

Mechanical System



**Seoul National Univ.
School of Mechanical
and Aerospace Engineering**

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Newton's Laws

1) First law : conservation of momentum

no external force

→ no momentum change

linear momentum : mv

angular momentum: $J\omega$

2) Second law : $\sum F = ma = m \frac{dv}{dt}$

$$\sum T = J\alpha = J \frac{d\omega}{dt}$$



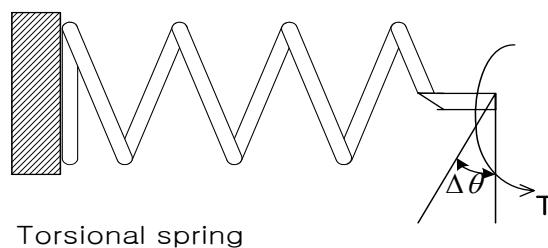
Three Basic Elements in Modeling Mechanical Systems

i) Inertial elements (kinetic energy):

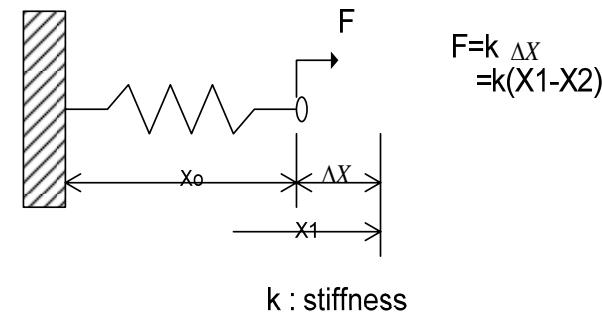
masses: M moments of inertial : J

ii) Spring elements (Potential energy)

$$T = k_{\theta} \cdot \Delta\theta = k(\theta_1 - \theta_0)$$



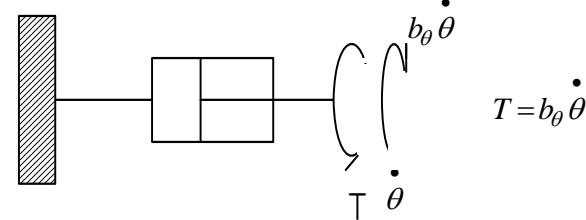
Torsional spring



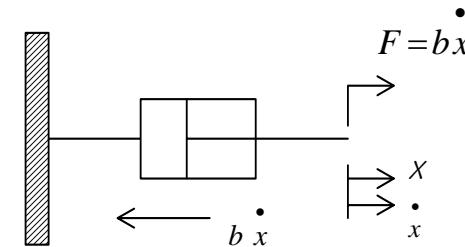
$$F = k \Delta X = k(x_1 - x_2)$$

k : stiffness

iii) Damper elements (energy dissipation)



$$T = b_{\theta} \dot{\theta}$$



$$F = b_x \dot{x}$$



Examples of Modeling Mechanical Systems

ex1) $t = 0, \omega(0) = \omega_0$

equation of motion :

$$J \frac{d\omega}{dt} = -b\omega$$

$$J \frac{d\omega}{dt} + b\omega = 0 \rightarrow \frac{d\omega}{dt} + \frac{b}{J}\omega = 0$$

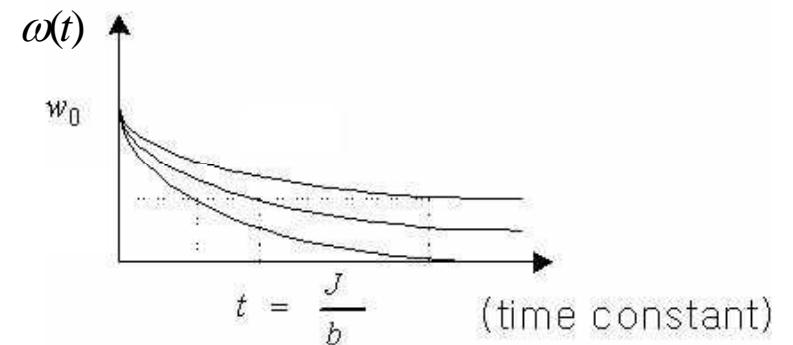
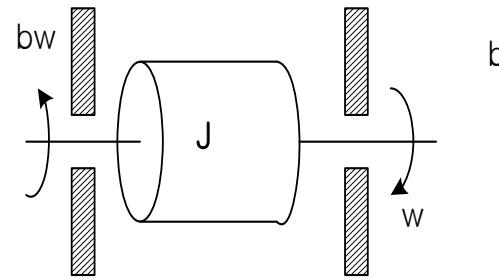
$$\text{let, } \omega(t) = ce^{\lambda t} \rightarrow \lambda e^{\lambda t} + \frac{b}{J}e^{\lambda t} = 0$$

$$\lambda = -\frac{b}{J}$$

$$\omega_0 e^{-1} = 0.368\omega_0$$

$$\omega(t) = ce^{-\frac{b}{J}t}, t = 0, \omega(0) = \omega_0 = C$$

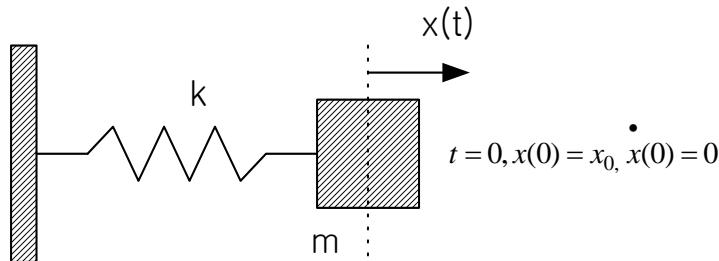
$$\therefore \omega(t) = \omega_0 e^{-\frac{b}{J}t}$$



Examples of Modeling Mechanical Systems

Spring-Mass

ex2)



$$m \frac{d^2x}{dt^2} = \sum F = -kx$$

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \frac{k}{m}x = 0$$

$$x(t) = A \cos \sqrt{\frac{k}{m}}t + B \sin \sqrt{\frac{k}{m}}t$$

$$x(0) = A = x_0$$

$$\dot{x}(0) = -A \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t + B \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}}t = B \sqrt{\frac{k}{m}} = 0$$

$$x(t) = x_0 \cos \sqrt{\frac{k}{m}}t = x_0 \cos \omega_n t$$

- Period $T = \frac{2\pi}{\sqrt{k/m}}$ [sec]

- Frequency $f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ [Hz]

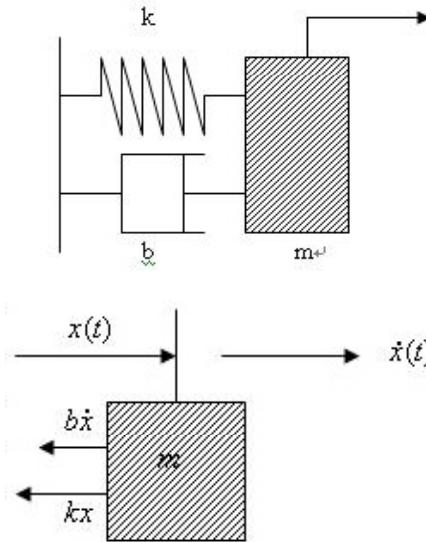
- Natural frequency

$$\omega_n = 2\pi f = \sqrt{\frac{k}{m}}$$
 [rad/sec]



Examples of Modeling Mechanical Systems

Spring-mass-damper



$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0$$

$$m \frac{d^2 x}{dt^2} = -kx - b\dot{x}$$

$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0$$

$$\text{let, } \frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n, \quad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad x(t) = c \cdot e^{\lambda t} \quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 x = 0$$

$$\therefore \lambda = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n$$



Examples of Modeling Mechanical Systems

$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0 \quad \begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 = 0 \end{aligned}$$

$$\text{let, } \frac{k}{m} = \omega_n^2, \quad \frac{b}{m} = 2\zeta\omega_n, \quad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

Laplace Transform

$$(s^2 X(s) - s \cdot x_0 - \dot{x}_0) + 2\zeta\omega_n (sX(s) - x_0) + \omega_n^2 X(s) = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) - s \cdot x_0 - 2\zeta\omega_n x_0 = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) = s \cdot x_0 + 2\zeta\omega_n x_0$$

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Examples of Modeling Mechanical Systems

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{s \cdot x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$s^2 + 2\zeta\omega_n s + \omega_n^2$: characteristic polynomial

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$: characteristic equation

$$s = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} : \text{characteristic roots}$$

1) $\zeta < 1$ underdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

$$X(s) = \frac{(s + \zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} + \frac{\zeta\omega_n x_0}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$x(t) = e^{-\zeta\omega_n t} \left(x_0 \cos \sqrt{1 - \zeta^2} \omega_n t + \frac{\zeta}{\sqrt{1 - \zeta^2}} x_0 \sin \sqrt{-\zeta^2} \omega_n t \right)$$



Examples of Modeling Mechanical Systems

2) $\zeta > 1$ overdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$X(s) = \frac{(s + 2\zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})}$$

$$x(t) = k_1 e^{(-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n)t} + k_2 e^{(-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n)t}$$

3) $\zeta = 1$ critically damped

$$s = -\zeta\omega_n$$

$$X(s) = \frac{(s + 2\zeta\omega_n) \cdot x_0}{(s + \zeta\omega_n)^2}$$

$$x(t) = k_1 e^{(-\zeta\omega_n)t} + k_2 t e^{(-\zeta\omega_n)t}$$



Examples of Modeling Mechanical Systems

1) $\zeta < 1$ Underdamped

$$x(t) = e^{-\zeta \omega_n t} \left(x_0 \cos \sqrt{1-\zeta^2} \omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} x_0 \sin \sqrt{1-\zeta^2} \omega_n t \right)$$

$$\begin{cases} \omega_n : \text{natural frequency} \\ \zeta : \text{damping ratio} \end{cases} \quad \omega_d = \sqrt{1-\zeta^2} \omega_n$$

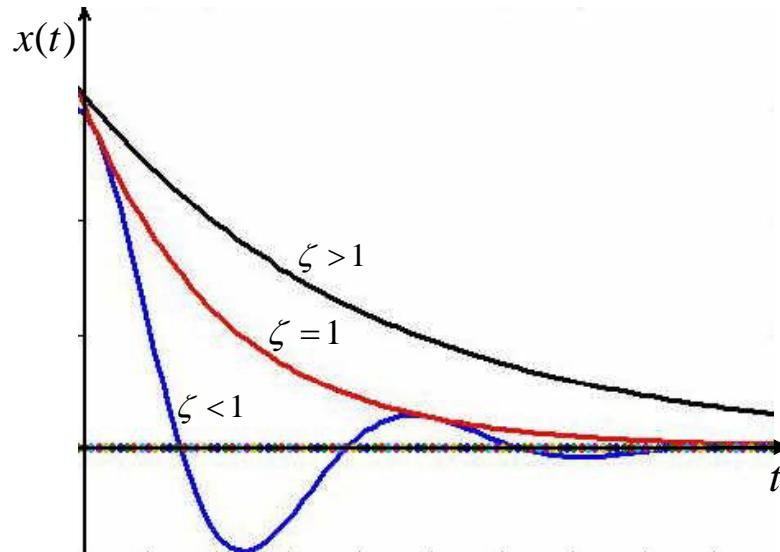
2) $\zeta > 1$ Overdamped

$$x(t) = k_1 e^{(-\zeta \omega_n + \sqrt{\zeta^2 - 1} \omega_n)t} + k_2 e^{(-\zeta \omega_n - \sqrt{\zeta^2 - 1} \omega_n)t}$$

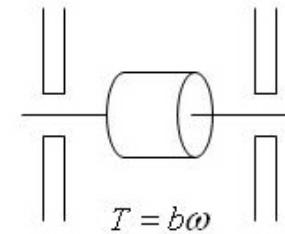
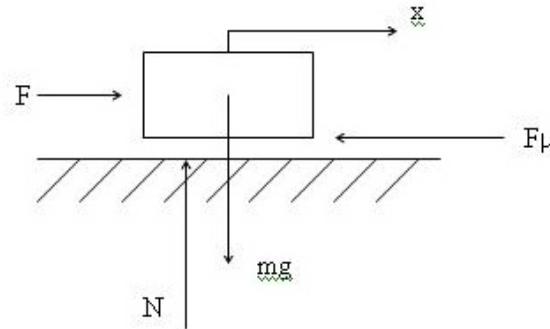
$$\lambda = -\zeta \omega_n \pm \sqrt{\zeta^2 - 1} \omega_n$$

3) $\zeta = 1$ $\lambda = -\zeta \omega_n$

$$x(t) = k_1 e^{-\zeta \omega_n t} + k_2 t e^{-\zeta \omega_n t}$$



Dry Friction (no lubricant)

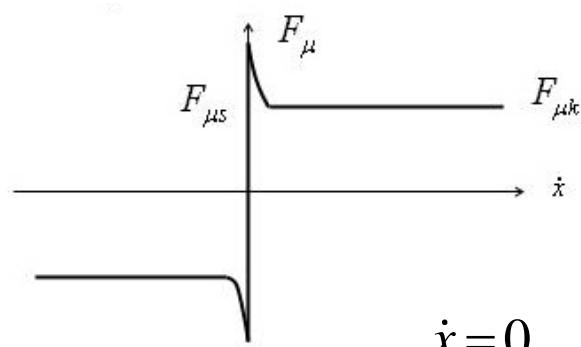


- $F_{\mu s}$ = Static Friction Force

- $F_{\mu k}$ = Kinetic Friction Force

- $F_{\mu s} = \mu s \cdot N$ μs : Static Friction Coefficient

- $F_{\mu k} = \mu k \cdot N$ μk : Kinetic Friction Coefficient



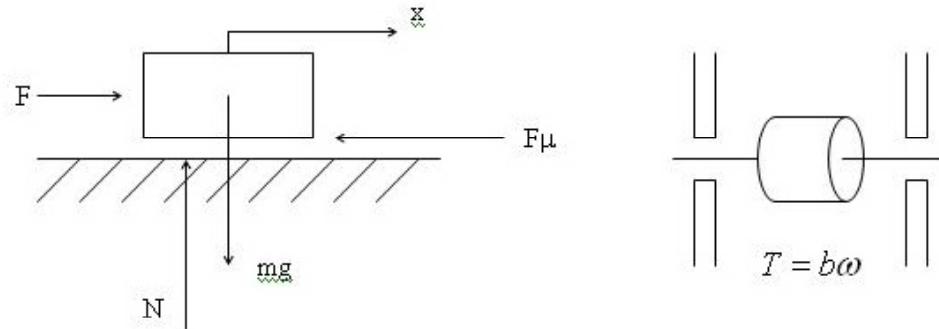
$$\dot{x} = 0$$

$$\dot{x} \neq 0$$

$$F_\mu = \begin{cases} F & \text{if } F \leq F_{\mu s} \\ F_{\mu s} \operatorname{sgn}(F) & \text{if } F > F_{\mu s} \end{cases} \quad F_\mu = F_{\mu k} \operatorname{sgn}(\dot{x})$$



Friction (with lubricant)



$$F_\mu = \begin{cases} b\dot{x} + G \cdot N \operatorname{sgn}(\dot{x}) & \forall \dot{x} > \varepsilon \\ F & \text{if } |\dot{x}| < \varepsilon \text{ and } |F| \leq (F_s + G \cdot N) \\ (F_s + G \cdot N) \operatorname{sgn}(F) & \text{otherwise} \end{cases} \quad \text{i.e., if } |\dot{x}| < \varepsilon \text{ and } |F| > (F_s + G \cdot N)$$

b: viscous friction coefficient

G: load-dependent factor

N: normal force

F_s: the maximum static friction

ε : a small bound for zero velocity detection



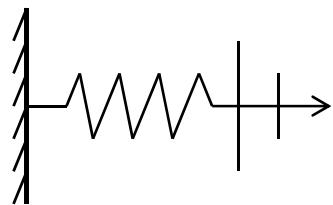
Work , Energy, and Power

- Mechanical work : $W = F \cdot x$ [N·m] = [Joule]
= Force \times displacement
- Energy : capacity or ability to do work. Electrical, Chemical, Mechanical, etc.
- Mechanical energy : Potential energy – position
Kinetic energy – velocity



Potential Energy

ex1)

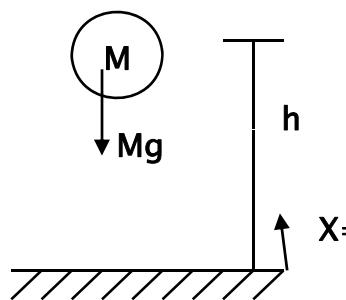


$$F = kx$$

$$dw = F \cdot dx = kx \cdot dx$$

$$\int_0^{x_1} dw = \int_0^{x_1} kx \, dx = \frac{1}{2} kx^2$$

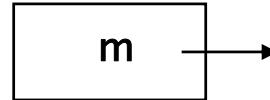
ex2)

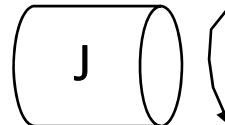


$$E_1 = mgx \quad (\text{Potential Energy})$$



Kinetic Energy


$$v : \frac{1}{2}mv^2$$

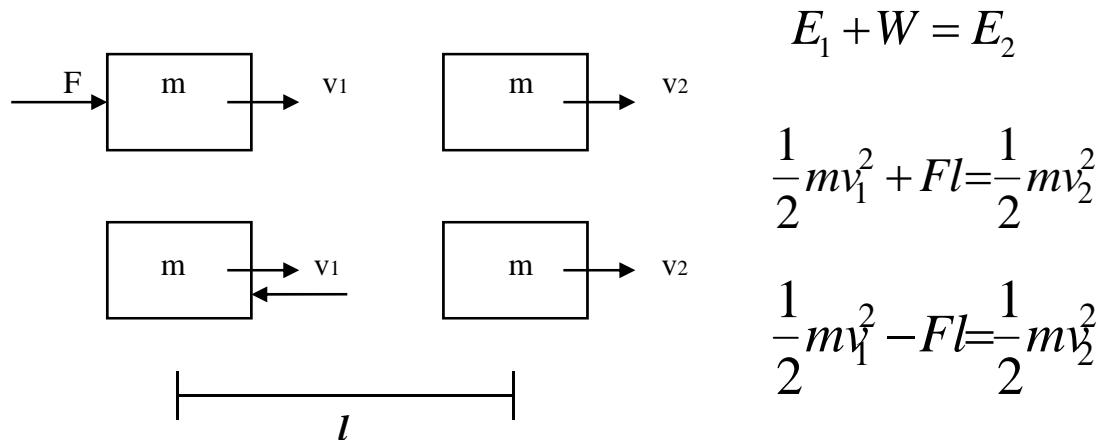

$$\omega : \frac{1}{2}J\omega^2$$



Work and Energy

$$\begin{array}{c} \text{System} \\ \text{Energy } E_1 \end{array} + \text{External Work } W = E_2$$

$$E_1 + W = E_2$$

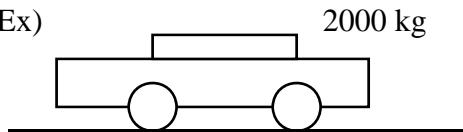


Power

Power : time rate & doing work

$$P = \frac{dw}{dt} \left[\frac{Nm}{sec} \right] = [Watt]$$

Ex)



$$V = 72 \text{ km/h} = 20 \text{ m/s} \text{ (in 10 sec)}$$

$$V_0 = 0$$

$$W = \frac{1}{2}(2000)(20)^2 = 400 \times 1000 \text{ Nm} = 400 \text{ kNm}[kJ]$$

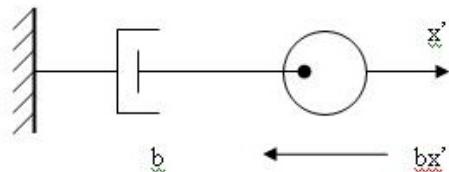
$$\frac{1}{2}mv_0^2 + W = \frac{1}{2}mv^2$$

$$P = \frac{W}{t} = \frac{400 \times 1000}{10} \frac{\text{Nm}}{\text{sec}} \\ = 40 \times 1000 \text{ Nm/s} = 40 \times 1000 \text{ W} = 40 \text{ W}$$

$$1 \text{ hp} = 745.7 \text{ W}$$

$$\therefore P = 54 \text{ hp}$$

Power dissipated in a damper



$$P = Fv = b\dot{x} \cdot \dot{x} = b\dot{x}^2$$



An Energy Method for Deriving Equations of Motion

- Conservative system : No energy dissipation

$$E_1 + W = E_2$$

$$E_2 - E_1 = W$$

- Kinetic Energy T

- Potential Energy U

$$\Delta(T+U) = \Delta W$$

(the change in the total energy)
= (the net work done on the system by the external force)

no external force ; $\Delta W = 0$

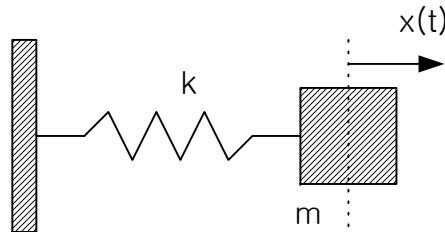
$$\Delta(T+U) = 0$$

$$T+U = \text{constant}$$



Examples of Energy Method

ex1)



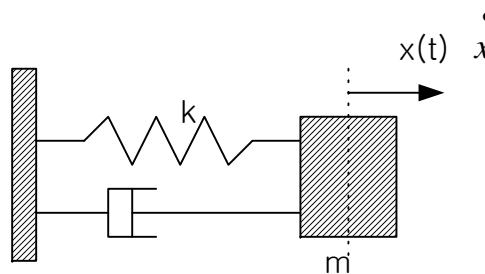
$$T = \frac{1}{2}m\dot{x}^2, \quad U = \frac{1}{2}kx^2, \quad T + U = C$$

$$\rightarrow \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = 0$$

$$\frac{d}{dt}(T + U) = 0, \quad m\ddot{x}\dot{x} + kx\dot{x} = (m\ddot{x} + kx)\dot{x} = 0$$

$$\dot{x} \neq 0, \quad \therefore (m\ddot{x} + kx) = 0$$

ex2)



$$T = \frac{1}{2}m\dot{x}^2, \quad U = \frac{1}{2}kx^2$$

$$\frac{d}{dt}(T + U) = -b\dot{x}^2, \quad m\ddot{x}\dot{x} + kx\dot{x} = -b\dot{x}\dot{x} = 0$$

$$(m\ddot{x} + kx + b\dot{x})\dot{x} = 0$$

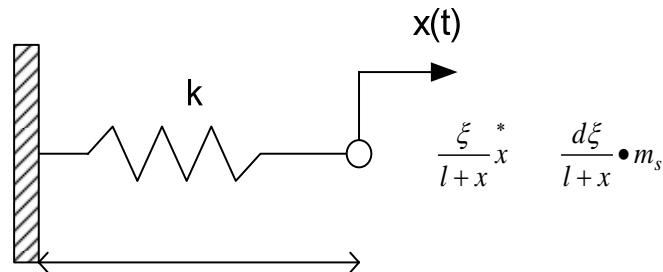
$$\dot{x} \neq 0, \quad \Rightarrow (m\ddot{x} + kx + b\dot{x}) = 0$$



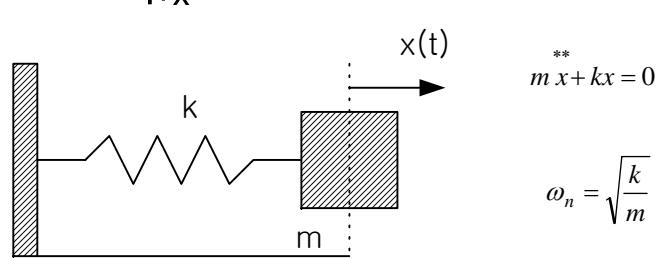
Examples of Energy Method

ex3)

Spring with mass



$$\frac{\xi}{l+x}x \quad \frac{d\xi}{l+x} \bullet m_s$$



$$m \ddot{x} + kx = 0 \quad \omega_n = \sqrt{\frac{k}{m}}$$

Potential E

$$U = \frac{1}{2}kx^2$$

Kinetic E

$$T = \frac{1}{2}m\dot{x}^2$$

$$dT = \frac{1}{2}m_s \cdot \frac{d\xi}{l+x} \cdot \left(\frac{\xi}{l+x}\dot{x}\right)^2$$

$$\int_0^{l+x} dT = \int_0^{l+x} \frac{1}{2}m_s \frac{1}{(l+x)^3} \dot{x}^2 \xi^2 d\xi$$

$$= \frac{1}{2}m_s \dot{x}^2 \frac{1}{(l+x)^3} \frac{1}{3}(l+x)^3$$

$$= \frac{1}{2} \left(\frac{1}{3}m_s\right) \dot{x}^2$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2} \left(\frac{1}{3}m_s\right) \dot{x}^2 = \frac{1}{2} \left(m + \frac{1}{3}m_s\right) \dot{x}^2$$

$$(m + \frac{1}{3}m_s)\ddot{x} + kx = 0$$

$$\omega_n = \sqrt{\frac{k}{m + \frac{1}{3}m_s}}$$

