\mu\varepsilon}}}$$

Is it a dominant mode?

- ***Eigenvalue Notation for a circular waveguide***

$$TM_{np}$$

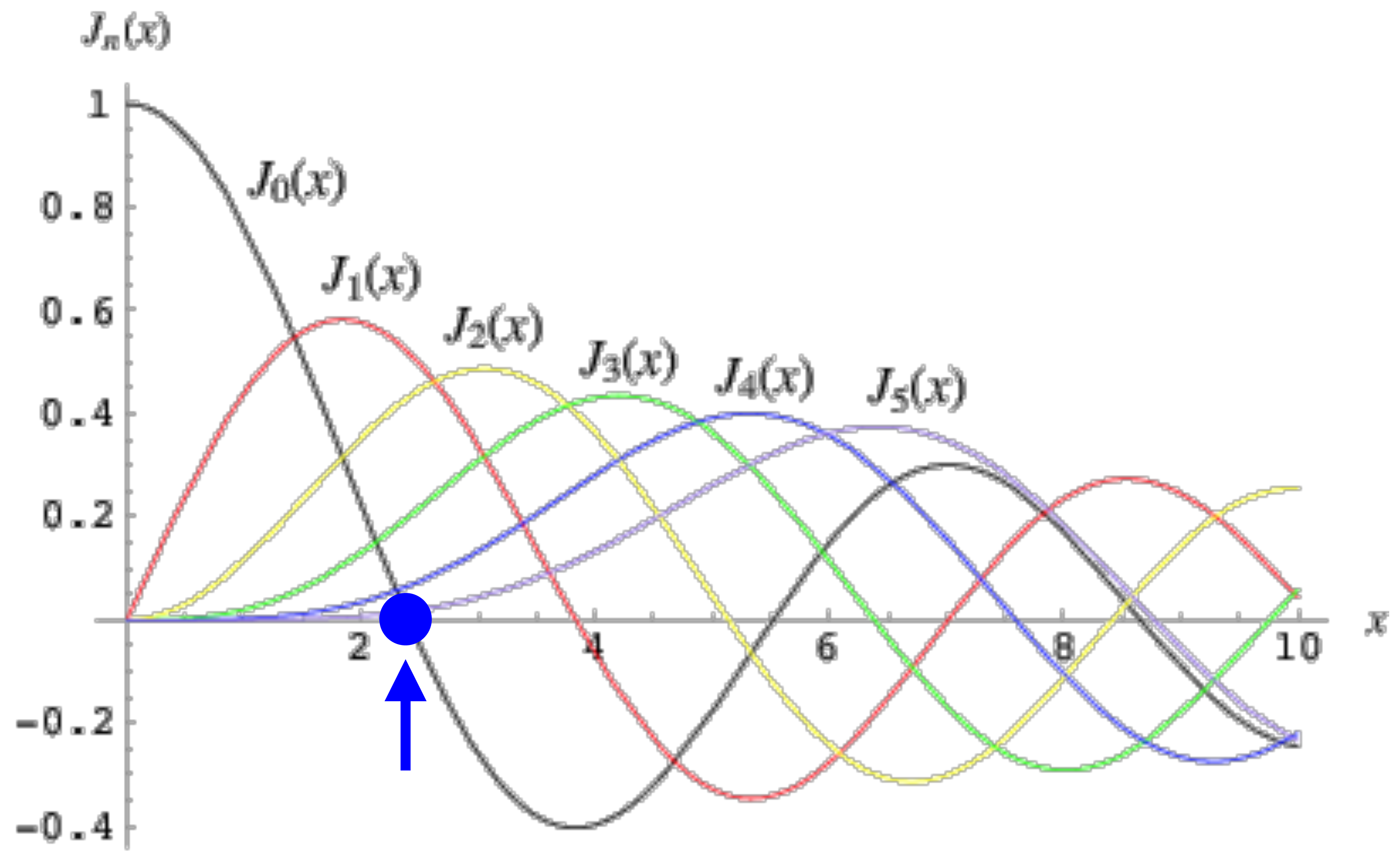
n: number of ***half-wave*** field variations ***in φ direction***

p: number of ***half-wave*** field variations ***in r direction*** $\leftarrow \because E_z^0(r,\phi) = C_n J_n(hr)\cos n\phi$

<*Table 1*>

Zeros of $J_n(x) = x_{np}$

p \ n	0	1	2	...
1	2.405	3.832	5.136	...
2	5.520	7.016	8.417	...
...	...	...	...	...



<*Bessel function of the 1st kind*>

TM waves in circular waveguide (5/5)

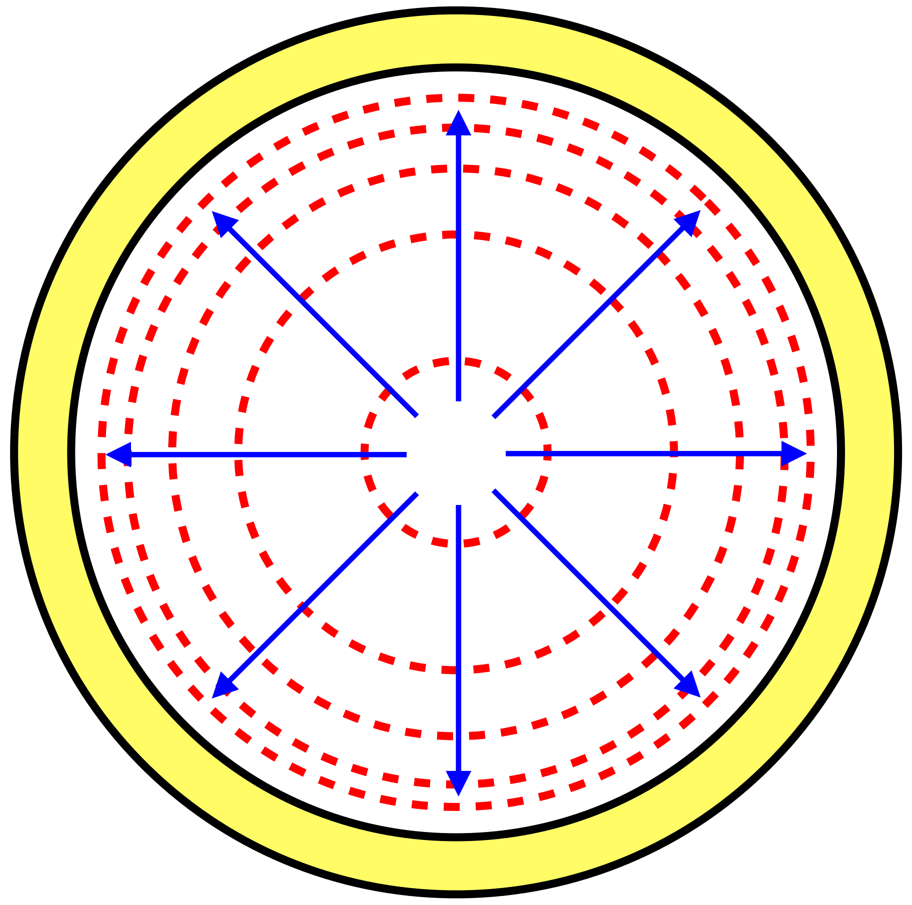
- *TM₀₁ mode*

E-field

$$\begin{cases} E_r^0 = -\dfrac{j\beta}{h} C_n J_n'(hr)\cos n\phi = -\dfrac{j\beta}{h} C_0 J_0'(hr) \ \text{(nonzero)} \\ E_\phi^0 = \dfrac{j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi = 0 \\ E_z^0 = C_n J_n(hr)\cos n\phi = C_0 J_0(hr) \ \text{(nonzero)} \end{cases}$$

H-field

$$\begin{cases} H_r^0 = -\dfrac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr)\sin n\phi = 0 \\ H_\phi^0 = -\dfrac{j\omega\varepsilon}{h} C_n J_n'(hr)\cos n\phi = -\dfrac{j\omega\varepsilon}{h} C_0 J_0'(hr) \ \text{(nonzero)} \\ H_z^0 = 0 \end{cases}$$



<E & H-field patterns in a polar plane>

* *E-field ⊥ H-field*

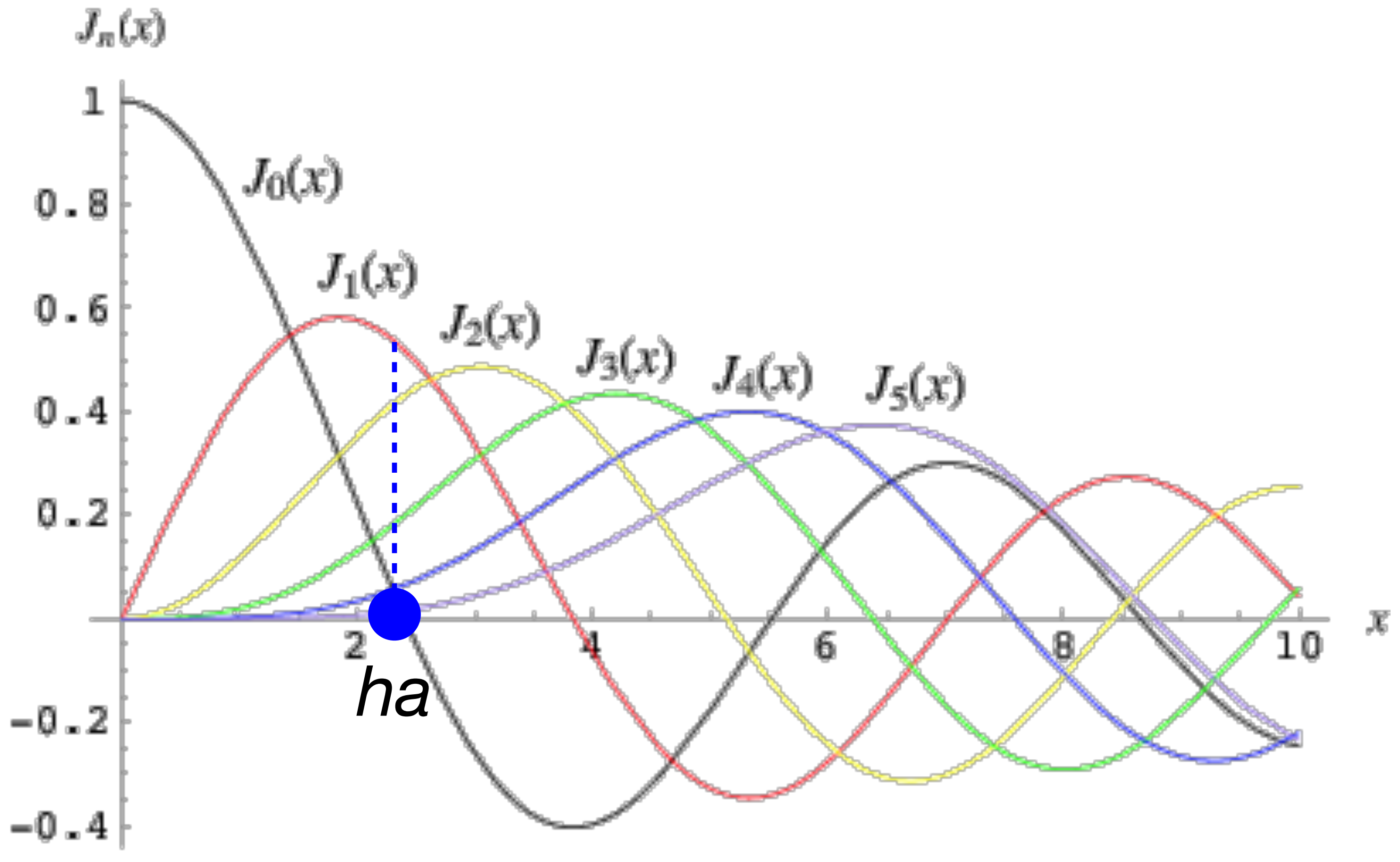
* *E-field lines form the radial pattern*

* *Density of H-field lines increases with "r" (from 0 to a)*

$$H_\phi^0 = -C_0 \frac{j\omega\varepsilon}{h} J_0'(hr) = C_0 \frac{j\omega\varepsilon}{h} J_1(hr)$$

$$\because J_0'(hr) = -J_1(hr)$$

why?



<Bessel function of the 1st kind>

Img src: Wolfram MathWorld

Chap. 10 | TE waves in circular waveguide (1/2)

- *Longitudinal field components*

$$\begin{cases} E_z = 0 \\ H_z(r,\phi,z) = H_z^0(r,\phi)e^{-\gamma z} \quad \text{where } \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0 \text{ and } H_z^0(r,\phi) = R(r)\Phi(\phi) \end{cases}$$

- Similarly to the TM case,

$$\boxed{\therefore H_z^0(r,\phi) = D_n J_n(hr)\cos n\phi \quad \text{(TE modes)}}$$

- *Transverse field components*

- Transverse magnetic fields:

$$\left[\left(\boldsymbol{H}_T^0\right)_{TE} = \boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0\right] = \left[-\frac{\gamma}{h^2}\nabla_T H_z^0 = -\frac{\gamma}{h^2}\left(\boldsymbol{a}_r \frac{\partial}{\partial r} + \boldsymbol{a}_\phi \frac{\partial}{r\partial\phi}\right)H_z^0\right]$$

- Transverse electric fields:

$$\left[\left(\boldsymbol{E}_T^0\right)_{TE} = \boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0\right] = \left[-Z_{TE}\left(\boldsymbol{a}_z \times \left(\boldsymbol{H}_T^0\right)_{TE}\right) = -\frac{j\omega\mu}{\gamma}\left(\boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0\right)\right]$$

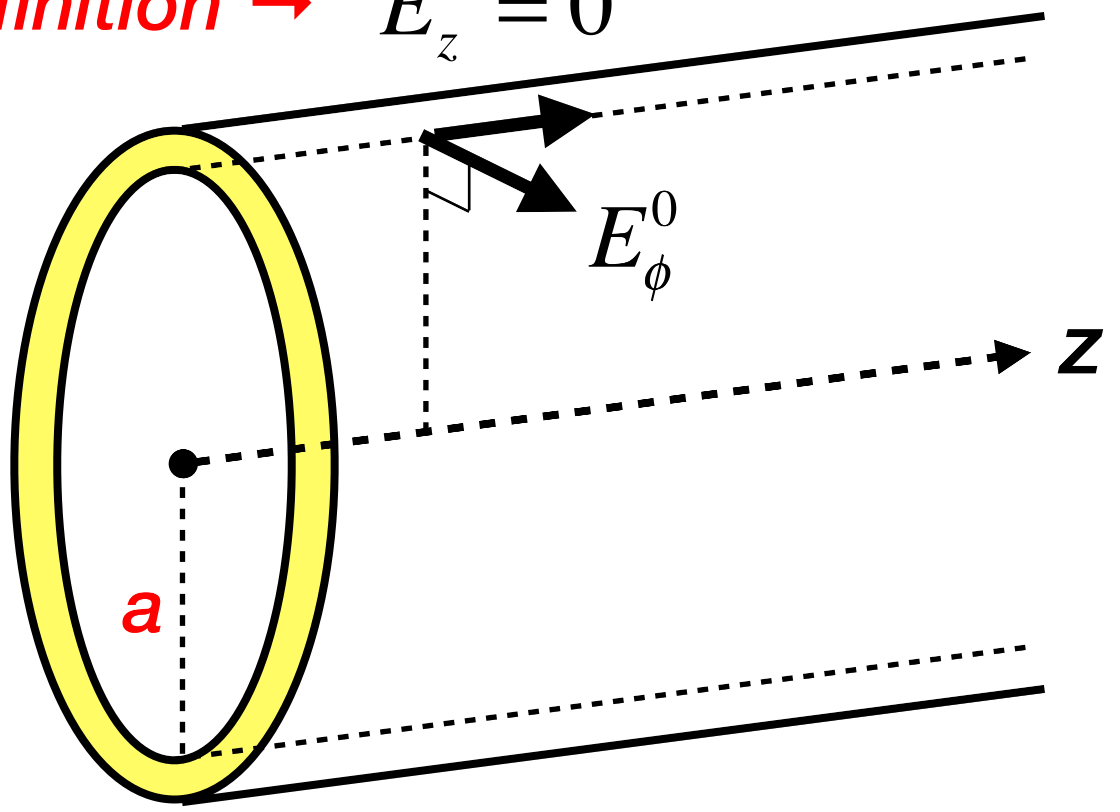
$$\boxed{\begin{cases} H_r^0 = -\dfrac{j\beta}{h^2}\dfrac{\partial H_z^0}{\partial r} = -\dfrac{j\beta}{h}D_n J_n'(hr)\cos n\phi \\ H_\phi^0 = -\dfrac{j\beta}{h^2 r}\dfrac{\partial E_z^0}{\partial\phi} = \dfrac{j\beta n}{h^2 r}D_n J_n(hr)\sin n\phi \\ E_r^0 = -\dfrac{\omega\varepsilon}{\beta}H_\phi^0 = -\dfrac{j\omega\varepsilon n}{h^2 r}D_n J_n(hr)\sin n\phi \\ E_\phi^0 = \dfrac{\omega\varepsilon}{\beta}H_r^0 = -\dfrac{j\omega\varepsilon}{h}D_n J_n'(hr)\cos n\phi \end{cases}}$$

Chap. 10 | TE waves in circular waveguide (2/2)

- ***Eigenvalues h***
  - Eigenvalues provided by B.C. where ***tangential E-fields = 0 at r = a***

$$\boxed{\therefore J_n'(ha) = 0}$$

by definition → $E_z^0 = 0$



$E_\phi^0 = 0$ for all ϕ at $r = a$

$$E_\phi^0 = \frac{\omega\varepsilon}{\beta} H_r^0 = -\frac{j\omega\varepsilon}{h} D_n J_n'(hr)\cos n\phi$$

- ***Lowest cutoff frequency for TE modes***
  - From <*Table 2*>, the lowest zero of $J'_n(x)$ is $x_{11} = 1.841$
  - Thus, the smallest ha for $= J'_1(ha) = 0 \rightarrow x_{11}$

$$ha = 1.841 \quad \rightarrow \quad h_{TE11} = \frac{1.841}{a}$$

  - Since cutoff frequency for TE modes is given by

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} \quad \rightarrow \quad \boxed{\therefore (f_c)_{TE11} = \frac{h_{TE11}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.293}{a\sqrt{\mu\varepsilon}}}$$

Is it a dominant mode?

  - Comparison between TE and TM modes

$$\because (f_c)_{TE11} = \frac{0.293}{a\sqrt{\mu\varepsilon}} < (f_c)_{TM01} = \frac{0.383}{a\sqrt{\mu\varepsilon}},$$

TE_{11} mode is a dominant mode of a circular waveguide!

<Table 2>

Zeros of $J_n'(x) = x'_{np}$

p \ n	0	1	2	...
1	3.832	1.841	3.054	...
2	7.016	5.331	6.706	...
...	...	...	...	...

TE waves in circular waveguide (Example)

Example 10-12 (a) A 10 GHz signal is transmitted inside a circular conducting pipe. Determine the *inside diameter* of the pipe such that its lowest f_c is 20% below this signal frequency. (b) If the pipe is to operate at 15 (GHz), what waveguide modes can propagate in the pipe?

(a) Lowest $f_c = (f_c)_{TE11}$

$$(f_c)_{TE11} = \frac{0.293}{a\sqrt{\mu\varepsilon}} = \frac{0.293c}{a} = \frac{0.293\times 3\times 10^8}{a} = \frac{0.0879}{a}\ \text{(GHz)} \qquad \therefore \frac{0.0879}{a}\ \text{(GHz)} = 10\times 0.8\ \text{(GHz)} \ \rightarrow\ \boxed{d = 2a = 2.2\ \text{(cm)}}$$

(b) Cutoff frequencies for various modes are given as

$$(f_c)_{TE11} = 8\ \text{(GHz)}$$

$$(f_c)_{TE21} = (f_c)_{TE11}\frac{3.054}{1.841} = 13.27\ \text{(GHz)}$$

$$(f_c)_{TE01} = (f_c)_{TE11}\frac{3.832}{1.841} = 16.25\ \text{(GHz)} > 15$$

$$\vdots$$

$$(f_c)_{TM01} = (f_c)_{TE11}\frac{2.405}{1.841} = 10.45\ \text{(GHz)}$$

$$(f_c)_{TM11} = (f_c)_{TE11}\frac{3.832}{1.841} = 16.65\ \text{(GHz)} > 15$$

$$\vdots$$

∴ **TE_{11}, TE_{21}, TM_{01}** modes can propagate at a given operating frequency 15 (GHz) and all other higher order modes should attenuate.

<Table 1> (for TM!)

Zeros of $J_n(x) = x_{np}$

p \ n	0	1	2	...
1	2.405	3.832	5.136	...
2	5.520	7.016	8.417	...
...	...	...	...	...

<Table 2> (for TE!)

Zeros of $J_n'(x) = x'_{np}$

p \ n	0	1	2	...
1	3.832	1.841	3.054	...
2	7.016	5.331	6.706	...
...	...	...	...	...

Electromagnetics

<Chap. 10> Waveguides and Cavity Resonators

Section 10.5 ~ 10.6

(2nd of **week 8**)

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Textbook: Field and Wave Electromagnetics, 2E, Addison-Wesley

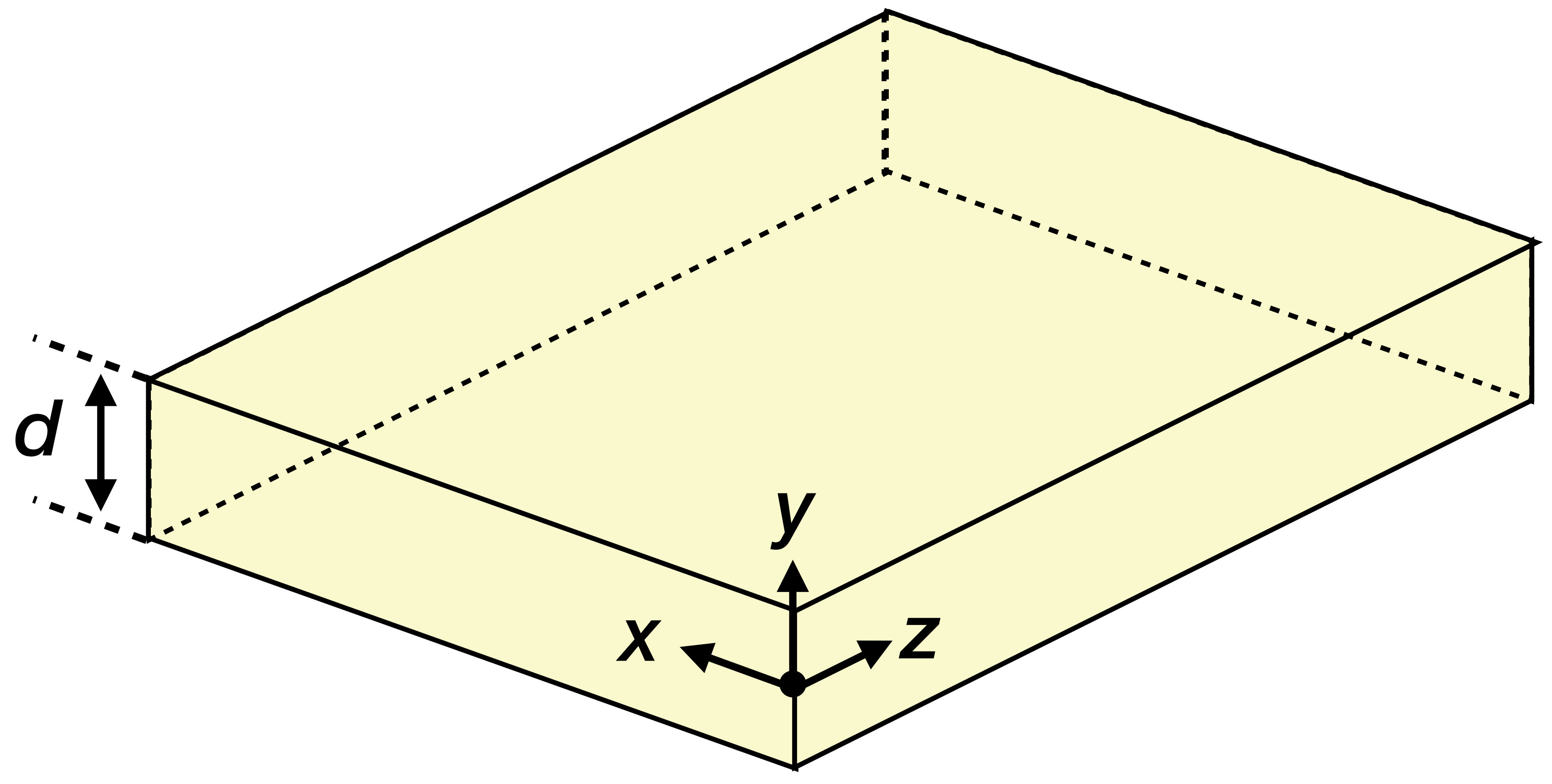
Chap. 10 | Contents for 2nd class of week 8

Sec 6. Dielectri-slab waveguide

- Odd and Even TE wave characteristics (*try TM case at home!*)
- Cutoff frequencies and possible modes

Chap. 10 | Introduction: Dielectric waveguide

- ***Dielectric-slab waveguide***
  - Thin dielectric slab (μ and ε) with thickness d situated in free space (μ_0 and ε_0)
  - Even without conducting walls, both TM and TE waves can be supported! (shown later)



<*Dielectric-slab waveguide*>

- ***Assumptions***
  - z: propagation direction
  - x: Infinite in extent and no variation of the fields $\longrightarrow$ $\dfrac{\partial \boldsymbol{E}}{\partial x} = 0, \quad \dfrac{\partial \boldsymbol{H}}{\partial x} = 0$
  - Lossless dielectric ($\sigma_d = 0$)

- ***Wave equations***

$$\begin{cases} \nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0 \\ \nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \end{cases} \quad \text{where} \quad \begin{cases} \boldsymbol{E} = \boldsymbol{a_x} E_x + \boldsymbol{a_y} E_y + \boldsymbol{a_z} E_z \\ \boldsymbol{H} = \boldsymbol{a_x} H_x + \boldsymbol{a_y} H_y + \boldsymbol{a_z} H_z \end{cases}$$

$$\therefore \begin{cases} \nabla_x^2 E_z = 0 \\ \nabla_x^2 H_z = 0 \end{cases}$$

In the z-direction,

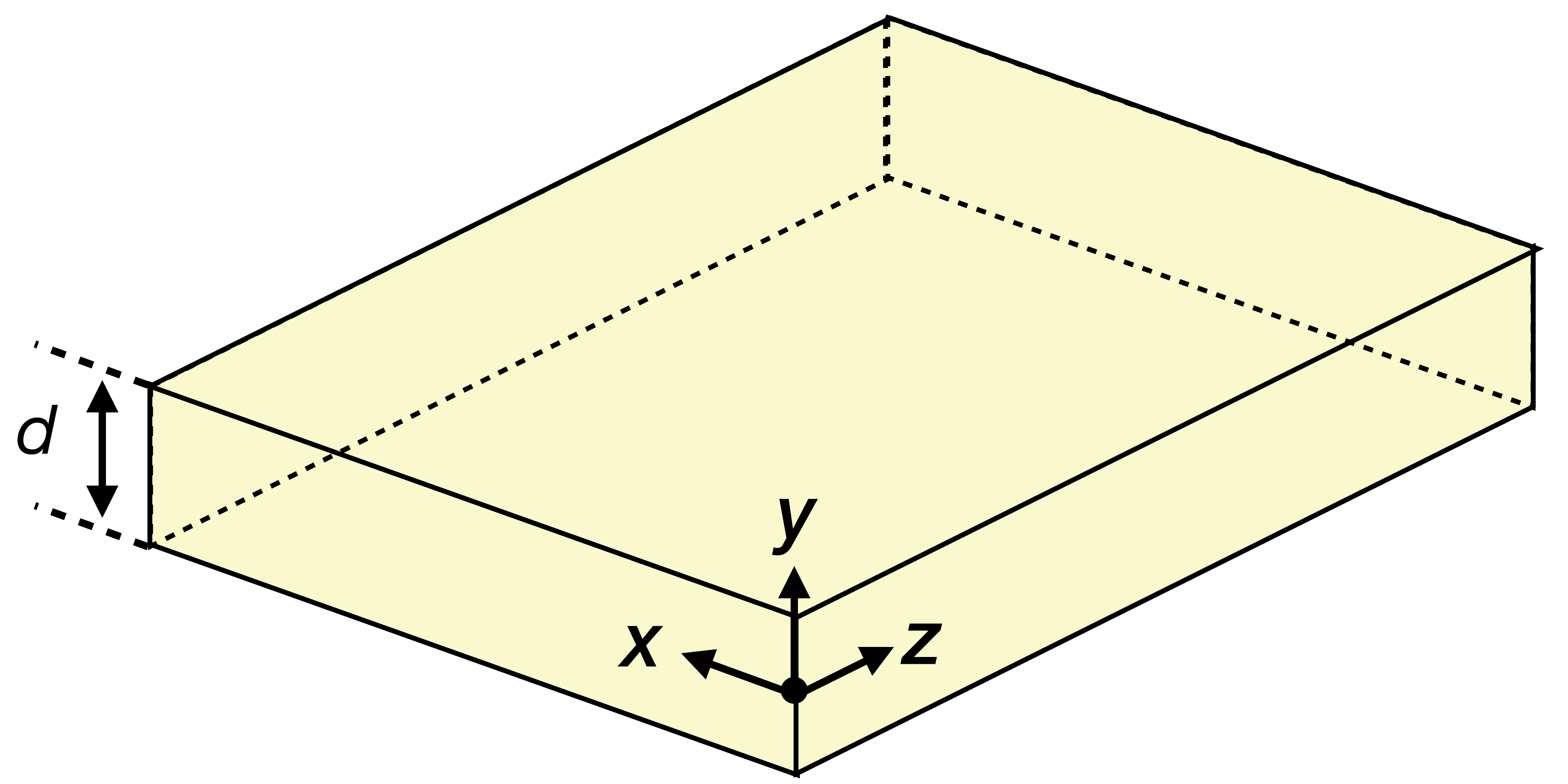
$$\begin{cases} \left(\nabla_y^2 + \nabla_z^2\right) E_z + k^2 E_z = 0 \\ \left(\nabla_y^2 + \nabla_z^2\right) H_z + k^2 H_z = 0 \end{cases} \quad \text{where} \quad \begin{cases} E_z(y,z) = E_z^0(y) e^{-\gamma z} \\ H_z(y,z) = H_z^0(y) e^{-\gamma z} \end{cases}$$

Wave equations for longitudinal fields

$$\therefore \begin{cases} \nabla_y^2 E_z^0 + \left(\gamma^2 + k^2\right) E_z^0 = 0 \quad \cdots \text{ for TM modes with } H_z^0 = 0 \\ \nabla_y^2 H_z^0 + \left(\gamma^2 + k^2\right) H_z^0 = 0 \quad \cdots \text{ for TE modes with } E_z^0 = 0 \end{cases}$$

Chap. 10 | TE waves along a dielectric slab

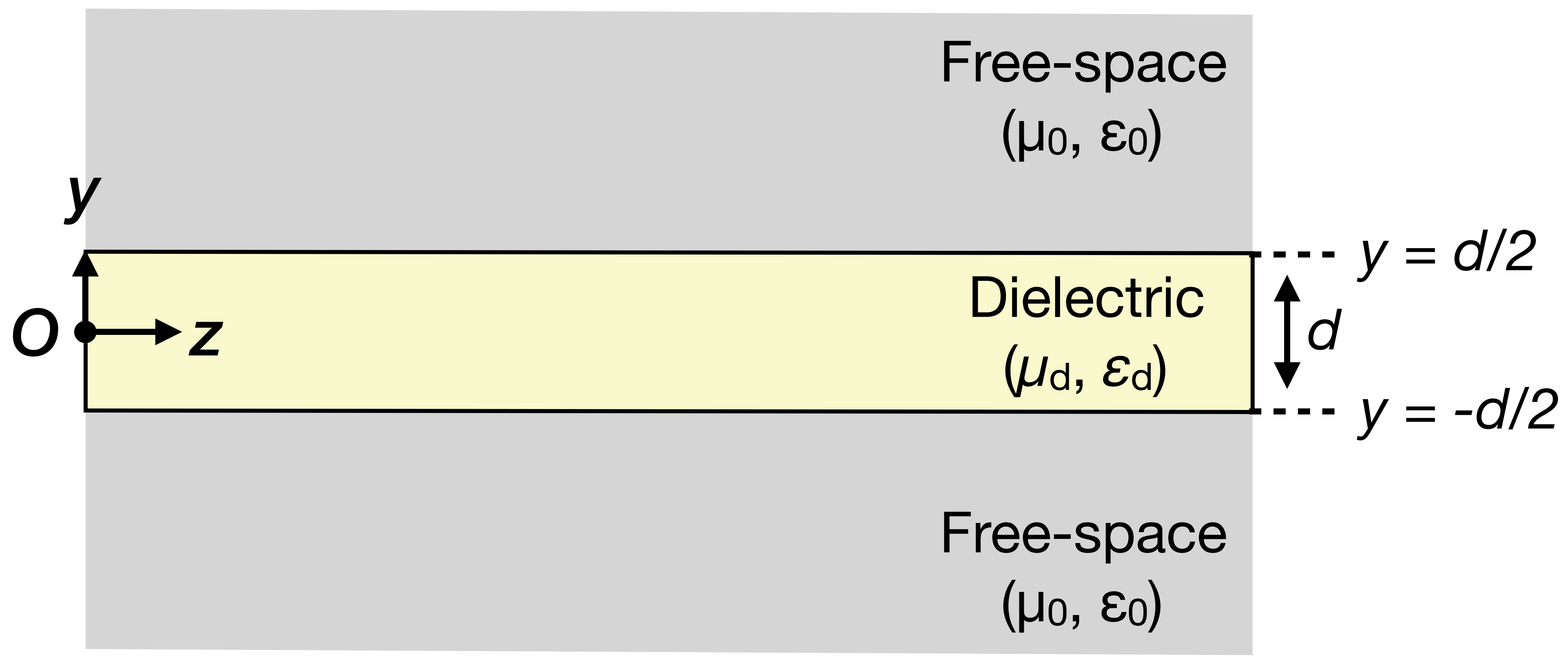
- ***Longitudinal field components***
  - $E_z = 0$
  - H_z satisfies wave equation: $\nabla_y^2 H_z^0 + \left(\gamma^2 + k^2\right) H_z^0 = 0$ where $H_z(y,z) = H_z^0(y) e^{-\gamma z}$



<Dielectric-slab waveguide>

- ***Transverse field components***

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases} \Rightarrow \begin{cases} E_x^0 = -\dfrac{j\omega\mu}{h^2}\dfrac{\partial H_z^0}{\partial y} \\ E_y^0 = 0 \\ H_x^0 = 0 \\ H_y^0 = -\dfrac{\gamma}{h^2}\dfrac{\partial H_z^0}{\partial y} \end{cases}$$



  - Fields must be considered ***both in dielectric (core) & free-space (cladding)*** regions
  - Field components should satisfy ***B.C. at y = d/2 and y = -d/2***
    (i.e. B.C. between two lossless dielectric)

Chap. 10 | TE waves along a dielectric slab (general solution)

- ***Solution for dielectric** (y ≤ |d|/2)*
  - Modes propagating in *z*-direction ***without attenuation (γ = jβ)***
  - A solution should be in ***a sinusoidal form***

    (i.e. a bounded standing wave)

$$\nabla_y^2 H_z^0 + \left(k^2+\gamma^2\right) H_z^0 = 0 \quad \rightarrow \quad \nabla_y^2 H_z^0 + h_d^2 H_z^0 = 0$$

Here, $\boxed{h_d{}^2 = k^2+\gamma^2 = \omega^2\mu_d\varepsilon_d - \beta^2 > 0.}$

→ Wavenumber for a bounded wave

$$\boxed{\therefore H_z^0(y) = H_o \sin h_d y + H_e \cos h_d y}$$

→ "Odd" & "Even" functions

- ***Solution for free-space** (y ≥ d/2 and y ≤ -d/2)*
  - Waves *decay exponentially in the y-direction* ("Evanescent wave")

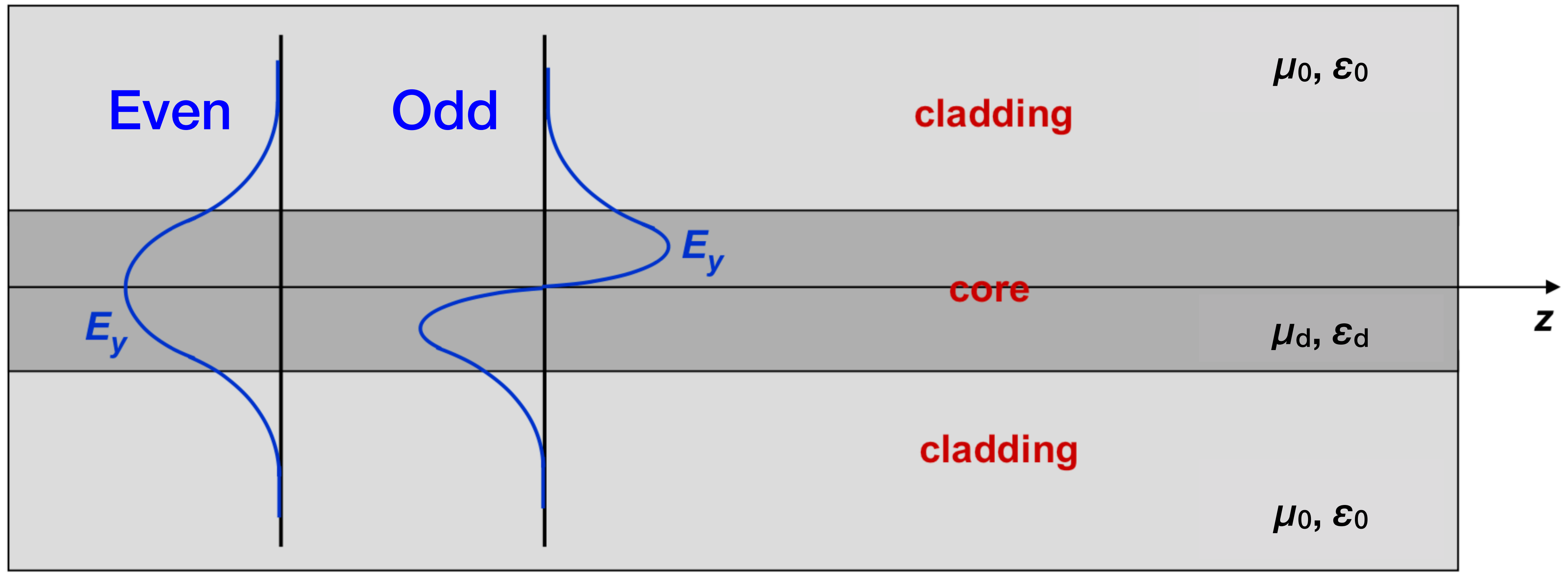
    → Waves bounded only within the guide (*total internal reflected*)

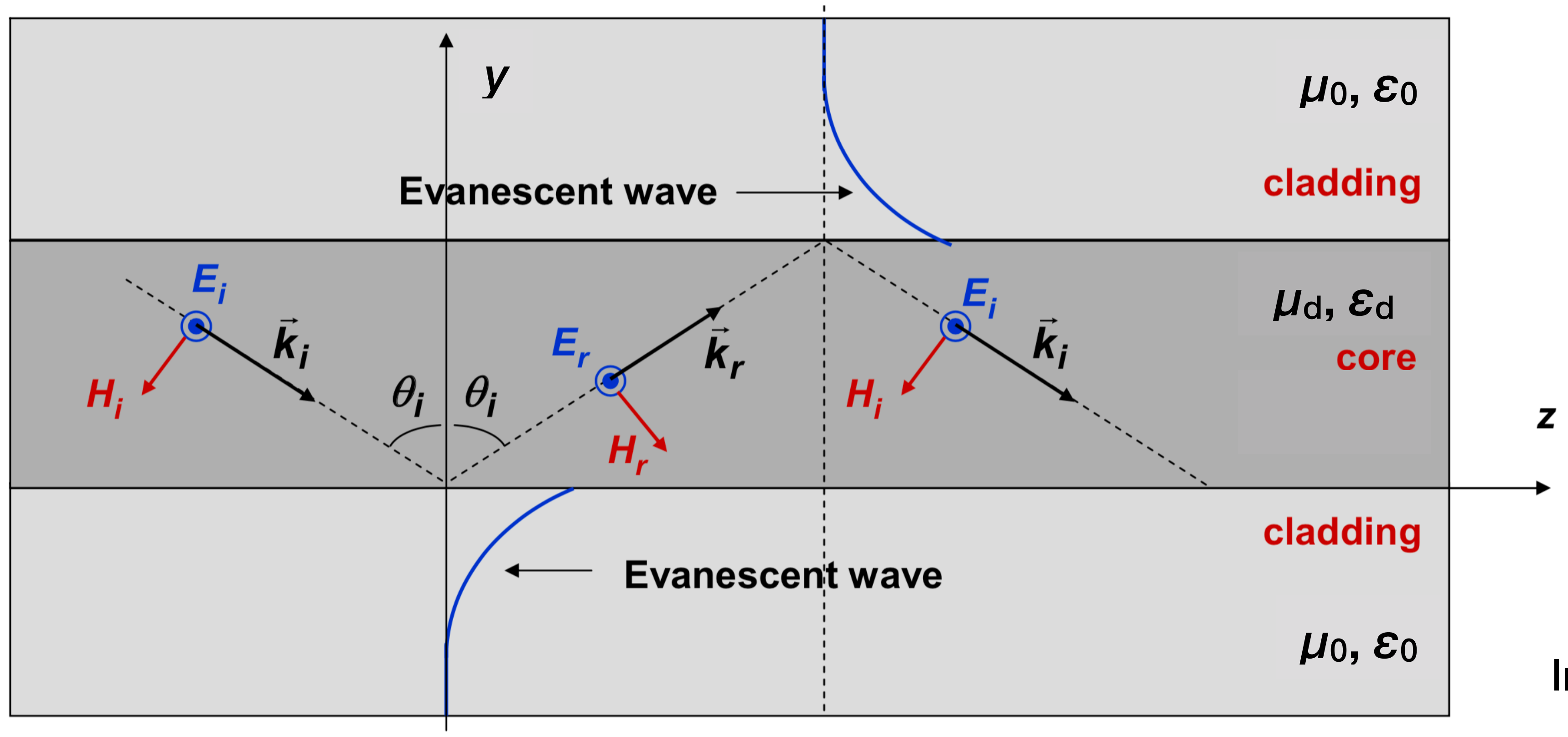
    → Waves not radiating away from it

$$\nabla_y^2 H_z^0 + \left(\gamma^2+k^2\right) H_z^0 = 0 \quad \rightarrow \quad \nabla_y^2 H_z^0 + h_0^2 H_z^0 = 0$$

Here, $\boxed{h_0{}^2 = k^2+\gamma^2 = \omega^2\mu_0\varepsilon_0 - \beta^2 < 0.\ \text{Thus, } h_0{}^2 \triangleq -\alpha^2}$

$$\boxed{\therefore \begin{cases} H_z^0(y) = C_u e^{-\alpha\left(y-\frac{d}{2}\right)} + \cancel{D_u e^{\alpha\left(y-\frac{d}{2}\right)}} & \text{where } y \geq d/2 \\ H_z^0(y) = C_l e^{\alpha\left(y+\frac{d}{2}\right)} + \cancel{D_l e^{-\alpha\left(y+\frac{d}{2}\right)}} & \text{where } y \leq -d/2 \end{cases}}$$





Img src: Cornell ECE 303 (Farhan Rana)

Chap. 10 | TE waves along a dielectric slab (Odd TE modes)

- ***Odd** TE modes in the dielectric ($|y| \le d/2$)*
  - Longitudinal components

$$E_z^0 = 0, \quad H_z^0(y) = H_o \sin h_d y$$

  - ***Nonzero*** Transverse components

$$\begin{cases} E_x^0(y) = -\dfrac{j\omega\mu_d}{h_d^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\omega\mu_d}{h_d} H_o \cos h_d y \\ H_y^0(y) = -\dfrac{\gamma}{h_d^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\beta}{h_d} H_o \cos h_d y \end{cases}$$

h_d : Wavenumber for the *bounded wave*

$$\begin{cases} H_z^0(y) = H_o \sin h_d y + H_e \cos h_d y & \text{where } |y| \le d/2 \\ H_z^0(y) = C_u e^{-\alpha\left(y-\frac{d}{2}\right)} & \text{where } y \ge d/2 \\ H_z^0(y) = C_l e^{\alpha\left(y+\frac{d}{2}\right)} & \text{where } y \le -d/2 \end{cases}$$

Here, $h_d{}^2 = \omega^2\mu_d\varepsilon_d - \beta^2$

$$\alpha^2 = -h_0{}^2 = \beta^2 - \omega^2\mu_0\varepsilon_0$$

- ***Odd** TE modes in the upper free-space ($y \ge d/2$)*
  - Longitudinal components

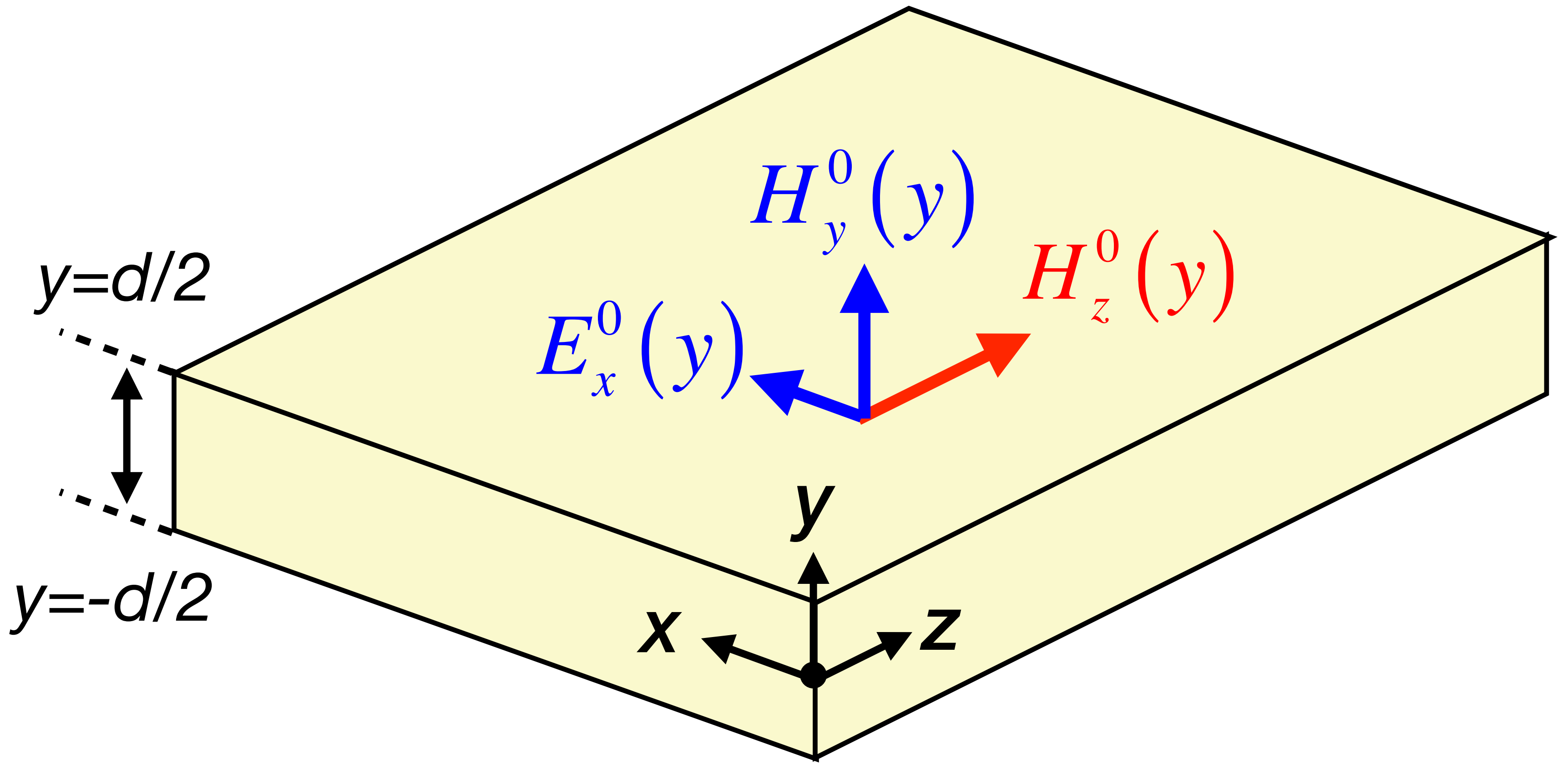
$$H_z^0(y) = C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \quad \text{where} \quad H_z^0\left(\frac{d}{2}\right) = C_u = H_o \sin\frac{h_d d}{2}$$

(∵B.C. Continuous tangential H-fields)

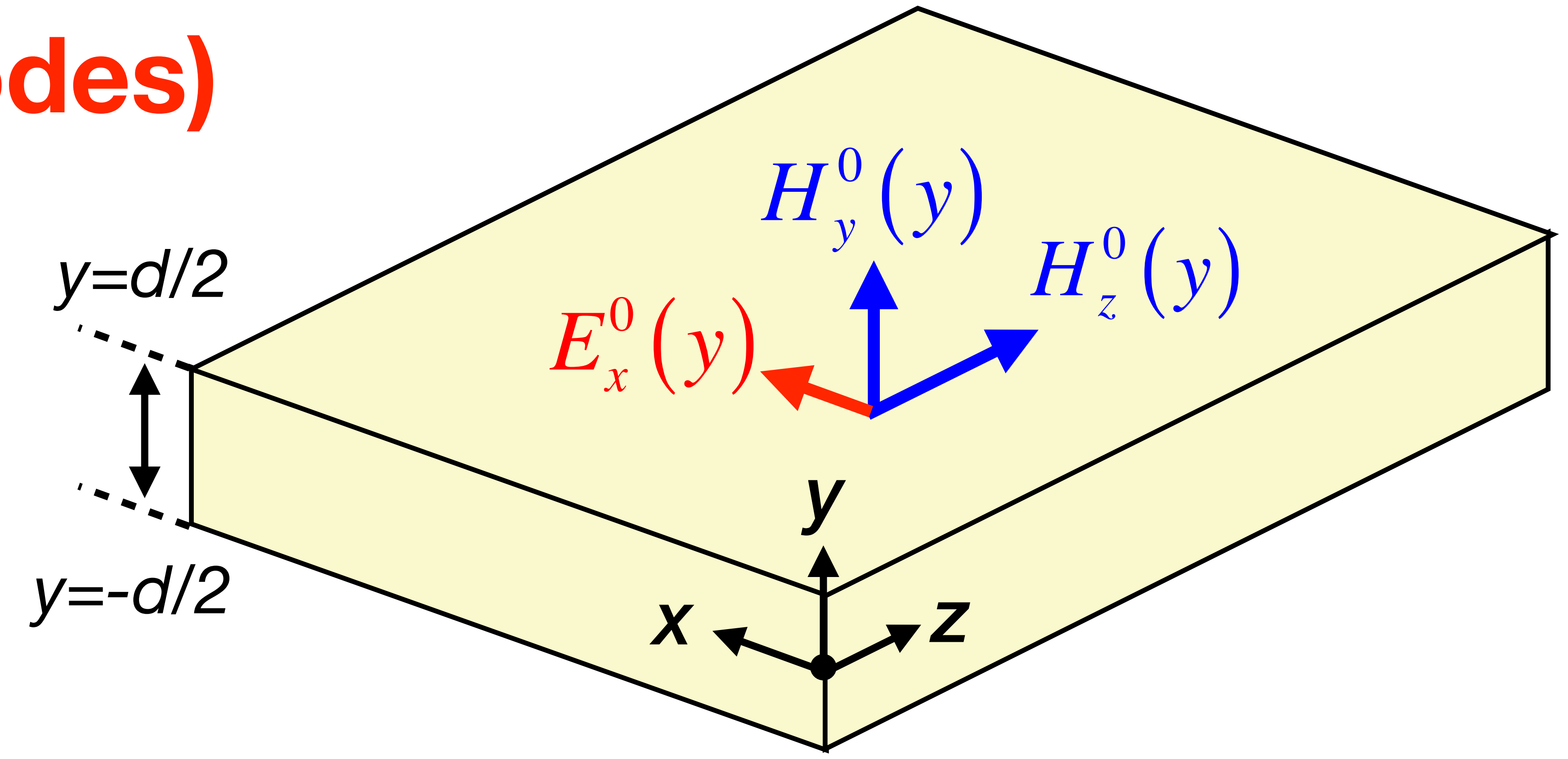
  - ***Nonzero*** Transverse components

$$\begin{cases} E_x^0(y) = -\dfrac{j\omega\mu_0}{h_0^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\omega\mu_0}{\alpha} C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \\ H_y^0(y) = -\dfrac{\gamma}{h_0^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\beta}{\alpha} C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \end{cases}$$

α : Attenuation coefficient for *evanescent wave*



Chap. 10 | TE waves along a dielectric slab (Odd TE modes)



- ***Relations between h_d and α***
  - Provided by B.C. such that tangential E-fields (at y = ±d/2) should be continuous

$$E_x^0\left(\frac{d}{2}\right) \rightarrow -\frac{j\omega\mu_d}{h_d}H_o\cos\frac{h_d d}{2} = -\frac{j\omega\mu_0}{\alpha}H_o\sin\frac{h_d d}{2}$$

$$\rightarrow \boxed{\frac{\alpha}{h_d} = \frac{\mu_0}{\mu_d}\tan\left(\frac{h_d d}{2}\right) \quad \cdots(1)}$$

:h_d-α relation for odd TE modes

  - By directly equating expressions for h_d and α,

$$\begin{cases} h_d^2 = \omega^2\mu_d\varepsilon_d - \beta^2 \\ \alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 \end{cases} \Rightarrow \quad h_d^2 + \alpha^2 = \omega^2\mu_d\varepsilon_d - \omega^2\mu_0\varepsilon_0$$

$$\rightarrow \boxed{\alpha = \sqrt{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - h_d^2} \quad \cdots(2)}$$

  - By substituting equation (2) into (1), we get

$$\frac{\sqrt{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - h_d^2}}{h_d} = \frac{\mu_0}{\mu_d}\tan\left(\frac{h_d d}{2}\right)$$

$$\rightarrow \boxed{\therefore \frac{\mu_d}{\mu_0}\sqrt{\frac{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0)d}{(h_d d)^2} - 1} = \tan\left(\frac{h_d d}{2}\right)}$$

:Transcendental equation for odd TE modes

B.C. between two lossless dielectrics

$$\begin{bmatrix} E_{1t} = E_{2t} \\ H_{1t} = H_{2t} \end{bmatrix} \text{(tangential)}$$

$$\begin{bmatrix} D_{1n} = D_{2n} \\ B_{1n} = B_{2n} \end{bmatrix} \text{(normal)}$$

$$\begin{cases} E_x^0(y) = -\dfrac{j\omega\mu_d}{h_d}H_0\cos h_d y & \cdots \text{ for } |y| \le \dfrac{d}{2} \\ E_x^0(y) = -\dfrac{j\omega\mu_0}{\alpha}C_u e^{-\alpha\left(y-\frac{d}{2}\right)} & \cdots \text{ for } y \ge d/2 \end{cases}$$

where $C_u = H_0\sin\dfrac{h_d d}{2}$

 TE waves along a dielectric slab (Even TE modes)

- ***Even*** ***TE modes in the dielectric (|y| ≤ d/2)***
  - Longitudinal components

$$E_z^0 = 0, \quad H_z^0(y) = H_e \cos h_d y$$

  - ***Nonzero*** Transverse components

$$\begin{cases} E_x^0(y) = -\dfrac{j\omega\mu_d}{h_d^2}\dfrac{\partial H_z^0}{\partial y} = \dfrac{j\omega\mu_d}{h_d} H_e \sin h_d y \\ H_y^0(y) = -\dfrac{\gamma}{h_d^2}\dfrac{\partial H_z^0}{\partial y} = \dfrac{j\beta}{h_d} H_e \sin h_d y \end{cases}$$

h_d : Wavenumber for the ***bounded wave***

$$\begin{cases} H_z^0(y) = E_o \sin h_d y + E_e \cos h_d y & \text{where } |y| \le d/2 \\ H_z^0(y) = C_u e^{-\alpha\left(y-\frac{d}{2}\right)} & \text{where } y \ge d/2 \\ H_z^0(y) = C_l e^{\alpha\left(y+\frac{d}{2}\right)} & \text{where } y \le -d/2 \end{cases}$$

Here, $h_d{}^2 = \omega^2\mu_d\varepsilon_d - \beta^2$

$$\alpha^2 = -h_0{}^2 = \beta^2 - \omega^2\mu_0\varepsilon_0$$

- ***Even*** ***TE modes in the upper free-space (y ≥ d/2)***
  - Longitudinal components

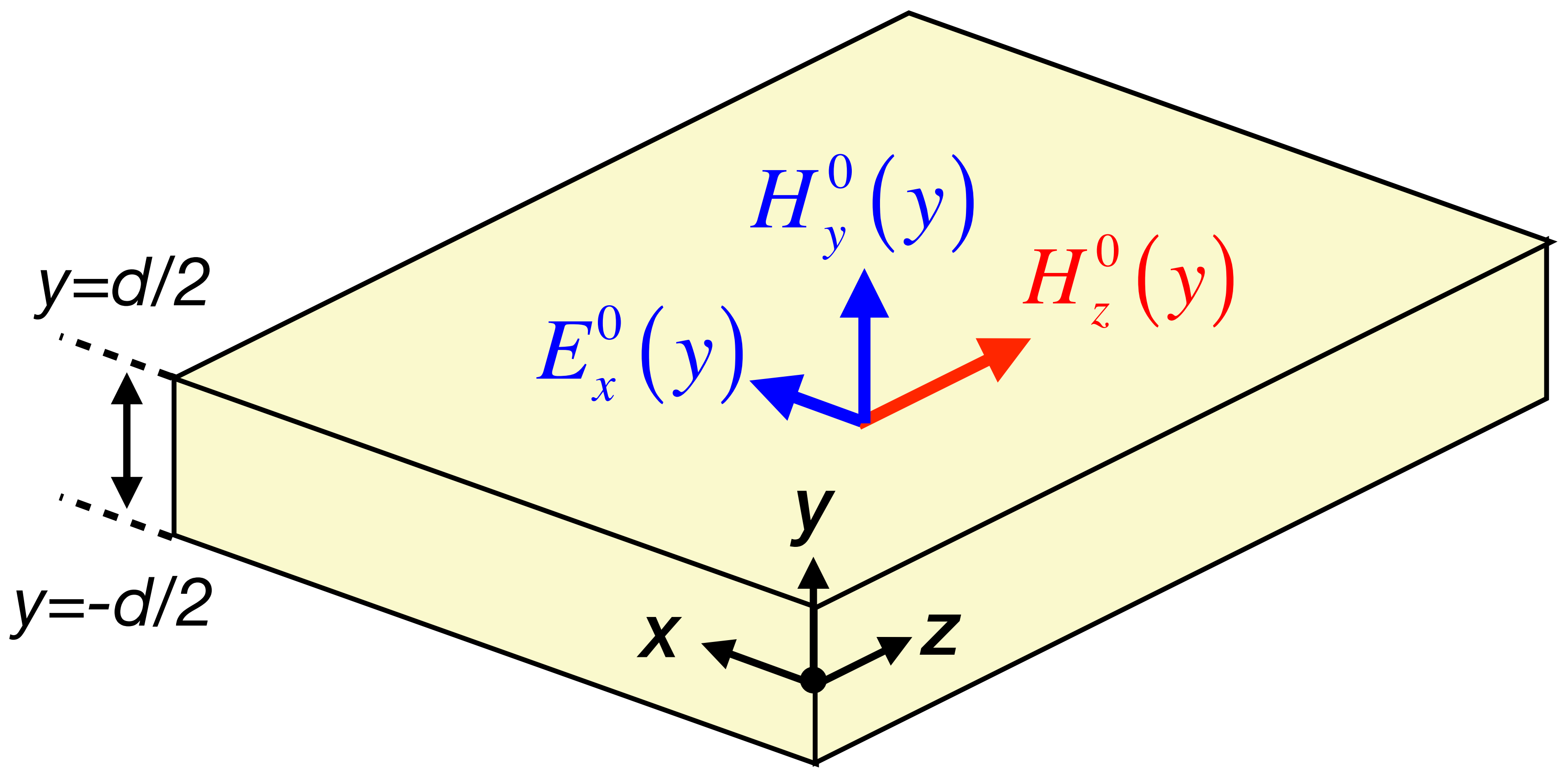
$$H_z^0(y) = C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \quad \text{where} \quad H_z^0\left(\frac{d}{2}\right) = C_u = H_e \cos\frac{h_d d}{2}$$

(∵B.C. Continuous tangential H-fields)

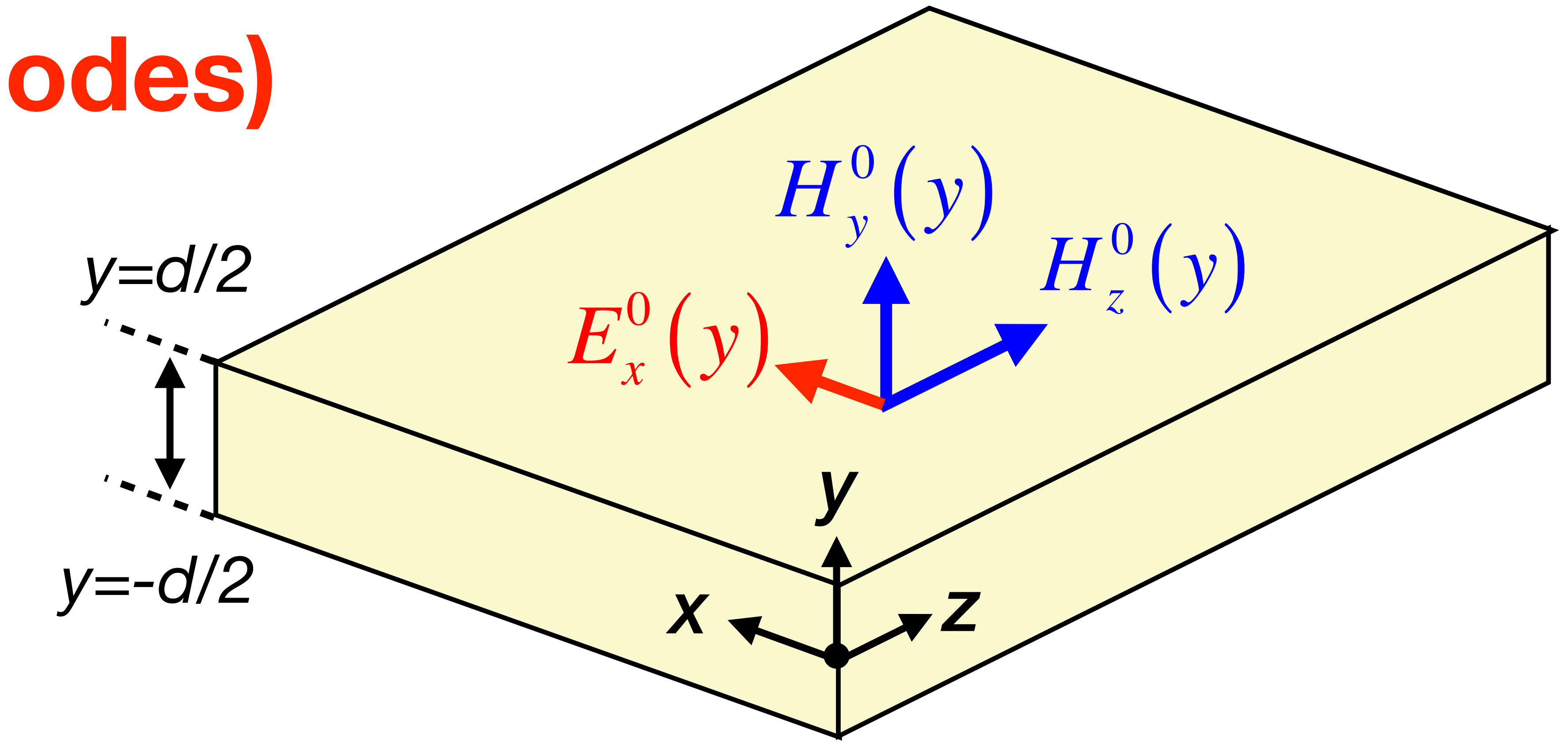
  - ***Nonzero*** Transverse components

$$\begin{cases} E_x^0(y) = -\dfrac{j\omega\mu_0}{h_0^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\omega\mu_0}{\alpha} C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \\ H_y^0(y) = -\dfrac{\gamma}{h_0^2}\dfrac{\partial H_z^0}{\partial y} = -\dfrac{j\beta}{\alpha} C_u e^{-\alpha\left(y-\frac{d}{2}\right)} \end{cases}$$

α : Attenuation coefficient for ***evanescent wave***



Chap. 10 | TE waves along a dielectric slab (Even TE modes)



- ***Relations between h_d and α***

- Provided by B.C. such that tangential E-fields (at y = ±d/2) should be continuous

$$E_x^0\left(\frac{d}{2}\right) \rightarrow \frac{j\omega\mu_d}{h_d} H_e \sin\frac{h_d d}{2} = -\frac{j\omega\mu_0}{\alpha} H_e \cos\frac{h_d d}{2}$$

$$\rightarrow \boxed{\frac{\alpha}{h_d} = -\frac{\mu_0}{\mu_d}\cot\left(\frac{h_d d}{2}\right) \quad \cdots(1)}$$

:h_d-α relation for even TE modes

- By directly equating expressions for h_d and α,

$$\begin{cases} h_d^{\,2} = \omega^2\mu_d\varepsilon_d - \beta^2 \\ \alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 \end{cases} \Rightarrow \begin{array}{l} h_d^{\,2} + \alpha^2 = \omega^2\mu_d\varepsilon_d - \omega^2\mu_0\varepsilon_0 \\ \rightarrow \boxed{\alpha = \sqrt{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - h_d^{\,2}} \quad \cdots(2)} \end{array}$$

- By substituting equation (2) into (1), we get

$$\frac{\sqrt{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - h_d^{\,2}}}{h_d} = -\frac{\mu_0}{\mu_d}\cot\left(\frac{h_d d}{2}\right)$$

$$\rightarrow \boxed{\therefore \frac{\mu_d}{\mu_0}\sqrt{\frac{\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0)d}{(h_d d)^2} - 1} = -\cot\left(\frac{h_d d}{2}\right)}$$

:Transcendental equation for even TE modes

B.C. between two lossless dielectrics

$$\begin{bmatrix} E_{1t} = E_{2t} \\ H_{1t} = H_{2t} \end{bmatrix} \text{(tangential)}$$

$$\begin{bmatrix} D_{1n} = D_{2n} \\ B_{1n} = B_{2n} \end{bmatrix} \text{(normal)}$$

$$\begin{cases} E_x^0(y) = \dfrac{j\omega\mu_d}{h_d} H_e \sin h_d y & \cdots \text{ for } |y| \le \dfrac{d}{2} \\ E_x^0(y) = -\dfrac{j\omega\mu_0}{\alpha} C_u e^{-\alpha\left(y - \frac{d}{2}\right)} & \cdots \text{ for } y \ge d/2 \end{cases}$$

where $C_u = H_0 \cos\dfrac{h_d d}{2}$

Chap. 10 | Cutoff frequencies for dielectric guides

(Note: Definition of cutoff frequency for dielectric waveguide is different from those for others [parallel-plate, single conductor, …])

- ***Cutoff frequencies***

: Frequencies where the waves are ***no longer bounded to the dielectric*** → ***Absence of attenuation, α = 0 (Not evanescent!)***

$$\alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0 = 0 \quad \rightarrow \quad \beta = \omega\sqrt{\mu_0\varepsilon_0} \quad \cdots(1)$$

- On the other hand, from the h_d-α relations we have

$$\begin{cases} \dfrac{\alpha}{h_d} = \dfrac{\mu_0}{\mu_d}\tan\left(\dfrac{h_d d}{2}\right) & \cdots\text{for odd TE modes} \\ \dfrac{\alpha}{h_d} = -\dfrac{\mu_0}{\mu_d}\cot\left(\dfrac{h_d d}{2}\right) & \cdots\text{for even TE modes} \end{cases} \Rightarrow \begin{cases} 0 = \dfrac{\mu_0}{\mu_d}\tan\left(\dfrac{h_d d}{2}\right) \rightarrow \dfrac{h_d d}{2} = n\pi \\ 0 = -\dfrac{\mu_0}{\mu_d}\cot\left(\dfrac{h_d d}{2}\right) \rightarrow \dfrac{h_d d}{2} = \left(n + \dfrac{1}{2}\right)\pi \end{cases} \quad (n = 0, 1, 2, \cdots)$$

- Since h_d is given by

$$h_d^{\ 2} = \omega^2\mu_d\varepsilon_d - \beta^2 = \omega^2\mu_d\varepsilon_d - \omega^2\mu_0\varepsilon_0$$

$$\rightarrow \quad h_d = \omega\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}$$

$$\Rightarrow \begin{cases} \dfrac{h_d d}{2} = \dfrac{\omega_{co} d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}{2} = n\pi \\ \dfrac{h_d d}{2} = \dfrac{\omega_{ce} d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}{2} = \left(n + \dfrac{1}{2}\right)\pi \end{cases} \quad (n = 0, 1, 2, \cdots)$$

$$\boxed{\therefore \begin{cases} f_{co} = \dfrac{n}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} & \text{for odd TE modes} \\ f_{ce} = \dfrac{n - 1/2}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} & \text{for even TE modes} \end{cases} \quad (n = 0, 1, 2, \cdots)}$$

Chap. 10 | Possible modes for dielectric guides

Equation requirements:
*L.H.S (green) and R.H.S (blue & red) should be **ALL POSITIVE!***

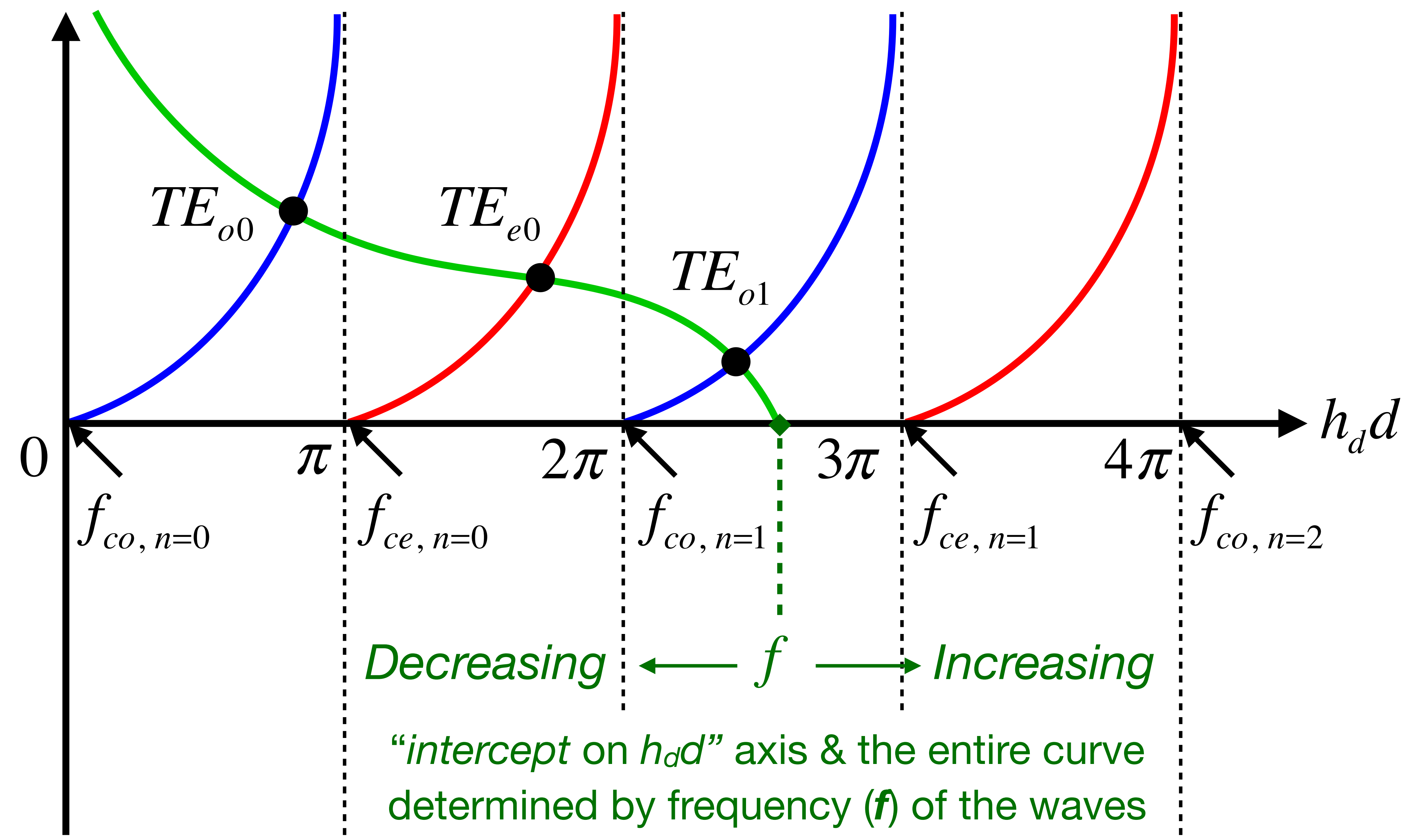
- ***Possible modes***

  - From transcendental equations for odd & even TE modes,

$$\begin{cases} \dfrac{\mu_d}{\mu_0}\sqrt{\dfrac{\omega^2(\mu_d\varepsilon_d-\mu_0\varepsilon_0)d}{(h_d d)^2}-1}=\tan\left(\dfrac{h_d d}{2}\right)>0 & \cdots\text{Odd TE modes} \\ \dfrac{\mu_d}{\mu_0}\sqrt{\dfrac{\omega^2(\mu_d\varepsilon_d-\mu_0\varepsilon_0)d}{(h_d d)^2}-1}=-\cot\left(\dfrac{h_d d}{2}\right)>0 & \cdots\text{Even TE modes} \end{cases}$$

Cutoff frequencies

$$\begin{cases} f_{co}=\dfrac{n}{d\sqrt{\mu_d\varepsilon_d-\mu_0\varepsilon_0}} & \cdots\text{Odd TE modes} \\ f_{ce}=\dfrac{n+1/2}{d\sqrt{\mu_d\varepsilon_d-\mu_0\varepsilon_0}} & \cdots\text{Even TE modes} \end{cases} \qquad (n=0,1,2,\cdots)$$



Example

- With a given width ***d*** of a slab and a frequency of the propagating waves (***f***), If $f_{co,n=1} < f < f_{ce,n=1}$
- There exist three possible modes, **TE_{o0}, TE_{e0} and TE_{o1}**
- Only ***a finite number of modes*** are allowed!

Dominant mode?

- If $n = 0 \rightarrow f_{co} = 0$
  - → TE_{o0} (lowest-order *odd* TE mode) = ***Dominant mode!***
  - → TE_{o0} can propagate along a waveguide ***with any thickness!***

Meaning of cutoff in dielectric waveguide

- ***Geometrical interpretation (Recall Section 8-10)***
  - Below cutoff → No total internal reflection → No propagation!

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_0}{u_d} = \frac{\sqrt{\mu_d\varepsilon_d}}{\sqrt{\mu_0\varepsilon_0}} \xrightarrow[\theta_i=\theta_c,\ \theta_t=\frac{\pi}{2}]{} \sin\theta_c = \frac{\sqrt{\mu_0\varepsilon_0}}{\sqrt{\mu_d\varepsilon_d}} \quad \cdots(1)$$

  - Below cutoff frequency, there is no total internal reflection since

$$\sin\theta_i < \sin\theta_c \quad \cdots(2) \quad \text{Here, } \sin\theta_i = \frac{\beta}{k} = \frac{\beta}{\omega^2\mu_d\varepsilon_d} \quad \cdots(3)$$

  - By plugging (1) and (3) into (2),

$$\beta < \omega\sqrt{\mu_0\varepsilon_0} \quad \cdots(4) \quad \text{where } \beta = \sqrt{k^2 - h_d^2} = \sqrt{\omega^2\mu_d\varepsilon_d - h_d^2} \quad \cdots(5)$$

  - By plugging (5) into (4),

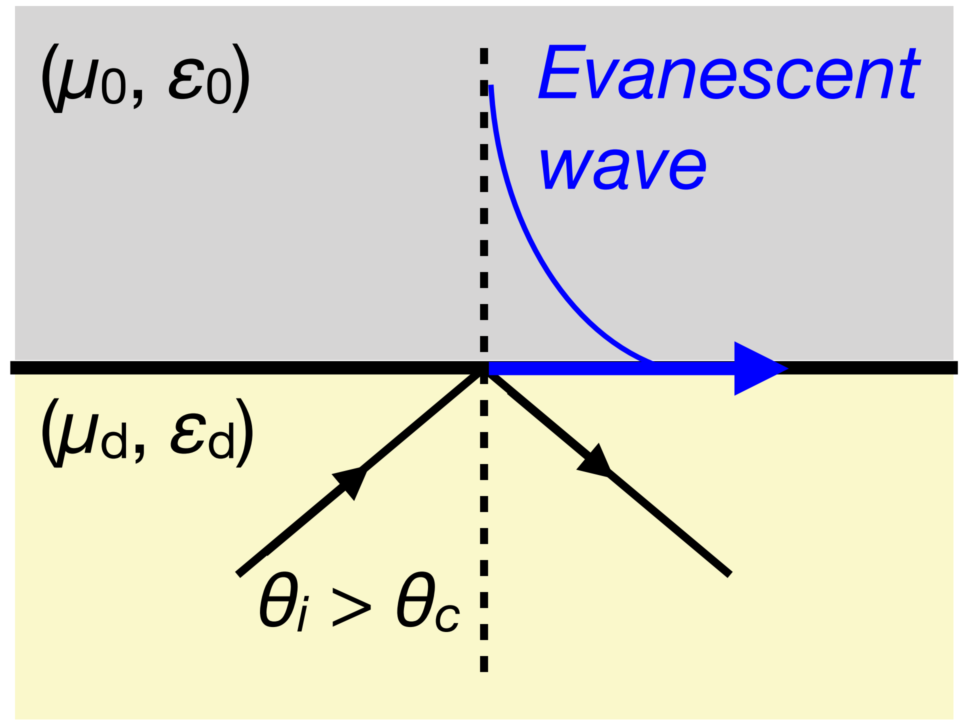
$$\omega\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0} < h_d \quad \rightarrow \quad f < \frac{h_d}{2\pi\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} \quad \cdots(6)$$

  - Smallest allowable h_d for n-th TE mode

$$\frac{h_d d}{2} = n\pi \quad \text{(for odd TE mode)} \quad \rightarrow \quad h_d = \frac{2n\pi}{d}$$

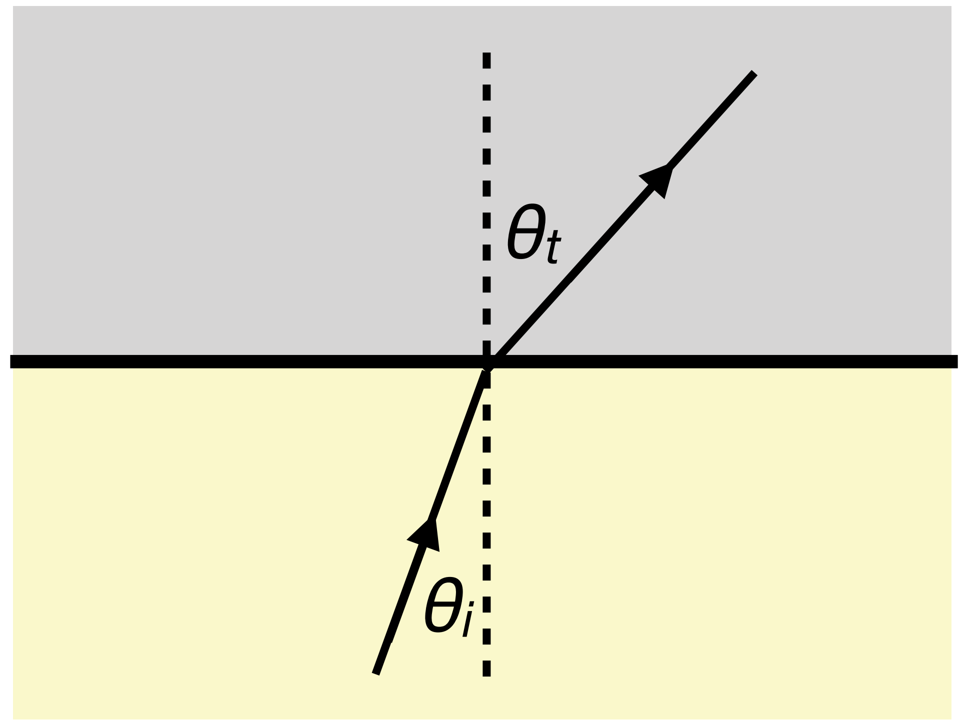
$$\boxed{\therefore f < \frac{n}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} = f_{co}}$$

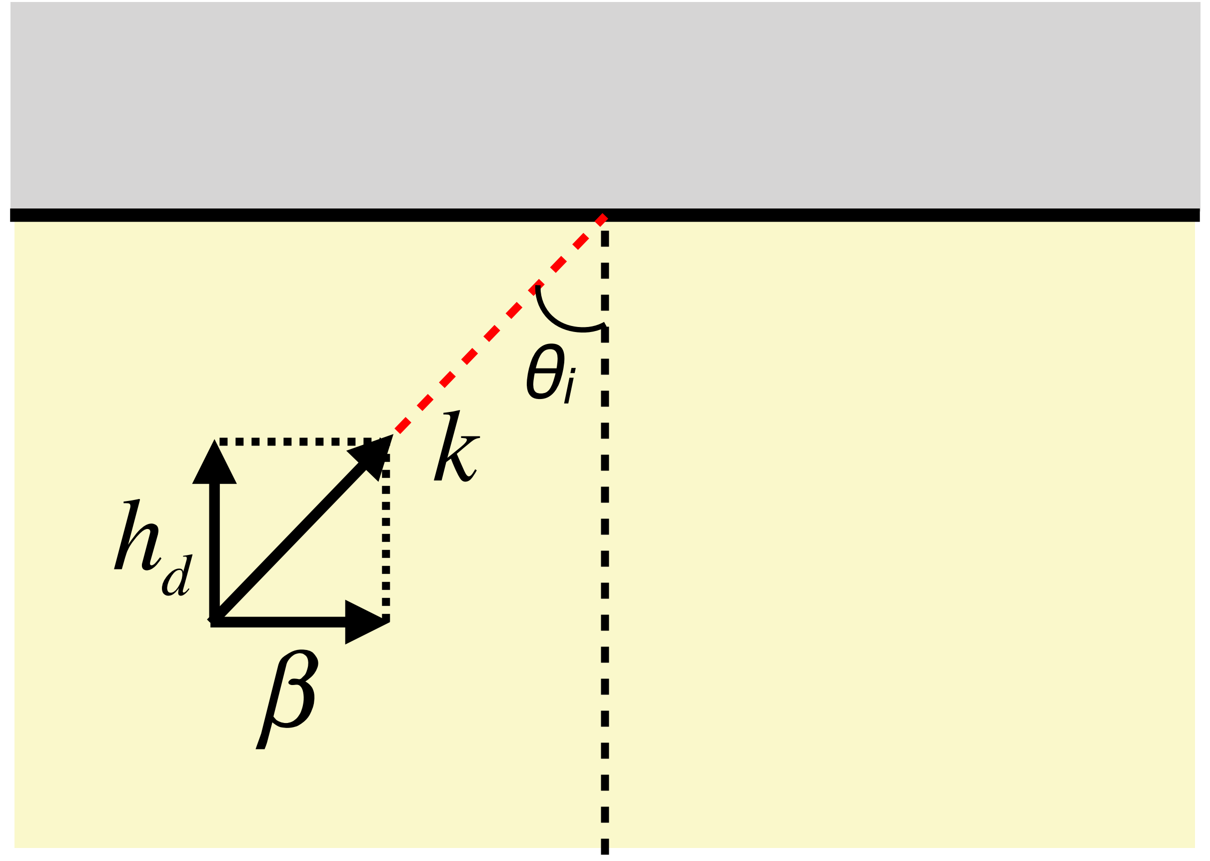
<Propagating (bounded) situation>
<u>*Above cutoff frequency*</u>



θ_c: *critical angle* when θ_t = 90°

<Unbounded situation>
<u>*Below cutoff frequency*</u>





$$\gamma^2 + k^2 = h_d^2$$

$$\rightarrow k^2 = h_d^2 + \beta^2 \quad (\because \gamma = j\beta)$$

• *Characteristic equations for dielectric-slab waveguide*

Mode		Characteristic equation	Cutoff frequency
TM	*Odd*	$\dfrac{\alpha}{h_d} = \dfrac{\varepsilon_0}{\varepsilon_d}\tan\left(\dfrac{h_d d}{2}\right)$	$f_{co} = \dfrac{n}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$
	Even	$\dfrac{\alpha}{h_d} = -\dfrac{\varepsilon_0}{\varepsilon_d}\cot\left(\dfrac{h_d d}{2}\right)$	$f_{ce} = \dfrac{n+1/2}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$
TE	*Odd*	$\dfrac{\alpha}{h_d} = \dfrac{\mu_0}{\mu_d}\tan\left(\dfrac{h_d d}{2}\right)$	$f_{co} = \dfrac{n}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$
	Even	$\dfrac{\alpha}{h_d} = -\dfrac{\mu_0}{\mu_d}\cot\left(\dfrac{h_d d}{2}\right)$	$f_{ce} = \dfrac{n+1/2}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$

Practice to derive at home!

Example 10-13 **|** *A dielectric-slab waveguide with* (μ_d, ε_d) = (μ_0, 2.5ε_0) is situated in free space. Determine the minimum thickness *d* such that a TM or TE wave of the "even" type at a frequency *f* = 20 GHz may propagate along the guide.

- Since $f_{ce,TM} = f_{ce,TE} = \dfrac{n+1/2}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} \rightarrow d = \dfrac{n+1/2}{f_{ce}\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} \rightarrow d_{\min} = \dfrac{1}{20\times10^9\times2\sqrt{\mu_0\varepsilon_0}\sqrt{2.5-1}} = 6.12(mm)$

$n = 1$