Aeroelasticity

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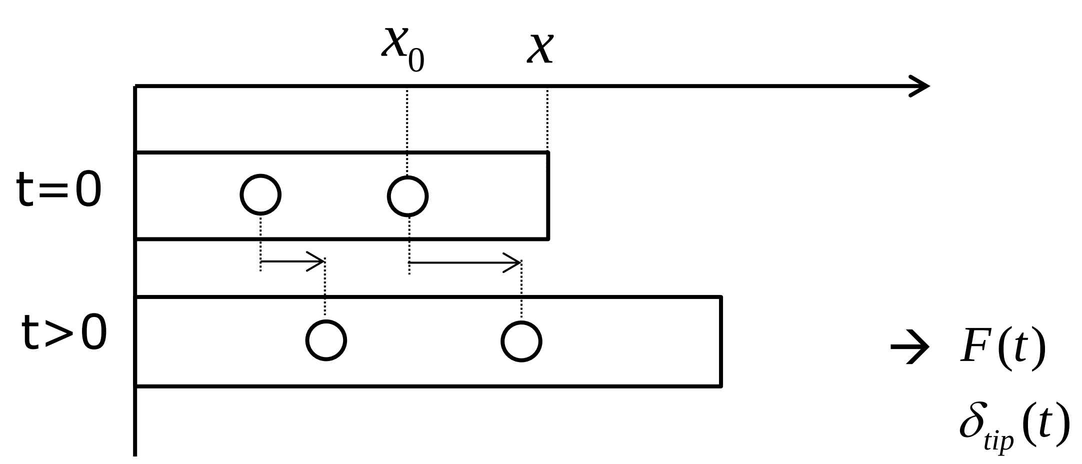
Index

Unsteady Aerodynamics

Field Description

I. Field Description

- Lagrangian method···particle point of view, position of any fluid particle P . $x = x_p(x_0, y_0, z_0, t), \cdots$ Abundance of information
- Eulerian method ···field point of view, spatial distribution of flow variables at each instant. $u = u(x, y, z, t)$



Governing Equations

II. Governing Eqns

i) Integral form

① Mass conservation

$$\frac{dm_{c.v.}}{dt} = \frac{\partial}{\partial t}\int_{c.v.} \rho dV + \int_{c.s.} \rho(\vec{q}\cdot\vec{n})dS = 0$$

Change in the mass within the c.v.

Rate of mass leaving across and normal to the surface

Governing Equations

② Momentum conservation

$$\frac{d(m\vec{q})_{c.v.}}{dt} = \frac{\partial}{\partial t}\int_{c.v.} \rho\vec{q}dV + \int_{c.s.} \rho\vec{q}(\vec{q}\cdot\vec{n})dS = \sum \vec{F}$$

$$\left(\sum \vec{F}\right)_i = \underbrace{\int_{c.v.} \rho f_i dV}_{\text{Body force}} + \underbrace{\int_{c.s.} n_j \tau_{ij} ds}_{\text{surface force}}$$

ii) Differential form

① Mass conservation

- divergence theorem

$$\int_{c.s.} n_j q_j dS = \int_{c.v.} \frac{\partial q_i}{\partial x_j} dV$$

Governing Equations

(1) $\therefore \rightarrow \int_{c.v.} \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q}\right)}_{} dV = 0$.

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho}_{} + \rho \nabla \cdot \vec{q} = 0$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q} \cdot \nabla$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{q} = 0$$

incompressible fluid⋯constant density $\Rightarrow \nabla \cdot \vec{q} = 0$

② Momentum Eqn.

From the divergence theorem.

$$\int_{c.s.} \rho q_i (\vec{q} \cdot \vec{n}) dS = \int_{c.v.} \nabla \cdot \rho q_i \vec{q} dV$$

$$\int_{c.s.} n_j \tau_{ij} dS = \int_{c.v.} \frac{\partial \tau_{ij}}{\partial x_j} dV$$

Governing Equations

$$\Rightarrow \int_{c.v.}\left[\frac{\partial}{\partial t}(\rho q_i)+\nabla\bullet pq_i\vec{q}-\rho f_i-\frac{\partial\tau_{ij}}{\partial x_j}\right]dV=0$$

$$\frac{\partial}{\partial t}(\rho q_i)+\nabla\bullet(\rho q_i\vec{q})=\rho\frac{Dq_i}{Dt}$$

$$\Rightarrow \rho\frac{Dq_i}{Dt}=\rho f_i+\frac{\partial\tau_{ij}}{\partial x_j}$$

For a Newtonian fluid

$$\Rightarrow \rho\left(\frac{\partial q_i}{\partial t}+\vec{q}\bullet\nabla q_i\right)=\rho f_i-\frac{\partial}{\partial x_i}(P+\frac{2}{3}\mu\nabla\bullet\vec{q})+\frac{\partial}{\partial x_j}\mu\left(\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right)$$

Incompressible fluid

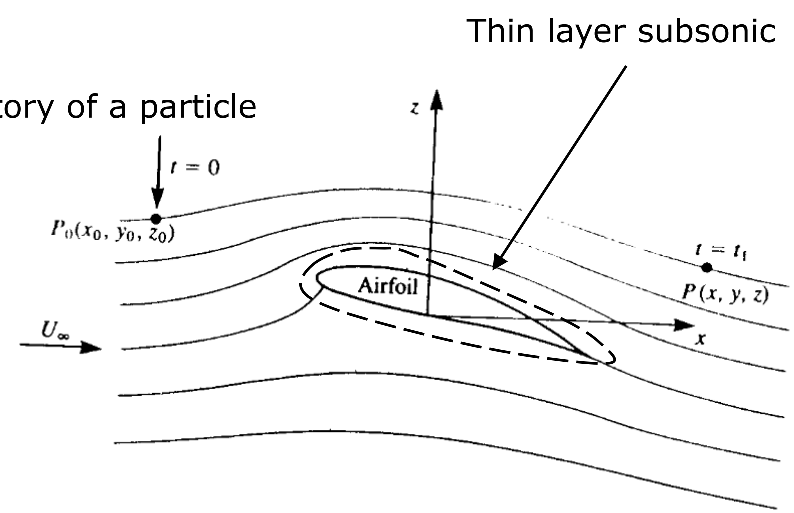
$$\Rightarrow \rho\left(\frac{\partial\vec{q}}{\partial t}+\vec{q}\bullet\nabla\vec{q}\right)=\rho\vec{f}-\nabla P+\mu\nabla^2\vec{q}$$

Inviscid fluid

$$\Rightarrow \frac{\partial\vec{q}}{\partial t}+\vec{q}\bullet\nabla\vec{q}=\vec{f}-\frac{\nabla P}{\rho}$$

III. Inviscid, Incompressible flow

- high Reynolds No. flow ⋯ effects of viscosity to thin boundary layers and thin wakes. Flow outside this regions ⋯ inviscid, incompressible



i) Vorticity Circulation

$$\vec{w} = \underbrace{\nabla \times \vec{q}}_{\text{Curl}} : \text{vorticity}$$

From Stoke's theorem

$$\int_s \nabla \times \vec{q} \bullet \vec{n} dS = \int_s \vec{w} \times \vec{n} dS = \underbrace{\oint_c \vec{q} \bullet d\vec{l}}_{} = \Gamma : \text{circulation}$$

Inviscid, Incompressible flow

ii) Velocity Potential

$$irrotational : \vec{q} = \nabla\Phi \qquad \frac{\partial\Phi}{\partial x} = v_x, \frac{\partial\Phi}{\partial y} = v_y$$

$$massconservation(continuity) : \nabla\vec{q} = \nabla\bullet\nabla\Phi = \nabla^2\Phi = 0$$

$$\Rightarrow Laplace's\ eqn \rightarrow thin\ layer\ external\ flow$$

Boundary condition
⋯ only one

$$\vec{n}(\vec{q} - \vec{q}_B) = 0$$

"Kutta condition"

iii) Bernoulli's Eqn

$$E + \frac{P}{\rho} + \frac{q^2}{2} + \frac{\partial\Phi}{\partial t_i} = C(t), \vec{F} = -\nabla E$$

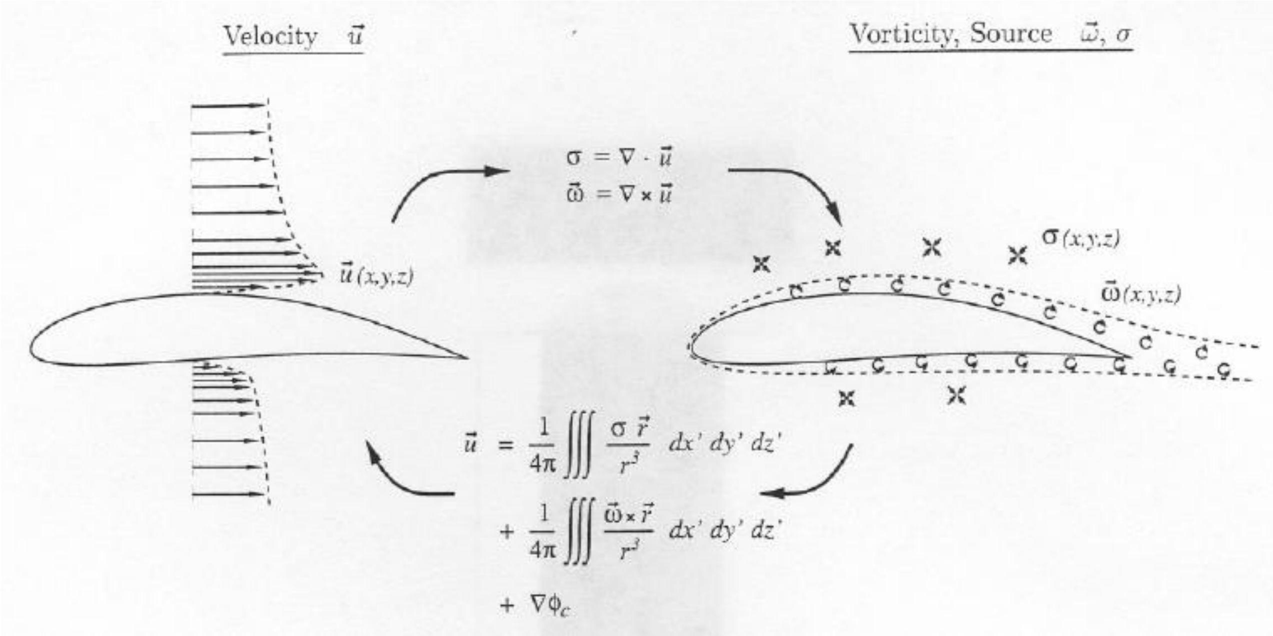
Inviscid, Incompressible flow

iv) Biot-Savart Law

Source/Sink

Doublet, Vortex

$$\vec{q} = \frac{1}{4\pi}\int_u \nabla \times \frac{\vec{w}}{|\vec{r}_0 - \vec{r}_1|}dV \qquad \vec{r}_0 : point, \vec{r}_1 : vorticity$$



General Solution of the incompressible, Potential Flow Eqns

IV. General Solution of the incompressible, Potential Flow Eqns.

i) Problem statement

- Incompressible, irrotational···continuity eqn. reduces to

$$\nabla^2\Phi = 0$$

- Velocity normal to the body's surface and solid boundaries must be zero

$$\nabla\Phi \bullet \vec{n} = 0$$

- Disturbance created by the motion should decay far from the body

$$\lim_{r\to\infty}(\nabla\Phi - \bar{v}) = 0$$

: relative velocity between the undisturbed fluid and the body

General Solution of the incompressible, Potential Flow Eqns

ii) Methodology of Solution

- Solution is obtained by distributing elementary solutions on the problem

- Potential specified on the boundaries -> Dirichlet problem

- Zero normal flow boundary condition -> Neumann problem

- Additional considerations are required(Kutta condition)

iii) Separation of thickness and lifting problems in wing

- Complete solution for the cambered wing with nonzero thickness at a certain angle of attack

General Solution of the incompressible, Potential Flow Eqns

= *symmetric wing with nonzero thick at zero angle of attack*(thickness effect)+*zero-thickness, uncambered wing at angle of attack*(effect of angle of attack)+*zero-thickness, cambered wing at zero angle of attack*(effect of camber)

iv) Zero-thickness cambered wing at angle of attack···Lifting surfaces

- Boundary condition requiring no flow across the surface

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = Q_\infty \left(\frac{\partial \eta_c}{\partial x} - \alpha \right)$$

- Can be solved by a doublet distribution or a vortex distribution
  For a vortex line distribution vortex elements cannot be terminated at the wing and must be shed into the flow

General Solution of the incompressible, Potential Flow Eqns

v) Vortex distribution for lifting surface

- Velocity due to a vortex element $d\vec{l}$ with a strength $\Delta\Gamma$

$$\Delta\vec{q} = \frac{-1}{4\pi}\frac{\Delta\Gamma\vec{r}\times d\vec{l}}{r^3}$$

- Downwash induced by γ_y (over wing) and γ_x(in the wake)

$$w(x,y,z) = \frac{-1}{4\pi}\int_{wing+wake}\frac{\gamma_y(x-x_0)-\gamma_x(y-y_0)}{r^3}dx_0 dy_0$$

- Helmholtz vortex theorem ⋯ vortex strength is const along a vortex line

$$\left|\frac{\partial\gamma_x}{\partial x}\right| = \left|\frac{\partial\gamma_y}{\partial y}\right| \rightarrow \quad \text{unknowns are one}$$

General Solution of the incompressible, Potential Flow Eqns

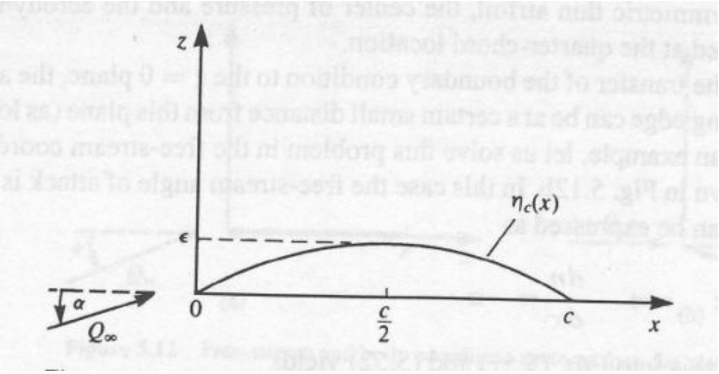
vi) Vortex wake

- Kutta condition ··· flow leaves the sharp T.E. smoothly and the velocity is finite

$$\gamma_{TE} = 0$$

Thin Airfoil Theory

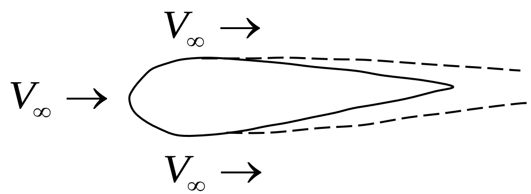
V. Thin Airfoil Theory



i) Zero-thickness airfoil at angle of attack

- Small-disturbance flow over thin airfoils ··· divided into a thickness problem and a lifting problem

Thin Airfoil Theory



= A.O.A effect + camber effect

- Due to linearity;

  Total effect = AOA effect + thickness effect + camber effect

- Lifting problem ··· thin, cambered airfoil, A.O.A α, inviscid, incompressible, irrotational, continuity eqn.

$$\nabla^2 \Phi = 0$$

: camber line of the airfoil is given by a known function $\eta_c(x)$

Thin Airfoil Theory

- Boundary condition requiring no flow across the surface

$$\frac{\partial \Phi}{\partial z}(x, 0\pm) = Q_\infty \left(\frac{d\eta_c}{dx} \cos\alpha - \sin\alpha \right) \simeq Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right)$$

$$\Rightarrow \frac{-1}{2\pi} \int_0^c \gamma(x_0) \frac{dx_0}{x - x_0} = Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right), 0 < x < c$$

- Kutta condition

$$\nabla^2 \Phi < 0 \Rightarrow \Gamma(x = c) = 0$$

ii) classical solution of the Lifting problem

- Glauert's approach ··· approximate $\gamma(x)$ by a trigonometric Expansion

Thin Airfoil Theory

Glauert transform ⋯ appropriate

$$\gamma = \frac{c}{2}(1-\cos\theta),$$

L.E. $\rightarrow x = 0 \Rightarrow \theta = 0$

T.E. $\rightarrow x = c \Rightarrow \theta = \pi$

① $$-\frac{1}{2\pi}\int_0^{\pi}\gamma(\theta_0)\frac{\sin\theta_0}{\cos\theta_0-\cos\theta}d\theta_0 = \theta_0\left[\frac{d\eta_c(\theta)}{dx}-\alpha\right], 0<\theta<\pi$$

② $\gamma(\pi) = 0$ ⋯ satisfied

- Trigonometric Expansion

$$\gamma(\theta) = 2V_\infty[A\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin(n\theta)]$$

$$\underset{\sim}{A_0}+\sum^{\infty}\underset{\sim}{A_n}\cos(n\theta) = \frac{d\eta_c(\theta)}{dx}-\alpha$$

Fourier expansion

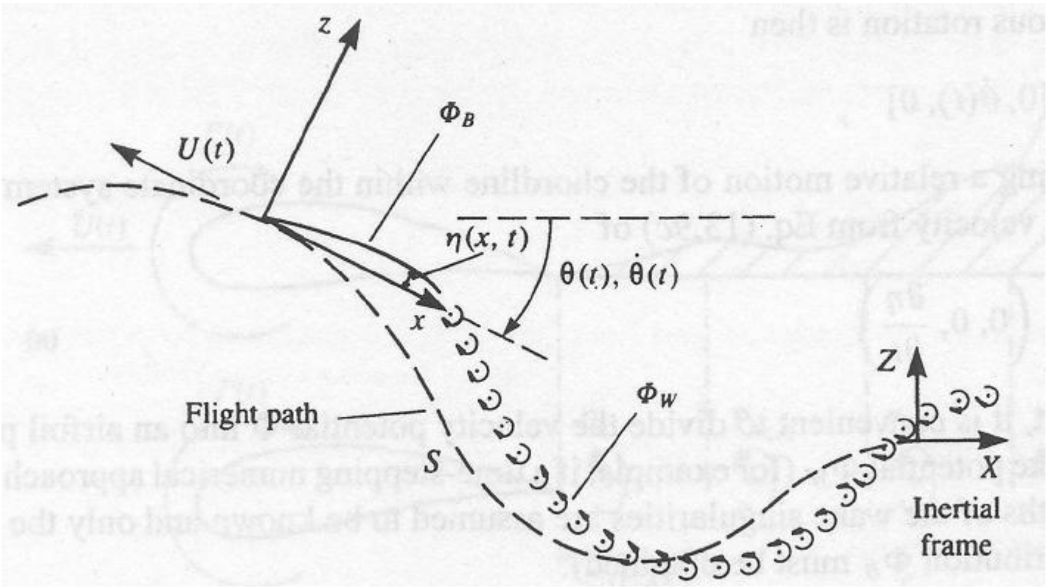
$$\begin{cases} A_0 = \alpha - \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{d\eta_c}{dx}d\theta \\ A_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\frac{d\eta_c}{dx}\cos n\theta d\theta \end{cases}$$

$$* \frac{w}{V_\infty} = -A_0+\sum_{n=1}^{\infty}A_n\cos(n\theta)$$

$$\frac{d\eta_c(\theta)}{dx} = \sum_{n=0}^{\infty}\beta_n\cos(n\theta)$$

Unsteady incompressible potential flow

VI. Unsteady incompressible potential flow



i) 2-D Thin airfoil

- modeling of the vortex wake's shape and strength ···

discretized vortex wake model

Unsteady incompressible potential flow

- Inertial frame *X*, *Z*, at $t > 0$, airfoil moves along a curved path *S*, the coordinates *x*, *z* are selected such that the origin is placed on *S*, *x* axis is always tangent to *S*

camberline $\eta(x,t)$

path radius of curvature

$$\zeta \cdots c/\zeta = \dot{\theta}c/U(t) << 1$$

- continuity eqn. in the moving frame of reference *x*, *z* system

$$\nabla^2\Phi = 0$$

Unsteady incompressible potential flow

- time dependent version of the boundary condition (no flow across the surface)

$$(\nabla\Phi - \vec{V}_0 - \vec{v}_{rel} - \vec{\Omega}\times\vec{r})\cdot\vec{n} = 0$$

$$\vec{n} = \frac{(-\partial\eta/\partial x, 0, 1)}{\sqrt{(\partial\eta/\partial x)^2+1}}, \vec{V}_0 = [-U(t),0,0], \vec{\Omega} = [0,\dot{\theta}(t),0], \vec{v}_{rel} = (0,0,\partial\eta/\partial t)$$

- velocity potential ⋯ $\Phi = \underbrace{\Phi_B}_{\text{Airfoil potential, assumed to be determined}} + \underbrace{\Phi_w}_{\text{Wake potential, assumed to be known}}$

$$\frac{\partial\Phi_B}{\partial z} = \left(\frac{\partial\Phi_B}{\partial x} + \frac{\partial\Phi_w}{\partial x} + U - \dot{\theta}z\right)\frac{\partial\eta}{\partial x} - \frac{\partial\Phi_w}{\partial z} - \dot{\theta}x + \frac{\partial\eta}{\partial t} \equiv w(x,t)$$

⋯ equivalent steady state flow problem at each time step, by exchanging the local downwash $w(x,t)$, the method developed in the steady state can be applied.

Unsteady incompressible potential flow

ii) Wake modeling

- Continuous vortex sheet shed from T.E. → discrete vortex model of strength Γ_{w_i} of each

$$\Gamma_{w_i} = \int_{t-\Delta t}^{t} \gamma_w(t) U(t) dt$$

- Location of discrete vortex element ⋯ place the latest vortex closer to T.E. (within 0.2~0.3$U(t)\Delta t$)

- Strength of the latest vortex element ⋯ Kelvin's theorem

$$\frac{d\Gamma}{dt} = \frac{d\Gamma(t)}{dt} + \frac{d\Gamma_w}{dt} = 0$$

At the i-th time step,

$$\Gamma_{w_i} = -[\Gamma(t_i) - \Gamma(t_{i-1})] = -[\Gamma(t_i) + \sum_{k=1}^{i-1} \Gamma_{w_k}]$$

- Helmholtz theorem ⋯ no vortex decay, vortex strength will be conserved (good approximation for high Reynolds No. flows)

Unsteady incompressible potential flow

iii) Solution by the time-stepping method

- Downwash induced by the airfoil bound circulation $\partial(x,t)$

$$\frac{\partial \Phi_B}{\partial z}(x,t)_{z=0} = \frac{-1}{2\pi}\int_0^c \gamma(x,t)\frac{dx_0}{x-x_0}$$

- Due to N_w discrete vortices of the wake ···(1)

$$\frac{\partial \Phi_B}{\partial z}(x,t)_{z=0} = \sum_{k=1}^{N_w}\frac{-\Gamma_k}{2\pi}\frac{x-x_k}{(x-x_k)^2+(z-z_k)^2}$$

k: counter of the wake vortices

- Boundary condition (no flow across the surface)

$$\frac{-1}{2\pi}\int_0^1 \gamma(x,t)\frac{dx_0}{x-x_0} = U(t)\frac{\partial \eta(x,t)}{\partial x} - \frac{\partial \Phi_w}{\partial z}(x,t) - \dot{\theta}(t)x + \frac{\partial \eta(x,t)}{\partial t}, 0 < x < c$$

- Kutta condition $\gamma(c,t)=0$

Unsteady incompressible potential flow

- Glauert transformation $x = \frac{c}{2}(1 - \cos\theta)$

$$\frac{w(x,t)}{U(t)} = -A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos(n\theta)$$

$$A_0(t) = -\frac{1}{\pi}\int_0^{\pi} \frac{w(x,t)}{v(t)} d\theta, n = 0$$

$$A_n(t) = \frac{2}{\pi}\int_0^{\pi} \frac{w(x,t)}{v(t)} \cos\theta d\theta, n = 1, 2, 3, \cdots$$

··· if momentary chordwise downwash $w(x,t)$ is known, then the momentary circulation distribution is known, too.

- Determine the strength of the latest vortex element ··· Kelvin's condition

$$f(\Gamma) = \underbrace{\Gamma(t)}_{\text{airfoil}} + \underbrace{\Gamma_{w_i}}_{\text{Latest vortex wake}} + \underbrace{\sum_{k=1}^{i-1} \Gamma_{w_k}}_{\text{Circulation of all the other wake vortices(known from the previous time steps)}}$$ {=0 for the converged sol.}

Unsteady incompressible potential flow

- Newton-Raphson iteration scheme ···(2)

$$(\Gamma_{w_i})_{j+1} = (\Gamma_{w_i})_j - \frac{f(\Gamma_{w_i})_j}{f'(\Gamma_{w_i})_j},\ f'(\Gamma)_j = \frac{[f(\Gamma)_j - f(\Gamma)_{j-1}]}{(\Gamma_w)_j - (\Gamma_w)_{j-1}}$$

- Iterative procedure

① At a given time step t_i, $w(x,t)$ is calculated by

$$w(x,t) \simeq U\frac{\partial \eta}{\partial x} - \frac{\partial \Phi_w}{\partial z} - \dot{\theta}x + \frac{\partial \eta}{\partial t}$$

② Assuming Γ_{w_i} for the most recently shed T.E. vortex, can calculate wake influence by (1)

③ Now $w(x,t)$ can be calculated at any point along the chord

→allows numerical computation of $A_n(t)$ and $f(\Gamma)$

④ Using (2), next value of the latest wake vortex is obtained

Unsteady incompressible potential flow

iv) Fluid dynamic loads

- Unsteady Bernoulli's eqn.

$$\frac{p_\infty - p}{\rho} = \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right] - (\vec{V}_0 + \vec{\Omega} \times \vec{r}) \cdot \nabla \Phi + \frac{\partial \Phi}{\partial t}$$

$$\simeq U(t)\frac{\partial \Phi}{\partial x} + \dot{\theta}(t)x\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial t} \simeq U(t)\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial t}$$

- Pressure difference across the airfoil Δp

$$\Delta p = p_l - p_u = 2\rho\left[U(t)\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial t}\right]_l = \rho\left[U(t)\frac{\partial}{\partial x}\Delta\Phi + \frac{\partial}{\partial t}\Delta\Phi\right]$$

where

$$\Delta\Phi = \Phi(x, 0+, t) - \Phi(x, 0-, t) = \int_0^x \gamma(x_0, t)\,dx_0 = \Gamma(x, t)$$

$$\Delta p = \rho\left[U(t)\gamma(x,t) + \frac{\partial}{\partial t}\int_0^x \gamma(x_0, t)\,dx_0\right]$$

Unsteady incompressible potential flow

- Lift

$$L' \equiv F_z = \int_0^c \Delta p dx = \int_0^c \rho \left[U(t)\gamma(x,t) + \rho \frac{\partial}{\partial t}\Gamma(x,t) \right] dx$$

$$= \rho U(t)\Gamma(t) + \rho \int_0^c \frac{\partial}{\partial t}\Gamma(x,t) dx$$

Instantaneous circulation (similar to steady-state circulatory term)

Contribution of time dependency

- Glauert transformation and integrals

$$\frac{\partial}{\partial t}\Delta\Phi(x,t) = \frac{\partial}{\partial t}\int_0^x \gamma(x_0,t)dx_0 = \frac{\partial}{\partial t}\int_0^\theta \gamma(\theta_0,t)\frac{c}{2}\sin\theta_0 d\theta_0$$

$$= \frac{\partial}{\partial t}\left\{ 2U(t)\int_0^\theta \left[A_0(t)\frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^\infty A_n(t)\sin(n\theta_0) \right] \frac{c}{2}\sin\theta_0 d\theta_0 \right\}$$

$$= 2\left\{ B_0(\theta+\sin\theta) + B_1\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right) + \sum_{n=2}^\infty B_n \left[\frac{\sin(n-1)\theta}{2(n-1)} - \frac{\sin(n+1)\theta}{2(n+1)} \right] \right\}$$

$$B_n = \frac{c}{2}\frac{\partial}{\partial t}[A_n(t)U(t)], n = 0,1,2,3,\cdots$$

Unsteady incompressible potential flow

- $$L^{'}(t) = \rho c\left\{\frac{3\pi}{2}B_0 + \frac{\pi}{2}B_1 + \frac{\pi}{4}B_0 + \pi U^2 A_0 + \frac{\pi}{2}U^2 A_1\right\}$$

$$= \pi\rho c\left\{\left[U^2 A_0 + \frac{3c}{4}\frac{\partial}{\partial t}(UA_0)\right] + \left[U^2\frac{A_1}{2} + \frac{c}{4}\frac{\partial}{\partial t}(UA_1) + \frac{c}{8}\frac{\partial}{\partial t}(UA_2)\right]\right\}$$

- Pitching moment

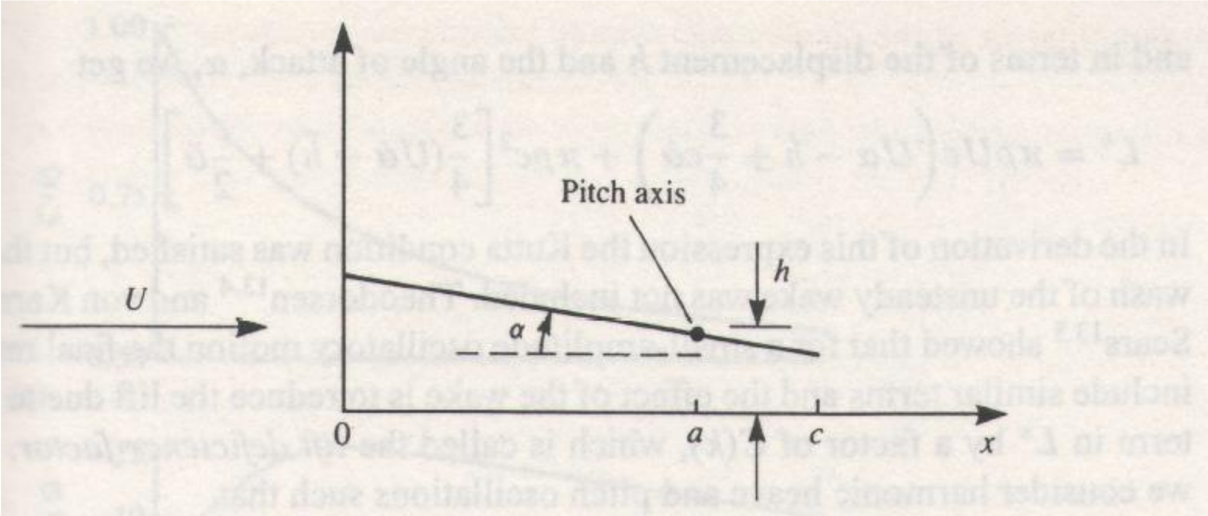
$$M_0(t) = -\int_0^c \Delta p x dx = -\int_0^c \rho\left[U(t)\frac{\partial}{\partial x}\Delta\Phi + \frac{\partial}{\partial t}\Delta\Phi\right]x dx$$

$$= -\rho c^2\frac{\pi}{2}\left[\frac{U^2}{2}(A_0 + A_1 - \frac{A_2}{2}) + \frac{7}{4}B_0 + \frac{3}{4}B_1 + \frac{1}{4}B_2 - \frac{1}{16}B_3\right]$$

$$= -\rho c^2\frac{\pi}{2}\left[\frac{U^2}{2}A_0 + \frac{7c}{8}\frac{\partial}{\partial t}(UA_0) + \frac{U^2}{2}A_1 + \frac{3c}{8}\frac{\partial}{\partial t}(UA_1) - \frac{U^2}{4}A_2 + \frac{c}{8}\frac{\partial}{\partial t}(UA_2) - \frac{c}{32}\frac{\partial}{\partial t}(UA_3)\right]$$

Unsteady incompressible potential flow

v) Small-amplitude oscillation of a thin airfoil (Theodorsen)



- $U(t) = U = const, (x, z)$ frame does not rotate $\rightarrow \ \theta = \dot{\theta} = 0$

Unsteady incompressible potential flow

- Time-dependent chordline position ··· represented by

$$\begin{cases} \text{vertical displacement } h(t) \\ \text{instantaneous a.o.a } \alpha(t) \end{cases}$$

$$\eta = h - \alpha(x - a)$$

Assume that the pitching axis is at the origin (a=0)

$$\eta = h - \alpha x, \frac{\partial \eta}{\partial t} = \dot{h} - \dot{\alpha} x, \frac{\partial \eta}{\partial x} = -\alpha$$

- Downwash $w(x,t)$

$$w(x,t) = -U\alpha + \dot{h} - \dot{\alpha} x - \frac{\partial \Phi_w}{\partial z}$$

Unsteady incompressible potential flow

- Loads due to the motion only $\rightarrow w^*(x,t)$

$$w^*(x,t) = -U\alpha + \dot{h} - \dot{\alpha}x = -U\alpha + \dot{h} - \frac{c}{2}\dot{\alpha} + \frac{c}{2}\alpha\cos\theta$$

$$A_0 = \frac{1}{U}(U\alpha - \dot{h} + \frac{c}{2}\dot{\alpha}), A_1 = \frac{\dot{\alpha}c}{2U}, A_2 = A_3 = \cdots = A_N = 0$$

- Circulation due to the downwash w^*

$$\Gamma^*(t) = \int_0^c \gamma(x,t)dx = \pi c U(A_0 + \frac{A_1}{2}) = \pi c(U\alpha - \dot{h} + \frac{3}{4}c\dot{\alpha})$$

$$L^* = \rho U\Gamma + \pi\rho c^2 U(\frac{3}{4}\frac{\partial A_0}{\partial t} + \frac{1}{4}\frac{\partial A_1}{\partial t})$$

$$= \pi\rho Uc(U\alpha - \dot{h} + \frac{3}{4}c\dot{\alpha}) + \pi\rho c^2\left[\frac{3}{4}(U\dot{\alpha} - \ddot{h}) + \frac{3}{2}\ddot{\alpha}\right]$$

$\cdots$ Kutta condition was satisfied, but the downwash of the unsteady wake is not included

Unsteady incompressible potential flow

- Theodorsen, von Karman, Sears ··· for a small-amplitude oscillatory motion, the final result will include similar terms, and the effect of wake is to reduce the lift due first term by a factor of $C(k)$: lift deficiency factor, k : reduced frequency $= \frac{wc}{2U} = \frac{wb}{U}$

- Harmonic heave and pitch oscillation

$$h = h_0 \sin wt, \alpha = \alpha_0 \sin wt$$

$$L' = \underbrace{\pi\rho U c C(k)\left[U_\alpha - \dot{h} + \frac{3}{4}c\dot{\alpha}\right]}_{L_1'} + \underbrace{\pi\rho\frac{c^2}{4}\left[U\dot{\alpha} - \dot{h} + \frac{c}{2}\ddot{\alpha}\right]}_{L_2'}$$

$$C(k) = F(k) + cG(k) \qquad H_2^{(1)} = J_n - iY_n$$

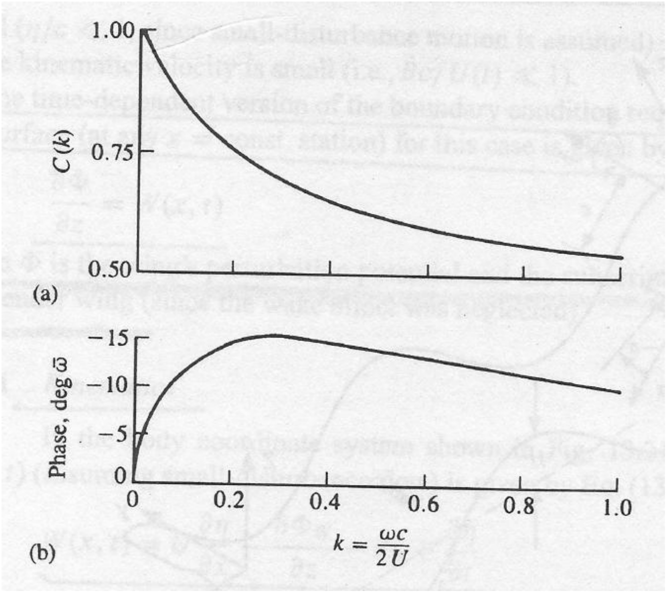
$$= \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + \overline{c}H_0^{(2)}(k)}$$

Unsteady incompressible potential flow

- Delaying effect

$$L_1'(t) = L_1' \sin(wt - \overline{w})$$

: time shift effect of the wake



- Pitching moment

$$M_0 = -\frac{\pi\rho c^2}{4}\left\{-\frac{c}{2}\ddot{h} + \frac{3Uc}{4}\dot{\alpha} + \frac{9}{32}c^2\ddot{\alpha} + UC(k)\left[-\dot{h} + U\alpha + \frac{3c}{4}\dot{\alpha}\right]\right\}$$

Unsteady Aerodynamics and Flutter

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

12.2 2-D Frequency-Domain Theories

1. Theodorsen function

   - "sinusoidally varying vorticity in wake"

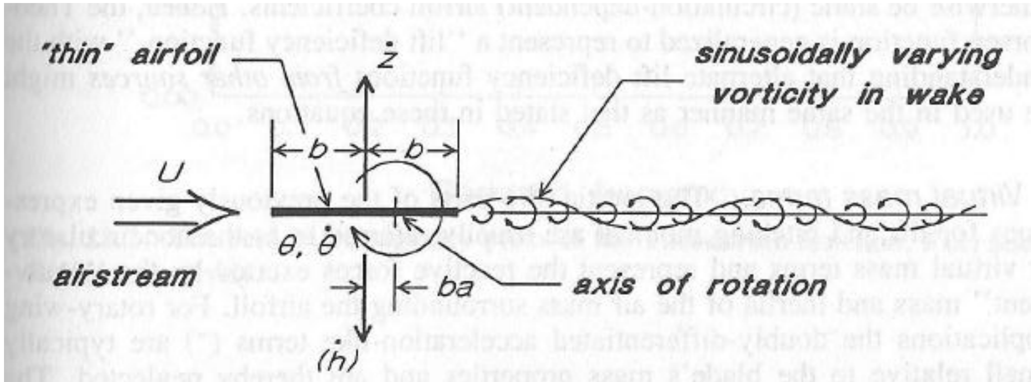


Fig. 12.1 Schematic of the Theodorsen problem.

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

2. Sears function

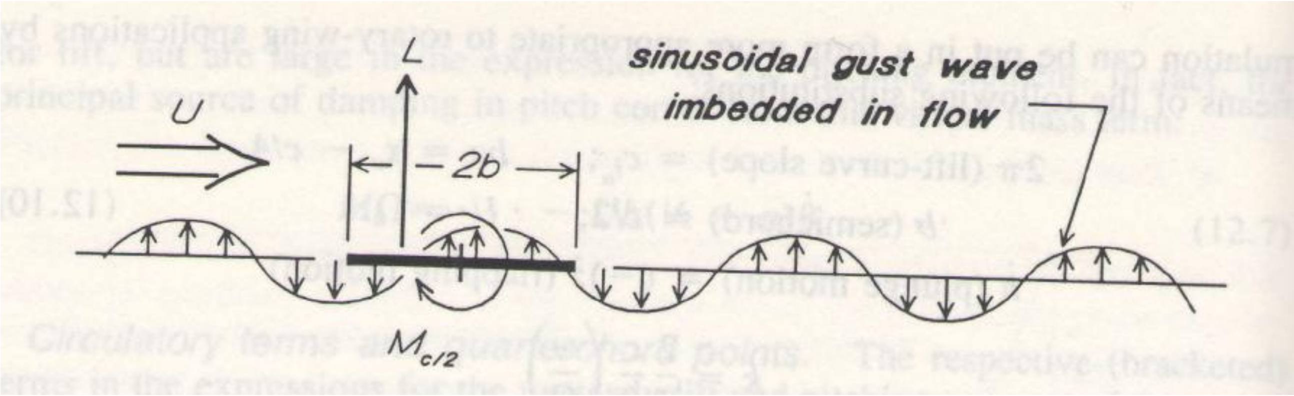
- Motionless airfoil + sinusoidal vertical gust pattern imbedded in the air mass

$$w(x,t) = We^{iw(t-x/U)}$$

$$L = \pi\rho c U W e^{iwt}\phi(k)$$

$$M_{c/2} = L \cdot \frac{c}{4}$$

where $\phi(k) = \left[J_0(k) - iJ_1(k)\right]C(k) + iJ_1(k)$



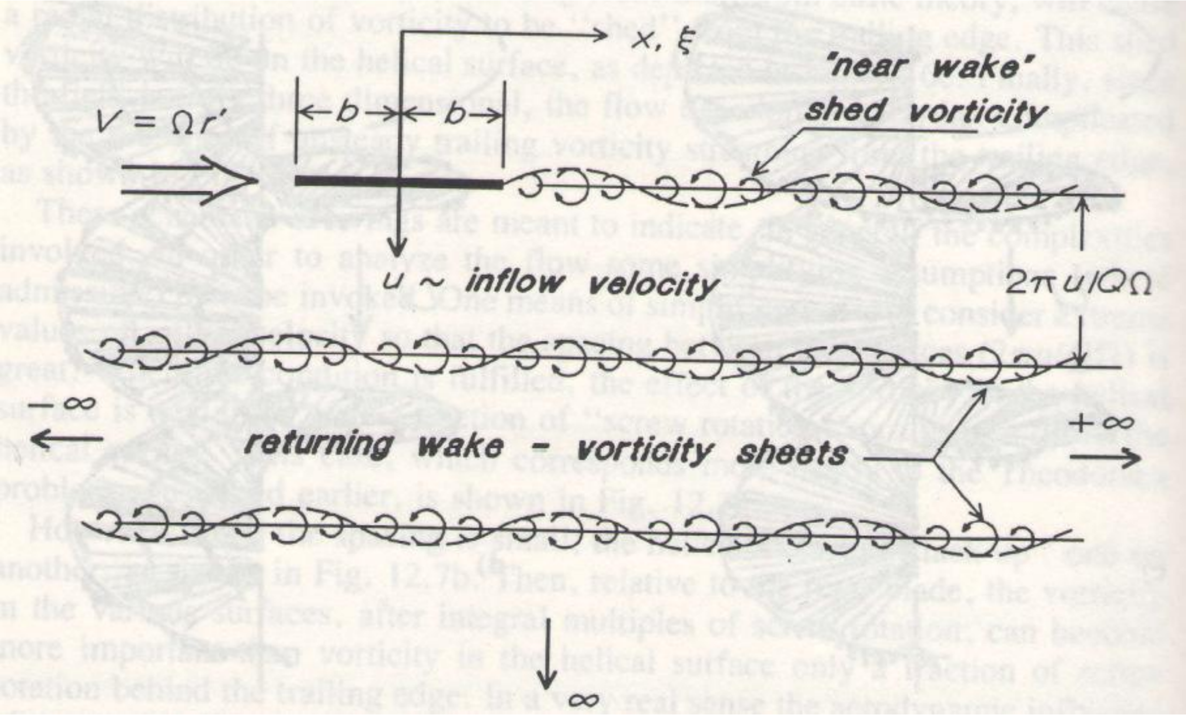
Unsteady Aerodynamics and Flutter

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

3. Loewy Function

   - Rotor in hover or vertical flight

   - Effect of the returning wake, trailing vorticity



Unsteady Aerodynamics and Flutter

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

$$C'(k,m,h,Q,\psi_q) = F'(k,m,h,Q,\psi_q) + iG'(k,m,h,Q,\psi_q)$$

where, $h \equiv h' / b = 2\pi u / bQ\Omega = 4|\lambda|\sigma$ σ : Rotor solidity

$r \equiv r' / b$

$m \equiv \omega / \Omega$

$Q \equiv$ Number of blades

$\Psi_q = 2\pi(q/Q)(\omega/\Omega)$

$q \equiv$ Blade index

Unsteady Aerodynamics and Flutter

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

12.3 2-D Arbitrary motion theories

Frequency-domain $\longleftrightarrow$	Time-domain
Theodorsen	Wagner
Sears	Kussner
Sinusoidal motion	Unit step motion

1. Wagner function

- Unit step change in A.O.A.

$$w_{3c/4}(t) = \begin{cases} 0, t < 0 \\ -U\alpha_0, t \geq 0 \end{cases}$$

Unsteady Aerodynamics and Flutter

- Bielawa, R. L., "Rotary Wing Structural Dynamics and Aeroelasticity," AIAA Education Series, 1992

- Fourier transformed (circulatory) lift

$$L_c(s) = \rho b U^2 \alpha_0 \int_{-\infty}^{+\infty} \frac{C(k)}{ik} e^{iks} dk = 2\pi\rho b U^2 \alpha_0 \phi(s)$$

where $s \equiv Ut/b$ (the aerodynamic time variable)

- Duhamel integration ··· any arbitrary angle of attack time history

$$L = \pi\rho b^2[\ddot{h} + U\dot{\theta} - ba\ddot{\theta}] - 2\pi\rho b U W_\phi$$

$$M_\theta = \pi\rho b^2[ba\ddot{h} - Ub(1/2-a)\dot{\theta} - b^2(1/8+a^2)\ddot{\theta}] - 2\pi\rho U b(1/2+a)W_\phi$$

where $W_\phi = w_{3c/4}(0)\phi(s) + \int_0^s \frac{dw_{3c/4}(\sigma)}{d\sigma}\phi(s-\sigma)d\sigma$

··· still complex to be used in a flutter(instability) analysis mainly in the calculation of transient motion

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

5.7 Arbitrary motion of thin airfoil in incompressible flow; the gust problem

- Rapid maneuvers, gust entry ··· more general small motions of airfoils
- Method of approach ··· Fourier-integral superposition of the linear results for incompressible flow
- Noncirculatory part ··· unchanged regardless of the nature of unsteady motion
- Instantaneous vertical velocity of the liquid particle in contact with the ¾ chord point of the airfoil.

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

$$w_{\frac{3}{4}c}(t) = -\left[\dot{h} + U\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right]$$

- $w_{\frac{3}{4}c}(t)$ for arbitrary motion by the Fourier integral

$$w_{\frac{3}{4}c}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{i\omega t}d\omega$$

- Inverse Fourier transformation

$$f(\omega) = \int_{-\infty}^{\infty} w_{\frac{3}{4}c}(t)e^{-i\omega t}dt$$

- Circulatory lift per unit span for any Fourier component with unit amplitude of $w_{\frac{3}{4}c}$

$$\Delta L_c = -2\pi\rho UbC(k)e^{i\omega t}$$

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Resultant lift

$$L_c = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)\Delta L_c d\omega = -\frac{2\pi\rho U b}{2\pi}\int_{-\infty}^{\infty} C(k) f(\omega) e^{i\omega t} d\omega$$

- Total running lift and moment (about the axis at $x = ba$)

$$L = \pi\rho b^2\left[\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}\right] - \rho U b\int_{-\infty}^{\infty} C\left(\frac{\omega b}{U}\right) f(\omega) e^{i\omega t} d\omega$$

- Fourier integral superposition ⋯ can be used with any function that has a finite number of finite discontinuities and whose absolute value has a finite integral in the range $t = -\infty \sim +\infty$

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Wagner's problem ··· step change in A.O.A.

$$w_{\frac{3}{4}c} = \begin{cases} 0, t < 0 \\ -U_{\alpha_0}, t > 0 \end{cases}$$

- Fourier integral formula

$$w_{\frac{3}{4}c} = -\frac{U_{\alpha_0}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{i\omega}d\omega$$

- Circulatory lift due to this motion → indicial lift

$$L = \rho U b U \alpha_0 \int_{-\infty}^{\infty}\frac{C(k)}{ik}e^{iks}dk \quad \text{, where } s = \frac{Ut}{b}$$

: is the distance in semichords traveled by the airfoil after the step.

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- In terms of Wagner's fn. $\phi(s)$: $L = 2\pi\rho U^2 b\alpha_0\phi(s)$
- Wagner's fn.

$$\phi(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{C(k)}{k}e^{iks}dk = 1(s) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{C(k)-1}{k}e^{iks}dk$$

- Separating e^{iks} and $C(k)$ into their real and imag. parts

$$\phi(s) = 1(s) + \frac{1}{2\pi}\int_{-\infty}^{\infty}[F(k)-1]\frac{\sin ks}{k}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}G(k)\frac{\cos ks}{k}dk +$$

$$\frac{i}{2\pi}\int_{-\infty}^{\infty}\left\{G(k)\frac{\sin ks}{k} - [F(k)-1]\frac{\cos ks}{k}\right\}dk = 1(s) + \frac{1}{\pi}\int_{0}^{\infty}\left\{[F(k)-1]\frac{\sin ks}{k} + G(k)\frac{\cos ks}{k}\right\}dk$$

$\cdots$ $F(k), \cos ks$:even fn. $G(k), \sin ks$:odd, $\phi(s)$ must have zero imag. part for all values of s. $C(k)$: complex admittance function for steady-state oscillation of a linear system, even real part, odd imag. part

Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Further through the vanishing of $\phi(s)$ and $1(s)$ when *s* is negative. For *s*<0,

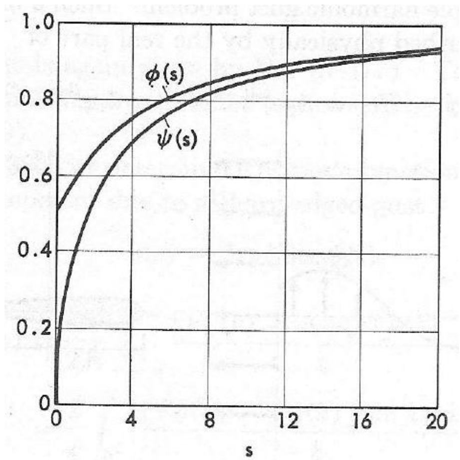
$$-\frac{1}{\pi}\int_0^\infty [F(k)-1]\frac{\sin ks}{k}dk = \frac{1}{\pi}\int_0^\infty G(k)\frac{\cos ks}{k}dk$$

$\sin ks$ is an odd fn.

$$\frac{1}{\pi}\int_0^\infty [F(k)-1]\frac{\sin k|s|}{k}dk = \frac{1}{\pi}\int_0^\infty G(k)\frac{\cos k|s|}{k}dk$$

: holds for all positive numbers |*s*|

$$\phi(s) = \frac{2}{\pi}\int_0^\infty \frac{F(k)}{k}\sin ks dk = 1 + \frac{2}{\pi}\int_0^\infty \frac{G(k)}{k}\cos ks dk$$



Arbitrary motion of thin airfoils in incompressible flow; the gust problem

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Duhamel or superposition integral ··· circulatory lift and moment due to arbitrary motion.

  For a rigid airfoil starting from rest at t=0,

$$L = \pi\rho b^2[\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}] - 2\pi\rho Ub\left[w_{\frac{3}{4}c}(0)\phi(s) + \int_0^s \frac{dw_{\frac{3}{4}c}(\sigma)}{d\sigma}\phi(s-\sigma)d\sigma \right]$$

- Convenient approximate representation

$$\phi(s) \cong 1 - 0.165e^{-0.0455s} - 0.335e^{-0.3s}$$

$$\phi(s) \cong \frac{s+2}{s+4}$$

Gust loading

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Atmospheric turbulence with normal velocity distribution $w_G \, (w_G << U)$
- B.C. ⋯ total vertical velocity due to the gust and the vortex sheet simulating the airfoil must vanish

$$w_G + w_a = 0$$

or

$$w_a(x,t) = -w_G \quad \text{for} \quad z = 0, -b \le x \le b$$

- Assumption ⋯ the turbulence moves post the airfoil at velocity *U* without an appreciable change in w_G

Gust loading

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Simple harmonic gust problem ⋯ sinusoidal gust

$$w_G(x-Ut) = \bar{w}_G e^{i\omega[t-(x/U)]} = \bar{w}_G e^{i\omega[s-x^*]}$$

* Suitable for use with Schwarz' solution

$$L = 2\pi\rho U b \bar{w}_G \{C(k)[J_0(k) - iJ_1(k)J + iJ_1(k)\}e^{i\omega t}$$

$$M_y = b(\frac{1}{2} + a)L$$

- Arbitrary w_G ⋯ sharp-edged gust striking the leading edge at the airfoil at *t*=0 (Kussner's problem)

$$w_G = \begin{cases} 0, x > Ut - b \\ w_0, x < Ut - b \end{cases} \qquad w_G = \frac{w_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-\frac{b}{U}-\frac{x}{U})}}{i\omega}d\omega = \frac{w_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(s-x^*-1)}}{ik}dk$$

$$L = \rho U b w_0 \int_{-\infty}^{\infty}\frac{\{C(k)[J_0(k) - iJ_1(k)] + iJ_1(k)\}e^{ik(s-1)}}{ik}dk$$

Gust loading

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Kussner's fn.⋯dimensionless lift development due to a sharp-edged gust

$$L(s) = 2\pi\rho U b w_0 \psi(s)$$

$$\psi(s) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[F_G(k) + iG_G(k)]e^{ik(s-1)}}{k} dk$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{[F_G(k) - G_G(k)]\sin ks \sin k}{k} dk$$

$F_G(k), G_G(k)$:real and imag. parts of the expression in braces in

$$L = 2\pi\rho U b \bar{w}_G \{C(k)[J_0(k) - iJ_1(k)J + iJ_1(k)\}e^{i\omega t}$$

$\psi(s)$ ⋯ increases from 0 at t=0 to 1 at t=∞

Gust loading

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Arbitrary $w_G(s)$ given, as the gust velocity encountered by the airfoil's leading edge at the instant $t = sb/U$, Duhamel's integral

$$L = 2\pi\rho Ub\{w_G(0)\psi(s) + \int_0^s \frac{dw_G(\sigma)}{d\sigma}\psi(s-\sigma)d\sigma\}$$

- Simple algebraic approximation

$$\psi(s) \cong 1 - 0.500e^{-0.130s} - 0.500e^{-s}$$

$$\psi(s) \cong \frac{s^2 + s}{s^2 + 2.82s + 0.80}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

5.6 Thin airfoil oscillating in incompressible flow

- Governing eqn. ⋯ small-disturbance theory

$$\rightarrow \text{Laplace's eqn. } \nabla^2 \phi' = 0$$

- 2-D B.C. ⋯ the surface of a body moving in a time-dependent fashion

$$F(x, y, z, t) = 0$$

→B.C.

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

⋯ the rate of change of the numerical value of *F* is zero when we follow the motion of a particular fluid element

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Steady flow

$$\vec{q} \cdot gradF = 0$$

··· component of velocity normal to *F* vanishes

- For an wing

$$F_v = z - z_v(x, y, t) = 0$$
$$F_L = z - z_L(x, y, t) = 0$$

- Vertical velocity *w* over the wing surface

$$w = \frac{\partial z_U}{\partial t} + u\frac{\partial z_U}{\partial x} + v\frac{\partial z_U}{\partial y}$$
$$w = \frac{\partial z_L}{\partial t} + u\frac{\partial z_L}{\partial x} + v\frac{\partial z_L}{\partial y}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Approximation

(1) The slopes $\frac{\partial z_U}{\partial x}, \frac{\partial z_U}{\partial y}$, etc. are very small compared to 1.

(2) The resultant fluid velocity $\vec{q}$ differs only slightly in direction and magnitude from the free-stream velocity *U*.

- Disturbance velocity potential $\phi^{'}$

$$\phi = \phi^{'} + U_x$$

- Disturbance velocity components

$$u - U = u^{'} = \frac{\partial \phi^{'}}{\partial x}, v = \frac{\partial \phi^{'}}{\partial y}, w = \frac{\partial \phi^{'}}{\partial z}$$

$$u^{'}, v, w << V$$

$\rightarrow u^{'}(\frac{\partial z_U}{\partial x}), v(\frac{\partial z_U}{\partial y})$ can be neglected by comparison with $U(\frac{\partial z_U}{\partial x})$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Refined B.C.

$$w=\frac{\partial z_U}{\partial t}+U\frac{\partial z_U}{\partial x} \quad \text{, for} \quad z=z_U,(x,y) \quad \text{in } R_a$$

$$w=\frac{\partial z_L}{\partial t}+U\frac{\partial z_L}{\partial x} \quad \text{, for} \quad z=z_L,(x,y) \quad \text{in } R_a$$

- Maclaurin series about the values just above and below the *xy*-plane

$$w(x,y,z_U,t)=w(x,y,0^+,t)+z_U\frac{\partial w(x,y,0^+,t)}{\partial z}+h.o.t.+\cdots$$

$$w(x,y,z_L,t)=w(x,y,0^-,t)+z_L\frac{\partial w(x,y,0^-,t)}{\partial z}+h.o.t.+\cdots$$

Neglecting h.o.t.,

$$w=\frac{\partial z_U}{\partial t}+U\frac{\partial z_U}{\partial x}: \quad \text{for} \quad z=0^+,(x,y) \quad \text{in } R_a$$

$$w=\frac{\partial z_L}{\partial t}+U\frac{\partial z_L}{\partial x}: \quad \text{for} \quad z=0^-,(x,y) \quad \text{in } R_a$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Actual B.C.

$$w = \frac{\partial z_a}{\partial t} + U\frac{\partial z_a}{\partial x}$$
$$= w_a(x,t): \quad \text{for} \quad z = 0, -b \le x \le b$$

- Kutta's hypothesis ··· finite continuous velocities and pressures at $x = b$

- Theodorsen's approach ··· dividing the solution into two parts

i) appropriate distribution of source and sink just above and below the line z=0

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

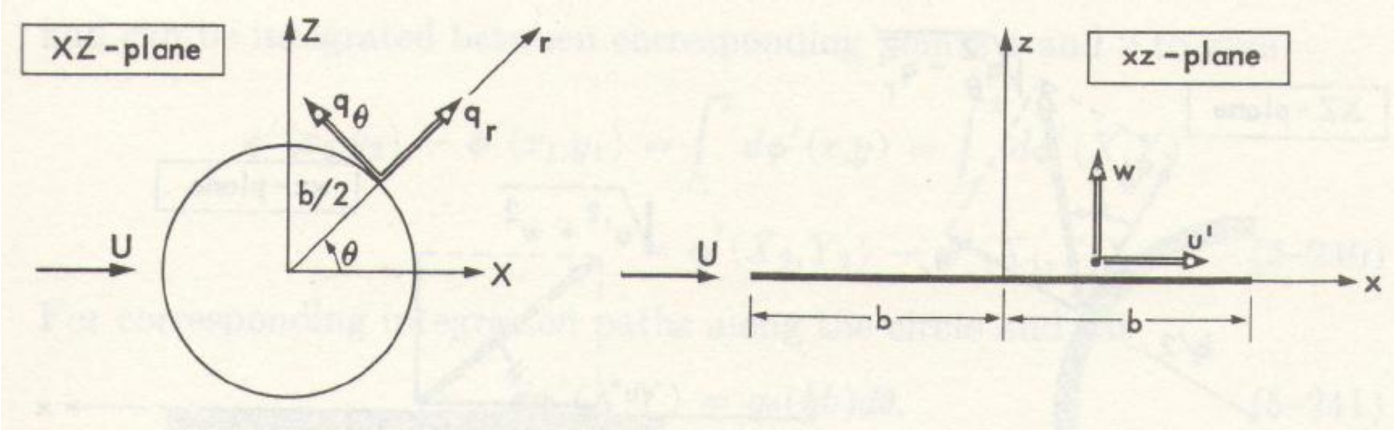
ii) a pattern of vortices put on the z=0 and counter vortices along the wake to infinity

→ Kutta's hypothesis is fulfilled without disturbing B.C.

Using Joukowski's conformal transformation to circle of radius $\dfrac{b}{2}$

Circle ⟷ line or "slit"

$r=\dfrac{b}{2}$ in the XZ-plane　　$-b \le x \le b,\ z=0$ in the xz-plane



Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Transformation between the velocity components

$$|u' - iw| = \sqrt{u'^2 + w^2} = \frac{\sqrt{q_x^2 + q_z^2}}{|2\sin\theta|} = \frac{\sqrt{q_\theta^2 + q_r^2}}{|2\sin\theta|}$$

q_r, q_θ :radial, tangential components in the *XZ*-plane

- Due to conformal property,

$$|u'| = \frac{|q_\theta|}{|2\sin\theta|}, |w| = \frac{|q_r|}{|2\sin\theta|}$$

On the upper surface, $\sin\theta \ge 0$

$$\left.\begin{aligned} q_\theta &= -2u'\sin\theta \\ q_r &= 2w\sin\theta \end{aligned}\right\} 0 \le \theta \le \pi$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Details of Theodorsen's approach (1)

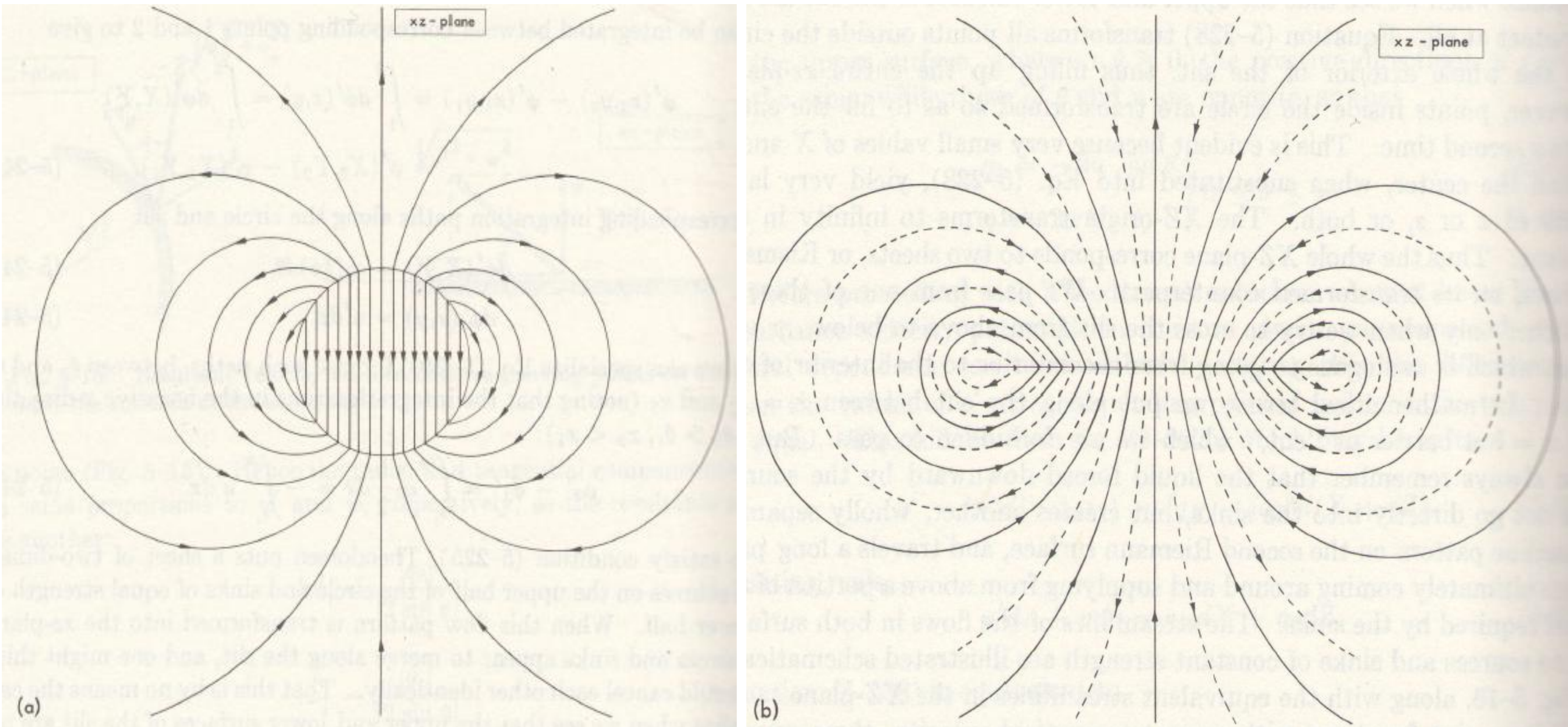
- A sheet of 2-D sources on the upper half of the circle sinks of the equal strength on the lower half

→ Does not cancel out each other since the upper and lower surfaces of the slit are not in contact at all

- Whole *XZ*-plane transforms to two sheets, or Riemann surfaces

→ Generates a streamline shown in Fig. 5-16(b)

Thin airfoil oscillating in incompressible flow



Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Single 2-D source of strength *H*, located at $x=\xi, z=\zeta$

$$\phi_{s_2} = \frac{H}{4\pi}\ln[(x-\xi)^2+(z-\zeta)^2]$$

- Continuous distribution of point source over the upper surface

$$\phi^{'}(x,z,t) = \frac{1}{4\pi}\int_{-b}^{b} H^{+}(\xi,t)\ln[(x-\xi)^2+z^2]d\xi$$

- Limit z>0 for correct Riemann surface

$$w(x,0^+,t) = \frac{\partial\phi^{'}}{\partial z}(x,0^+,t) = \frac{1}{4\pi}\lim_{z\to 0^+}\frac{\partial}{\partial z}\int_{-b}^{b} H^{+}(\xi,t)\ln[(x-\xi)^2+z^2]d\xi$$

$$= \frac{1}{2\pi}\lim_{z\to 0^+} z\int_{-b}^{b}\frac{H^{+}(\xi,t)d\xi}{[(x-\xi)^2+z^2]}$$

⋯ as *z* gets smaller, integral tends to vanish, except in the vicinity of point $\xi = x$, the integral tends to infinity

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Isolate the singularity with a short line of length 2ε

$$w(x,0^+,t) = \frac{1}{2\pi}\lim_{z\to 0^+} z\int_{x-\varepsilon}^{x+\varepsilon}\frac{H^+(\xi,t)d\xi}{[(x-\xi)^2+z^2]}$$

$$= \frac{H^+(x,t)}{2\pi}\lim_{z\to 0^+}\left[\tan^{-1}\left(\frac{t}{z}\right)-\tan^{-1}\left(-\frac{t}{z}\right)\right]$$

$$= \frac{1}{2}H^+(x,t)$$

Actual B.C. $\rightarrow H^+(x,t) = 2w_a(x,t)$

$\cdots$ sources discharge $H^+ ft^3$ of liquid per unit time per unit area of the sheet

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

Half of it going upward with normal velocity

The other half disappear downward onto the other Riemann surface

H^- strength of sinks just below the slit produces an equal upward velocity w_a

$$H^-(x,t) = -2w_a(x,t)$$

- Circle $r = \frac{b}{2}$ is itself a streamline
- Velocity induced by a source or sink upon the circle

Normal velocity dq_r is zero $\rightarrow$ circle is a streamline

$$q_\theta(\theta,t) = \frac{2}{\pi}\int_0^\pi \frac{w_a \sin^2\phi \, d\phi}{(\cos\phi - \cos\theta)}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Disturbance velocity potential $\phi_U^{'}$ at an arbitrary point on the upper half of the circle (and at the corresponding point on top of the slit)

$\cdots$ antisymmetric flow distribution w.r.t. *X*-axis q_θ is symmetric on the upper and lower halves of the circle

$$\phi^{'}(\pi,t)-\phi_U^{'}(\pi,t)=\phi_L^{'}(-\theta,t)-\phi^{'}(\pi,t)$$
$$\rightarrow \phi_L^{'}(-\theta,t)=-\phi_U^{'}(\theta,t)$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Pressure distribution along the slit

- linearized Bernoulli eqn. for unsteady flow

$$p - p_\infty = -\rho U u' - \rho \frac{\partial \phi'}{\partial t} = -\rho \left[U \frac{\partial \phi'}{\partial x} + \frac{\partial \phi'}{\partial t} \right]$$

$$p_U - p_L = -\rho \left[U \left(\frac{\partial \phi_U'}{\partial x} - \frac{\partial \phi_L'}{\partial x} \right) + \left(\frac{\partial \phi_U'}{\partial t} - \frac{\partial \phi_L'}{\partial t} \right) \right]$$

$$= -2\rho \left[U \frac{\partial \phi_U'}{\partial x} + \frac{\partial \phi_U'}{\partial t} \right] = -2\rho \left[\frac{\partial \phi_U'}{\partial t} - \frac{U}{b \sin\theta} \frac{\partial \phi_U'}{\partial \theta} \right]$$

- Lift and moment (about $x=ba$) per unit span

$$L_{NC} = -\int_{-b}^{b} (p_U - p_L) dx = 2\rho \int_{-b}^{b} \frac{\partial \phi_U'}{\partial t} dx + 2\rho U \int_{-b}^{b} \frac{\partial \phi_U'}{\partial x} dx$$

$$= 2\rho \frac{\partial}{\partial t} \int_{-b}^{b} \phi_U' dx = 2\rho b \frac{\partial}{\partial t} \int_{0}^{\pi} \phi_U' \sin\theta d\theta$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

$$M_{yNC} = \int_{-b}^{b} (p_U - p_L)\left[x - ba\right] dx$$

$$= 2\rho U b \int_0^{\pi} \phi_U' \sin\theta d\theta - 2\rho b^2 \frac{\partial}{\partial t} \int_0^{\pi} \phi_U' [\cos\theta - a] \sin\theta d\theta$$

where $\phi_U'(\theta, t) = -\frac{b}{\pi} \int_\theta^{\pi} \int_0^{\pi} \frac{w_a \sin^2 \phi d\phi d\theta}{(\cos\phi - \cos\theta)}$

- ϕ_U' vanishes both at the leading and trailing edges

$$\phi_U'(\theta = 0, t) = \phi_L'(\theta = 0, t) = 0$$

- Subscript “NC” ··· noncirculatory character, there would be no lift in any steady flow

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Chordwise-rigid airfoil conducting plunge $h(t)$ and rotation $\alpha(t)$, $\beta(t)\cdots$ angular displacement of the control surface, hinged at *x=bc*

- $z_\alpha(x,t)$ representing the instantaneous small displacement of the chordline

$$z_\alpha(x,t) = -h - \alpha[x - ba] \quad \text{for } -b \le x \le b$$

$$w_\alpha(x,t) = -\dot{h} - U\alpha - \dot{\alpha}[x - ba]$$

$$L_{NC} = \pi\rho b^2[\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}]$$

$$M_{yNC} = \pi\rho b^2[U\dot{h} + ba\dot{h} + U^2\alpha - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha}]$$

$\cdots$ reactive forces exerted by virtual("apparent") mass of cylinder of liquid with diameter equal to the wing chord

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- "NC" solution is capable of fulfilling Kutta's hypothesis, by itself.

- Disturbance velocity at the trailing edge $x = b, \theta = 0$

$$|u'| = \frac{|q_\theta|}{|2\sin\theta|}$$

this will $\rightarrow\infty$ where $\sin\theta = 0,$ unless q_θ also vanishes there.

- $q_\theta = 0$ only for a very special motion which satisfies

$$\int_0^\pi \frac{w_a(\phi, t)\sin\phi d\phi}{(\cos\phi - 1)} = 0$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

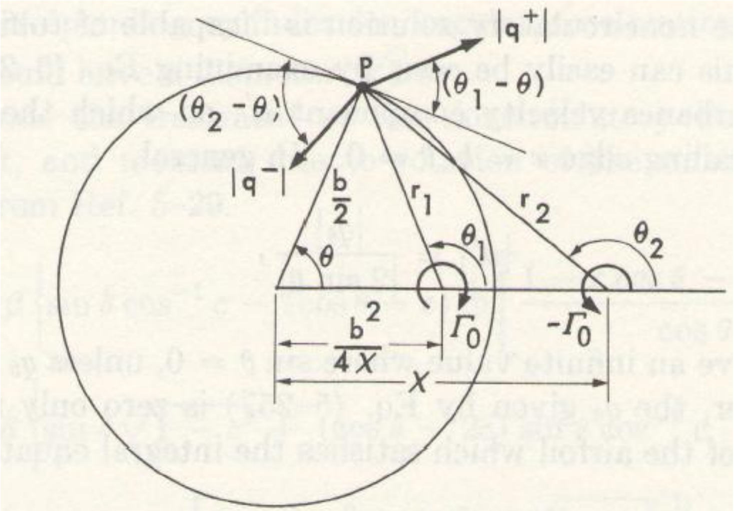
→ It is therefore necessary to superimpose some additional flow pattern that just cancels the noncirculatory $q_\theta(0,t)$

⋯ Theodorsen’s approach (2): bound vortices + wake of shed counter vortices continually moving away from the airfoil at the free-stream velocity (along the positive *x*-axis beyond *x*=*b*)

- Put the image of a vortex at point χ on the *X*-axis, $X = \dfrac{b^2}{4\chi}$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955



- Velocity induced by two vortices Γ_0 and $-\Gamma_0$ on the circle

$$q_\theta = -\frac{\Gamma_0}{\pi b}\left[\frac{\chi^2 - \left(\frac{1}{2}b\right)^2}{\chi^2 + \left(\frac{1}{2}b\right)^2 - \chi b \cos\theta}\right]$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Velocity potential on the upper surface of the circle or slit

$$\phi_U^{'}(\theta,t) = -\int_0^{\pi} q_\theta \frac{b}{2} d\theta = \frac{\Gamma_0}{\pi} \tan^{-1}\left[\frac{\chi - \frac{1}{2}b}{\chi + \frac{1}{2}b} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \right]$$

- Location χ of the wake vortex ··· assume the location $x = \xi$ moves downward with velocity U

$$\frac{d\xi}{dt} = U$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Pressure distribution due to the vortex pair

$$\left(p_U - p_L\right)_{\Gamma_0} = \frac{-\rho U \Gamma_0 [\xi + b\cos\theta]}{\pi b \sin\theta \sqrt{\xi^2 - b^2}}$$

→ cannot use the formula obtained previously for noncirculatory flow to compute lift and moment, since $\phi_U^{'}$ no longer vanishes at the trailing edge when there is circulation.

- Lift and moment due to Γ_0

$$L_{\Gamma_0} = \frac{\rho U \Gamma_0 \xi}{\sqrt{\xi^2 - b^2}}, M_{y\Gamma_0} = \frac{\rho U \Gamma_0 b^2}{\sqrt{\xi^2 - b^2}}\left[\frac{\xi}{b} a - \frac{1}{2}\right]$$

… As ξ become large, flow approaches that of a single bound vortex Γ_0

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Concentrated vortex $-\Gamma_0 \rightarrow$shed vortex sheet $\gamma_w d\xi$

$$\Gamma_0 = -\gamma_w d\xi$$

- Pressure distribution ··· integrating over the complete wake from $\xi = b$ to $\xi = \infty$

$$p_U - p_L = \frac{\rho U}{\pi b \sin\theta} \times \int_b^\infty \left[\frac{\xi}{\sqrt{\xi^2 - b^2}}(1 - \cos\theta) + \sqrt{\frac{\xi + b}{\xi - b}} \cos\theta \right] \gamma_w(\xi, t) d\xi$$

$$L_c = -\rho U \int_b^\infty \frac{\xi}{\sqrt{\xi^2 - b^2}} \gamma_w(\xi, t) d\xi$$

$$M_{y_c} = \rho U b \int_b^\infty \left[\frac{1}{2}\sqrt{\frac{\xi + b}{\xi - b}} - \left(a + \frac{1}{2}\right) \frac{\xi}{\sqrt{\xi^2 - b^2}} \right] \gamma_w(\xi, t) d\xi$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

- Kutta's hypothesis ··· assemble the source and vortex flows by making $q_\theta(\theta,t)$ to zero

$$\frac{2}{\pi}\int_0^\pi \frac{w_a \sin^2\phi d\phi}{(\cos\phi - 1)} + \frac{1}{\pi b}\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w(\xi,t)d\xi = 0$$

··· integral eqn. for the wake circulation γ_w, when w_α is given. First term of R.H.S =2Q

$$(p_U - p_L) = (p_U - p_L)_{NC} - 2\rho UQ\left\{\cot\theta + \left[\frac{1-\cos\theta}{\sin\theta}\right]\frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w(\xi,t)d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w(\xi,t)d\xi}\right\}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

$$L = L_{NC} + 2\pi\rho UbQ\left\{\frac{\displaystyle\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w(\xi,t)d\xi}{\displaystyle\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w(\xi,t)d\xi}\right\}$$

$$M_y = M_{yNC} - 2\pi\rho Ub^2Q\left\{\frac{1}{2}-\left(a+\frac{1}{2}\right)\underbrace{\frac{\displaystyle\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w(\xi,t)d\xi}{\displaystyle\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w(\xi,t)d\xi}}\right\}$$

Particular ratio of two integrals
$\rightarrow$ influence of the wake circulation

- Simple harmonic oscillation of the wing

$$z_a(x,t) = \bar{z}_a(x)e^{i\omega t}$$

$$w_a(x,t) = \bar{w}_a(x)e^{i\omega t}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

$\rightarrow$ simple harmonic wake

$$\gamma_w(\xi,t) = \bar{\gamma}_w e^{i\omega[t-(\xi/U)]} = \bar{\gamma}_w e^{i(\omega t - k\xi^*)}$$

where, $k = \dfrac{\omega b}{U}$ ⋯ reduced frequency of oscillation

- Influence of the wake circulation ⋯ ration of two integrals

$$C(k) = \frac{\displaystyle\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w(\xi,t)d\xi}{\displaystyle\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w(\xi,t)d\xi}$$

$$C(k) = F(k) + iG(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)}$$

Thin airfoil oscillating in incompressible flow

- Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., " Aeroelasticity," Addison-Wesley, 1955

where, $H_n^{(2)}$ is a combination of Bessel functions of the first and second kinds. Useful in radiation problems.

$H_n^{(2)} = J_n - iY_n$: Hankel function of the 2nd kind

- Final form

$$L = \pi\rho b^2[\ddot{h} + U\dot{\alpha} - ba\ddot{\alpha}] + 2\pi\rho UbC(k)\left[\dot{h} + U\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right]$$

$$M_y = \pi\rho b^2\left[ba\ddot{h} - Ub\left(\frac{1}{2} - a\right)\dot{\alpha} - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha}\right] + 2\pi\rho Ub^2\left(a + \frac{1}{2}\right)C(k)$$
$$\left[\dot{h} + U\alpha + b\left(\frac{1}{2} - a\right)\dot{\alpha}\right]$$

Dynamic Aeroelasticity

Dynamic aeroelasticity

- Two principal phenomena
- Dynamic instability (flutter)
- Responses to dynamic load, or modified by aeroelastic effects

- Flutter ··· self-excited vibration of a structure arising from the interaction of aerodynamic elastic and internal loads

  "response" ··· forced vibration

  "Energy source" ··· flight vehicle speed

- Typical aircraft problems
- Flutter of wing
- Flutter of control surface
- Flutter of panel

Dynamic aeroelasticity

- Stability concept

If solution of dynamic system may be written or

$$y(x,t) = \sum_{k=1}^{N} \bar{y}_k(x) \cdot e^{(\sigma_k + i\omega_k)t}$$

a) $\sigma_k < 0, \omega_k \neq 0 \Rightarrow$ Convergent solution : “stable”

b) $\sigma_k = 0, \omega_k \neq 0 \Rightarrow$ Simple harmonic oscillation : ”stability boundary”

c) $\sigma_k > 0, \omega_k \neq 0 \Rightarrow$ Divergence oscillation : “unstable”

d) $\sigma_k < 0, \omega_k = 0 \Rightarrow$ Continuous convergence : “stable”

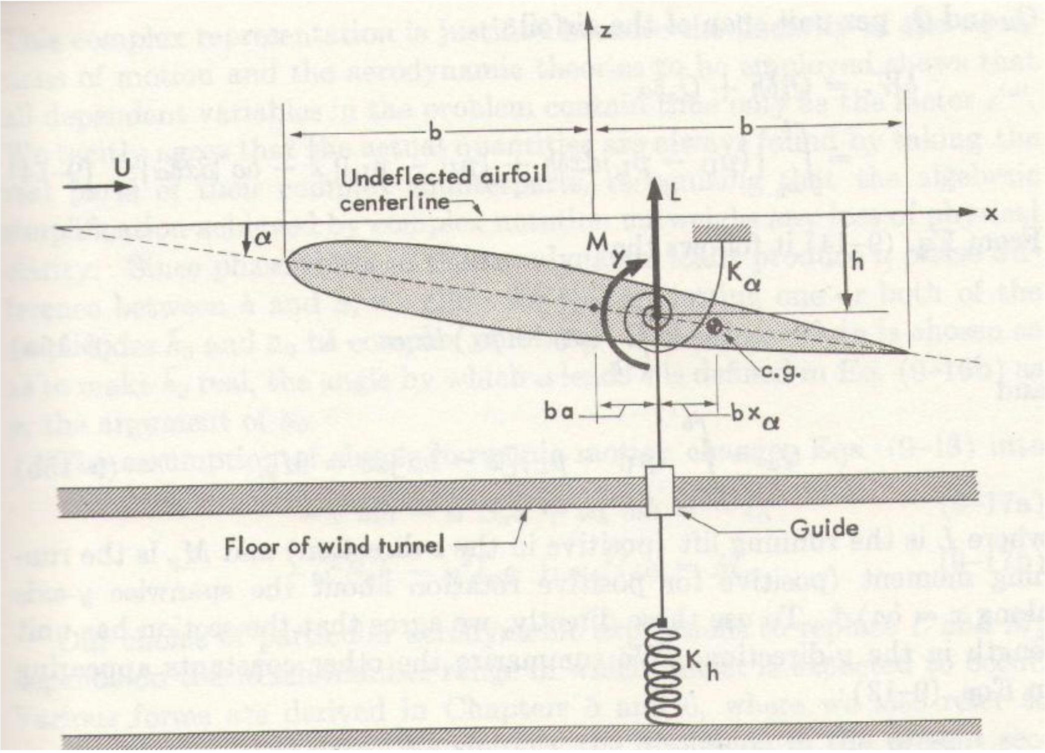
e) $\sigma_k = 0, \omega_k = 0 \Rightarrow$ Time independent solution : ”stability boundary”

f) $\sigma_k > 0, \omega_k = 0 \Rightarrow$ Continuous divergence : “unstable”

Dynamic aeroelasticity

- Flutter of a wing

Typical section with 2 D.O.F



K_α, K_h : torsional, bending stiffness

Dynamic aeroelasticity

- First step in flutter analysis

- Formulate eqns of motion
- The vertical displacement at any point along the mean aerodynamic chord from the equilibrium $z=0$ will be taken as $z_a(x,t)$

$$z_a(x,t) = -h - (x - x_{ea})\alpha$$

- The eqns of motion can be derived using Lagrange's eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$L = T - U$$

Dynamic aeroelasticity

- The total kinetic energy(T)

$$T = \frac{1}{2}\int_{-b}^{b} \rho \left(\frac{\partial z_a}{\partial t}\right)^2 dx$$

$$= \frac{1}{2}\int_{-b}^{b} \rho \left[\dot{h} + (x - x_{ea})\dot{\alpha}\right]^2 dx$$

$$= \frac{1}{2}\dot{h}^2 \underbrace{\int_{-b}^{b} \rho dx}_{\substack{m \\ \text{(airfoil mass)}}} + \dot{h}\dot{\alpha} \underbrace{\int_{-b}^{b} \rho (x - x_{ea}) dx}_{\substack{S_\alpha \\ \text{(static unbalance)}}} + \frac{1}{2}\dot{\alpha}^2 \underbrace{\int_{-b}^{b} (x - x_{ea})^2 dx}_{\substack{I_\alpha \\ \text{(mass moment of inertia about c.g.)}}}$$

*Note) if $x_{ea} = x_{cg},$ then $S_\alpha = 0$ by the definition of c.g.

Therefore,

$$T = \frac{1}{2} m\dot{h}^2 + \frac{1}{2} I\dot{\alpha}^2 + S_\alpha \dot{h}\dot{\alpha}$$

Dynamic aeroelasticity

- The total potential energy (strain energy)

$$U = \frac{1}{2} k_h h^2 + \frac{1}{2} k_\alpha \alpha^2$$

- Using Lagrange's eqns with $L = T - U$

$$q_1 = h_1, q_2 = \alpha$$

$$\Rightarrow \begin{cases} m\ddot{h} + S_\alpha \ddot{\alpha} + k_h h = Q_h \\ S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + k_\alpha \alpha = Q_\alpha \end{cases}$$

Where Q_h, Q_α are generalized forces associated with two d.o.f's h, α respectively.

Dynamic aeroelasticity

$$Q_h = -L = -L(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

$$Q_\alpha = M_{ea} = M_{ea}(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

Governing eqn.

$$\Rightarrow \begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -L \\ M_{ea} \end{Bmatrix}$$

- For approximation, let's use quasi-steady aerodynamics

$$L = qSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty})$$

$$M_{ac} = qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

$$M_{ea} = (x_{ea} - x_{ac}) \bullet L + M_{ac} = eqSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty}) + qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

Dynamic aeroelasticity

*Note) Three basic classifications of unsteadiness (linearized potential flow)

i) Quasi-steady aero: only circulatory terms due to the bound vorticity. Used for characteristic freq. below $2Hz$ (e.g., conventional dynamic stability analysis)

ii) Quasi-unsteady aero: includes circulatory terms from both bound and wake vorticities. Satisfactory results for $2Hz < \omega_\alpha, \omega_h < 10Hz$. Theodorsen is one that falls into here. (without apparent mass terms)

iii) Unsteady aero: "quasi-unsteady"+"apparent mass terms" (non-circulatory terms, inertial reactions: $\dot{\alpha}$, $\ddot{h}$)
For $\omega > 10Hz$, for conventional aircraft at subsonic speed.

Dynamic aeroelasticity

Then, aeroelastic systems of equations becomes

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSeC_{L_\alpha}}{U_\infty} & -qS_c C_{m_{\dot{\alpha}}} \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- For stability, we can obtain characteristic eqn. of the system and analyze the roots.

neglect damping matrix for first,

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Much insight can be obtained by looking at the undamped system (Dowell, pp. 83)

Dynamic aeroelasticity

Set $\alpha = \bar{\alpha} e^{pt}, h = \bar{h} e^{pt}$

$$\Rightarrow \begin{bmatrix} (mp^2 + K_h) & (S_\alpha p^2 + qSC_{L\alpha}) \\ S_\alpha p^2 & (I_\alpha p^2 + K_\alpha - qSeC_{L_\alpha}) \end{bmatrix} \begin{Bmatrix} \bar{h} \\ \bar{\alpha} \end{Bmatrix} e^{pt} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For non-trivial solution,

Characteristic eqn., $\det(\Delta) = 0$

$$\underbrace{(mI_\alpha - S_\alpha)}_{A} p^4 + \underbrace{[K_h I_\alpha + (K_\alpha - qSeC_{L_\alpha})m - qSC_{L\alpha} S_\alpha]}_{B} p^2 + \underbrace{K_h (K_\alpha - qSeC_{L_\alpha})}_{C} = 0$$

$$\therefore p^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Dynamic aeroelasticity

The signs of *A*, *B*, *C* determine the nature of the solution.

$A > 0, C > 0$ (if $q < q_D$)

B Either (+) or (-)

$B = mK_\alpha + K_h I_\alpha - [me + S_\alpha] qSC_{L_\alpha}$

- If $[me + S_\alpha] < 0, B > 0$ for all q

- Otherwise $B < 0$ when

$$\frac{K_\alpha}{e} + \frac{K_h I_\alpha}{me} - \left[1 + \frac{S_\alpha}{me}\right] qSeC_{L_\alpha} < 0$$

Dynamic aeroelasticity

- Two possibilities for *B* (*B*>0 and *B*<0)

i) *B*>0:

① $B^2 - 4AC > 0, P^2$ are real, negative, so P is pure imaginary → neutrally stable

② $B^2 - 4AC < 0, P^2$ is complex, at least one value should have a positive real part → unstable

③ $B^2 - 4AC = 0$ → stability boundary

- Calculation of q_F

$Dq_F^2 + Eq_F + F = 0$ ←(from $B^2 - 4AC = 0,$ stability boundary)

$$q_F = \frac{-E \pm \sqrt{E^2 - 4DF}}{2D}$$

Dynamic aeroelasticity

where,

$$D \equiv \left\{ \left[me + S_\alpha \right] SC_{L_\alpha} \right\}^2$$

$$E \equiv \left\{ -2\left[me + S_\alpha \right]\left[mK_\alpha + K_h I_\alpha \right] + 4\left[mI_\alpha - S_\alpha^2 \right] eK_h \right\} SC_{L_\alpha}$$

$$F \equiv \left[mK_\alpha + K_h I_\alpha \right]^2 - 4\left[mI_\alpha - S_\alpha^2 \right] K_h K_\alpha$$

① At least, one of the q_F must be real and positive in order for flutter to occur.

② If both are, the smaller is the more critical.

③ If neither are, flutter does not occur.

④ If $S_\alpha \le 0$ (c.g. is ahead of e.a), no flutter occurs(mass balanced)

Dynamic aeroelasticity

ii) B<0: B will become (-) only for large q

$B^2 - 4AC = 0$ will occur before B=0 since A>0,C>0

$\therefore$ To determine q_F, only B>0 need to be calculated.

Examine p as q increases

Low $q \rightarrow p = \pm i\omega_1, \pm i\omega_2 (B^2 - 4AC > 0)$

Higher $q \rightarrow p = \pm i\omega_1, \pm i\omega_2 (B^2 - 4AC = 0) \rightarrow$ stability boundary

More higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2 (B^2 - 4AC < 0) \rightarrow$

dynamic instability

Even more higher $q \rightarrow p = 0, \pm i\omega_1 (C = 0) \rightarrow$ stability boundary

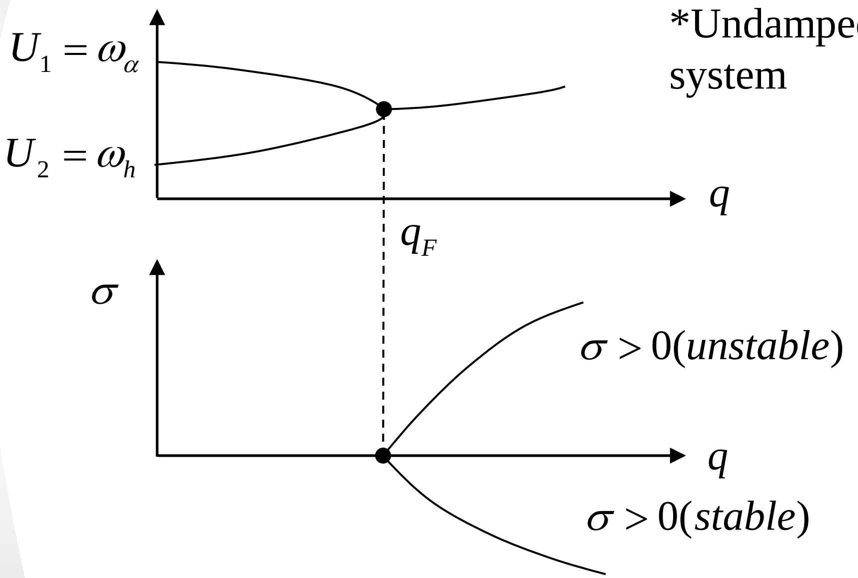
$\therefore$ Flutter condition: $B^2 - 4AC = 0$

Torsional divergence: $C = 0$

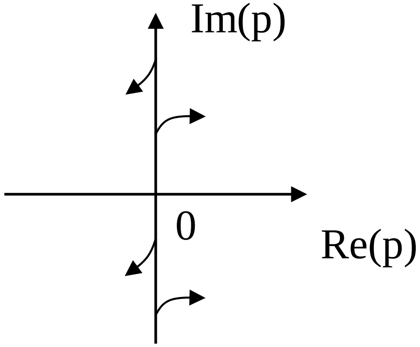
Dynamic aeroelasticity

Graphically,

$$\omega_\alpha^2 = \frac{K_\alpha}{I_\alpha}, \omega_h^2 = \frac{K_h}{m}$$



*Undamped system



- Effect of static unbalance

In Dowell's book, after Pines[1958]

$S_\alpha \le 0 \rightarrow$ avoid flutter, if $S_\alpha = 0, \dfrac{q_F}{q_D} = 1 - \dfrac{\omega_h^2}{\omega_\alpha^2}$

Dynamic aeroelasticity

If $q_D < 0 (e < 0)$ $\dfrac{\omega_h}{\omega_\alpha} < 1.0 \Rightarrow q_F < 0$ no flutter

If $q_D > 0$ and $\dfrac{\omega_h}{\omega_\alpha} > 1.0 \Rightarrow$ no flutter

- Inclusion of damping$\rightarrow$ "can be a negative damping"

for better accuracy,

$m\ddot{q} + c\dot{q} + Kq = 0,$ where

$$c = \begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSC_{L_\alpha}}{U_\infty} & -qScC_{m_{\dot{\alpha}}} \end{bmatrix}$$

The characteristic equation is now in the form of

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0$$

Dynamic aeroelasticity

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0 \cdots *$$

- Routh criteria for stability

; At critical position, the system real part becomes zero, damping becomes zero.

Substitute $p = i\omega$ into (*), we get,

$$\begin{cases} A_4\omega^4 - A_2\omega^2 + A_0 = 0 \\ i(-A_3\omega^3 + A_1\omega) = 0 \end{cases}$$

From the second eqn, $\omega^2 = \frac{A_1}{A_3}$, substitute into first equation, then,

$$A_4\left(\frac{A_1}{A_3}\right)^2 - A_2\left(\frac{A_1}{A_3}\right) + A_0 = 0 \quad \text{or} \quad A_4A_1^2 - A_1A_2A_3 + A_0A_3^2 = 0$$

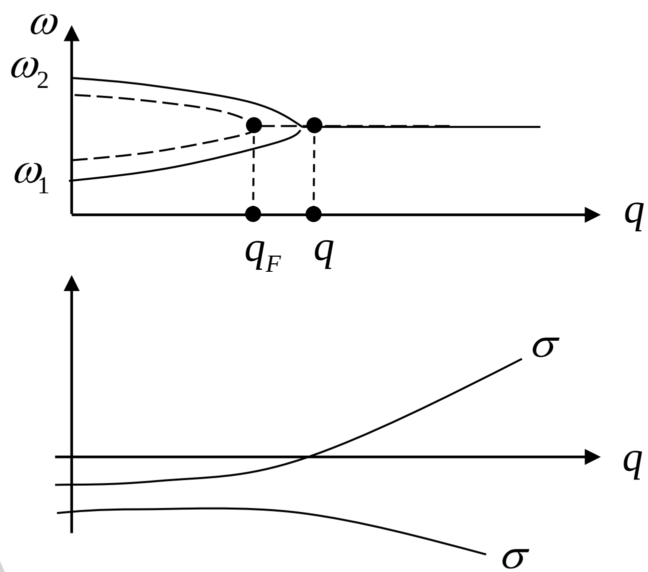
Dynamic aeroelasticity

And, we can examine p as q increases,

Low $q \rightarrow p = -\sigma_1 \pm i\omega_1, -\sigma_2 \pm i\omega_2 \rightarrow$ damped natural freq.

Higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm i\omega_2$

More higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm\sigma_2 \pm i\omega_2 \rightarrow$ dynamic instability.



- Static instability $\cdots |\kappa| = 0$
- Dynamic instability
  a) frequency coalescence (unsymmetric κ)
  b) Negative damping ($c_{ij} < 0$)
  c) Unsymmetric damping (gyroscopic)