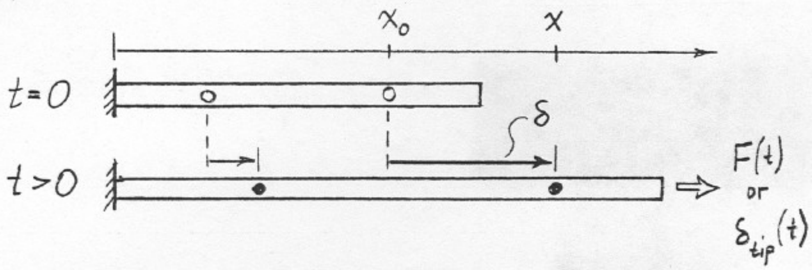
FIELD DESCRIPTIONS

Example: Elastic rod in extension (1-D "Flow")



Displacement field δ can be given via

a) original location: $\bar{\delta}(t, x_0)$ Lagrangian Description, material-based

OR b) current location: $\delta(t, x)$ Eulerian Description, spatially-based

Velocity of particle

a) $\bar{u}(t, x_0) = \left.\dfrac{\partial \bar{\delta}}{\partial t}\right|_{\text{fixed } x_0}$ since x_0 never changes for the particle

b) $u(t,x) = \lim\limits_{\substack{\Delta t \to 0 \\ \Delta x \to u \Delta t}} \dfrac{\delta(t+\Delta t, x+\Delta x) - \delta(t,x)}{\Delta t} = \dfrac{\partial \delta}{\partial t} + u \dfrac{\partial \delta}{\partial x}$ Not useful for getting u.

Better to define $\delta(t,x)$ in terms of $u(t,x)$

Acceleration of particle

a) $\bar{a}(t, x_0) = \left.\dfrac{\partial \bar{u}}{\partial t}\right|_{x_0} = \left.\dfrac{\partial^2 \bar{\delta}}{\partial t^2}\right|_{x_0}$

b) $a(t,x) = \lim\limits_{\substack{\Delta t \to 0 \\ \Delta x \to u \Delta t}} \dfrac{u(t+\Delta t, x+\Delta x) - u(t,x)}{\Delta t} = \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} \equiv \dfrac{Du}{Dt}$

Rate of change of any quantity Q in the particle's view is $\dfrac{DQ}{Dt}$ or $\left.\dfrac{\partial \bar{Q}}{\partial t}\right|_{x_0}$

$\dfrac{DQ}{Dt} \equiv \dfrac{\partial Q}{\partial t} + u \dfrac{\partial Q}{\partial t}$, has same numerical value in any inertial t, x system

In 3D: $\dfrac{D}{Dt} Q(t,x,y,z) = \dfrac{\partial Q}{\partial t} + \vec{q} \cdot \nabla Q$; $\vec{q} = \hat{\imath} u + \hat{\jmath} v + \hat{k} w$ (velocity)

GOVERNING EQUATIONS (Eulerian Description), Inviscid Flow

Flow quantities:
$v(x,y,z,t)$ volume/mass $= \dfrac{1}{\text{density}} = \dfrac{1}{\rho}$

$\vec{q}(x,y,z,t)$ momentum/mass = velocity

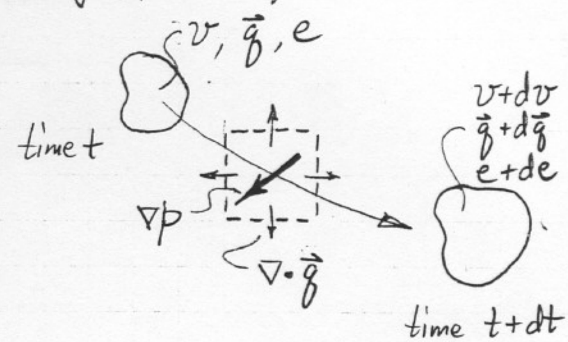
$e(x,y,z,t)$ internal-energy/mass $= \int c_v\,dT$

Rates of change seen by "blob" of given identity (fixed mass):

$$\frac{Dv}{Dt} = v\,\nabla\cdot\vec{q} \quad \text{or} \quad \boxed{\frac{D\rho}{Dt} = -\rho\,\nabla\cdot\vec{q} \quad (*)}$$

$$\frac{D\vec{q}}{Dt} = -v\,\nabla p \quad \text{or} \quad \boxed{\rho\frac{D\vec{q}}{Dt} = -\nabla p \quad (**)}$$

$$\frac{De}{Dt} = -p\frac{Dv}{Dt} \quad \text{or} \quad \boxed{\rho\frac{De}{Dt} = -p\,\nabla\cdot\vec{q} \quad (***)}$$



state equation for pressure: $p = p(\rho, e)$

Combine $\frac{1}{\rho}(***) - \frac{p}{\rho^2}(*) \rightarrow \dfrac{De}{Dt} - p\dfrac{D(1/\rho)}{Dt} = 0$

Entropy change for blob is $T\,ds \equiv de - p\,d\left(\frac{1}{\rho}\right)$ (definition for ds)

so $\dfrac{Ds}{Dt} = 0$ entropy is constant for a given blob in inviscid flow.

For ideal gas: $p = (\gamma-1)\rho e \rightarrow (\gamma-1)\,de = \left(\frac{1}{\rho}\right)dp + p\,d\left(\frac{1}{\rho}\right)$

but for isentropic flow $de = p\,d\left(\frac{1}{\gamma}\right) \rightarrow \dfrac{dp}{p} = \gamma\dfrac{d\rho}{\rho} \rightarrow \boxed{p = \text{¢}\,\rho^{\gamma}}$ (for given blob)

Take $\nabla\times(**) \rightarrow \dfrac{D}{Dt}\left(\dfrac{\vec{\omega}}{\rho}\right) = \dfrac{1}{\rho}\vec{\omega}\cdot\nabla\vec{q} + \dfrac{1}{\rho^2}\nabla\rho\times\nabla p$ → 0 for isentropic flow since $\nabla p \times \nabla\rho \sim \nabla\rho\times\nabla\rho = 0$

If $\vec{\omega} = 0$ initially, $\dfrac{D}{Dt}\left(\dfrac{\vec{\omega}}{\rho}\right) = 0$ henceforth.

Irrotational flow stays irrotational. In 2-D, $\vec{\omega}\cdot\nabla\vec{q} = 0$ always, so $\dfrac{D}{Dt}\left(\dfrac{\vec{\omega}}{\rho}\right) \equiv 0$

POTENTIAL FLOW RELATIONS

Assumes x,y,z,t is an inertial frame

Assuming $\vec{\omega} \equiv \nabla\times\vec{q} = 0$, can write $\vec{q} = \nabla\phi$ $\quad \phi(x,y,z,t)$ = velocity potential

Momentum Eqn: $\frac{\partial}{\partial t}(\nabla\phi) + \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi) = \frac{-\nabla p}{\rho}$, uses identity $\vec{q}\cdot\nabla\vec{q} = \frac{1}{2}\nabla(\vec{q}\cdot\vec{q}) - \vec{q}\times\overbrace{(\nabla\times\vec{q})}^{0}$

or $\boxed{\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2\right] + \frac{\nabla p}{\rho} = 0}$ (*) , $q \equiv |\vec{q}| = |\nabla\phi|$

Incompressible flow: $\rho = \text{const.} \rightarrow \boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \frac{p}{\rho} = C = \frac{1}{2}q_{ref}^2 + \frac{p_{ref}}{\rho}}$ (typ. freestream)

Isentropic flow: $\rho = \rho_{ref}\left(\frac{p}{p_{ref}}\right)^{1/\gamma} \rightarrow \frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \frac{\gamma}{\gamma-1}\frac{p_{ref}}{\rho_{ref}}\left(\frac{p}{p_{ref}}\right)^{\frac{\gamma-1}{\gamma}} = C$ (**)

(uniformly isentropic)

or $\boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + \frac{1}{\gamma-1}a_{ref}^2\left(\frac{p}{p_{ref}}\right)^{\frac{\gamma-1}{\gamma}} = \frac{a_{ref}^2}{\gamma-1}\left[1 + \frac{\gamma-1}{2}M_{ref}^2\right]}$

Take $\frac{\partial}{\partial t}$(**) $\rightarrow \frac{\partial^2\phi}{\partial t^2} + \vec{q}\cdot\frac{\partial\vec{q}}{\partial t} + \frac{1}{\rho_{ref}}\left(\frac{p}{p_{ref}}\right)^{-1/\gamma}\frac{\partial p}{\partial t} = 0$

or $\frac{\partial p}{\partial t} = -\rho\left[\frac{\partial^2\phi}{\partial t^2} + \vec{q}\cdot\frac{\partial\vec{q}}{\partial t}\right]$

$\rightarrow \frac{\partial\rho}{\partial t} = -\frac{\rho}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \vec{q}\cdot\frac{\partial\vec{q}}{\partial t}\right]$

From (*) $\rightarrow \nabla\rho = -\frac{\rho}{a^2}\left[\frac{\partial\vec{q}}{\partial t} + \frac{1}{2}\nabla(q^2)\right]$

since $p\sim\rho^\gamma$, then $\frac{1}{p}\frac{\partial p}{\partial t} = \frac{\gamma}{\rho}\frac{\partial\rho}{\partial t}$, $\frac{1}{p}\nabla p = \frac{\gamma}{\rho}\nabla\rho$

Require continuity: $\frac{\partial\rho}{\partial t} + \vec{q}\cdot\nabla\rho + \rho\nabla\cdot\vec{q} = 0$

or $-\frac{\rho}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(q^2) + \vec{q}\cdot\nabla\left(\frac{1}{2}q^2\right)\right] + \rho\nabla\cdot\vec{q} = 0$, $\nabla\cdot\vec{q} = \nabla^2\phi$

Summary: Equations governing $\phi(x,y,z,t)$ and related quantities...

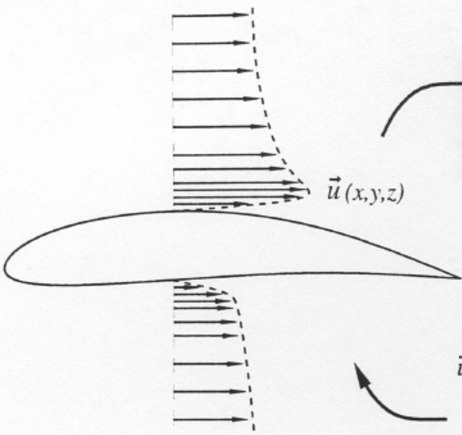
Incompressible Flow.	$\nabla^2\phi = 0$
ρ = const.	$p = p_{ref} + \frac{1}{2}\rho q_{ref}^2 - \rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2\right]$, $q^2 = \nabla\phi\cdot\nabla\phi$

Isentropic Ideal-Gas Flow:	$\nabla^2\phi = \frac{1}{a^2}\left[\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(q^2) + \nabla\phi\cdot\nabla\left(\frac{1}{2}q^2\right)\right]$; $a^2 = \frac{\gamma p}{\rho}$
$\frac{\rho}{\rho_{ref}} = \left(\frac{p}{p_{ref}}\right)^{1/\gamma}$	$\frac{p}{p_{ref}} = \left\{1 + \frac{\gamma-1}{2}M_{ref}^2\left[1 - \frac{2}{q_{ref}^2}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2\right)\right]\right\}^{\frac{\gamma}{\gamma-1}}$

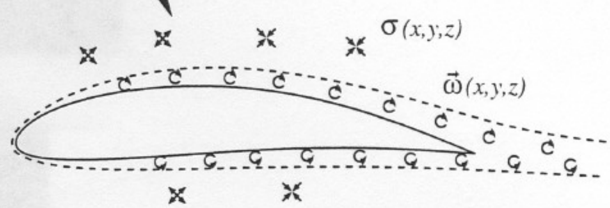
Velocity / Vorticity, Source Duality

Alternative flowfield representations

Velocity $\vec{u}$ | Vorticity, Source $\vec{\omega}$, σ



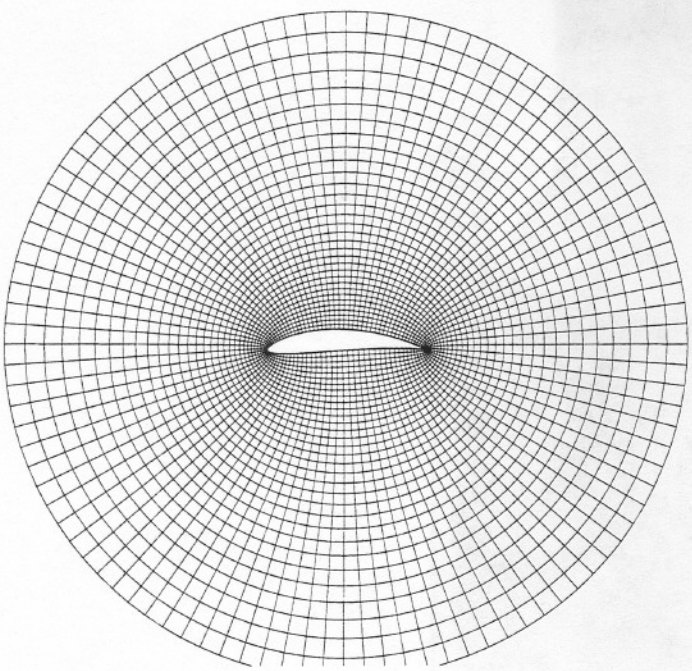
$$\sigma = \nabla \cdot \vec{u}$$
$$\vec{\omega} = \nabla \times \vec{u}$$

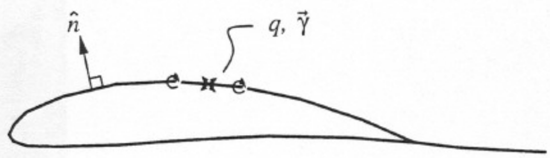


$$\vec{u} = \frac{1}{4\pi}\iiint \frac{\sigma\,\vec{r}}{r^3}\, dx'\, dy'\, dz' + \frac{1}{4\pi}\iiint \frac{\vec{\omega}\times\vec{r}}{r^3}\, dx'\, dy'\, dz' + \nabla\phi_c$$

Alternative calculation approaches

Grid method | Panel method





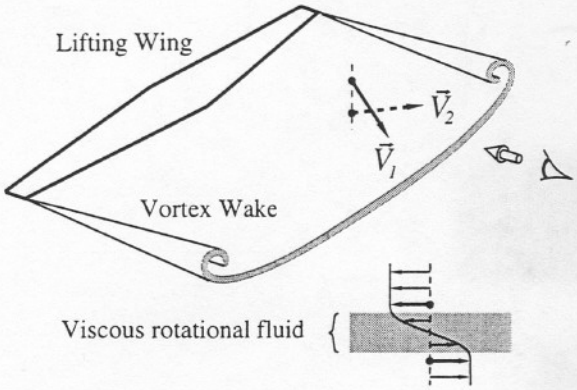
$$q \equiv \int \sigma\, dn$$
$$\vec{\gamma} \equiv \int \vec{\omega}\, dn$$

Velocity $\vec{u}$ or potential ϕ defined at *grid nodes.*

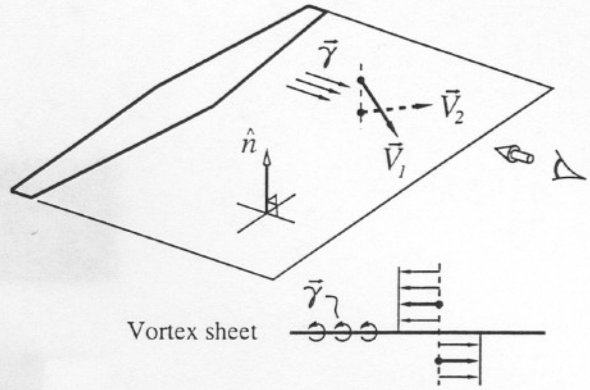
Sheet strengths q, $\vec{\gamma}$ defined on *surface panels.*

Modeling with Vortex and Source Sheets

Reality



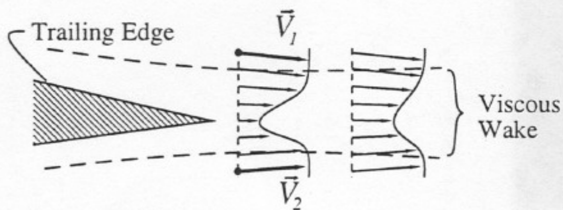
Vortex Sheet Model



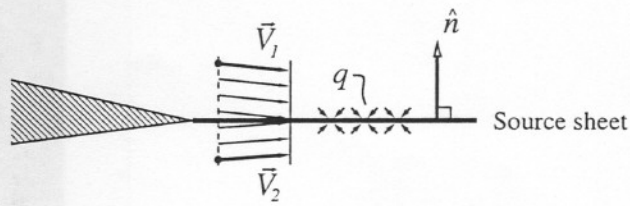
$$\boxed{\text{Vortex sheet strength} \quad \vec{\gamma} \;=\; \hat{n} \times \left(\vec{V}_1 - \vec{V}_2\right)}$$

Note: $|\vec{\gamma}| \;=\; \Delta V_{\text{tangential}}$ across sheet.

Reality



Source Sheet Model



$$\boxed{\text{Source sheet strength} \quad q \;=\; \hat{n} \cdot \left(\vec{V}_1 - \vec{V}_2\right)}$$

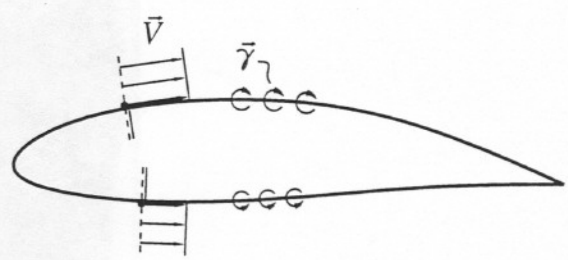
Note: $q \;=\; \Delta V_{\text{normal}}$ across sheet.

"Non-Physical" Uses

Vortex sheet represents solid surface.

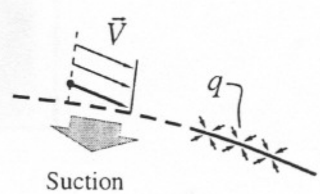
$$\vec{\gamma} \;=\; \hat{n} \times \vec{V}_{\text{surface}}$$

in 2D: $\gamma \;=\; V_{\text{tangential}}$



Source sheet represents wall suction.

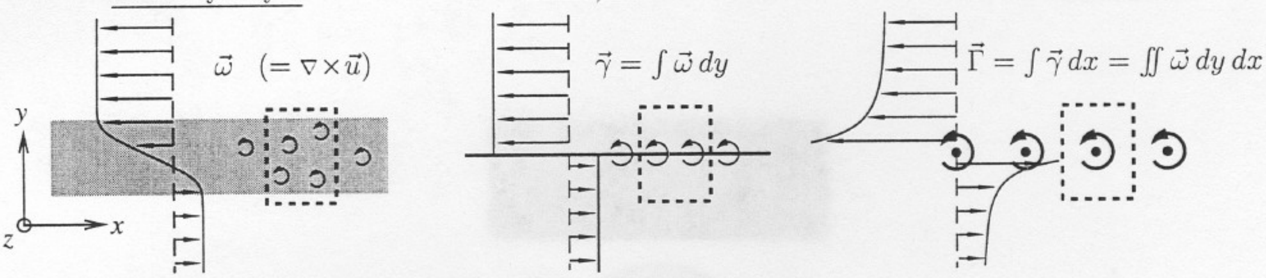
$$q \;=\; \hat{n} \cdot \vec{V}_{\text{surface}} \;=\; V_{\text{normal}}$$



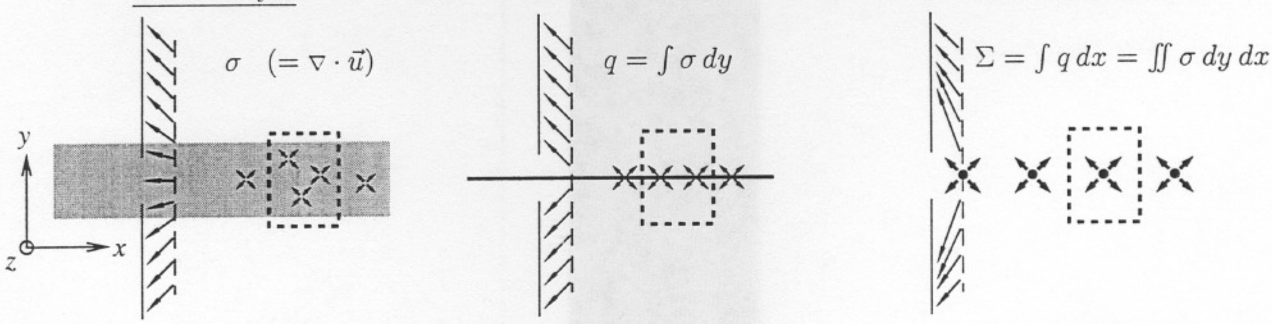
Vorticity $\vec{\omega}$ and Source density σ can be lumped from ...

$\boxed{\text{Volume}}$ to $\boxed{\text{Surface}}$ to $\boxed{\text{Line}}$

Vorticity Layer $\longrightarrow$ Vortex sheet $\longrightarrow$ Vortex Filaments

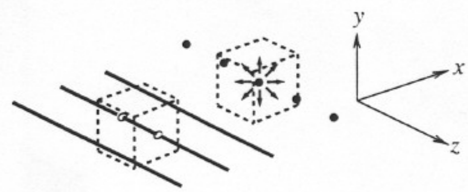


Source Layer $\longrightarrow$ Source sheet $\longrightarrow$ Source Filaments



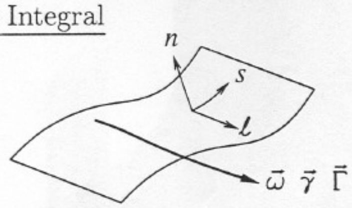
$\longrightarrow$ Point Sources

$\Sigma_p = \int \Sigma\, dz = \iiint \sigma\, dx\, dy\, dz$



Lumped forms of Biot-Savart Integral

$$\vec{u} \;=\; \vec{u}_v \,+\, \vec{u}_s \,+\, \nabla\phi_c$$



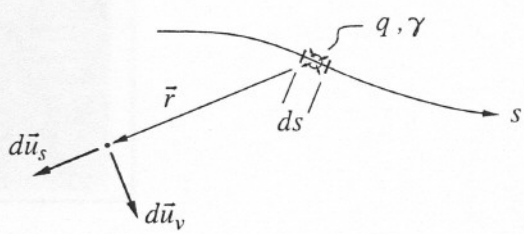
$$\vec{u}_v \;=\; \frac{1}{4\pi}\iiint \frac{\vec{\omega}\times\vec{r}}{r^3}dx\,dy\,dz \;=\; \frac{1}{4\pi}\iint \frac{\vec{\gamma}\times\vec{r}}{r^3}ds\,d\ell \;=\; \frac{1}{4\pi}\int \frac{\vec{\Gamma}\times\vec{r}}{r^3}d\ell \;=\; \frac{\Gamma}{4\pi}\int \frac{d\vec{\ell}\times\vec{r}}{r^3} \quad , \text{ since } \vec{\Gamma} \parallel \vec{\ell}$$

$$\vec{u}_s \;=\; \frac{1}{4\pi}\iiint \frac{\sigma\,\vec{r}}{r^3}\,dx\,dy\,dz \;=\; \frac{1}{4\pi}\iint \frac{q\,\vec{r}}{r^3}\,ds\,d\ell \;=\; \frac{1}{4\pi}\int \frac{\Sigma\,\vec{r}}{r^3}\,d\ell \;=\; \frac{\Sigma_p}{4\pi r^2}\,\hat{r} \qquad , \quad \hat{r} \;=\; \frac{\vec{r}}{r}$$

In 2-D:

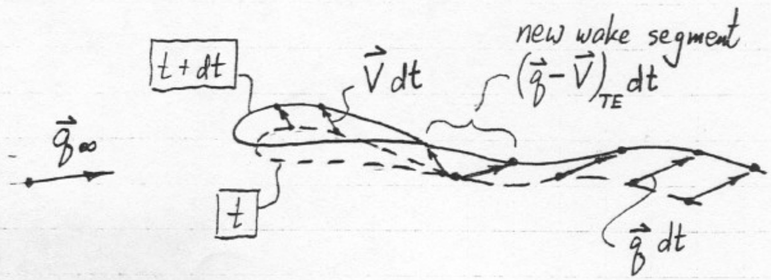
$$\vec{u}_v \;=\; \frac{1}{2\pi}\int \hat{k}\times\hat{r}\,\frac{\gamma}{r}\,ds \;=\; \hat{k}\times\hat{r}\,\frac{\Gamma}{2\pi r}$$

$$\vec{u}_s \;=\; \frac{1}{2\pi}\int \hat{r}\,\frac{q}{r}\,ds \;=\; \hat{r}\,\frac{\Sigma}{2\pi r}$$



UNSTEADY PANEL METHODS

Flowfield Representation



$\vec{q}(x,y,z,t)$ flow velocity

$\vec{V}(t)$ airfoil velocity (prescribed)

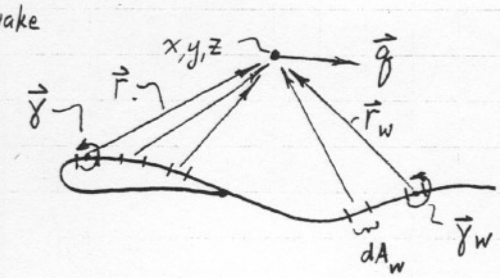
<u>Airfoil model:</u> "bound" vortex + source sheet at given location (Eulerian) } hybrid

<u>Wake model:</u> "free" vortex + source sheet convecting freely (Lagrangian) } description

snapshot: $$\vec{q}(x,y,z,t) = \vec{q}_\infty + \frac{1}{4\pi}\iint\limits_{\text{airfoil}} \frac{\vec{\gamma}\times\vec{r}}{r^3}\,dA + \frac{1}{4\pi}\iint\limits_{\text{wake}} \frac{\vec{\gamma}_w\times\vec{r}_w}{r_w^3}\,dA_w + \iint \text{sources}$$

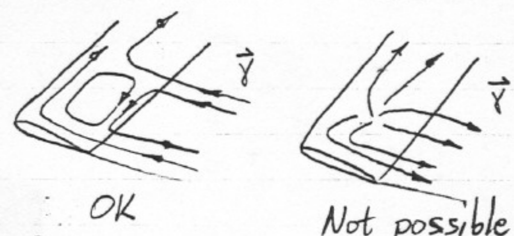
$\vec{\gamma}(t;A)$ unknown, at given location on airfoil

$\vec{\gamma}_w(A_w)$ known, at known location on wake



In 2-D, $\vec{\gamma} = \hat{k}\gamma$. γ is a scalar

In 3-D, $\vec{\gamma}$ must satisfy $\nabla_A \cdot \vec{\gamma} = 0$



Preferable to use <u>doublet sheet:</u> $\vec{\mu} = \hat{n}\mu$

Equivalence: $\vec{\gamma} = \nabla_A \mu$

$$\vec{\gamma} = \hat{n}\times\Delta\vec{V} \qquad \mu = \Delta\phi$$

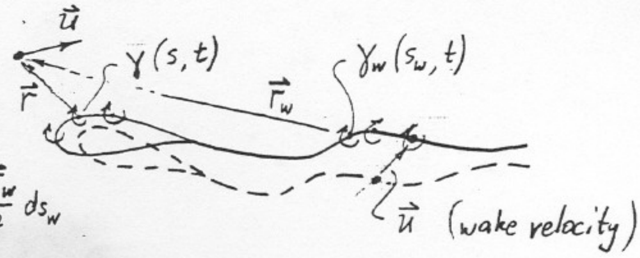
Note: $\nabla_A\cdot\vec{\gamma} = \nabla_A^2\mu = \Delta(\nabla_A^2\phi) = 0$

$$\phi(x,y,z,t) = \frac{1}{4\pi}\iint\limits_{\text{Airfoil}} \mu\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dA + \frac{1}{4\pi}\iint\limits_{\text{Wake}} \mu_w\,\frac{\partial}{\partial n}\left(\frac{1}{r_w}\right)dA + \iint \text{sources}$$

Unsteady Airfoil Flow — Approximation Levels

Airfoil BC in all cases: $\vec{u}\cdot\hat{n} = \vec{V}\cdot\hat{n}$

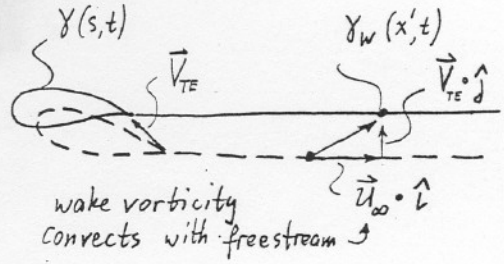
$\vec{u}$ = flow velocity, $\vec{V}$ = airfoil velocity



Exact, nonlinear ($\vec{r}_w$ depends on $\vec{u}$ history)

$$\vec{u}(x,y,t) = \vec{u}_\infty + \frac{1}{2\pi}\oint -\gamma\,\frac{\hat{k}\times\vec{r}}{r^2}\,ds + \frac{1}{2\pi}\int -\gamma_w\,\frac{\hat{k}\times\vec{r}_w}{r_w^2}\,ds_w$$

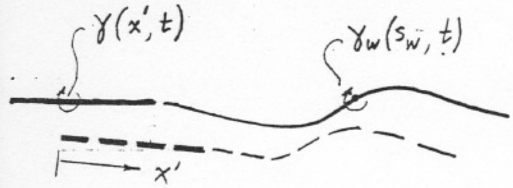
Prescribed-wake, (linear, $\vec{r}_w$ given)



$$\vec{u}(x,y,t) = \text{same as above, } r_w \text{ prescribed}$$

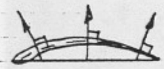
$$\gamma_w(x,t) = \gamma_{TE}\left(t - x/u_\infty\right)$$

Thin-airfoil approximation (nonlinear)

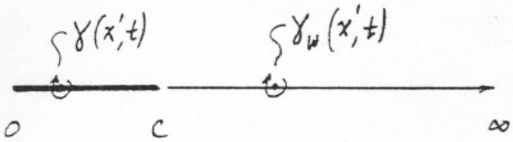


$$\vec{u}(x,y,t) = \vec{u}_\infty + \frac{1}{2\pi}\int_0^c -\gamma\,\frac{\hat{k}\times\vec{r}}{r^2}\,dx' + \frac{1}{2\pi}\int -\gamma_w\,\frac{\hat{k}\times\vec{r}}{r^2}\,ds$$

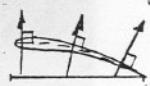
$\hat{n}$ defined via camber line



Thin-airfoil approximation (linear)



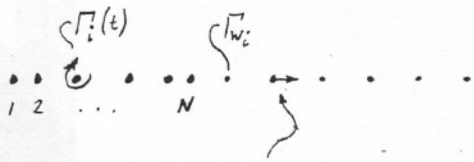
$$\vec{u}(x,y,t) = \vec{u}_\infty + \frac{1}{2\pi}\int_0^c -\gamma\,\frac{\hat{k}\times\vec{r}}{r^2}\,dx' + \int_c^\infty -\gamma_w\,\frac{\hat{k}\times\vec{r}}{r^2}\,dx'$$



$\hat{n}$ defined via camber line, α

on airfoil: $\vec{r} = (x-x')\hat{i} + 0\hat{j}$

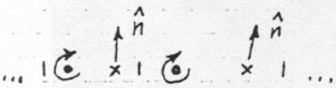
Lumped-vorticity (Vortex Lattice)



$$\vec{u}(x,y,t) = \vec{u}_\infty + \frac{1}{2\pi}\sum_{i=1}^{N} -\Gamma\,\frac{\hat{k}\times\vec{r}}{r^2} + \frac{1}{2\pi}\sum_{i=1}^{N_w} -\Gamma_{w_i}\,\frac{\hat{k}\times\vec{r}_w}{r_w^2}$$

Vortex placed at $\frac{1}{4}$ point

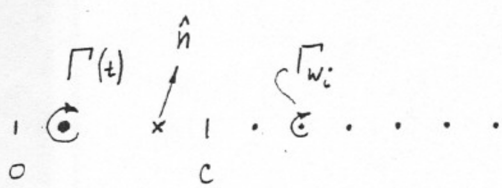
$\vec{u}\cdot\hat{n}$ imposed at $\frac{3}{4}$ point



wake vortices convect with freestream

$$\vec{r}_w = \left[x - (x_{TE} + u_\infty t)\right]\hat{i}$$

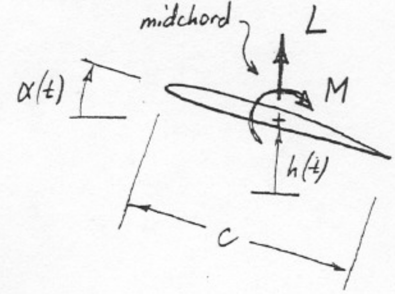
Single-vortex VL



$$\vec{u}(x,y,t) = \vec{u}_\infty + \frac{-\Gamma}{2\pi}\,\frac{\hat{k}\times\vec{r}}{r^2} + \frac{1}{2\pi}\sum_{i=1}^{N_w} -\Gamma_{w_i}\,\frac{\hat{k}\times\vec{r}_w}{r_w^2}$$

$$\hat{n} = (\alpha - \alpha_0)\hat{i} + \hat{j}\quad,\ \alpha_0 = \text{zero-lift } \alpha$$

TYPICAL - SECTION FLUTTER ANALYSIS



MD

Quasi-Steady Aero:

$$L_Q = \frac{1}{2}\rho V^2 c\, 2\pi\left(\alpha - \frac{\dot h}{V} + \frac{c}{4}\frac{\dot\alpha}{V}\right)$$

$$M_Q = \frac{c}{4} L_Q$$

Apparent - Mass Aero:

$$L_A = \rho\pi\left(\frac{c}{2}\right)^2\left(V\dot\alpha - \ddot h\right) = \frac{1}{2}\rho V^2\frac{c}{2}\cdot 2\pi\left(\frac{c}{2}\frac{\dot\alpha}{V} - \frac{c}{2}\frac{\ddot h}{V^2}\right)$$

$$M_A = \rho\pi\left(\frac{c}{2}\right)^3\left(-\frac{V}{2}\dot\alpha - \frac{c}{16}\ddot\alpha\right) = \frac{1}{2}\rho V^2\left(\frac{c}{2}\right)^2 2\pi\left(-\frac{c}{4}\frac{\dot\alpha}{V} - \frac{1}{8}\left(\frac{c}{2}\right)^2\frac{\ddot\alpha}{V^2}\right)$$

Assumed harmonic motion: $\alpha(t) = \bar\alpha e^{i\omega t}$ $\quad h(t) = \bar h e^{i\omega t}$

Total Aero Forces:

$$L = L_Q C(k) + L_A = \frac{1}{2}\rho V^2\frac{c}{2}\cdot 2\pi\underbrace{\left\{\left[\left(2 + i\frac{\omega c}{2V}\right)C(k) + i\frac{\omega c}{2V}\right]\bar\alpha + \left[-\frac{4}{c}i\frac{\omega c}{2V}C(k) + \frac{2}{c}\left(\frac{\omega c}{2V}\right)^2\right]\bar h\right\}}_{e^{i\omega t}}$$

$$= \frac{1}{2}\rho V^2\frac{c}{2}\, 2\pi\left\{\left[(2+ik)C(k) + ik\right]\bar\alpha + \left[-2ik\,C(k) + k^2\right]\frac{\bar h}{c/2}\right\}e^{i\omega t}$$

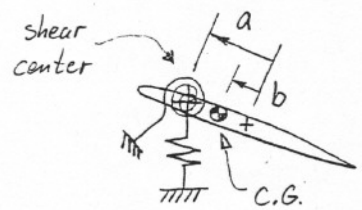
$$M = M_Q C(k) + M_A = \frac{1}{2}\rho V^2\left(\frac{c}{2}\right)^2 2\pi\left\{\left[\left(1 + \frac{i}{2}\frac{\omega c}{2V}\right)C(k) - \frac{i}{2}\frac{\omega c}{2V} + \frac{1}{8}\left(\frac{\omega c}{2V}\right)^2\right]\bar\alpha + \left[-\frac{2}{c}i\frac{\omega c}{2V}C(k)\right]\bar h\right\}e^{i\omega t}$$

$$= \frac{1}{2}\rho V^2\left(\frac{c}{2}\right)^2 2\pi\left\{\left[\left(1 + i\frac{k}{2}\right)C(k) - i\frac{k}{2} + \frac{1}{8}k^2\right]\bar\alpha + \left[-ik\,C(k)\right]\frac{\bar h}{c/2}\right\}e^{i\omega t}$$

curve fit: $C(k) \simeq \left[0.75 + 0.25\cos f(k)\right] - i\left[0.20\sin f(k)\right]$ Theodorsen function (aero-lags)

where $k \equiv \frac{\omega c}{2V}$ reduced frequency, $f(k) = \pi\left(1 - e^{-4k/\pi}\right)^{1/2}$

Structural + Inertial Response:



$$\Sigma F: \quad m(\ddot h + b\ddot\alpha) + K_h(h + a\alpha) - L = 0$$

$$\Sigma M_{c/2}: \quad I\ddot\alpha + mb\ddot h + K_h(h + a\alpha)a + K_\alpha\alpha - M = 0$$

$$\left[m - \frac{K_h}{\omega^2}\right]\frac{\bar h}{c/2} + \left[m\frac{2b}{c} - \frac{K_h}{\omega^2}\frac{2a}{c}\right]\bar\alpha + \frac{1}{c/2}\frac{1}{\omega^2}\bar L = 0$$

$$\left[m\frac{2b}{c} - \frac{K_h}{\omega^2}\frac{2a}{c}\right]\frac{\bar h}{c/2} + \left[I\left(\frac{2}{c}\right)^2 - \frac{K_h}{\omega^2}\left(\frac{2a}{c}\right)^2 - \frac{K_\alpha}{\omega^2}\left(\frac{2}{c}\right)^2\right]\bar\alpha + \frac{1}{(c/2)^2}\frac{1}{\omega^2}\bar M = 0$$

m = mass

I = moment of inertia (about midchord)

K_h = spring constants

K_α = spring constants

per span

$$\omega^2_{bending} \simeq \frac{K_h}{m}$$

$$\omega^2_{torsion} \simeq \frac{K_\alpha}{I}$$

$$\frac{1}{c/2}\frac{1}{\omega^2}L = \frac{1}{k^2}\rho\left(\frac{c}{2}\right)^2\pi\left\{\quad\right\}e^{i\omega t}$$

$$\frac{1}{(c/2)^2}\frac{1}{\omega^2}M = \frac{1}{k^2}\rho\left(\frac{c}{2}\right)^2\pi\left\{\quad\right\}e^{i\omega t}$$