Integral Equation for Neutron Transport

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Reference

- George I. Bell, Samuel Glasstone, “Nuclear Reactor Theory,” Van Nostrand Reinhold Company, NY (1970).

Boltzmann Transport Equation

- An integral equation equivalent to the integro-differential Boltzmann transport equation will be derived.

- The BTE serves to precisely describe particle balance in which the rate of accumulation of particles is equal to the difference between their rates of production and removal.

$$\frac{1}{v}\frac{\partial \Phi(\mathbf{r},E,\mathbf{\Omega},t)}{\partial t}+\mathbf{\Omega}\cdot\nabla\Phi(\mathbf{r},E,\mathbf{\Omega},t)+\Sigma_t(\mathbf{r},E,\mathbf{\Omega},t)\Phi(\mathbf{r},E,\mathbf{\Omega},t)=S(\mathbf{r},E,\mathbf{\Omega},t); \quad \cdots\cdots \quad (1)$$

$$S(\mathbf{r},E,\mathbf{\Omega},t)=S_s(\mathbf{r},E,\mathbf{\Omega},t)+S_F(\mathbf{r},E,\mathbf{\Omega},t)+Q(\mathbf{r},E,\mathbf{\Omega},t),$$

$$S_s(\mathbf{r},E,\mathbf{\Omega},t)=\int_{E'}dE'\int_{4\pi}d\mathbf{\Omega}'\Sigma_s(\mathbf{r},E'\to E,\mathbf{\Omega}'\to\mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t),$$

$$S_F(\mathbf{r},E,\mathbf{\Omega},t)=\int_{4\pi}d\mathbf{\Omega}'\int_{E'}dE'\chi(\mathbf{r},E'\to E)\nu_f(E)\Sigma_f(\mathbf{r},E',\mathbf{\Omega}'\to\mathbf{\Omega},t)\Phi(\mathbf{r},E',\mathbf{\Omega}',t)$$

- In deriving the transport equation it was necessary to consider the neutron angular density in the immediate (space-time) vicinity only of the point under consideration, whereas the whole range of energies and angles had to be included in the transport equation for the angular density <u>at a particular energy and angle</u>.
- Hence, **the formulation is local, involving derivatives, in space and time, but it is extended, involving integrals, in energy and angle**.

Method of Characteristics

- By a standard procedure known as the method of characteristics, the neutron transport equation which is a linear first order partial differential-integral equation can be converted into an integral equation.

- The first two derivative terms on the LHS of Eq. (1) may be written, in a cartesian coordinate system, as

$$\left(\frac{1}{v}\frac{\partial}{\partial t}+\Omega_x\frac{\partial}{\partial x}+\Omega_y\frac{\partial}{\partial y}+\Omega_z\frac{\partial}{\partial z}\right)\Phi$$

and the upper terms can be regarded as the total derivative of Φ with respect to *s* <u>at fixed values of Ω and *E*</u>:

$$\frac{d\Phi}{ds}=\left(\frac{1}{v}\frac{\partial}{\partial t}+\Omega_x\frac{\partial}{\partial x}+\Omega_y\frac{\partial}{\partial y}+\Omega_z\frac{\partial}{\partial z}\right)\Phi \qquad \text{.........} \quad (2)$$

- Note that dΦ/d*s* can be expressed as

$$\frac{d\Phi}{ds}=\frac{\partial\Phi}{\partial t}\cdot\frac{dt}{ds}+\frac{\partial\Phi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\Phi}{\partial y}\cdot\frac{dy}{ds}+\frac{\partial\Phi}{\partial z}\cdot\frac{dz}{ds} \qquad \text{.........} \quad (3)$$

Derivation of the Integral Equation

- Then by comparing Eqs. (2) and (3), it is found that

$$\frac{d\Phi}{ds} = \left(\frac{1}{v}\frac{\partial}{\partial t} + \Omega_x \frac{\partial}{\partial x} + \Omega_y \frac{\partial}{\partial y} + \Omega_z \frac{\partial}{\partial z} \right)\Phi \qquad \frac{d\Phi}{ds} = \frac{\partial \Phi}{\partial t}\cdot\frac{dt}{ds} + \frac{\partial \Phi}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial \Phi}{\partial y}\cdot\frac{dy}{ds} + \frac{\partial \Phi}{\partial z}\cdot\frac{dz}{ds}$$

$$\frac{dt}{ds} = \frac{1}{v}, \quad \frac{dx}{ds} = \Omega_x, \quad \frac{dy}{ds} = \Omega_y, \quad \frac{dz}{ds} = \Omega_z \qquad \cdots\cdots \quad (4)$$

- For an arbitrary time t_0 and position x_0, y_0, and z_0, the solutions of Eq. (4) becomes

$$t = t_0 + \frac{s}{v},$$

$$\left.\begin{aligned} x &= x_0 + \Omega_x s, \\ y &= y_0 + \Omega_y s, \\ z &= z_0 + \Omega_z s \end{aligned}\right\} \mathbf{r} = \mathbf{r}_0 + s\mathbf{\Omega} \qquad \cdots\cdots \quad (5)$$

- From Eq. (5), it is found that s means the travel distance along the direction $\mathbf{\Omega}$ of neutron having energy E.

Derivation of the Integral Equation (Contd.)

- Hence, the transport equation (1) can be written as

$$\frac{d}{ds}\Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) + \Sigma_t \Phi = S(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \quad \cdots\cdots (6)$$

- The <u>$\mathbf{r}(s)$</u> and <u>$t(s)$</u> curves are called the characteristic curves of the differential equation, and for every $\mathbf{r}_0$ and t_0 at fixed values of $\mathbf{\Omega}$ and E, there is one curve passing through that point.
- <u>The derivative in Eq. (6) is a derivative along a characteristic curve</u>.

- Eq. (6) is seen to be a linear first-order ordinary differential equation which may be integrated.
- By introducing an integrating factor, Eq. (6) becomes

$$\frac{d}{ds}\left[\Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \cdot \exp\left(\int^s \Sigma_t(\mathbf{r}_0 + s'\mathbf{\Omega}, E)ds'\right)\right]$$

$$= \exp\left(\int^s \Sigma_t(\mathbf{r}_0 + s'\mathbf{\Omega}, E)ds'\right) \cdot S(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \quad \cdots\cdots (7)$$

Derivation of the Integral Equation (Contd.)

- Now Eq. (7) will be integrated from s=-∞, and as a result the integral terms will include earlier times, to some upper limit s. And it is assume that

$$\Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \rightarrow 0 \ \text{ as } \ s \rightarrow -\infty \qquad \cdots\cdots (8)$$

as would be true, for example, if there were no neutrons in the system at times long past.

- Then an integration of Eq. (7) from s=-∞ yields

$$\Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \cdot \exp\left(\int^{s} \Sigma_t(\mathbf{r}_0 + s'\mathbf{\Omega}, E) ds'\right) - \cancel{\Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v}) \cdot \exp\left(\int^{s} \Sigma_t(\mathbf{r}_0 + s'\mathbf{\Omega}, E) ds'\right)}\Bigg|_{s=-\infty}$$

$$= \int_{-\infty}^{s} \exp\left(\int^{s'} \Sigma_t(\mathbf{r}_0 + s''\mathbf{\Omega}, E) ds''\right) \cdot S(\mathbf{r}_0 + s'\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s'}{v}) ds'$$

$$\Rightarrow \Phi(\mathbf{r}_0 + s\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s}{v})$$

$$= \exp\left(-\int^{s} \Sigma_t(\mathbf{r}_0 + s'\mathbf{\Omega}, E) ds'\right) \cdot \int_{-\infty}^{s} \exp\left(\int^{s'} \Sigma_t(\mathbf{r}_0 + s''\mathbf{\Omega}, E) ds''\right) \cdot S(\mathbf{r}_0 + s'\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s'}{v}) ds'$$

$$= \int_{-\infty}^{s} \exp\left(\int_{s'}^{s} -\Sigma_t(\mathbf{r}_0 + s''\mathbf{\Omega}, E) ds''\right) \cdot S(\mathbf{r}_0 + s'\mathbf{\Omega}, E, \mathbf{\Omega}, t_0 + \frac{s'}{v}) ds' \qquad \cdots\cdots (9)$$

Derivation of the Integral Equation (Contd.)

- Equation (9) can be simplified to some extent by setting

$$\mathbf{r}_0 + s\mathbf{\Omega} = \mathbf{r}, \ \ t_0 + \frac{s}{v} = t$$

as

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t) = \int_{-\infty}^{s} \exp\left(\int_{s'}^{s} -\Sigma_t(\mathbf{r}_0 + s''\mathbf{\Omega},E)ds''\right) \cdot S(\mathbf{r}_0 + s'\mathbf{\Omega},E,\mathbf{\Omega},t_0 + \frac{s'}{v})ds' \quad \text{(10.a)}$$

- By changing the sign of the variable as

$$s = -\ell$$

we obtain

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t) = \int_{-\infty}^{-\ell} \exp\left(\int_{s'}^{-\ell} -\Sigma_t(\mathbf{r}_0 + s''\mathbf{\Omega},E)ds''\right) \cdot S(\mathbf{r}_0 + s'\mathbf{\Omega},E,\mathbf{\Omega},t_0 + \frac{s'}{v})ds' \quad \text{(10.b)}$$

- By substituting $\mathbf{r}_0$ and t_0 as

$$\mathbf{r}_0 = \mathbf{r} + \ell\mathbf{\Omega}, \ \ t_0 = t + \frac{\ell}{v}$$

Eq. (10.b) can be expressed as

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t) = \int_{-\infty}^{-\ell} \exp\left(\int_{s'}^{-\ell} -\Sigma_t(\mathbf{r} + (\ell + s'')\mathbf{\Omega},E)ds''\right) \cdot S(\mathbf{r} + (\ell + s')\mathbf{\Omega},E,\mathbf{\Omega},t + \frac{\ell + s'}{v})ds' \quad \text{(10.c)}$$

Derivation of the Integral Equation (Contd.)

- By changing the variable as

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t)=\int_{-\infty}^{-\ell}\exp\left(\int_{s'}^{-\ell}-\Sigma_t(\mathbf{r}+(\ell+s'')\mathbf{\Omega},E)ds''\right)\cdot S(\mathbf{r}+\underbrace{(\ell+s')}_{\ell+s'=-\ell'}\mathbf{\Omega},E,\mathbf{\Omega},t+\frac{\ell+s'}{v})ds'$$

Eq. (10.c) can be written as

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t)=\int_{0}^{\infty}\exp\left(\int_{-\ell-\ell'}^{-\ell}-\Sigma_t(\mathbf{r}+\underbrace{(\ell+s'')}_{\ell+s''=-\ell''}\mathbf{\Omega},E)ds''\right)\cdot S(\mathbf{r}-\ell'\mathbf{\Omega},E,\mathbf{\Omega},t-\frac{\ell'}{v})d\ell' \quad \text{(10.d)}$$

- By changing the variable as

Eq. (10.d) can be written as

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t)=\int_{0}^{\infty}\exp\left(\int_{0}^{\ell'}-\Sigma_t(\mathbf{r}-\ell''\mathbf{\Omega},E)d\ell''\right)\cdot S(\mathbf{r}-\ell'\mathbf{\Omega},E,\mathbf{\Omega},t-\frac{\ell'}{v})d\ell' \quad \text{(10.e)}$$

$$\boxed{\Phi(\mathbf{r},E,\mathbf{\Omega},t)=\int_{0}^{\infty}\exp\left(\int_{0}^{s'}-\Sigma_t(\mathbf{r}-s''\mathbf{\Omega},E)ds''\right)\cdot S(\mathbf{r}-s'\mathbf{\Omega},E,\mathbf{\Omega},t-\frac{s'}{v})ds'} \quad \text{(10)}$$

Meaning of the Derived Integral Equation

$$\Phi(\mathbf{r},E,\mathbf{\Omega},t)=\int_0^{\infty}\exp\left(-\int_0^{s'}\Sigma_t(\mathbf{r}-s''\mathbf{\Omega},E)ds''\right)\cdot S(\mathbf{r}-s'\mathbf{\Omega},E,\mathbf{\Omega},t-\frac{s'}{v})ds' \qquad (10)$$

- Equation (10) implies that the flux at **r** is made up of neutrons which appeared in the direction $\mathbf{\Omega}$ and energy E at all other possible position $\mathbf{r}-s'\mathbf{\Omega}$, with all positive values of s', multiplied by the attenuation factor

$$\exp\left(-\int_0^{s'}\Sigma_t(\mathbf{r}-s''\mathbf{\Omega},E)ds''\right)$$

Reformulation of Integral Equation

- For the further derivations, Eq. (10) and the S($\mathbf{r}$,E,$\boldsymbol{\Omega}$,t) can be expressed as

$$\Phi(\mathbf{r},E,\boldsymbol{\Omega},t)=\int_0^\infty e^{-\eta(s')}\cdot S(\mathbf{r}-s'\boldsymbol{\Omega},E,\boldsymbol{\Omega},t-\frac{s'}{v})ds'; \qquad (11)$$

$$\eta(s')=\int_0^{s'}\Sigma_t(\mathbf{r}-s''\boldsymbol{\Omega},E)ds'' \qquad (12)$$

$$\begin{aligned} S(\mathbf{r},E,\boldsymbol{\Omega},t) &= S_s(\mathbf{r},E,\boldsymbol{\Omega},t)+S_F(\mathbf{r},E,\boldsymbol{\Omega},t)+Q(\mathbf{r},E,\boldsymbol{\Omega},t)\\ &=\int_{E'}dE'\int_{4\pi}d\boldsymbol{\Omega}'\Sigma_s(\mathbf{r},E'\to E,\boldsymbol{\Omega}'\to\boldsymbol{\Omega},t)\Phi(\mathbf{r},E',\boldsymbol{\Omega}',t)\\ &\quad+\int_{4\pi}d\boldsymbol{\Omega}'\int_{E'}dE'\chi(\mathbf{r},E'\to E)\nu_f(E)\Sigma_f(\mathbf{r},E',\boldsymbol{\Omega}'\to\boldsymbol{\Omega},t)\Phi(\mathbf{r},E',\boldsymbol{\Omega}',t)\\ &\quad+Q(\mathbf{r},E,\boldsymbol{\Omega},t) \end{aligned}$$

$$\Rightarrow S(\mathbf{r},E,\boldsymbol{\Omega},t)=\int_{E'}dE'\int_{4\pi}d\boldsymbol{\Omega}'\Sigma_t(\mathbf{r},E')f(\mathbf{r};E',\boldsymbol{\Omega}'\to E,\boldsymbol{\Omega})\Phi(\mathbf{r},E',\boldsymbol{\Omega}',t)+Q(\mathbf{r},E,\boldsymbol{\Omega},t) \qquad (13)$$

- The transition probability can be expressed by the probability f_α for each reaction type α as

$$\Sigma_t(\mathbf{r},E')f(\mathbf{r};E',\boldsymbol{\Omega}'\to E,\boldsymbol{\Omega})=\sum_\alpha\Sigma_\alpha(\mathbf{r},E')f_\alpha(\mathbf{r};E',\boldsymbol{\Omega}'\to E,\boldsymbol{\Omega}) \qquad (14)$$

Collision Density Equation

- The collision density can be written, from its definition, as

$$\psi(\mathbf{r},E,\mathbf{\Omega},t) \equiv \Sigma_t(\mathbf{r},E)\Phi(\mathbf{r},E,\mathbf{\Omega},t)$$

$$= \Sigma_t(\mathbf{r},E)\int_0^{\infty} e^{-\eta(s')} \cdot S(\mathbf{r}-s'\mathbf{\Omega},E,\mathbf{\Omega},t-\frac{s'}{v})ds' \qquad (15)$$

- We transform Eq. (15) into the three-dimensional form to describe the collision density in the kernel form:

$$\psi(\mathbf{r},E,\mathbf{\Omega},t) = \int d\mathbf{r}'\Sigma_t(\mathbf{r},E)\cdot\exp\left(\int_0^{|\mathbf{r}-\mathbf{r}'|}\Sigma_t(\mathbf{r}-s''\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},E)ds''\right)\frac{\delta\left(\mathbf{\Omega}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}-1\right)}{|\mathbf{r}-\mathbf{r}'|^2}$$

$$\times S(\mathbf{r}',E,\mathbf{\Omega},t'); \qquad (16)$$

$$\mathbf{r}' = \mathbf{r}-s'\mathbf{\Omega}, \quad t' = t-\frac{s'}{v} \qquad (17)$$

where $\delta(x)$ is the Dirac delta function.

Collision Density Equation (Contd.)

- By introducing the free-flight kernel, Eq. (16) can be expressed

$$\psi(\mathbf{r},E,\mathbf{\Omega},t)=\int d\mathbf{r}' T(E,\mathbf{\Omega};\mathbf{r}'\to\mathbf{r})S(\mathbf{r}',E,\mathbf{\Omega},t'); \quad \cdots\cdots (18)$$

$$T(E,\mathbf{\Omega};\mathbf{r}'\to\mathbf{r})=\frac{\Sigma_t(\mathbf{r},E)}{|\mathbf{r}-\mathbf{r}'|^2}\exp\left[-\int_0^{|\mathbf{r}-\mathbf{r}'|}\Sigma_t\left(\mathbf{r}-s\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},E\right)ds\right]\delta\left(\mathbf{\Omega}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}-1\right) \quad \cdots\cdots (19)$$

- Then the insertion of Eq. (13) into Eq. (18) yields

$$\begin{aligned}\psi(\mathbf{r},E,\mathbf{\Omega},t)&=\int d\mathbf{r}' T(E,\mathbf{\Omega};\mathbf{r}'\to\mathbf{r})\\&\times\left[\int_{E'}dE'\int_{4\pi}d\mathbf{\Omega}'\Sigma_t(\mathbf{r},E')f(\mathbf{r};E',\mathbf{\Omega}'\to E,\mathbf{\Omega})\Phi(\mathbf{r}',E',\mathbf{\Omega}',t')+Q(\mathbf{r}',E,\mathbf{\Omega},t')\right]\\&=\int d\mathbf{r}' T(E,\mathbf{\Omega};\mathbf{r}'\to\mathbf{r})\\&\times\left[\int_{E'}dE'\int_{4\pi}d\mathbf{\Omega}'C(\mathbf{r}';E',\mathbf{\Omega}'\to E,\mathbf{\Omega})\psi(\mathbf{r}',E',\mathbf{\Omega}',t')+Q(\mathbf{r}',E,\mathbf{\Omega},t')\right];\end{aligned} \quad \cdots\cdots (20)$$

$$C(\mathbf{r}';E',\mathbf{\Omega}'\to E,\mathbf{\Omega})=\sum_\alpha\frac{\Sigma_\alpha(\mathbf{r}';E',\mathbf{\Omega}')}{\Sigma_t(\mathbf{r}',E')}f_\alpha(E',\mathbf{\Omega}'\to E,\mathbf{\Omega}) \quad \cdots\cdots (21)$$

Collision Density Equation (Contd.)

- By introducing the first-collision source defined by

$$\hat{Q}(\mathbf{r},E,\mathbf{\Omega},t)=\int T(E,\mathbf{\Omega};\mathbf{r}'\rightarrow\mathbf{r})Q(\mathbf{r}',E,\mathbf{\Omega},t')d\mathbf{r}' \quad \text{(22)}$$

Eq. (20) can be expressed as

$$\psi(\mathbf{r},E,\mathbf{\Omega},t)=\hat{Q}(\mathbf{r},E,\mathbf{\Omega},t)+\int d\mathbf{r}'\int_{E'}dE'\int_{4\pi}d\mathbf{\Omega}'K(\mathbf{r}',E',\mathbf{\Omega}'\rightarrow\mathbf{r},E,\mathbf{\Omega})\psi(\mathbf{r}',E',\mathbf{\Omega}',t'); \quad \text{(23)}$$

$$K(\mathbf{r}',E',\mathbf{\Omega}';\rightarrow\mathbf{r},E,\mathbf{\Omega})=T(E,\mathbf{\Omega};\mathbf{r}'\rightarrow\mathbf{r})\cdot C(\mathbf{r}';E',\mathbf{\Omega}'\rightarrow E,\mathbf{\Omega}) \quad \text{(24)}$$

- For simplicity, Eq. (23) can be expressed as

$$\psi(\mathbf{P})=\hat{Q}(\mathbf{P})+\int d\mathbf{P}'K(\mathbf{P}'\rightarrow\mathbf{P})\psi(\mathbf{P}') \quad \text{(25)}$$

where $\mathbf{P}=(\mathbf{r},E,\Omega,t)$.

Cf. Collision Density Eq. for Eigenvalue Calculations

- The integral equation for the collision density $\psi(\mathbf{P})$ defined by $\Sigma_t(\mathbf{r},E)\phi(\mathbf{P})$ can be written as

$$\psi(\mathbf{P}) = \int d\mathbf{r}' T(E,\mathbf{\Omega};\mathbf{r}' \to \mathbf{r}) S(\mathbf{r}',E,\mathbf{\Omega}) + \int d\mathbf{P}' K_s(\mathbf{P}' \to \mathbf{P}) \psi(\mathbf{P}') \quad \text{(B.1)}$$

K_s is defined by the product of the scattering collision kernel, C_s and the transition kernel [B.1] (or the free flight kernel), T:

$$K_s(\mathbf{P}' \to \mathbf{P}) = T(E,\mathbf{\Omega};\mathbf{r}' \to \mathbf{r}) \cdot C_s(\mathbf{r}';E',\mathbf{\Omega}' \to E,\mathbf{\Omega}); \quad \text{(B.2)}$$

$$C_s(\mathbf{r}';E',\mathbf{\Omega}' \to E,\mathbf{\Omega}) = \sum_{r \neq fis.} \nu_r \frac{\Sigma_r(\mathbf{r}';E',\mathbf{\Omega}')}{\Sigma_t(\mathbf{r}',E')} f_r(E',\mathbf{\Omega}' \to E,\mathbf{\Omega}) \quad \text{(B.3)}$$

$$T(E,\mathbf{\Omega};\mathbf{r}' \to \mathbf{r}) = \frac{\Sigma_t(\mathbf{r},E)}{|\mathbf{r}-\mathbf{r}'|^2} \exp\left[-\int_0^{|\mathbf{r}-\mathbf{r}'|} \Sigma_t\left(\mathbf{r} - s\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, E\right) ds\right] \delta\left(\mathbf{\Omega} \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} - 1\right) \quad \text{(B.4)}$$

ν_r is the average number of neutrons produced from a reaction type r and f_r is the probability that a collision of type r by a neutron of direction $\mathbf{\Omega}'$ and energy E' will produce a neutron in direction interval $d\Omega$ about $\mathbf{\Omega}$ with energy in dE about E.

([B.1] I. Lux, L. Koblinger, "Monte Carlo Particle Transport Methods: Neutron and Photon Calculations," CRC Press (1991).)

Series Solution

$$\boxed{\psi(\mathbf{P}) = \hat{Q}(\mathbf{P}) + \int d\mathbf{P}' K(\mathbf{P}' \to \mathbf{P}) \psi(\mathbf{P}')} \qquad \cdots\cdots \quad (25)$$

- Consider the solution of Eq. (25) obtained by iteration; thus

$$\psi_0(\mathbf{P}) = \hat{Q}(\mathbf{P})$$
$$\psi_1(\mathbf{P}) = \int d\mathbf{P}' K(\mathbf{P}' \to \mathbf{P}) \cdot \psi_0(\mathbf{P}')$$
$$\vdots \qquad \vdots$$
$$\psi_n(\mathbf{P}) = \int d\mathbf{P}' K(\mathbf{P}' \to \mathbf{P}) \cdot \psi_{n-1}(\mathbf{P}')$$

Clearly ψ_0 is the first-collision source. ψ_1 means the collision density from the second-collision neutrons. Similarly, ψ_2 indicates the contribution of the third-collision neutrons, and so on. If the series $\sum_{j=0}^{\infty} \psi_j(\mathbf{P})$ converges, it represents a solution to Eq. (25).

Neumann Series Solution

$$\boxed{\psi(\mathbf{P}) = \hat{Q}(\mathbf{P}) + \int d\mathbf{P}' K(\mathbf{P}' \to \mathbf{P})\psi(\mathbf{P}')} \quad \text{(25)}$$

- The solution of Eq. (25) can be expressed by the Neumann series:

$$\boxed{\psi(\mathbf{P}) = \sum_{j=0}^{\infty} \int d\mathbf{P}' K_j(\mathbf{P}' \to \mathbf{P})\hat{Q}(\mathbf{P}');} \quad \text{(26a)}$$

$$\begin{aligned}
K_0(\mathbf{P}' \to \mathbf{P}) &= \delta(\mathbf{P}' - \mathbf{P}), \\
K_1(\mathbf{P}' \to \mathbf{P}) &= K(\mathbf{P}' \to \mathbf{P}), \\
K_2(\mathbf{P}' \to \mathbf{P}) &= \int d\mathbf{P}_1 K(\mathbf{P}_1 \to \mathbf{P}) K(\mathbf{P}' \to \mathbf{P}_1), \\
&\vdots \\
K_j(\mathbf{P}' \to \mathbf{P}) &= \int d\mathbf{P}_1 \cdots \int d\mathbf{P}_{j-1} K(\mathbf{P}_{j-1} \to \mathbf{P}) K(\mathbf{P}_{j-2} \to \mathbf{P}_{j-1}) \cdots K(\mathbf{P}' \to \mathbf{P}_1)
\end{aligned} \quad \text{(26b)}$$

- From Eq. (26), we can find that the collision density is the sum of the contribution from particles colliding at **P** first and after a collision or more.
- The Monte Carlo particle transport analysis is based on Eqs. (25) & (26).
- Normalizing the source to unity, Eq. (25) is the probability density for the number of collision at **P**.

Cf. Neumann Series Sol for Eigenvalue Eq.

- From the Neumann series solution for the integral transport equation, the neutron flux can be written as

$$\boxed{\begin{aligned}\phi(\mathbf{r},E,\mathbf{\Omega}) = \frac{1}{\Sigma_t(\mathbf{r},E)}\sum_{j=0}^{\infty}\int d\mathbf{r}'\int dE_0\int d\mathbf{\Omega}_0 K_{s,j}(\mathbf{r}',E_0,\mathbf{\Omega}_0 \to \mathbf{r},E,\mathbf{\Omega})\\ \times\int d\mathbf{r}_0 T(E_0,\mathbf{\Omega}_0;\mathbf{r}_0 \to \mathbf{r}')S(\mathbf{r}_0,E_0,\mathbf{\Omega}_0),\end{aligned}}$$

$$\begin{aligned}K_{s,j}(\mathbf{r}',E_0,\mathbf{\Omega}_0 \to \mathbf{r},E,\mathbf{\Omega}) = \int d\mathbf{r}_1\int dE_1\int d\mathbf{\Omega}_1\cdots\int d\mathbf{r}_{j-1}\int dE_{j-1}\int d\mathbf{\Omega}_{j-1}\\ \times K_s(\mathbf{r}_{j-1},E_{j-1},\mathbf{\Omega}_{j-1} \to \mathbf{r},E,\mathbf{\Omega})\cdots K_s(\mathbf{r}',E_0,\mathbf{\Omega}_0 \to \mathbf{r}_1,E_1,\mathbf{\Omega}_1)\end{aligned}$$

$$K_s(\mathbf{r}',E',\mathbf{\Omega}' \to \mathbf{r},E,\mathbf{\Omega}) = T(E',\mathbf{\Omega}';\mathbf{r}' \to \mathbf{r})C_s(\mathbf{r};E',\mathbf{\Omega}' \to E,\mathbf{\Omega})$$

$$C_s(\mathbf{r}';E',\mathbf{\Omega}' \to E,\mathbf{\Omega}) = \sum_{r\neq fis.}\nu_r\frac{\Sigma_r(\mathbf{r}';E',\mathbf{\Omega}')}{\Sigma_t(\mathbf{r}',E')}f_r(E',\mathbf{\Omega}' \to E,\mathbf{\Omega})$$

$$T(E,\mathbf{\Omega};\mathbf{r}' \to \mathbf{r}) = \frac{\Sigma_t(\mathbf{r},E)}{|\mathbf{r}-\mathbf{r}'|^2}\exp\left[-\int_0^{|\mathbf{r}-\mathbf{r}'|}\Sigma_t(\mathbf{r}-s\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|},E)ds\right]\delta\left(\mathbf{\Omega}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}-1\right)$$