457.649 Advanced Structural Analysis

Part II:
Displacement method

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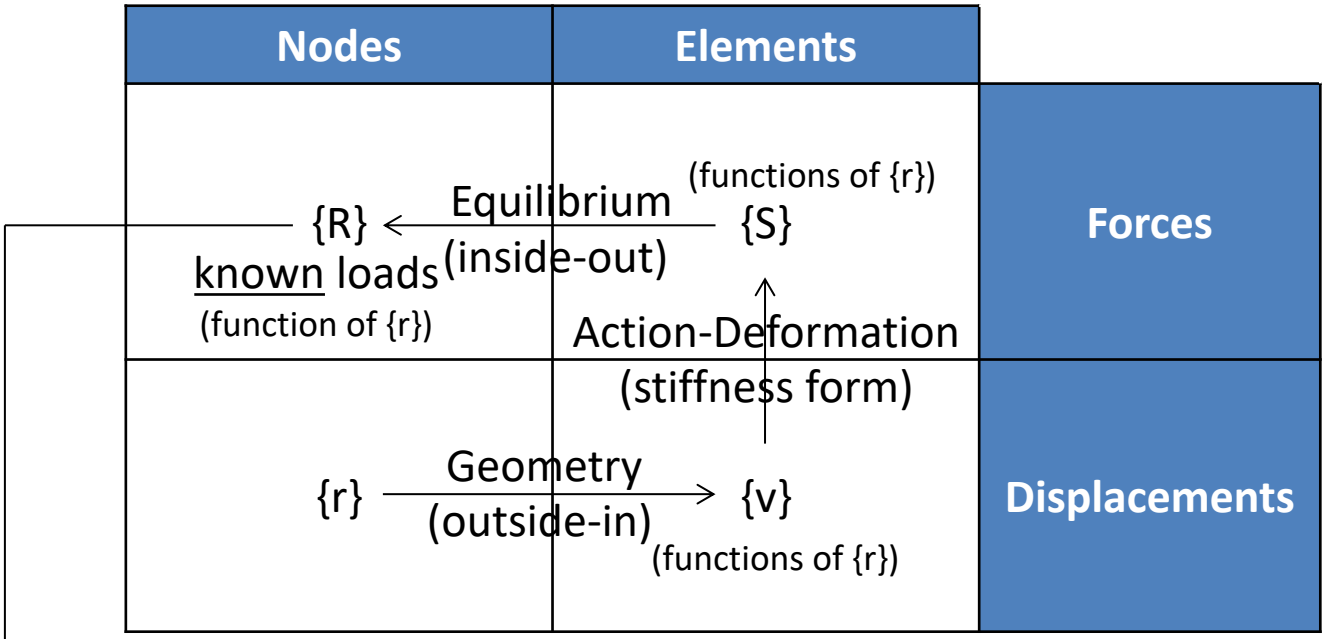
Displacement Method

- Reverse of force method
- Also called “stiffness method” or “equilibrium method”

See “Step in the displacement method”



These are “equilibrium equations”. Solve for {r}.

If linear, can use superposition. → greatly simplifies the calculation

{r}, and {S}=[k]{v}

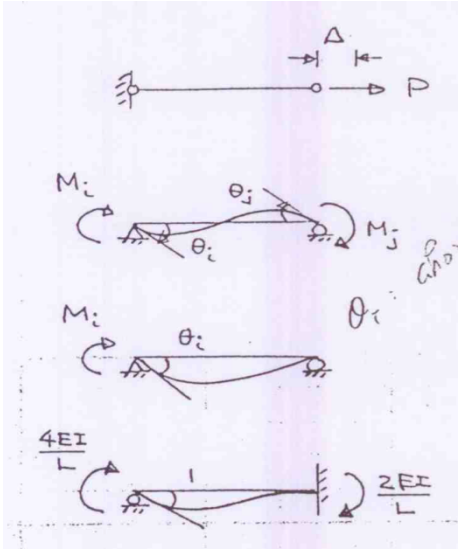
Degree of Freedom

- Primary unknown in the displacement method

	Truss	Frame
2D(X-Y plane)	Δ_x, Δ_y	Δ_x, Δ_y, θ_z
3D	Δ_x, Δ_y, Δ_z	Δ_x, Δ_y, Δ_z, θ_x, θ_y, θ_z

- Kinematic constraints(support conditions, axially rigid members, etc.) could reduce the number of DOF

Action-Deformation Relations



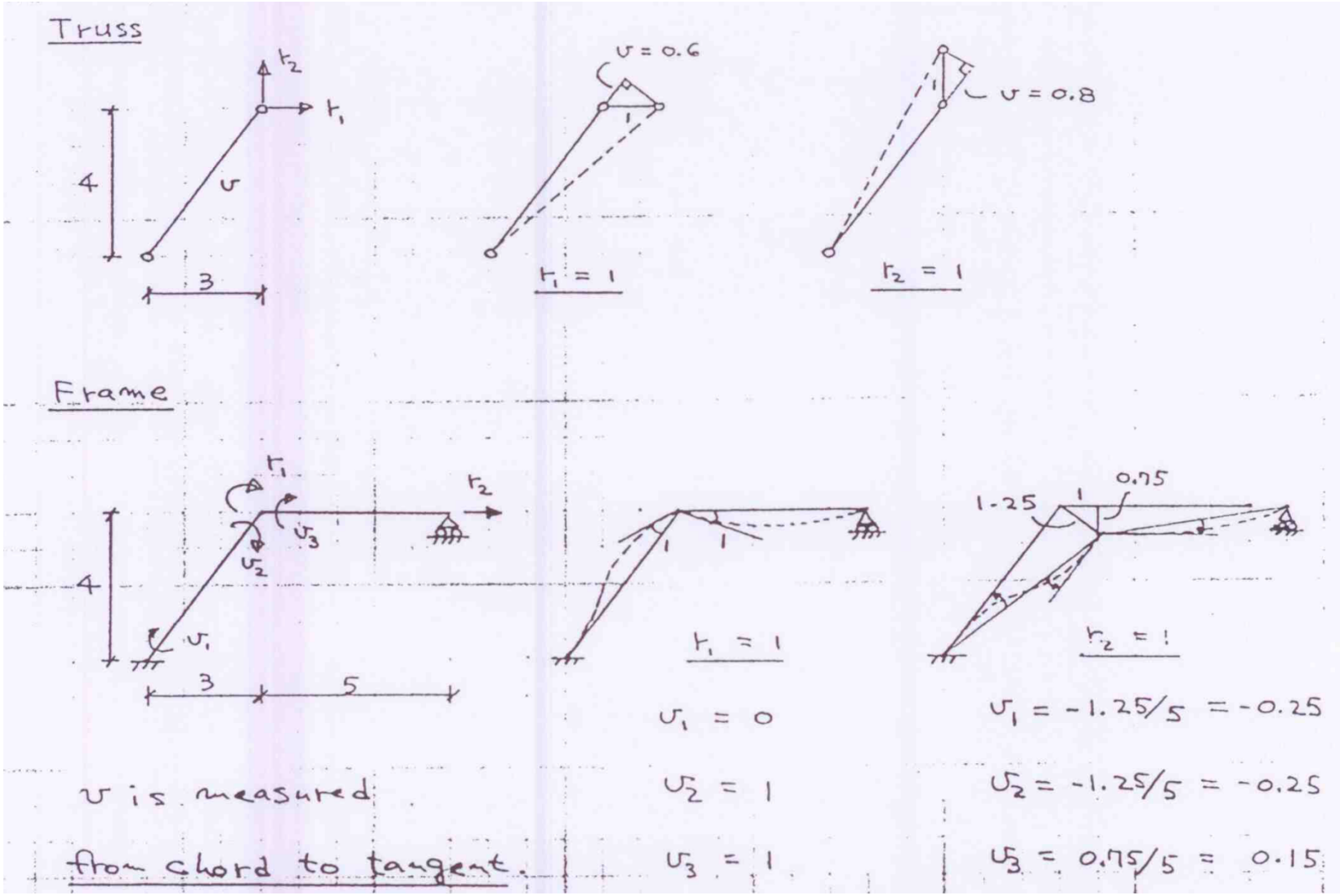
$$\{S\} = [k]\{v\} \qquad \{v\} = [f]\{S\}$$

$$[k] = \frac{EA}{L} \qquad [f] = \frac{L}{EA}$$

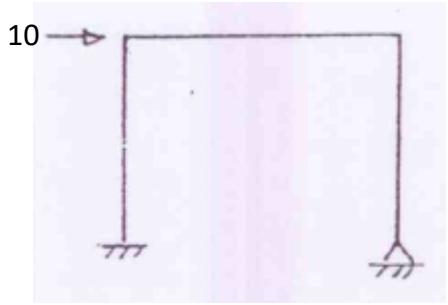
$$[k] = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad [f] = \frac{L}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[k] = \frac{3EI}{L} \qquad [f] = \frac{L}{3EI}$$

Geometry of Deformation

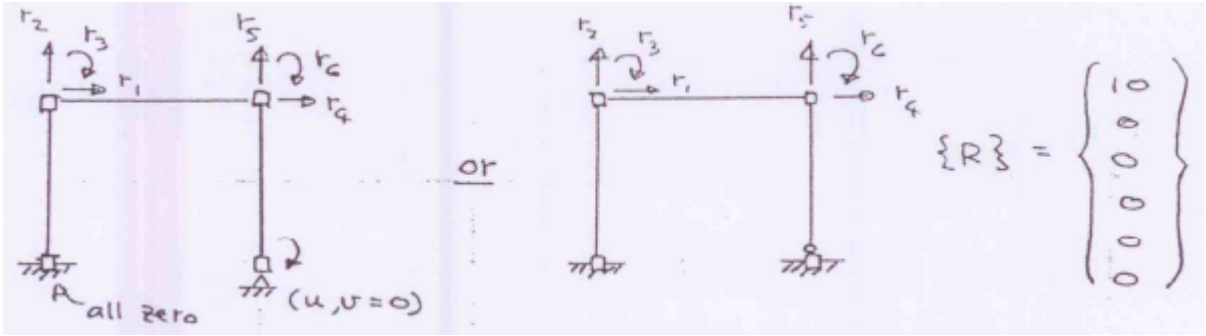


Modeling options



can model this several ways.

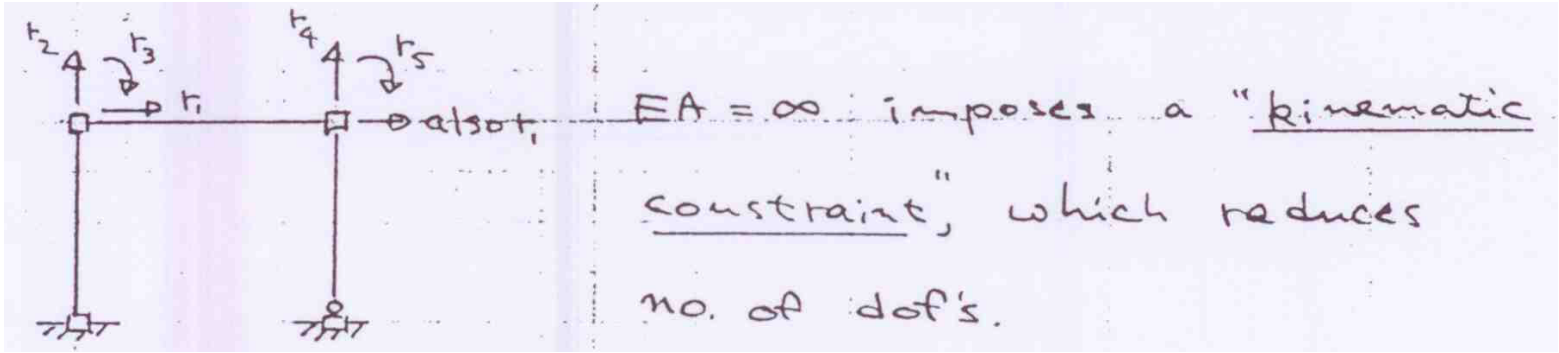
(1) Assume EA= finite (i.e. consider axial deformations)



Essentially, if we allow axial deformations, each node has →, ↑and ↷ dofs (unless restrained by displacement B.C.)

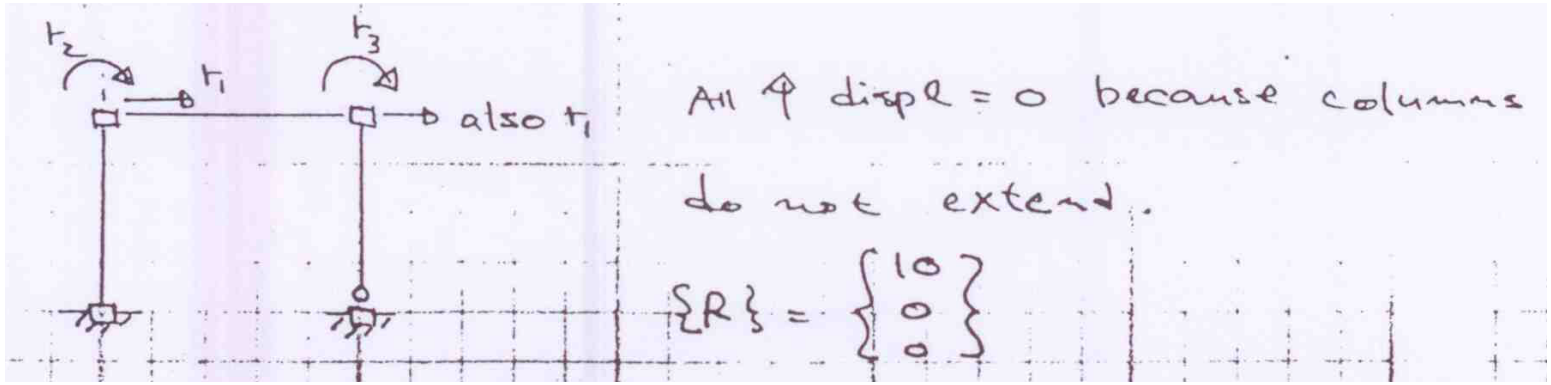
(2) Assume EA = finite for columns, infinite for girders.

(a common assumption for tall building frames)



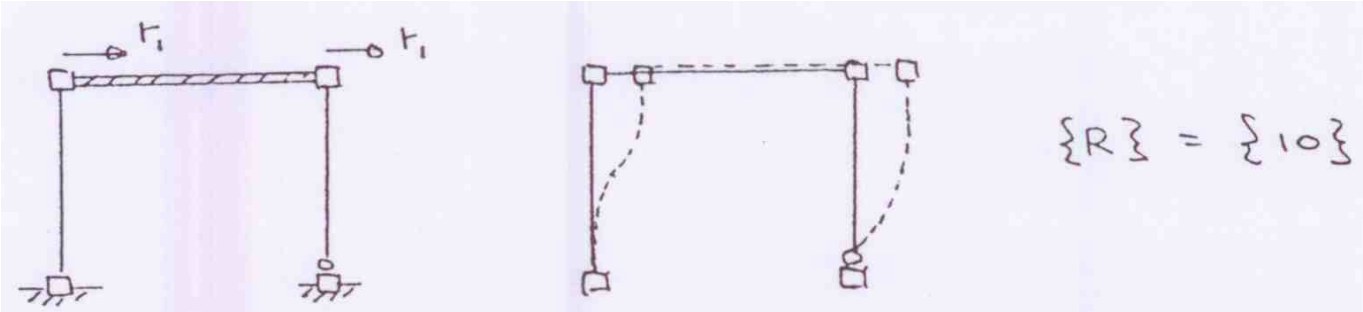
(3) Assume EA = ∞ for both girders and columns.

Reduces number of unknown displacements.

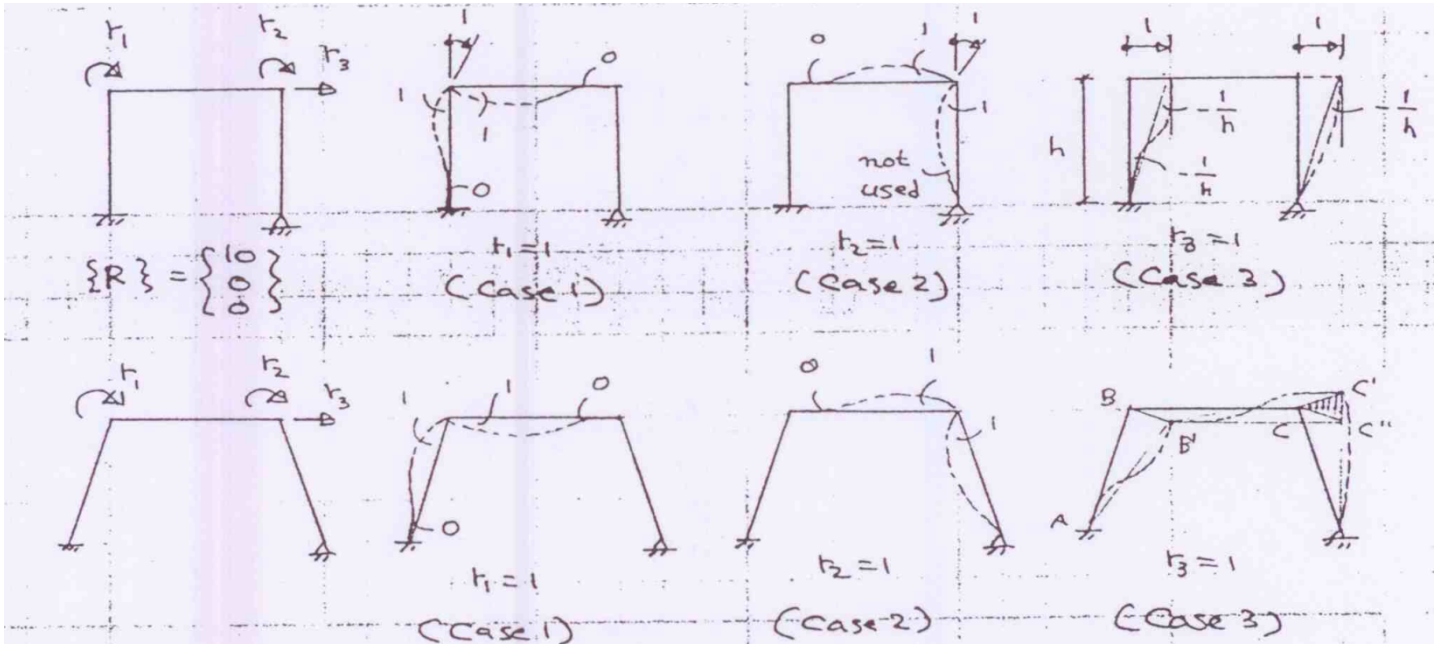


(4) Assume girder is also rigid in bending (and EI = ∞ throughout).

All rotations are zero (“shear building”).



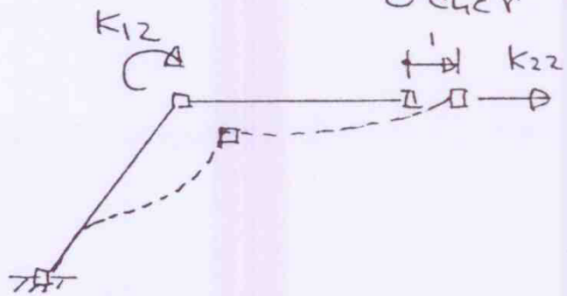
- **Compare**



Definition of Stiffness & Flexibility Coefficients

Stiffness Coefficients

K_{ij} ($\equiv R_{ij}$) = force corresp. to dof i when $r_j = 1$ & all other $r = 0$. unit: force per unit displ.



K_{12} : moment per unit translational displ.

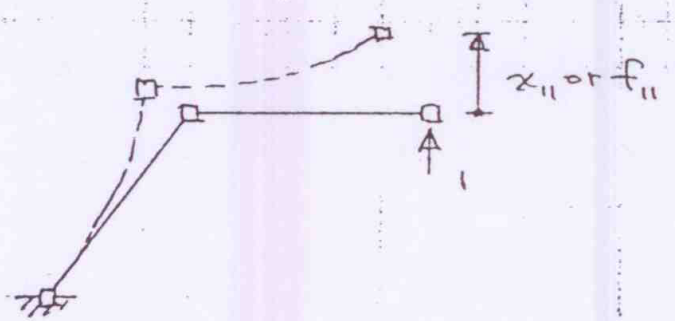
units : $F \cdot L / L = F$

K_{22} : force per unit transktional displ.

units : F/L

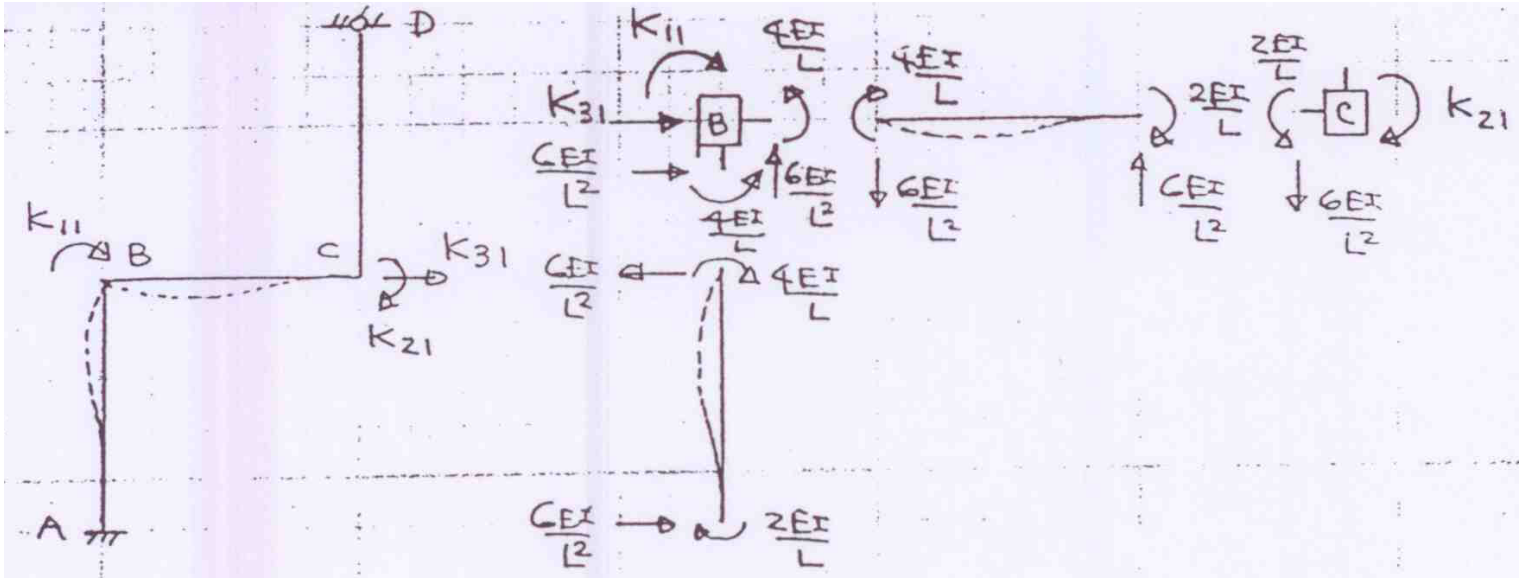
Flexibility Coefficients

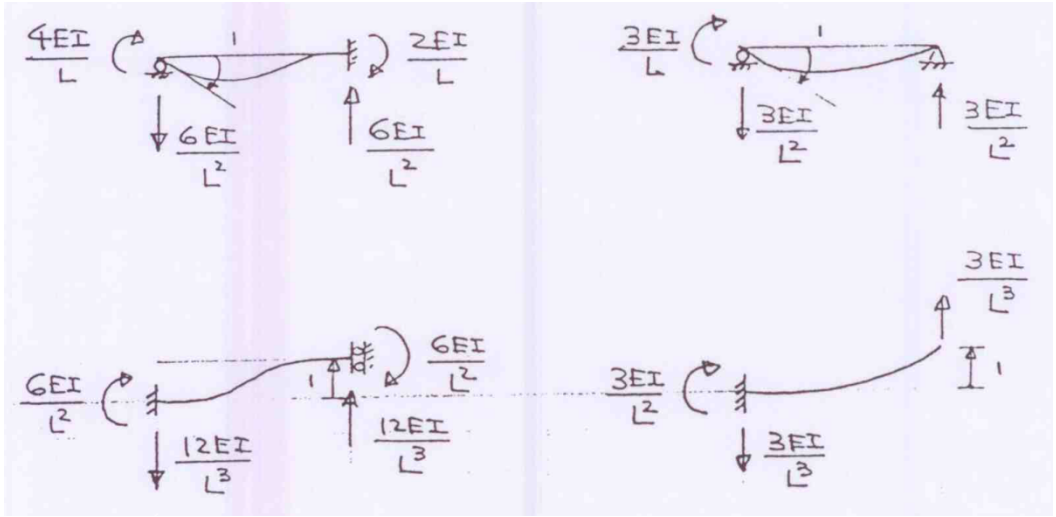
f_{ij} ($\equiv x_{ij}$) = displ. corresp. to X_i when $X_j = 1$ and all other $X = 0$.



units : translation or rotation per unit force or moment

Determination of Stiffness Coefficients by Equilibrium





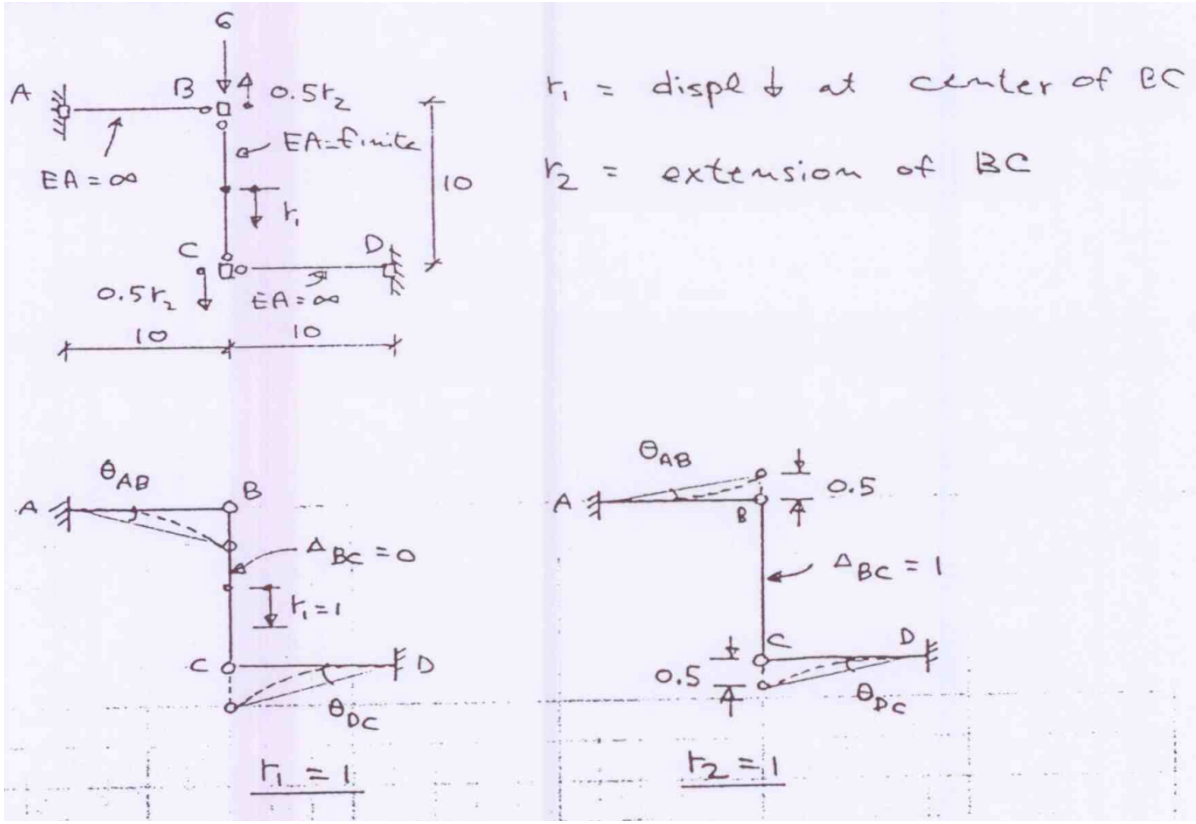
$$k_{11} = \frac{4EI}{L} + \frac{4EI}{L} = \frac{8EI}{L}$$

$$k_{21} = \frac{2EI}{L}$$

$$k_{31} = -\frac{6EI}{L^2}$$

{R} acting on Non-DOFs

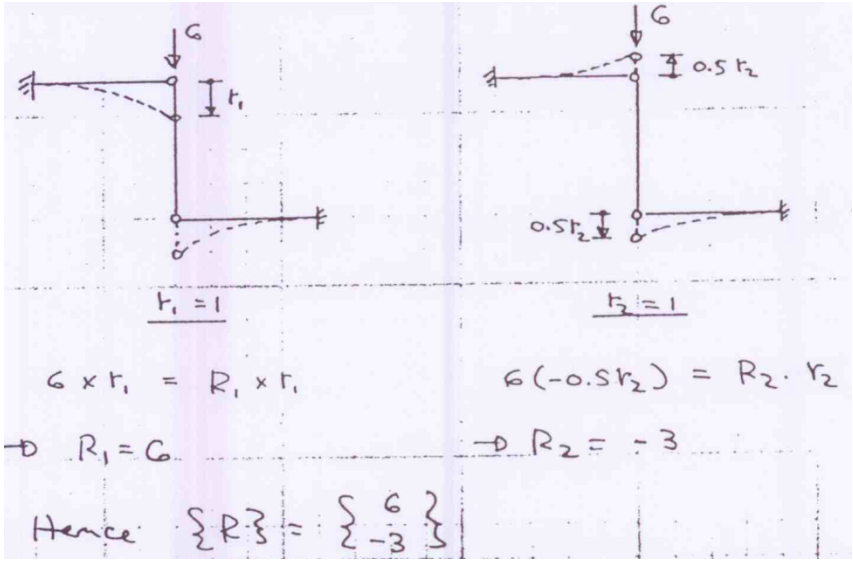
- **Work equivalence for calculating {R} acting on non-DOFs**

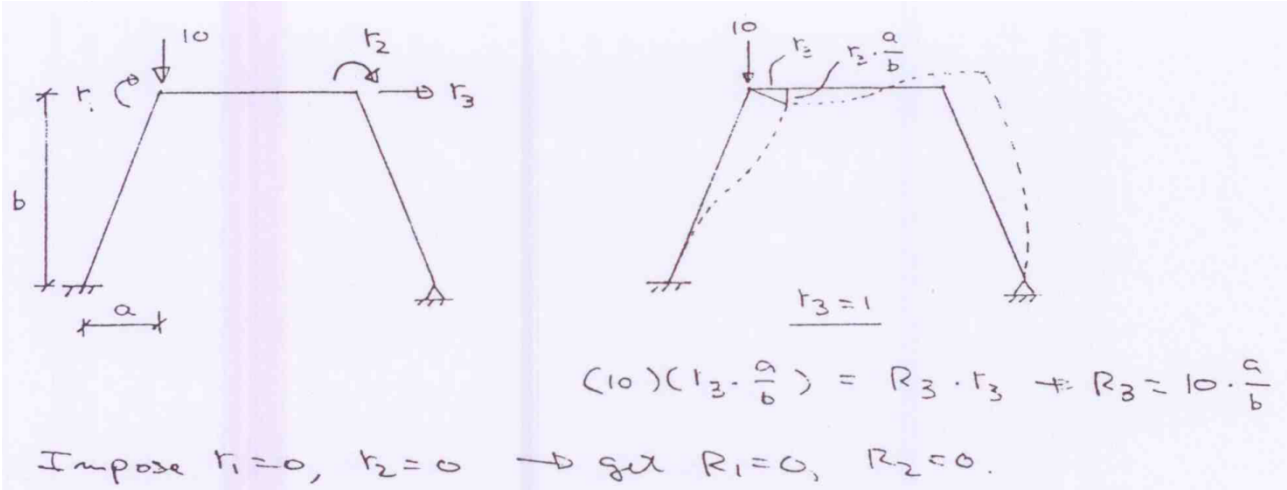


- But what is $\{R\} = \begin{Bmatrix} ? \\ ? \end{Bmatrix}$
- For cases as this, we can not get {R} simply by inspection, but must use a work approach.

{R} acting on Non-DOFs

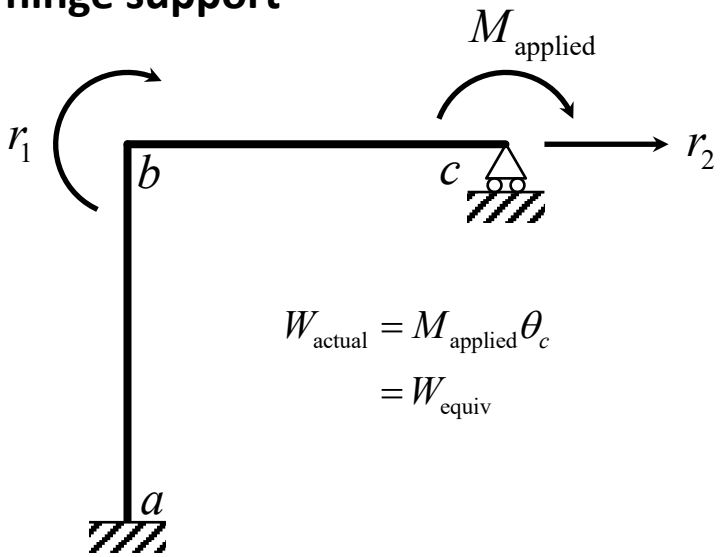
- Procedure: Impose r_1 with all other r=0.





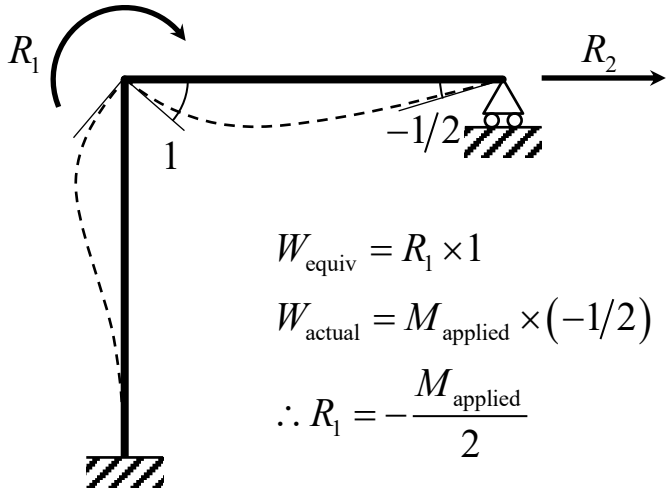
{R} acting on Non-DOFs

- **E.g. 2-DOF frames with a hinge support**



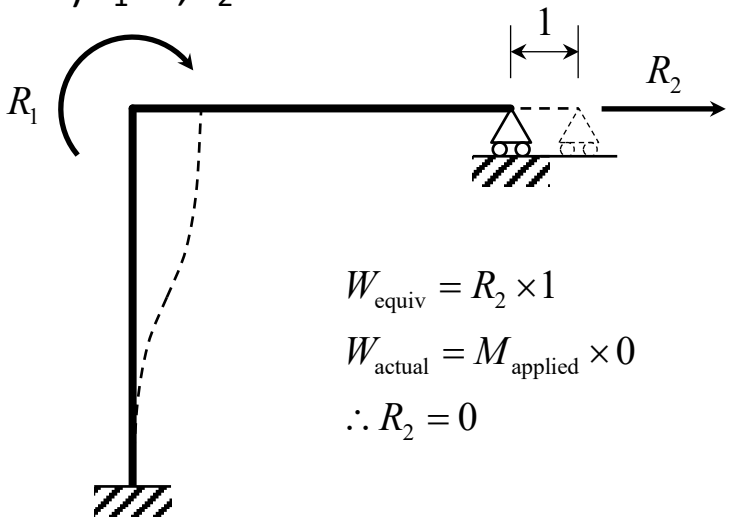
$$W_{\text{actual}} = M_{\text{applied}}\theta_c$$
$$= W_{\text{equiv}}$$

- Case 1) $r_1=1$, $r_2=0$



$$W_{\text{equiv}} = R_1 \times 1$$
$$W_{\text{actual}} = M_{\text{applied}} \times (-1/2)$$
$$\therefore R_1 = -\frac{M_{\text{applied}}}{2}$$

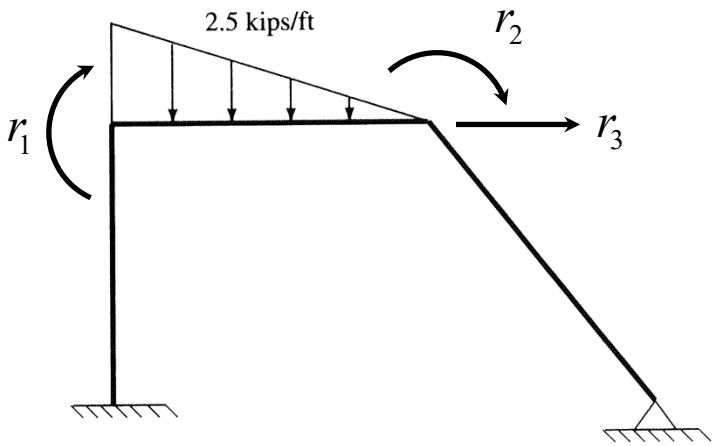
- Case 2) $r_1=0$, $r_2=1$

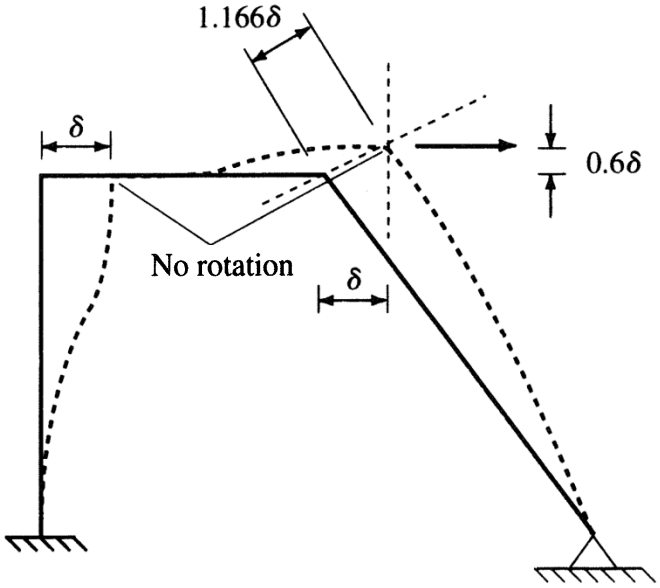


$$W_{\text{equiv}} = R_2 \times 1$$
$$W_{\text{actual}} = M_{\text{applied}} \times 0$$
$$\therefore R_2 = 0$$

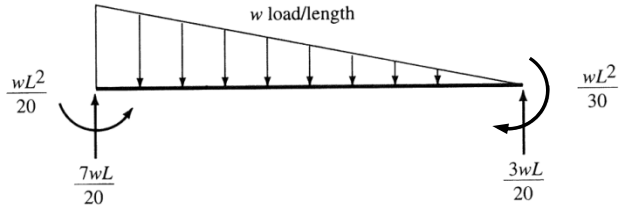
{R} for Distributed Loads

E.g. Distributed load for a 3-DOF frame

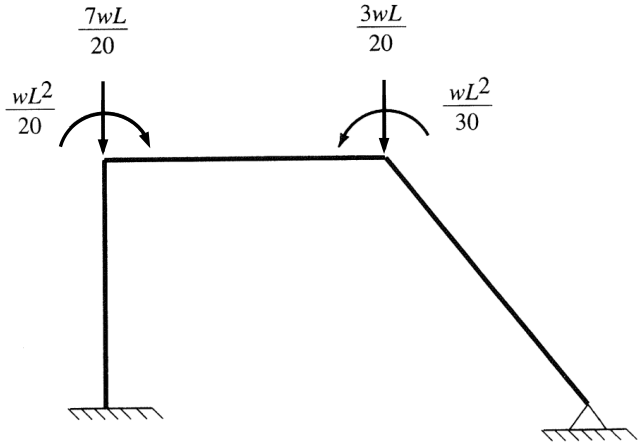




- Fixed-End-Forces



- Equivalent concentrated loads on nodes



$$R_1 = \frac{wL^2}{20}(1) = \frac{(2.5/12)(96)^2}{20} = 96 \text{ kip-in}$$

$$R_2 = -\frac{wL^2}{30}(1) = \frac{-(2.5/12)(96)^2}{30} = -64 \text{ kip-in}$$

$$R_3 = -\frac{3wL}{20}[0.6(1)] = \frac{-3(2.5/12)96(0.6)}{20} = -1.8 \text{ kips}$$

Matrix Formulation of the Displacement Method

Notations

- {r} = Unknown node displacements
- {R} = Known forces corresponding to {r}
- {S} = Element actions
- {v} = Element deformations
- [a] = Displacement transformation matrix
- [k] = Element stiffness matrix
- [K] = Structure stiffness matrix

Geometry

$$\{v\} = [a]\{r\} \quad (1)$$

Action-deformation

$$\{S\} = [k]\{v\} \quad (2)$$

$$\{S\} = [k][a]\{r\} \quad (3)$$

Matrix Formulation of the Displacement Method

- **Equilibrium using the Principle of Virtual Displacements**

To get equation for $\{R\}$, impose arbitrary $\{\bar{r}\}$

Then $\{\bar{v}\} = [a]\{\bar{r}\}$; $W_E = \{\bar{r}\}^T\{R\}$;

$W_I = \{\bar{v}\}^T\{S\} = \{\bar{r}\}^T[a]^T\{S\}$

Put $W_E = W_I$ and cancel arbitrary $\{\bar{r}\}$ to get

$$\{R\} = [a]^T\{S\} \tag{4}$$

$$\{R\} = [a]^T[k][a]\{r\} = [K]\{r\} \tag{5}$$

Eq. (5) is the equilibrium equation to be solved.

Matrix Formulation of the Displacement Method

Steps in the displacement method

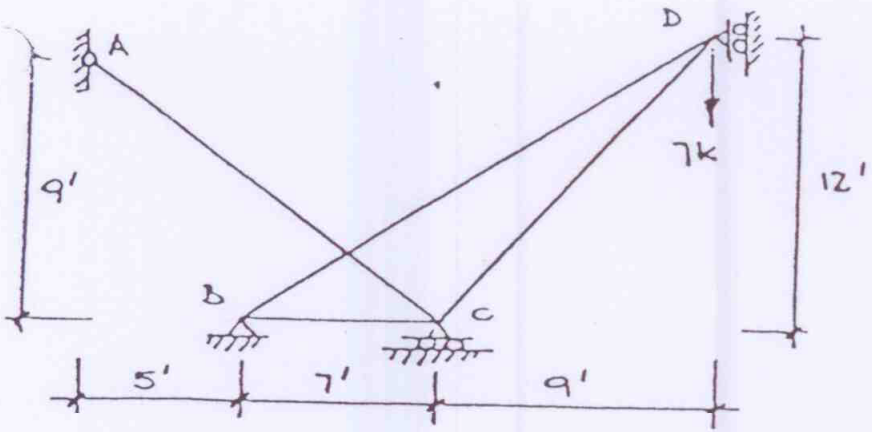
1. Define the joints, members and boundary conditions of the structure.
2. Identify the displacement degrees of freedom (r_i, i=1,…,N) of the structure (the unknowns of the problem). Take account of all kinematic constraints (support conditions, axially rigid members, etc.).
3. Define the measures of member deformation which are to be used.
4. Consider N separate displaced states (cases). For "Case i", impose r_i=1 with all other r=0. For each Case, determine the member deformations. This is a geometry problem.
5. For each Case multiply the member deformations by the member stiffnesses to get the member actions.
6. For each Case calculate the joint loads (R_i, i=1,…,N, corresponding to r_i) required to hold the structure in equilibrium. This is an "inside-out" static problem. It will usually be convenient to use the Virtual Displacement Principle. For Case j, let the load corresponding to displacement r_j be k_{ij}. The k_{ij} values are stiffness coefficients for the structure.
7. The actual value, R_i, of the load corresponding to r_i will always be known. Hence, by superposition .

$$R_i = \sum_j k_{ij} r_j$$

   These are the equilibrium equations.
8. Solve for the r values.
9. Obtain the final state of the structure as a linear combination of the separate Cases.

Matrix Formulation of the Displacement Method

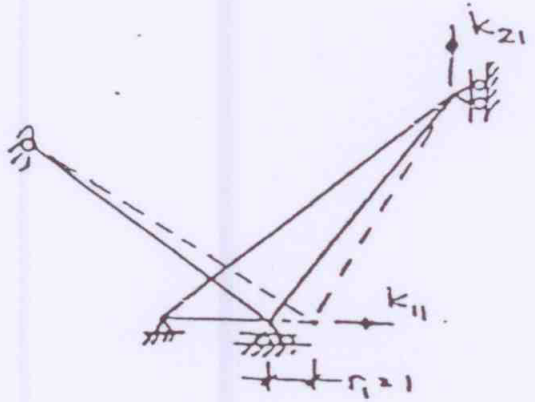
- **Example: Truss by displacement method**



EA values: AC : $3 \cdot 10^4$ k.
BC : $1 \cdot 10^4$ k
BD : $4 \cdot 10^4$ k
CD : $5 \cdot 10^4$ k

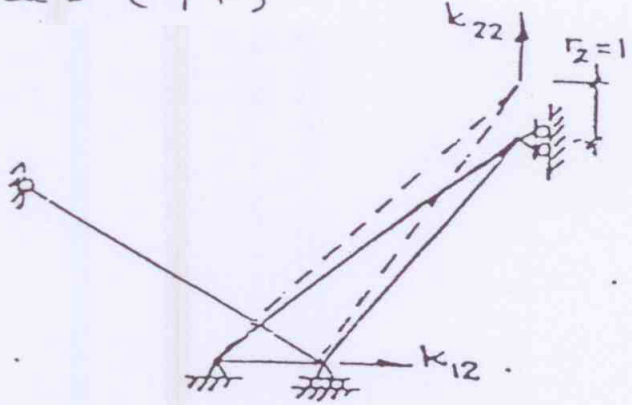
Unknown displacements:
r_1 = horiz. at C (→ +)
r_2 = vert. at D (↓ +)

KINEMATICS :



Case 1.

	v_1
AC	0.8
BC	1.0
BD	0
CD	-0.6



Case 2.

v_2
0
0
0.6
0.8

Matrix Formulation of the Displacement Method

- **Action-deformation:** $S = \frac{EA}{L} v$

- **Statics: Virtual displacements procedure**

To get	Real act/force	Imag displ./def	W_E	W_I
k_{11}	1	1	k_{11}	$\Sigma \bar{v}_1 EAv_1/L$
k_{21}	1	2	k_{21}	$\Sigma \bar{v}_2 EAv_1/L$
k_{12}	2	1	k_{12}	$\Sigma \bar{v}_1 EAv_2/L$
k_{22}	2	2	k_{22}	$\Sigma \bar{v}_2 EAv_2/L$

Member	EA/L ($\cdot 10^{-4}$)	V_1	V_2	$V_1 \cdot EA/L \cdot V_1$	$V_2 \cdot EA/L \cdot V_2$	$V_1 \cdot EA/L \cdot V_2$
AC	.2	.8	0	.128	0	0
BC	.143	1.0	0	.143	0	0
BD	.2	0	.6	0	.072	0
CD	.333	-.6	.8	.120	.213	-.160
				$.391 \cdot 10^4$	$.285 \cdot 10^4$	$-.160 \cdot 10^4$
				k_{11}	k_{22}	k_{12} or k_{21}

- **Equilibrium equations:**

$$.391 \cdot 10^4 r_1 - .160 \cdot 10^4 r_2 = R_1 = 0.$$
$$-.160 \cdot 10^4 r_1 + .285 \cdot 10^4 r_2 = R_2 = -7.$$

Hence: $r_1 = -13.04 \cdot 10^{-4}$ ft, $r_2 = -31.89 \cdot 10^{-4}$ ft.

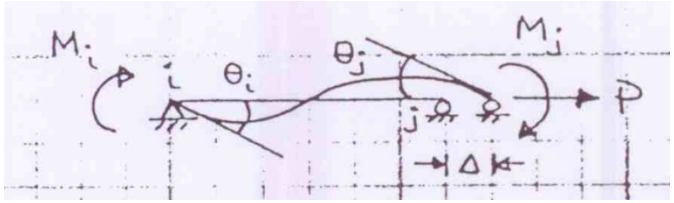
Stiffness Coefficients by Virtual Displacement Procedure

$$W_E = \bar{r}^T \mathrm{R} = 1 \times R = 1 \times k_{ij}$$

$$W_I = \sum \bar{v}^T S = \sum \bar{v}^T k v$$

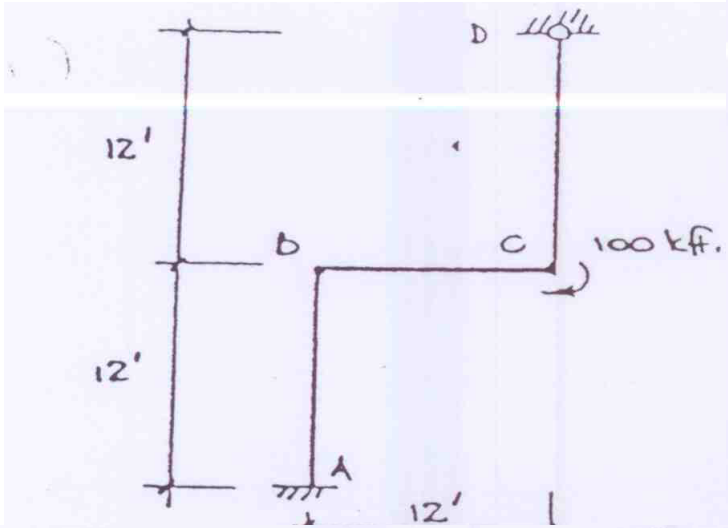
$$= \sum \left(\bar{\Delta} P + \bar{\theta}_i M_i + \bar{\theta}_j M_j \right)$$

$$= \sum \left[\bar{\Delta} \cdot \frac{EA}{L} \Delta + \bar{\theta}_i \left(\frac{4EI}{L} \theta_i + \frac{2EI}{L} \theta_j \right) + \bar{\theta}_j \left(\frac{2EI}{L} \theta_i + \frac{4EI}{L} \theta_j \right) \right]$$



Matrix Formulation of the Displacement Method

- **Example: Frame by displacement method**



$EI = 3.10^6$ k.in^2

Unknown displacements:

r_1 = rotation at B (↻ +)

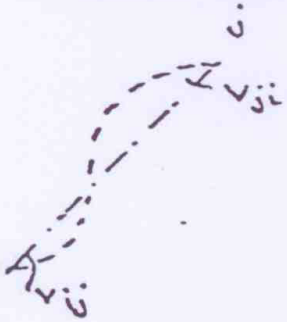
r_2 = rotation at C (↻ +)

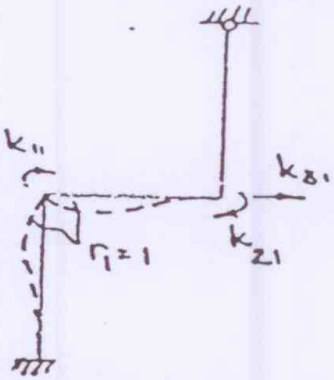
r_3 = horiz. displ at B (→ +)

Ignore axial, shear deformations.

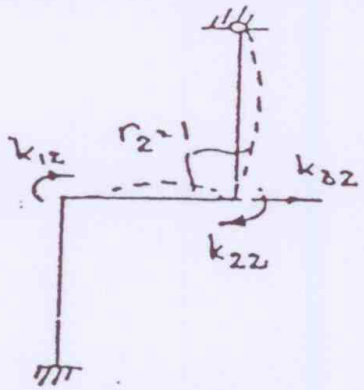
Rotation at D need not be treated as a separate unknown. Regard as use modified beam stiffness.

KINEMATICS :

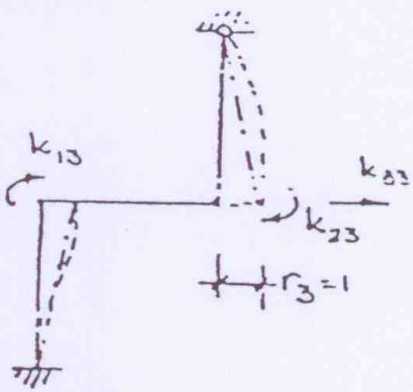




Case 1



Case 2



Case 3

	Case 1	Case 2	Case 3
V_{AB}	0	0	-1/12
V_{BA}	1	0	-1/12
V_{BC}	1	0	0
V_{CB}	0	1	0
V_{CD}	0	1	1/12

Matrix Formulation of the Displacement Method

- **Action-deformation:**

$$\begin{Bmatrix} M_i \\ M_j \end{Bmatrix} = \frac{2EI}{L}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \end{Bmatrix} = 3.47\cdot10^{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \end{Bmatrix} \quad :AB, BC$$

$$M_i = \frac{3EI}{L}v_i = 5.20\,10^{3}v_i \quad : CD$$

- **Statics: Given member moments in each case, can find k_{11}, k_{21}, etc. by equilibrium equations. Alternatively, can use virtual displacements. For example:**

To get	Real act/force	Imag displ./def	W_E	W_I
k_{11}	1	1	k_{11}	$\Sigma \{\bar{v}_1\}^T[k]\{v_1\}$
k_{13}	3	1	k_{13}	$\Sigma \{\bar{v}_1\}^T[k]\{v_3\}$

Get: $k_{11} = 13.88.10^3$ kft/rad.; $k_{22} = 12.14.10^3$ kft/rad.; $k_{33} = .1445.10^3$ k/ft

$k_{12} = k_{21} = 3.47.10^3$ kft/rad.; $k_{13} = k_{31} = -.867.10^3$ k/rad.; $k_{23} = k_{32} = .433.10^3$ k/rad.

- **Equilibrium equations:**

$$13.88.10^3 r_1 + 3.47.10^3 r_2 - .867.10^3 r_3 = 0$$
$$3.47.10^3 r_1 + 12.14.10^3 r_2 + .433.10^3 r_3 = 100$$
$$-.867.10^3 r_1 + .433.10^3 r_2 + .1445.10^3 r_3 = 0$$

Hence: $r_1 = -7.20.10^{-3}$ rad; $r_2 = 12.58.10^{-3}$ rad; $r_3 = -64.9.10^{-3}$ ft.