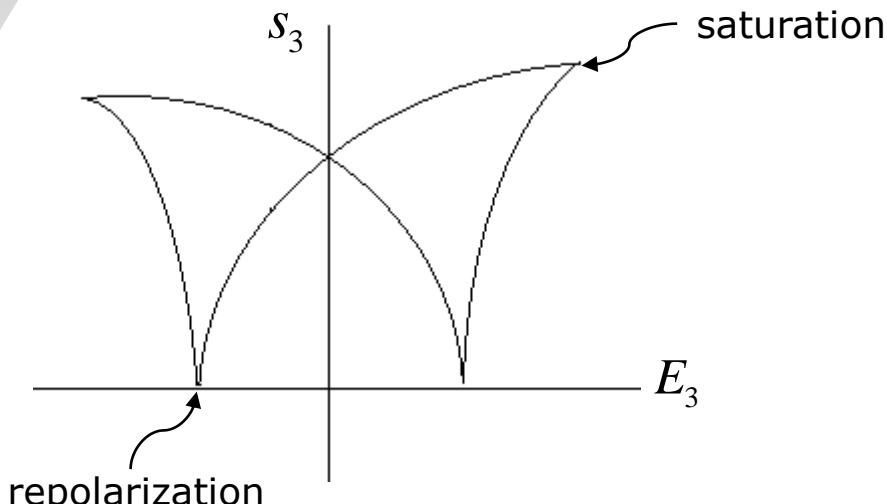
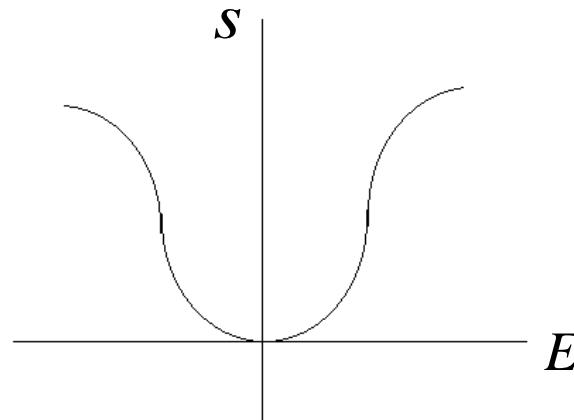


Introduction to Material Behavior

❖ Piezoelectric



❖ Electrostrictive Ceramics



❖ Shape Memory Alloys

❖ Magnetostrictive

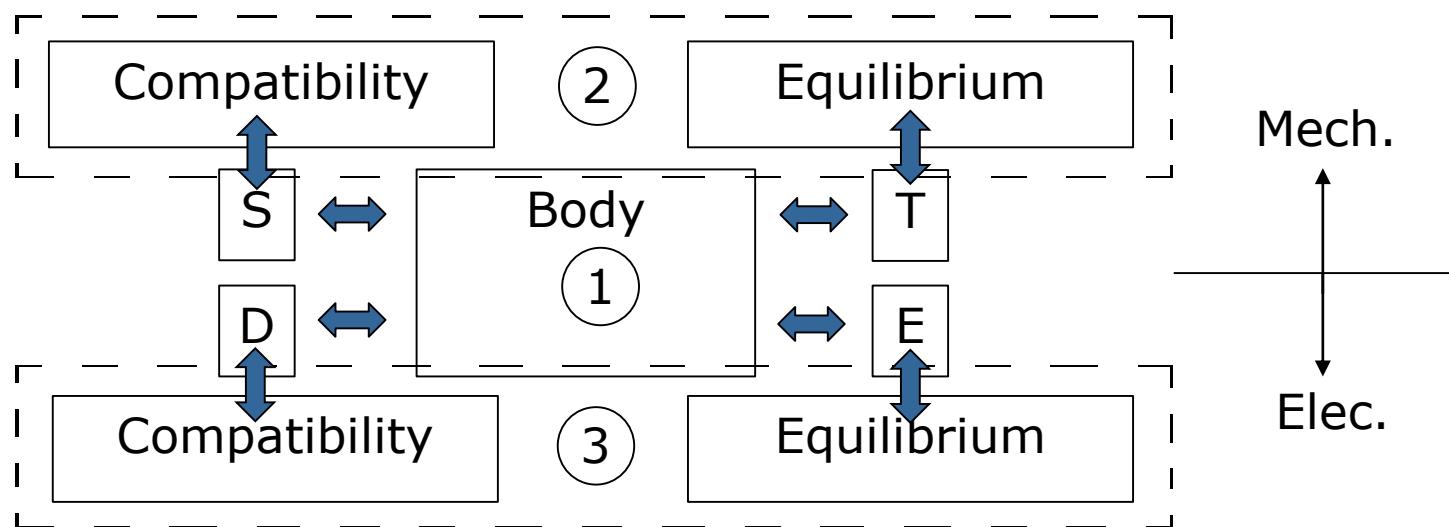
- Terfenol-D
Iron – Terbium - Dysprosium

Introduction to Material Behavior

❖ Continuum Mechanics

- 1) Kinematics ... compatibility
- 2) Kinetics ... Equilibrium
- 3) Constitutive Relation { Differential Equation
 Integrated Equation }

- Roadmap for Analysis of Coupled Continuum



Introduction to Material Behavior

- Mechanical Field

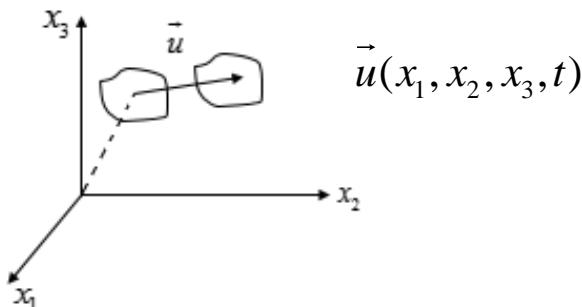
S : strain , S_{ij} $i = 1, 2, 3$

T : stress , T_{ij}

$T_{ij} = T_{ji}$ 6 independent components

Voight or Contracted Notion

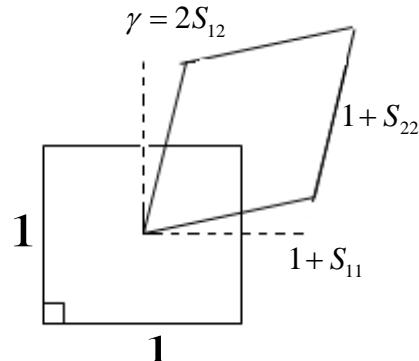
- Displacement Field



$$\vec{S} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} S_x \\ S_y \\ S_z \\ 2S_{yz} \\ 2S_{zx} \\ 2S_{yx} \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{31} \\ 2S_{21} \end{bmatrix}, \quad \vec{T} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{21} \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Introduction to Material Behavior

- Strain: relative deformation

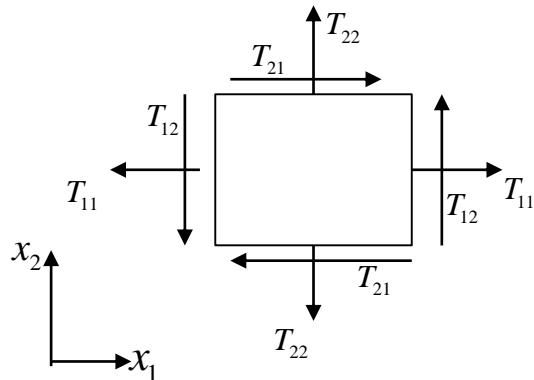


Strain-Displacement Relation

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

\Rightarrow 6 independent eqn

- Newton's Law



$$\sum F = ma$$

$$T_{ij} = \frac{\text{Force}}{\text{Area}} = \text{on } i\text{-th face, } j\text{-th direction}$$

Introduction to Material Behavior

- Equilibrium Equation

Differential Form

$$\frac{\delta T_{ij}}{\delta X_j} + f_i = \rho a_i \quad \Rightarrow \quad 3 \text{ Eqns.}$$

$$= \left\{ \begin{array}{l} \frac{\delta T_{11}}{\delta X_1} + \frac{\delta T_{12}}{\delta X_2} + \frac{\delta T_{13}}{\delta X_3} + f_1 = \rho a_1 \\ \frac{\delta T_{21}}{\delta X_1} + \frac{\delta T_{22}}{\delta X_2} + \frac{\delta T_{23}}{\delta X_3} + f_2 = \rho a_2 \end{array} \right.$$

Unknowns

6 strains

6 stresses

3 displacements

15 Eqn.

3 Equilibrium

6 strain-displacement

6 constitutive Relations

15 Eqn.

- Constitutive Relations

tensor $T_{ij} = E_{ijmn} S_{mn}$

$$S_{ij} = C_{ijmn} T_{mn}$$

E_{ijmn} : stiffness tensor, elasticity

$$\tilde{T} = \tilde{c} \tilde{S}, \quad \tilde{S} = \tilde{s} \tilde{T}$$

↓

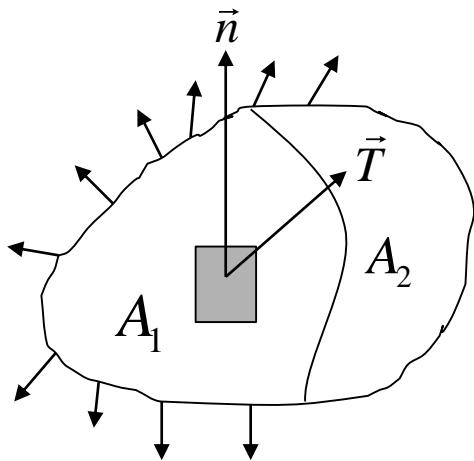
6X6

compliance

stiffness matrix

Introduction to Material Behavior

- Boundary Conditions



on A_1
prescribed stress vector

traction

$$\vec{T}_s = (T_{sn})\vec{i}_n = [T_{mn} \cos(Nm)]\vec{i}_n$$

$$\begin{bmatrix} T \\ T \\ T \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ & & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Differential Form

$$\frac{\partial T_{ij}}{\partial X_j} + f_i - \rho a_i = 0$$

Principle of Minimum Potential Energy $\delta(U - W) = 0$

Integral Form

\Rightarrow Principle of Total Minimum Potential Energy

$$\int_V \left\{ \left[\frac{\partial T_{ij}}{\partial X_j} + f_i \right] \cdot \delta \vec{u} \right\} dV = 0$$

$$\int_V \frac{T \delta S \, dV}{\delta U : \text{Energy}} = \frac{\int_V (\vec{f} \cdot \delta u) dV}{\delta W : \text{virtual work}} + \int_s (\vec{f}_s \cdot \delta u) dS$$

Introduction to Material Behavior

- Electric Fields

$$E, D \text{ (electrical displacement)} \quad \vec{E} = \lim_{q \rightarrow 0} \frac{\vec{F}}{q} = \frac{\text{volts}}{m}$$

Coulomb's Law

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\vec{r}}{|r|^3} = \frac{q_1 q_2}{4\pi\epsilon_0} \nabla \frac{1}{|\vec{r}|} \quad \epsilon_0 = 8.84 \times 10^{-12} \text{ Farad / meter} \quad \text{"permittivity"} \\ \text{(誘電率)}$$

$$\vec{E} = \frac{-q}{4\pi\epsilon_0} \nabla \frac{1}{|\vec{r}|}$$

superposition gives field associated with the other charge distribution

- Vector field facts

Vector field is uniquely defined by its circulation density + source density

$$\begin{aligned} \nabla \cdot \vec{v} &= S \\ \nabla \times \vec{v} &= \vec{c} \end{aligned} \quad \left. \right\} \quad \vec{v} = -\nabla \varphi + \nabla \times \vec{A}$$

$$\varphi(r): \text{scalar potential} \quad = \frac{1}{4\pi} \int \frac{S(r')}{|r - r'|} dr'$$

$$\vec{A}(r): \text{vector potential} \quad = \frac{1}{4\pi} \int \frac{\vec{c}(r')}{|r - r'|} dr'$$

Introduction to Material Behavior

- Gauss's Flux Theorem

$$\int_S \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}$$

-Differential Form

divergence theorem

$$\int_S \vec{E} \cdot d\vec{S} = \int_V \nabla \cdot \vec{E} dV = \frac{q}{\epsilon_0} = \int_V \frac{\rho}{\epsilon_0} dV$$

$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \vec{E} &= 0\end{aligned}$$

$$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \oint \vec{E} \cdot d\vec{l} = 0 \quad \text{"irrotationality"}$$

- Electric Potential

$$\vec{E} = -\nabla \varphi \quad \begin{array}{l} \downarrow \\ \text{gradient operator} \end{array}$$

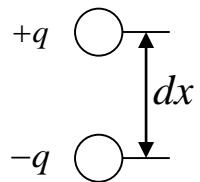
$$E_1 = -\frac{\delta \varphi}{\delta x_1} \Rightarrow \nabla^2 \varphi = -\frac{\rho}{\epsilon_0} \quad \text{Poisson's Eq.}$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(r')}{|r - r'|} dr'$$

$$\text{point charge } \varphi(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Introduction to Material Behavior

- Polarization Field



$$\vec{P} = \text{dipole moment} = q d \vec{x}$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{P} \cdot \vec{r}}{r^3}$$

Volume distribution

$$\vec{P} = \vec{p} / \text{volume}$$

Potential associated with volume distribution of dipoles

$$\varphi = \frac{1}{4\pi\epsilon_0} \int \vec{P} \cdot \nabla \left(\frac{1}{r} \right) dr \quad \vec{\nabla} \cdot \left(\frac{\vec{P}}{r} \right) = \frac{1}{r} \vec{\nabla} \cdot \vec{P} + \vec{P} \cdot \nabla \left(\frac{1}{r} \right)$$

$$\varphi = \frac{1}{4\pi\epsilon_0} \left[\int \nabla \cdot \frac{\vec{P}}{r} dV - \int \frac{1}{r} \nabla \cdot \vec{P} dV \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\underbrace{\int_s \frac{\vec{P} \cdot d\vec{S}}{r}}_{\text{bound charge}} - \underbrace{\int \frac{\nabla \cdot \vec{P}}{r} dV}_{\text{free charge}} \right]$$

Potential equivalent to
surface charge
distribution (free charge)

bound charge

$$\rho^B = -\nabla \cdot \vec{P}$$

$$\sigma_\rho = \rho^S$$

: ρ consists of $\rho^S + \rho^B$, $\rho^B = -\nabla \cdot \vec{P}$

Introduction to Material Behavior

- Electrical Displacement

$$\nabla^2 \varphi = -\nabla \cdot \vec{E} = -\frac{\rho^{tot}}{\epsilon_0} = -\frac{(\rho^f + \rho^B)}{\epsilon_0}$$

let $\rho^B = -\nabla \cdot \vec{P}$

$$\nabla(\vec{E} + \frac{\vec{P}}{\epsilon_0}) = \frac{\rho^f}{\epsilon_0}$$

Describe a new vector field

$$\vec{D} = \text{charge / area} = \epsilon_0 \vec{E} + \vec{P}$$

$$\nabla \cdot \vec{D} = \rho^f$$

$$\int_S \vec{D} \cdot dS = q^f$$

- Electrical Constitutive Relations

Polarization is dependent on electrical field

electrical susceptibility, polarization coeff.

$$\vec{P} = \epsilon_0 \chi \downarrow \vec{E}$$

$$\vec{P} = \epsilon_0 X \vec{E} \quad (3 \times 3 \text{ matrix})$$

$$P_i = \epsilon_0 X_{ij} E_j$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + X) \vec{E}$$

$$\vec{D} = \kappa \epsilon_0 \vec{E} = \epsilon \vec{E}$$

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

dielectric tensor

Introduction to Material Behavior

-Equations

Compatibility

$$\vec{E} = -\nabla \varphi \quad 3$$

Equilibrium

$$\nabla \cdot \vec{D} = \rho^f \quad 1$$

$$\nabla \times \vec{D} = 0 \quad 3$$

$$\vec{D} = \varepsilon \vec{E} \quad 3$$

Variables

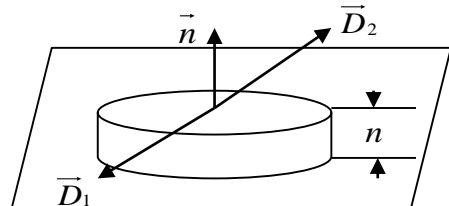
$$\varphi \quad 1$$

$$E \quad 3$$

$$\frac{D}{3}$$

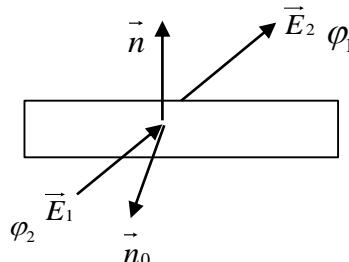
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-Boundary Conditions⁷



$$\int \vec{D} \cdot d\vec{S} = q^f$$

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma$$



$$\varphi_1 = \varphi_2$$

Introduction to Material Behavior

- Differential Form

$$\nabla \cdot \vec{D} = \sigma \quad \text{applied free charge}$$

$$\int_V [\nabla \cdot \vec{D}] \delta\phi dV = \int_V \sigma \delta\phi dV$$

$$\int_V \vec{D} \cdot \delta \vec{E} = \sum q_i \delta\phi_i$$

$$\delta U = \delta W$$

❖ Magnetism

- Current

- Current density = amps/area
- Conservation of the charge $\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$
- Stationary current $\nabla \cdot \vec{j} = 0$
- Constitutive relation $\vec{j} = \sigma \vec{E}$

Introduction to Material Behavior

- Electromotive force



$$j = \sigma(\vec{E} + \vec{E}')$$

where $\nabla \times \vec{E} = 0$
 $\nabla \times \vec{E}' \neq 0$

$$\oint \frac{j}{\sigma} \cdot d\vec{l} = \int (\vec{E} + \vec{E}') d\vec{l} = \int \vec{E}' dl = \varepsilon$$

$$\varepsilon = \oint \frac{j}{\sigma} = I \int \frac{dl}{ds} = IR \quad R = \int \frac{dl}{ds} \Rightarrow V = IR$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot (\sigma \vec{E}) = -\nabla \cdot (\sigma \vec{E}')$$

$$\nabla \cdot j = 0$$

$$j = \sigma \vec{E}$$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot (\varepsilon \vec{E}) = 0$$

E

j

$j = \sigma \vec{E}$

\vec{D}

$\vec{D} = \varepsilon \vec{E}$

B.C's

$n(\vec{D}_2 - \vec{D}_1)$

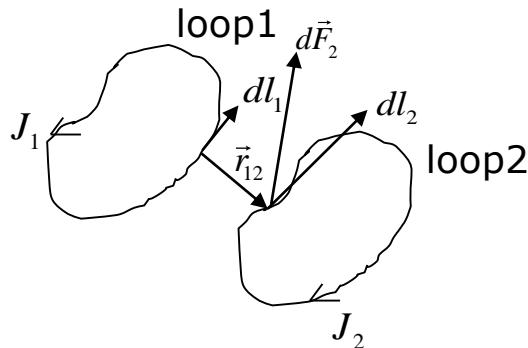
$n(\vec{J}_2 - \vec{J}_1)$

$n \times (\vec{E}_2 - \vec{E}_1) = 0$

Introduction to Material Behavior

- Magnetic Field

Ampere's Law



$$F_2 = \frac{\mu_0}{4\pi} J_1 J_2 \int_1 \int_2 \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{|r_{12}|^3} \quad \mu_o = 4\pi \times 10^{-7} H/m$$

permeability of a vacuum

$$F = \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{|\vec{r}|}{|r|^3} \quad (\text{Coulomb's Law})$$

$$F_2 = J_2 \oint d\vec{l}_2 \times \vec{B}_2 \quad B_2 = \frac{\mu_0}{4\pi} J_1 \oint \frac{d\vec{l}_1 \times \vec{r}_{12}}{|r|^3}$$

$$F = \int_V \vec{j} \times \vec{B} dV \quad B = \frac{\mu_0}{4\pi} \int_V \frac{\vec{j} \times \vec{r}}{|r|^3} dV$$

$$\nabla \cdot \vec{B} = 0 \quad \begin{matrix} | \\ \oplus \\ | \end{matrix} \quad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{B} = \mu_0 \hat{j} \quad | \quad \nabla \cdot \vec{E} = \frac{q}{\epsilon_0}$$

vector potential

$$\vec{B} = \nabla \times \vec{A} \quad \vec{E} = -\nabla \varphi$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{j}}{|r|} dV \quad \varphi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho}{|r|} dV$$

Introduction to Material Behavior

- Currents

- free, bound
- polarization current $\frac{d\mathbf{P}}{dt}$ neglect
- magnetization current j_m
- displacement current $\frac{\partial \mathbf{E}}{\partial t}$

Just as for polarization

$$\vec{p} = \int \vec{P} dV = \int \rho \vec{\epsilon} dV$$

$$\vec{m} = \int \vec{M} dV = \frac{1}{2} \int (\xi \times j_m) dV$$

$$\rho_b = -\nabla \cdot \vec{P}$$

$$j_m = \nabla \times \vec{M} \quad (\vec{M} : \text{magnetization})$$

Introduction to Material Behavior

- Magnetic Fields in matter

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}_{tot} = \mu_0 (\vec{j}_{true} + \nabla \times \vec{M})$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{E} = \frac{\rho_{tot}}{\epsilon_0} = \frac{1}{\epsilon_0} (\rho_{true} - \nabla \cdot \vec{P})$$

$$\nabla \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{j}_{true}$$

or $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$, Coercive field

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\begin{cases} \nabla \times \vec{H} = j_{true} \\ \nabla \cdot \vec{D} = \rho_{true} \end{cases}$$

$$\begin{cases} \nabla \times \vec{B} = J_{tot} \mu_0 \\ \nabla \cdot E = \frac{\rho_{tot}}{\epsilon_0} \end{cases}$$

$$\oint \vec{H} \cdot d\vec{l} = I_{true}$$

Introduction to Material Behavior

- Permeable Material

$$\vec{M} = \chi \vec{H}$$

χ : magnetic susceptibility

$$\vec{B} = \mu_0 (\chi_m + 1) \vec{H}$$

$$B = \kappa_m \mu_0 \vec{H} = \mu \vec{H}$$

μ : absolute permeability

κ_m : relative permeability

$$\mu_0 = 4\pi \times 10^{-9} \text{ Henry/m}$$

Terfenol κ_m : 7.9

Iron κ_m : 1,000

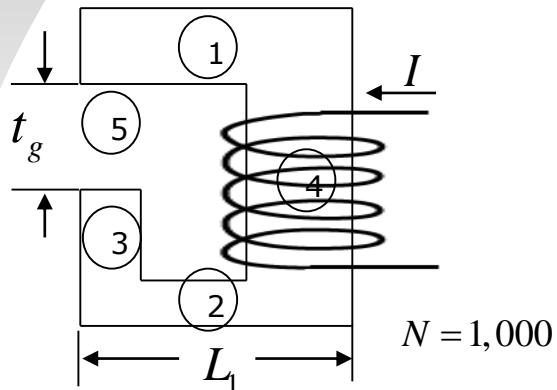
B.C's $\nabla \cdot \vec{B} = 0$ $n(\vec{B}_2 - \vec{B}_1) = 0$

$$\nabla \times \vec{H} = j_{true} \quad n(\mu_2 \vec{H}_2 - \mu_1 \vec{H}_1) = 0$$

$$n \times (\vec{H}_2 - \vec{H}_1) = \kappa \quad : \text{surface current density}$$

Introduction to Material Behavior

- Magnetic Circuit Analysis



$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{H} \cdot d\vec{l} = I_{true}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{B} \Leftrightarrow \vec{j}$$

$$\vec{H} \Leftrightarrow \vec{E'}$$

$$\mu \Leftrightarrow \sigma \quad \varepsilon_i = I_i R_i$$

$$I_1 = \int_A \vec{j} dA, \quad R_1 = \frac{l_1}{\sigma_1 S_1}$$

$$dH_1 = \varphi R, \quad \varphi = \int_A \vec{B} da, \quad R_i = \frac{l_i}{\mu_i S_i}$$

$$R_1 = \frac{L_1}{A_1 \mu_1}, \quad R_2 = R_1, \quad R_3 = \frac{L_3}{A_3 \mu_3}, \quad R_4 = \frac{L_2}{A_2 \mu_2}, \quad R_5 = \frac{t_g}{A_5 \mu_0}$$

$$V_{gap} = H_g t_g = \frac{R_5}{R_{tot}} V_{tot} = \frac{R_5}{R_{tot}} H_{tot} l_{tot}$$

$$H_{gap} = \frac{NI}{A_5 \mu_0 R_{tot}} \longrightarrow H_{gap} = \frac{NI}{t_g}$$

$$B_{gap} = \mu_0 H_{gap}$$

$$\nabla \cdot j = 0$$

$$\oint E' dl = \varepsilon$$

$$\vec{j} = \sigma \vec{E}$$