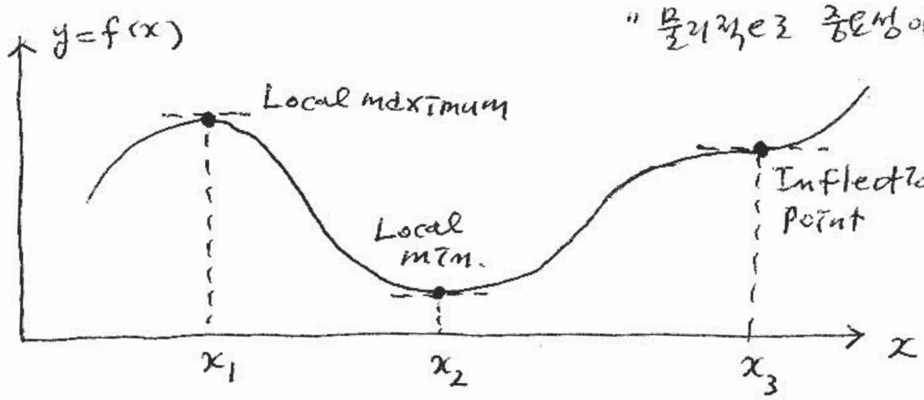
I. The theory of local extrema

"물리적으로 중요성이 큰 point 들"



Taylor series expansion about a position $x=a$ with assuming that $f(x)$ has continuous derivatives at $x=a$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

or

$$f(x) - f(a) = \left[f'(a)\right](x-a) + \left[\frac{f''(a)}{2!}\right](x-a)^2 + \left[\frac{f'''(a)}{3!}\right](x-a)^3 + \cdots$$

Note : ① "a"의 근방 (the neighborhood of "a") ← "admissible values of x"

② the dominant term, higher order term

(1) $f(a)$가 극값을 갖기 위한 필요조건은? ← $f'(a)=0$
극소, 극대

(2) $f(a)$가 극소값을 갖기 위한 충분조건은?
(극대값)
$f'(a)=0$ & $f''(a) > 0$ for $\{f(x)-f(a)\} > 0$

(3) $f'(a) = f''(a) = 0$ 이되 $f'''(a) \neq 0$ 인 경우,
$x=a$ 에 대응하는 점의 물리적 의미는? (변곡점)
↳ 좀더 일반적으로 saddle point

Thus we see that point "a", for which $f'(a)=0$, may correspond to a local min., to a local max., or to an inflection point. Such points as a group are often of much physical interest.
stationary points

Note : 만일 $f'(a) = f''(a) = f'''(a) = 0$ 이면 극값에 대한 규정은?

함수해석학

Functionals (범함수)

plainly speaking, functions of functions

ex) $$I = \int_{x_1}^{x_2} F(x, y, y')\,dx$$ ← 가령 total potential energy expression

"Scalar" (정적분)

function → scalar로 가는 mapping

① The value of I for a given set of end points x_1 and x_2 will depend on the function $y(x)$. Thus, just as $f(x)$ depends on the value of x, so does the value of I depend on the form of the function $y(x)$

② Just as we were able to set up "necessary conditions" for a local extreme of $f(x)$ at some point "a" by considering admissible values of x (i.e., x in the neighborhood of "a"), so can we find necessary conditions for extremizing I with respect to an admissible set of functions $y(x)$.

one of the cornerstones of the calculus of variation.

continum

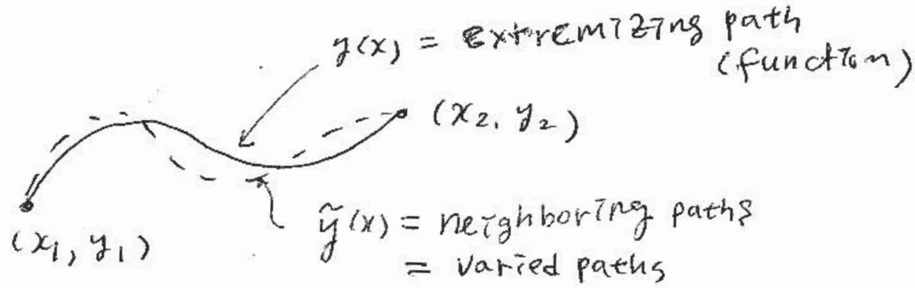
③ In the theory of elasticity, extremization of the total potential energy of a body with respect to an admissible family of displacement fields to satisfy the equations of equilibrium for the body will be of primary concern.

"functional"

Some very powerful approximate procedure (one important benefit)

The First Variation (제1차 변분)

Example :



$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \qquad \text{----} \quad (1)$$

where F = a known function

$y(x)$ = path extremizing I with respect to other neighboring paths $(= \tilde{y}(x))$

admissible function

$$\tilde{y}(x) = y(x) + \varepsilon\, \eta(x) \qquad \text{------} \quad (2)$$

single-parameter family of varied paths (an infinity of varied paths by adjusting ε)

where ε = a small parameter

$\eta(x)$ = a differentiable function with the requirement that

$$\eta(x_1) = \eta(x_2) = 0$$

$$\tilde{I} = \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}')\, dx = \int_{x_1}^{x_2} F(x,\, y + \varepsilon\eta,\, y' + \varepsilon\eta')\, dx \quad \text{---} (3)$$

Since $\tilde{I}$ is now a function of ε, we are able to use the extremization criteria of simple function theory as presented earlier.

$$\tilde{I} = f(\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} = (\tilde{I})_{\varepsilon=0} + \left(\frac{d\tilde{I}}{d\varepsilon}\right)_{\varepsilon=0} \times \varepsilon + \overbrace{\left(\frac{d^2\varepsilon}{d\varepsilon^2}\right)_{\varepsilon=0}}^{\frac{1}{2!}} \times \varepsilon^2 + \cdots$$

Hence :

$$\tilde{I} - I = \left(\frac{d\tilde{I}}{d\varepsilon}\right)_{\varepsilon=0} \times \varepsilon + \left(\frac{d^2\tilde{I}}{d\varepsilon^2}\right)_{\varepsilon=0} \times \varepsilon^2 + \cdots$$

④

For $\tilde{I}$ to be extreme when $\varepsilon = 0$,

$\left(\dfrac{d\tilde{I}}{d\varepsilon}\right)_{\varepsilon=0} = 0$ is a necessary condition, or

$$\left(\frac{d\tilde{I}}{d\varepsilon}\right)_{\varepsilon=0} = \frac{d}{d\varepsilon}\left[\int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}')\,dx\right]_{\varepsilon=0}$$

$$= \left[\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \tilde{y}} \underbrace{\frac{d\tilde{y}}{d\varepsilon}}_{\eta} + \frac{\partial F}{\partial \tilde{y}'} \underbrace{\frac{d\tilde{y}'}{d\varepsilon}}_{\eta'}\right) dx\right]_{\varepsilon=0}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \underbrace{\frac{\partial F}{\partial y'}\eta'}_{\text{부분적분 수행}}\right) dx = 0 \quad \text{---- (4)}$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'\,dx = \underbrace{\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}}_{\text{zero, } \because\ \eta(x_1)=\eta(x_2)=0} - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] \times \eta\,dx$$

$$= \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] \right\} \times \eta\,dx = 0$$

With $\eta(x)$ "arbitrary" between end points,

$$\therefore\ \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial y} = 0 \quad \text{---- (5)} \qquad F = T - \bar{V}$$

Hamilton eq. in dynamics

The famous Euler-Lagrange equation, or the condition required for $y(x)$ to be the extremizing function.

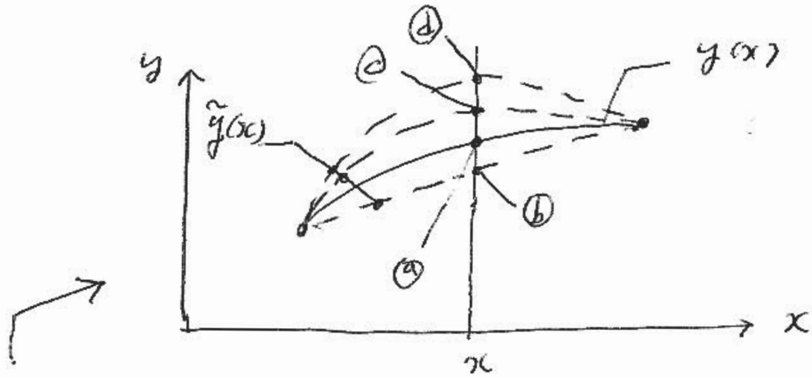
↳ 일반적으로 미지함수 $y(x)$에 대한 2계 상미분 방정식이 얻어짐. ← 정해를 구하기 어려운 경우가 많음 → Approximate solution

The delta operator (δ) ← 변분기호

↳ To give a formalism to the procedure of obtaining the first variation ← mechanical skill in handling variational problem.

Define $\delta[y(x)]$:

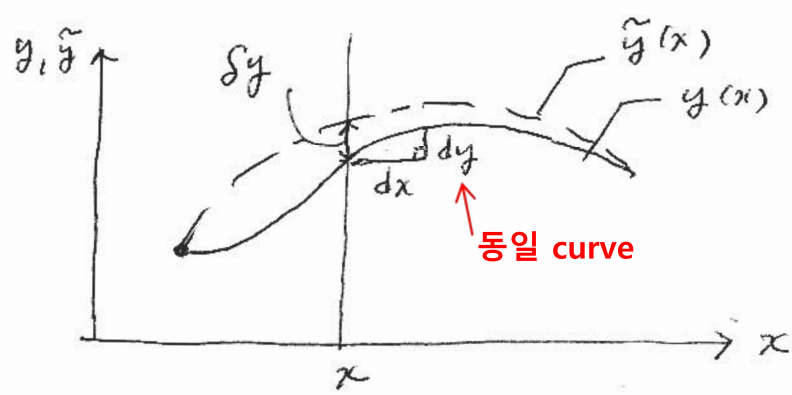
$$\delta[y(x)] = \tilde{y}(x) - y(x) \quad \text{------} \quad (6)$$



1) "The delta operator represents a small arbitrary change in the dependent variable y for a fixed value of the independent variable x."

2) Any of increments (b-a), (c-a), or (d-a) may be considered as δy.

3) δy 와 dy 의 차이



4) We can generalized the δ operator to represent a small change of a function wherein the independent variable is kept fixed.

"도함수의 변분" : $\delta\left[\frac{dy}{dx}\right] \equiv \left(\frac{d\tilde{y}}{dx} - \frac{dy}{dx}\right) = \frac{d}{dx}(\tilde{y} - y) = \frac{d}{dx}(\delta y) = \frac{d(\delta y)}{dx}$

↳ 따라서 미분연산자와 δ 연산자는 가환적!

"적분의 변분" : 동일 논리에 의해 $\delta\left[\int y\,dx\right] = \int \tilde{y}\,dx - \int y\,dx = \int (\tilde{y} - y)\,dx = \underline{\int \delta y\,dx}$

Go back to Eq. (1) in page 3:

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \quad \text{-----} \quad (1)$$

$$\tilde{y} = y + \varepsilon\eta$$
$$\tilde{y}' = y' + \varepsilon\eta'$$

Note: one-parameter family approach로서 예를 들고 있음.

$$\delta y = \tilde{y} - y = \varepsilon\eta \quad \text{--} \quad (a)$$

$$\delta y' = \tilde{y}' - y' = \varepsilon\eta' \quad \text{---} \quad (b)$$

We can give F along a varied path using the δ notation:

$$F(x, \underbrace{y+\delta y}_{\tilde{y}}, \underbrace{y'+\delta y'}_{\tilde{y}'})$$

Taylor series expansion about y and y'

$$= F(x, y, y') + \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] + \underbrace{O(\delta^2)}_{\text{negligible higher order}}$$

negligible higher order ($(\delta y)^2, (\delta y')^2, (\delta y)^3 \cdots$ etc)

$$\underbrace{\Big[F(x, y+\delta y, y'+\delta y') - F(x, y, y')\Big]}_{\text{total variation } \delta^{(T)}F} = \underbrace{\left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]}_{\text{the first variation } \delta^{(1)}F} + \underbrace{O(\delta^2)}_{\text{higher order}} \quad \text{---} \quad (7)$$

$\therefore$ On integrating Eq. (7) from x_1 to x_2,

$$\underbrace{\int_{x_1}^{x_2} F(x, y+\delta y, y'+\delta y')\, dx}_{\tilde{I}} - \underbrace{\int_{x_1}^{x_2} F(x, y, y')\, dx}_{I}$$

$$= \underbrace{\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] dx}_{\delta^{(1)} I} + O(\delta^2)$$

or

$$\tilde{I} - I = \delta^{(T)} I$$
$$= \delta^{(1)} I + O(\delta^2)$$

$$\delta^{(1)} I = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \underbrace{\frac{\partial F}{\partial y'} \delta y'}_{\text{이 항에 대하여서 부분적분}} \right) dx$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y} \delta y \, dx + \underbrace{\frac{\partial F}{\partial y'} \delta y \Big|_{x_1}^{x_2}}_{\text{zero}} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \delta y \, dx$$

($\delta y(x_1) = \varepsilon \eta(x_1) = 0$

$\delta y(x_2) = \varepsilon \eta(x_2) = 0$ 이므로)

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] \delta y \, dx$$

$$\tilde{I} - I = \delta^{(T)} I = \delta^{(1)} I + O(\delta^2)$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \underbrace{\delta y}_{\text{any value for any } x \in (x_1, x_2)} dx + O(\delta^2) \quad \text{---- (8)}$$

"I 값이 (최대 또는 최소의) 극값을 갖기 위해서는, 적분구간 내의 가능한 모든 변분값 δy 에 대해서 동일부호를 유지할 수 있어야 한다. 이 가능성은 (8)식의 적분기호 내의 괄호 안의 식이 zero가 되어야 확보된다.; 앞에서 유도한 Euler-Lagrange equation이 얻어짐."

↳ 곧 I가 극값을 갖기 위한 필요조건은 1차 변분값이 zero가 되어야 함을 알 수 있다:

$$\delta^{(1)} I = 0 \quad \text{---- (9)}$$

pp. 3~4의 방법보다 조금 간편하게 결과가 얻어졌음!

Note: 변분법에 관한 참고문헌으로는, 가령,

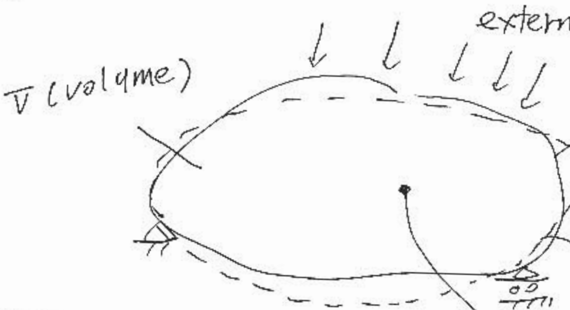
"Solid Mechanics: a variational approach", by C. L. Dym and I. H. Shames, McGraw-Hill, 1973

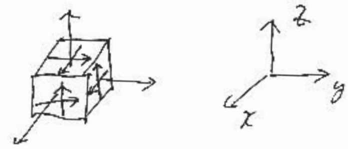
II. Work and Energy Related Principles and Theorems

(Brief review)

↳ stability analysis와 관련된 부분을 중심으로

1. Strain energy





: stress at a point
(6 components in general)

$\tau_{xx} \leftrightarrow \varepsilon_{xx}$
$\tau_{yy} \leftrightarrow \varepsilon_{yy}$ normal stress (strain)
$\tau_{zz} \leftrightarrow \varepsilon_{zz}$

$\tau_{xy} \leftrightarrow \gamma_{xy}$
$\tau_{yz} \leftrightarrow \gamma_{yz}$ shear stress (strain)
$\tau_{zx} \leftrightarrow \gamma_{zx}$

$$\underset{6\times1}{\tau} = \underset{6\times6}{C}\ \underset{6\times1}{\varepsilon}$$

"constitutive matrix"
↓
등방성 균질의 재료는 독립된 물질상수 2개
(E, G, ν ; $G = \dfrac{E}{2(1+\nu)}$)

$$W_e = U_i$$

↑ external work

↑ internal work (strain energy)
↑ the energy that can be converted into mechanical work (reversable energy)
↑ 보존계

$$W_e = U_i = \int_V \left\{ \underbrace{\int_{I}^{II} [\tau_{xx}\, d\varepsilon_{xx} + \cdots + \tau_{zx}\, d\gamma_{zx}]}_{\text{strain energy density function } \bar{U}_i} \right\} dV$$

or

$$\bar{U}_i = \int_{I}^{II} (\tau_{xx}\, d\varepsilon_{xx} + \cdots + \tau_{zx}\, d\gamma_{zx}) \quad \text{--- (1)}$$

Since the strain energy density at a point depends on the state of strain, $\varepsilon_{xx} \sim \gamma_{zx}$

$d\bar{U}_i$ (the incremental strain energy density)

$$= \frac{\partial \bar{U}_i}{\partial \varepsilon_{xx}} d\varepsilon_{xx} + \frac{\partial \bar{U}_i}{\partial \varepsilon_{yy}} d\varepsilon_{yy} + \cdots + \frac{\partial \bar{U}_i}{\partial \gamma_{zx}} d\gamma_{zx} \quad \leftarrow \text{total differential}$$

$$\bar{U}_i = \int_{I}^{II} d\bar{U}_i = \int_{I}^{II} \left(\frac{\partial \bar{U}_i}{\partial \varepsilon_{xx}} d\varepsilon_{xx} + \cdots + \frac{\partial \bar{U}_i}{\partial \gamma_{zx}} d\gamma_{zx} \right) \quad \text{--- (2)}$$

(1) 및 (2) 식을 비교하면,

$$\tau_{xx} = \frac{\partial \bar{U}_i}{\partial \varepsilon_{xx}} \; ; \; \tau_{yy} = \frac{\partial \bar{U}_i}{\partial \varepsilon_{yy}} \; ; \; \cdots \; ; \; \tau_{zx} = \frac{\partial \bar{U}_i}{\partial \gamma_{zx}} \quad \text{--- (3)}$$

$$\tau_{ij} = \frac{\partial \bar{U}_i}{\partial \varepsilon_{ij}}$$
(tensor notation)

When the material follows Hooke's law,

$$(\tau_{xx} = E\varepsilon_{xx}, \; \cdots, \; \tau_{zx} = G\gamma_{zx})$$

$$\bar{U}_i = \int_{I}^{II} [\underbrace{E\varepsilon_{xx}}_{\tau_{xx}} d\varepsilon_{xx} + \cdots + G\gamma_{zx}\, d\gamma_{zx}]$$

$$= \frac{1}{2} E\varepsilon_{xx}^2 + \cdots + \frac{1}{2} G\gamma_{zx}^2$$

$$= \frac{1}{2} (\underbrace{E\varepsilon_{xx}}_{\tau_{xx}} \cdot \varepsilon_{xx} + \cdots + \underbrace{G\gamma_{zx}}_{\tau_{zx}} \cdot \gamma_{zx})$$

$$= \frac{1}{2} (\tau_{xx}\varepsilon_{xx} + \cdots + \tau_{zx}\gamma_{zx}) \quad \text{---- (4)}$$

If the linear stress-strain relations are used in Eq. (4) in terms of Poisson's ratio (ν) and Young's modulus (E), the strain-energy density can be expressed solely in terms of strains or in terms of stresses.

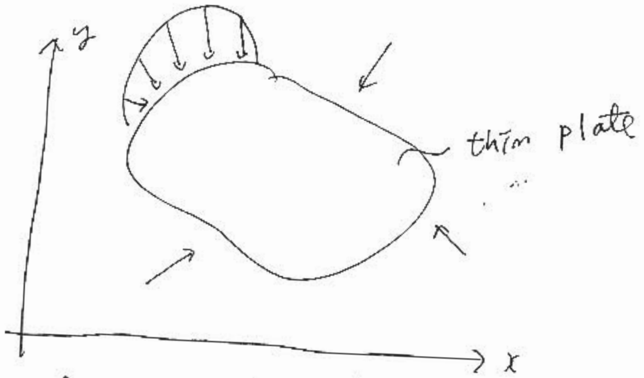
HW # 1. (생략!)

아래의 조건에 대해 $\bar{U}_i$를 순전히 strain을 사용하여 표시해 볼 것.

(1) three-dimensional case (가장 일반적인 경우)

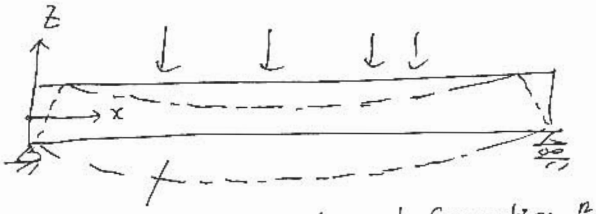
(2) two-dimensional case

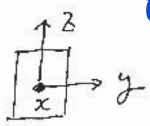
plane stress (xy-plane) : $\tau_{xz} = \tau_{yz} = \tau_{zz} = 0$



: An example of plane stress state

Ex) one-dimensional case ; a Bernoulli-Euler beam with the xz-plane as the plane of structural symmetry





$$U=\int_0^L \frac{M^2(x)}{2EI}\,dx \qquad \tau_{xx}=\frac{M(x)}{I}y$$

$$U=\int_V \bar{U}_i\, dV=\int_0^L\int_A \frac{\tau_{xx}^2}{2E}\,dA\,dx$$

동일결과 얻어짐.

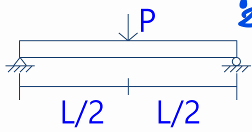
B.-E. beam (shear deformation 무시)

↳ 재축 방향의 normal strain 만 존재 (flexural strain energy)

$$\bar{U}_i=\frac{1}{2}\tau_{xx}\varepsilon_{xx}=\frac{1}{2}\cdot E\varepsilon_{xx}^2 \quad \text{(in terms of strain)}$$

$$=\frac{1}{2}\tau_{xx}\frac{\tau_{xx}}{E}=\frac{1}{2}\frac{\tau_{xx}^2}{E} \quad \text{(in terms of stress)}$$

Specifically



$\int \bar{U}_i d\bar{V} = ?$

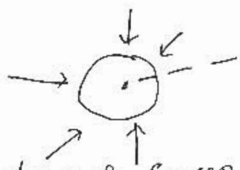
2. Principle of Virtual Displacements (or Virtual work)

Before stating this principle:

① A virtual displacement = a "hypothetical" displacement which must be "compatible" with the constraints for a given problem.

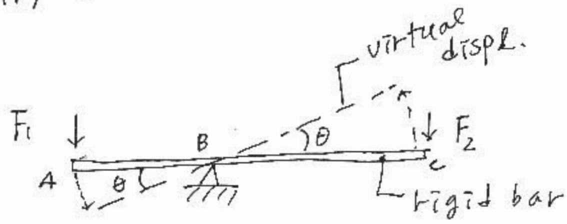
displ. field의 variation으로 생각 할 수 있음

(i) With a particle,

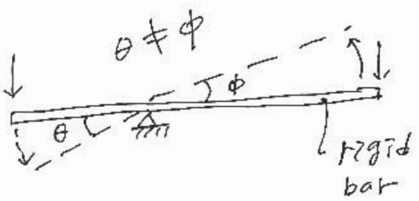


virtual displ. (a single vector without any limitations on magnitude and direction)

(ii) With a rigid body,



(a) Compatible



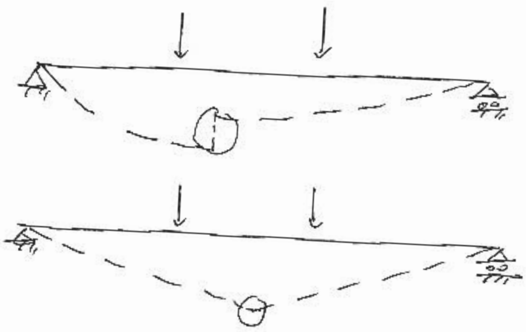
(b) Incompatible

; A virtual displacement is a displacement field which must be compatible with the requirement that the body be rigid.

(iii) With a cohesive deformable continuum,

Kinematically admissible deformation field의 요건

- the virtual displ. components must be single-valued continuous function of positions with continuous derivatives.



continuous and smooth

: Incompatible (not continuous and not smooth)

- the virtual displacements must be consistent with the theory and its related kinematic assumptions
  (Ex: small deformation theory)

(2) The reason the virtual displacement is called hypothetical :
During virtual displacement, the forces, internal and external, are kept constant, which is not compatible with the behavioral response of systems, in general.

The principle of virtual displacements (or virtual work) :

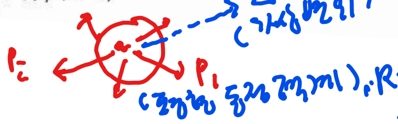
"A body is in equilibrium, under a given system of loads, if and only if for any virtual displacement the work done by the external forces is equal to the strain energy." ⇄ 평형조건식과 동치임 입증가능 (Cauchy's formula 및 발산정리 이용)

↳ 가상일의 원리는 탄·소성계 모두에 적용가능

탄·소성계 모두에서 만족되어야

Notes :
(1) 역학의 원리(principle)는 수학의 공리(axiom)와 흡사하다 :
(원리가 다른 원리나 법칙과 동등하다는 것을 보일 수는 있지만 직접 원리를 증명할 수는 없다)
(2) 입자, 강체, deformable body 모두에 적용 가능

Mathematical expression → $\delta W_e = \delta U_i$ ------------- (1)
δU_i ← 입자 및 강체의 경우는 zero

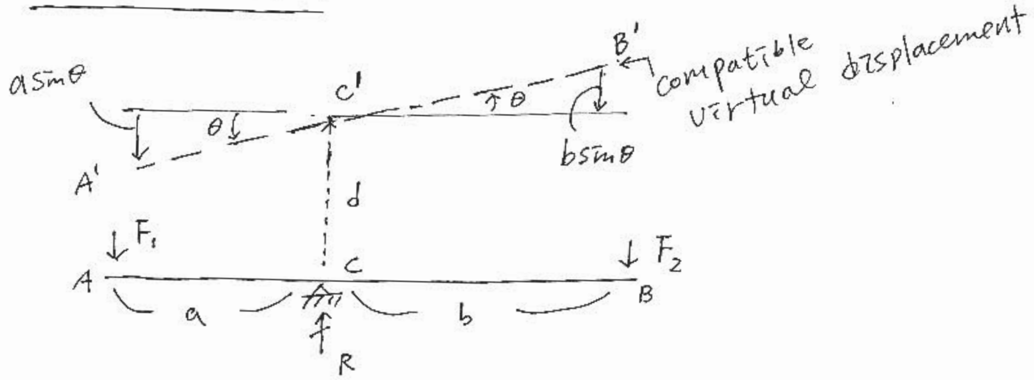


(평형상태 등가평형계), $R = \sqrt{\Sigma X^2 + \Sigma Y^2} = 0$, or $\Sigma X = 0$, $\Sigma Y = 0$

$\delta W_e = \underbrace{\Sigma X}_{0} \cdot \Delta_x + \underbrace{\Sigma Y}_{0} \cdot \Delta_y = 0$

반대로, 임의 가상 변위 Δx, Δy 에 항상 δW_e가 zero이면?

* 강체 바에 적용예 (the fulcrum problem)



: 평형상태에 있는 fulcrum

$$\delta \bar{W}_e = \delta U_i \ (=0)$$

평형상태에 있다면 가상일의 원리가 성립해야 할 것임.

$$= -F_1(d - a\sin\theta) + Rd - F_2(d + b\sin\theta)$$

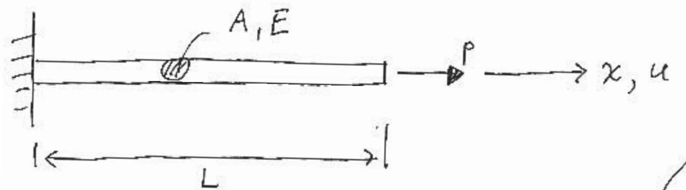
$$= (F_1 \cdot a - F_2 \cdot b)\sin\theta + (R - F_1 - F_2)\, d = 0$$

Independent and arbitrary

따라서 $\begin{pmatrix} F_1 a - F_2 b = 0 \\ R - F_1 - F_2 = 0 \end{pmatrix} \rightleftarrows \begin{pmatrix} \Sigma M_{about\ c} = 0 \\ \Sigma Y_i = 0 \end{pmatrix}$

($\Sigma X_i = 0$ 은 자동만족)

☆ 인장봉 (a deformable body)에 적용예



$$\varepsilon_{xx} = \frac{du}{dx} \ ; \ \tau_{xx} = E\,\varepsilon_{xx}$$ ← 평형조건식을 적용하여 τ_{xx} 및 ε_{xx}를 구하고 strain-displ. 관계식을 적분하면 $u(x)$를 구할 수 있음.

($\bar{u}(x)$: 가상변위 $\bar{u}(x=0)=0$ 을 만족해야 kinematically admissible

$\bar{\varepsilon}_{xx} = \dfrac{d\bar{u}}{dx}$

$$\delta \bar{W}_e = \delta U_i$$ ← 가상일의 원리 성립 (평형조건을 만족해야 하므로)

$$\delta \bar{W}_e = P \times \bar{u}(L)$$

$$\delta \bar{U}_i = \int_V \tau_{xx} \cdot \underbrace{\bar{\varepsilon}_{xx}}_{\frac{d\bar{u}}{dx}} \underbrace{dV}_{A\,dx}$$

$$= \int_0^L \tau_{xx} \cdot \frac{d\bar{u}}{dx} \cdot A\,dx$$

$$= \int_0^L (A\tau_{xx})\left(\frac{d\bar{u}}{dx}\right) dx \quad \leftarrow \text{부분적분 적용}$$

$$= A\tau_{xx}\,\bar{u}\Big|_0^L - \int_0^L \frac{d}{dx}(A\tau_{xx})\,\bar{u}\cdot dx = \overbrace{P \times \bar{u}(L)}^{\delta \bar{W}_e}$$

or $$\left[(P - A\tau_{xx})\bar{u}\right]_{x=L} - \int_0^L \frac{d}{dx}(A\tau_{xx}) \times \bar{u}\cdot dx = 0 \quad \cdots\ (*)$$

$\bar{u}(x)$ 및 $\bar{u}(L)$는 각각 가상변위이고 임의 값을 가지므로
(*) 식이 성립하려면,

i) $\frac{d}{dx}(\tau_{xx} A) = 0$ 이고 동시에 ii) $P - A\tau_{xx} = 0$ 가 만족되어야.

$\downarrow$ $\tau_{xx} \cdot A = \text{Constant}$ $\qquad$ $\downarrow$ $\tau_{xx} = \frac{P}{A}$

평형조건식을 적용해서 얻은 결과와 동일.

3. 가상일의 원리에서 파생되는 몇가지 원리, 정리 및 방법 등
principle theorem method

3.1 The principle of the Stationary Value of the Total Potential
실제로는 theorem 이 타당

Linear elastic system

에너지 보존계에서 Zero deformation (strain-free 위치) 상태에서 최종 상태에 도달할 때까지 하중이 한 일의 크기는 외력의 총 포텐셜의 감소값과 같다:

$$\bar{W}_e = -U_p$$

energy loss 없이

가상변위 동안에 행해진 일량의 변동값은 외력의 포텐셜의 변동값으로 표시할 수 있다.

$$\delta W_e = -\delta U_p \quad \text{---} \quad (2)$$

(2)식을 (11) 쪽 하단의 가상일의 원리 수식에 대입하면

임의의 가상변위에 대해 total P.E.의 변동량은 zero이다.

$$-\delta U_p = \delta U_i$$

내부 P.E.로 생각가능

평형조건식 ↔ or $\delta(U_i + U_p) = \boxed{\delta U_T} = 0 \quad \text{----} \quad (3)$

↑ total potential energy

「An "elastic" system is in equilibrium if and only if the first variation of the total potential energy (U_T) vanishes for every virtual displacement.」

이때의 보존계

↑ 가상일의 원리의 보존계에 대한 특화된 표현 (only to elastic bodies)

---

discrete system

만일 U_T가 N개의 deformation parameter q_i ($i=1,2,\cdots N$)의 함수로 표시될 수 있다면, 가상변위는 임의의 q_i에 대하여 취할 수 있으므로

generalized displ.

$$\delta U_T = \frac{\partial U_T}{\partial q_i} \times \delta q_i \quad (i=1,2,\cdots N) \quad \text{---} \quad (4)$$

가상일의 원리

δq_i는 임의의 값이므로 평형상태에 있기 위해서는, (3)식에 의해

$$\frac{\partial U_T}{\partial q_i} = 0 \quad (i=1,2,3,\cdots N) \quad \text{---} \quad (5)$$

따라서 U_T의 total differential (dU_T) 역시 zero가 된다:

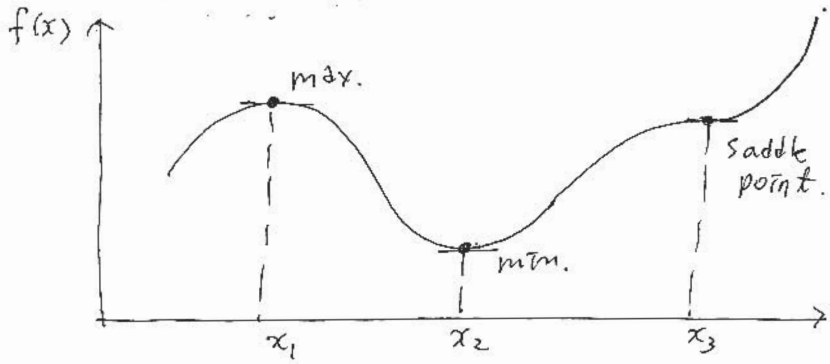
$$dU_T = \sum_{i=1}^{N} \frac{\partial U_T}{\partial q_i} \times dq_i = 0 \quad \text{(증분값이 zero)} \quad \text{---} \quad (6)$$

즉 「1차변분값 (δU_T)이 zero이면 곧 $dU_T = 0$ 라는 의미이고, dU_T가 zero가 되는 경우는 stationary points (정류점, 곧 극대점, 극소점 또는 안장점)에 해당한다.

local maximum, local minimum, saddle point

↳ 곧 U_T가 정류값을 갖는다고 말함.」

평형 ↔ 조건식



: stationary points (정류점)

앞 쪽의 논의를 기준으로 (3)식은 다음과 같이 해석될 수 있다:

" An elastic system (deformable) is in equilibrium if and only if the total potential energy has a stationary value."

↑ (II)쪽의 가상일의 원리를 더욱 특화시킨 것 ("theorem")

stable, unstable? 다음 문제

(The principle of the stationary value of the total potential energy)

3.2 The principle of the minimum total potential energy.

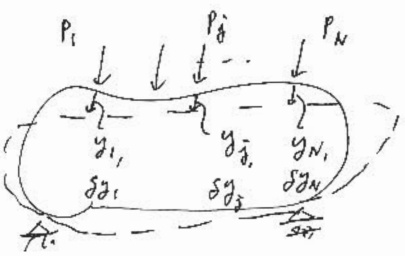
stress problem (자명, stable equm.)

" Of all possible kinematically admissible deformation fields in an elastic conservative system, only those that make the total potential assume a minimum value correspond to a stable equilibrium."

↑ variational calculus를 이용하여 입증가능 : 2nd variation의 부호 검토.

stability problem. → dominant term

3.3 Castigliano's First Theorem



$$\delta \overline{W}_e = \sum_{j=1}^{N} P_j \, \delta y_j \quad \text{--- (7)}$$

U_i = function (of $y_1, y_2, \cdots y_N$; structural geometry ; elastic behavior)

$$\delta U_i = \sum_{j=1}^{N} \frac{\partial U_i}{\partial y_j} \times \delta y_j \quad \text{--- (8)}$$

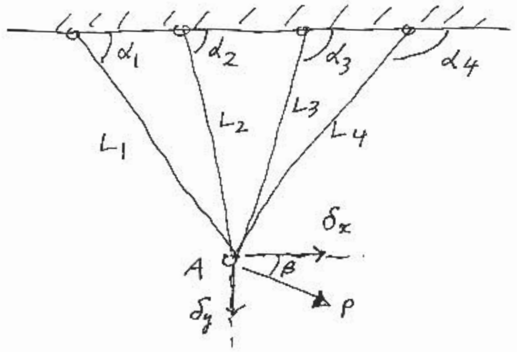
By the principle of virtual work ((7) = (8))

$$\sum_{j=1}^{N} \left(\frac{\partial U_i}{\partial y_j} - P_j \right) \delta y_j = 0 \;\rightarrow\; \frac{\partial U_i}{\partial y_j} = P_j$$

↑ independent

카스틸리아노 제1정리.
(displ. method)

Summary Example :



단면적 / 길이 / 영계수

: A system of pin-connected rods (A_i, L_i, E)

elastic

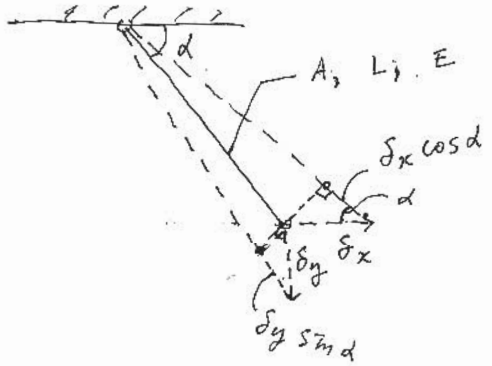
Find the deflection of pin joint A from the force P assuming linear elastic behavior(with small deformation).

(Solution)

$$\delta (U_i + U_p) = 0 \qquad ----- (*)$$

$U_T = U_i + U_p$ 를 2개의 deformation parameter δx, δy로 우선 표시해야 할 것임.

$$U_p = - (P\cos\beta \cdot \delta x + P\sin\beta \cdot \delta y)$$



$$\varepsilon_{xx} = \frac{\delta_x \cos\alpha}{L} + \frac{\delta_y \sin\alpha}{L} = 상수$$

$$\text{strain energy density} = \frac{1}{2} \tau_{xx} \varepsilon_{xx} = \frac{1}{2E} \varepsilon_{xx}^2$$

: deformation parameter vs. strain relationship.

$$U_i = \sum_{p=1}^{4} \int_V \left(\frac{1}{2E} \varepsilon_{xx}^2 \right) dv = \sum_{p=1}^{4} \frac{1}{2E} \left(\frac{\delta_x \cos\alpha_p}{L_p} + \frac{\delta_y \sin\alpha_p}{L_p} \right)^2 \times A_p L_p$$

따라서 $$U_T = \sum_{p=1}^{4} \frac{A_p}{2E L_p} (\delta_x \cos\alpha_p + \delta_y \sin\alpha_p)^2 - (\cos\beta \cdot \delta_x + \sin\beta \cdot \delta_y) \times P$$

$$----- (**)$$

$\frac{\partial U_T}{\partial \delta_i} = 0$ (in general)
⑭쪽 (5)식

U_T 값은 정류값 (실제로 최소값)을 가지므로,

$$\frac{\partial U_T}{\partial \delta_x} = \sum_{p=1}^{4} \frac{2A_p}{2EL_p}(\delta_x \cos\alpha_p + \delta_y \sin\alpha_p)\cos\alpha_p - P\cdot\cos\beta = 0$$

$$= \sum_{p=1}^{4} \frac{A_p}{EL_p}(\cos^2\alpha_p\, \delta_x + \cos\alpha_p\cdot\sin\alpha_p\cdot\delta_y) - P\cos\beta = 0, \text{ or}$$

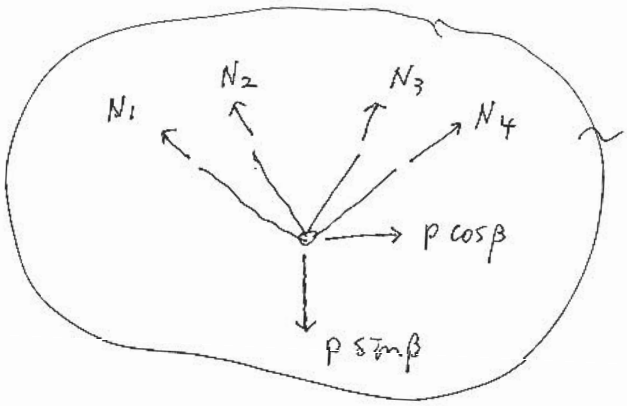
$$\left(\sum_{p=1}^{4} \frac{A_p\cos^2\alpha_p}{EL_p}\right)\delta_x + \left(\sum_{p=1}^{4} \frac{A_p\cos\alpha_p\cdot\sin\alpha_p}{EL_p}\right)\delta_y = P\cos\beta \quad \text{--- (1)}$$

$$\frac{\partial U_T}{\partial \delta_y} = \sum_{p=1}^{4} \frac{2A_p}{2EL_p}(\delta_x \cos\alpha_p + \delta_y \sin\alpha_p)\sin\alpha_p - P\cdot\sin\beta = 0$$

$$\left(\sum_{p=1}^{4} \frac{A_p\cos\alpha_p\cdot\sin\alpha_p}{EL_p}\right)\delta_x + \left(\sum_{p=1}^{4} \frac{A_p\sin^2\alpha_p}{EL_p}\right)\delta_y = P\sin\beta \quad \text{--- (2)}$$

(1), (2) 식을 연립하여 δ_x, δ_y를 구하면 될 것임.

(1), (2) 식은 실제로 pin joint A를 고립시켜서 얻어진 평형조건식과 동일할 것이다.



$\overset{+}{\rightarrow}$ $\Sigma X_i = 0$ --- (1) 식

$+\downarrow$ $\Sigma Y_i = 0$ --- (2) 식

Note : 카스틸리아노 제1정리를 즉시 이용하면

$$U_i = \sum_{p=1}^{4} \frac{A_p}{2EL_p}(\delta_x \cos\alpha_p + \delta_y \sin\alpha_p)^2 \text{ 이므로}$$

$$\frac{\partial U_i}{\partial \delta_x} = \sum_{p=1}^{4} \frac{A_p}{EL_p}(\delta_x \cos\alpha_p + \delta_y \sin\alpha_p)\cos\alpha_p = P\cos\beta$$

↑ 위의 (1)식과 동일.

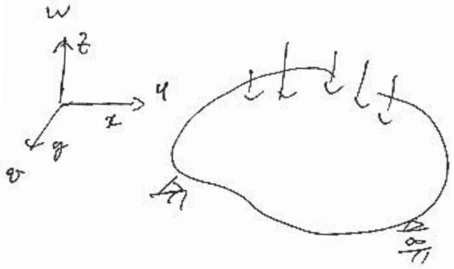
$$\frac{\partial U_i}{\partial \delta_y} = \qquad \qquad = P\sin\beta$$

↑ 위의 (2)식과 동일.

3.4 The Rayleigh-Ritz Method ← Interpolation/shape function 개념 도입

1) This method may be considered as a derivative of the principle of the stationary value of the total potential energy when applied to elastic systems under static loads.

2) The Basic Idea : ∞ dofs $\longrightarrow$ finite dofs. (유한요소법의 뿌리)



$$
\left.\begin{aligned}
u(x,y,z) &= \sum_{i=1}^{N} a_i f_i(x,y,z) \\
v(x,y,z) &= \sum_{i=1}^{N} b_i g_i(x,y,z) \\
w(x,y,z) &= \sum_{i=1}^{N} c_i h_i(x,y,z)
\end{aligned}\right\} \quad \text{--- (9)}
$$

A finite series of kinematically admissible functions multiplied by undetermined constants. → 3N

[single-valued, continuous, differentiable, and satisfying the kinematic boundary conditions.]

3) The total potential energy (a functional, originally) becomes a function of the 3N undetermined constants a_i, b_i, and c_i ;

$$\underbrace{U_T[u,v,w]}_{\text{functional}} = \underbrace{U_T(a_i, b_i, c_i)}_{\text{function of 3N constants}} \quad \text{---- (10)}$$

Rigorous calculus of variation 적용 → 지배방정식이 (미분방정식) 유도되고, not solution → 이 미방을 다시 풀어야... (ex. Euler-Lagrange equation)

4) Since the functions f_i, g_i, and h_i are kinematically admissible, the virtual displacement can be taken as

$$\delta u = \delta a_i \cdot f_i \; ; \; \delta v = \delta b_i \cdot g_i \; ; \; \delta w = \delta c_i \cdot h_i \quad \text{--- (11)}$$

The variation in the total potential is given by

$$\delta U_T = \sum_{i=1}^{N} \left[\frac{\partial U_T}{\partial a_i} \times \delta a_i + \frac{\partial U_T}{\partial b_i} \times \delta b_i + \frac{\partial U_T}{\partial c_i} \times \delta c_i \right] = 0$$

The elastic system is in equilibrium if

$$\frac{\partial U_T}{\partial a_i} = 0 \; ; \; \frac{\partial U_T}{\partial b_i} \; ; \; \frac{\partial U_T}{\partial c_i} = 0 \quad (i = 1, 2, \ldots N) \qquad \text{---- (12)}$$

Weaker formulation to extremize (minimize) the functional (or total potential energy) $U_T(u, v, w)$ in Eq. (10).

→ 3N linearly independent algebraic equations

↓ solution

the approx. expressions for u, v, w

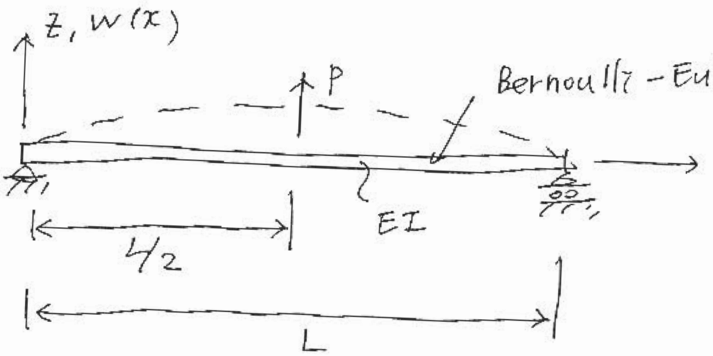
↓ kinematic relations

strains

↓ constitutive relations

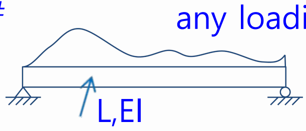
stress, thus the analysis is complete.

Ex)



Bernoulli-Euler beam (flexural strain energy only)

HW#



strain E. of bending

$$U_i = \int_0^L \frac{M^2}{2EI} dx$$

$$U_T = U_i + U_p$$

$$= \int_0^L \frac{1}{2} EI (w'')^2 dx - P \times \overline{w}\left(x = \frac{L}{2}\right) \qquad \text{---- } (*) \quad = \int_0^L \frac{1}{2} EI (W'')^2 dx$$

↓ strain energy density 에서 시작

$$U_i = \int_{\bar{V}} \frac{1}{2} \sigma_x \varepsilon_x d\bar{V}$$

$$w = \sum_{m=1}^{N} a_m \sin\left(\frac{m\pi}{L} x\right) \qquad \text{---- } (**)$$

satisfies the kinematic BCs; $w(x=0) = 0$, $w(x=L) = 0$

(**) → (*) 대입하여 정리하면,

$$U_T = \frac{EIL}{4} \sum_{m=1}^{N} a_m^2 \left(\frac{m\pi}{L}\right)^4 - P \times \sum_{m=1}^{N} a_m \sin\left(\frac{m\pi}{2}\right) \qquad \text{--- } (***)$$

$$\frac{\partial U_T}{\partial a_m} = 0 \quad (m = 1, 2, \ldots N)$$

(the principle of the min. of the total potential E.)

$$\left(\frac{EIL}{2}\right) a_m \left(\frac{m\pi}{L}\right)^4 - P \times \sin\left(\frac{m\pi}{2}\right) = 0 \quad (m = 1, 2, \cdots N)$$

} a decoupled system of N linear algebraic equation in a_m

The solution is

$$a_m = \frac{2P}{LEI}\left(\frac{L}{m\pi}\right)^4 \sin\left(\frac{m\pi}{2}\right)$$

$$w = \sum_{m=1}^{N} \frac{2P}{LEI}\left(\frac{L}{m\pi}\right)^4 \sin\left(\frac{m\pi}{2}\right) \times \sin\left(\frac{m\pi}{L}x\right)$$

The convergence for the deformation is very rapid.

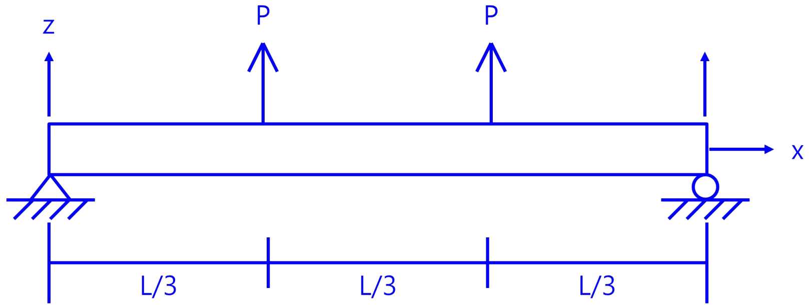
$$M = EIw''$$
$$V = -EIw'''$$

the convergence not so rapid.
(higher derivatives of w)

— The End — .

HW#

위 예제와 동일 조건, but



N=10 까지만 truncation하여
BMD 그려 볼 것 (excel, matlab?)