[Gradient, Jacobian, Hessian]

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

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Key Questions

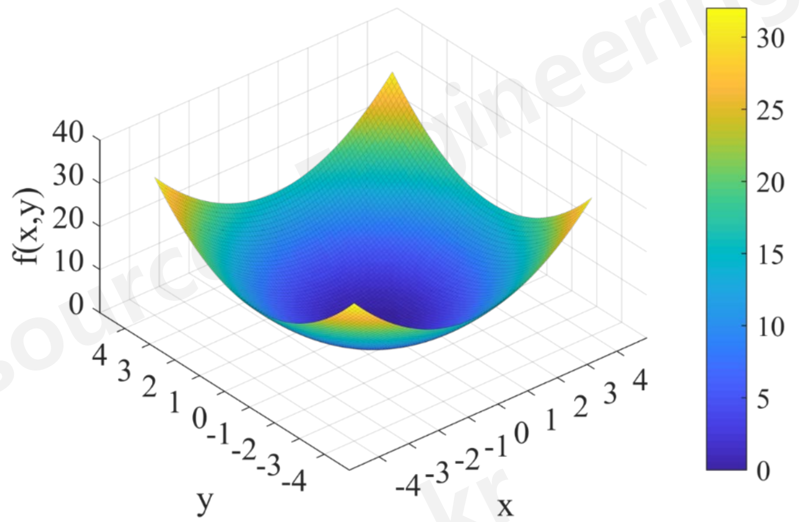
- What is the formula of ∇f?
- What is the physical meaning of ∇f?
- What is the formula of a Jacobian matrix?
- How is a Jacobian matrix utilized?
- What is the formula of a Hessian matrix?
- How is a Hessian matrix utilized?

Differences between Gradient, Jacobian, Hessian?

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Gradient Vector

- Single scalar function: f
- Multivariate function: $f(x_1, \ldots, x_n)$
- 1st derivative
- Physical meaning?
  - ✓ Example
    - $f = x^2 + y^2$
    - $\nabla f = ?$
    - Calculate ∇f for (1, 1)
    - Calculate and plot $f(0,2), f(1,2), f(2,2), f(0,1), f(1,1), f(2,1), f(0,0), f(1,0), f(2,0)$
    - Calculate $f([0,1,2],[0,1,2]) - f(1,1)$
    - Which one has maximum increase and decrease of function values from $f(1,1)$?
  - ✓ ∇f: (x, y) direction locally maximizing f, steepest ascent direction
  - ✓ $-\nabla f$: (x, y) direction locally minimizing f, steepest descent direction
- How utilize a gradient vector?



$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Example matlab code for gradient vector

- See L2_gradient.pdf

Jacobian Matrix

- Vector-valued function: $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$
- Multivariate function
- 1st derivative
- Gradients for multiple functions
- Physical meaning?
  - ✓ Example
    - $f = [x^2 + y^2 \quad xy]$
    - $J =?$
    - Calculate J for (1, 1)
- How utilize a Jacobian matrix?

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Hessian Matrix

- Single scalar function: f
- Multivariate function: $f(x_1, \dots, x_n)$
- 2nd derivative
- Physical meaning?
  - ✓ 1st example
    - $f = x^2 + y^2$
    - $H = ?$
  - ✓ 2nd example
    - $f = x(x-1)(x+1)$
    - $H = ?$
    - Plot f and H
  - ✓ $f''(x) > 0$: Concave up, $f''(x) < 0$: Concave down
  - ✓ Eigenvalues of H > 0 : Concave up, Eigenvalues of H < 0: Concave down
  - ✓ If ∇f is a zero vector and H contains both positive and negative eigenvalues, it is a saddle point
- How utilize a Hessian Matrix?
  - ✓ $f'(x) = 0, f''(x) > 0$ ➔ $f(x)$ local minimum: $\nabla f = 0$, eigenvalues of H > 0
  - ✓ $f'(x) = 0, f''(x) < 0$ ➔ $f(x)$ local maximum: $\nabla f = 0$, eigenvalues of H < 0

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$