16 COMBINED AND SPECIAL FOOTINGS

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16.1 GENERAL CONSIDERATIONS

Chapter 15 has considered the commonly encountered spread footing carrying a single column load. This chapter will consider footings carrying more than one column in a line (combined footings) as well as spread footings with eccentricity, with holes or notches, and ring and chimney foundations.

Combined footings will be treated both with the conventional (rigid member) design method and as the problem of a beam on an elastic foundation.

Concrete design will use the ultimate strength (USD) method based on ACI 318–71 as presented in Chapter 15.

It may be necessary to place more than one column load on a footing for a number of reasons such as:

(a) Insufficient area near a property line with the result that a spread footing (Chapter 15) would be eccentrically loaded (Fig. 16.1a).
(b) Building equipment necessitating that the spread footings be rectangular when the space between adjacent footings is relatively small (Fig. 16.1b).
(c) The column loads and/or soil conditions are such that the resulting footings occupy most of the site (Fig. 16.1c).

When isolated footings are subjected to very large eccentric loadings we are faced with the possibility of excessive footing rotation, differential settlements, or exceeding of the allowable bearing capacity of the soil. This situation may be rectified by placing two or more of the columns in a line on a single continuous footing. Proper proportioning of the footing can result in a uniform pressure distribution (conventional analysis) on the soil beneath the footing. The footings may be *rectangular* or *trapezoidal*, or (rarely) of another form as shown in Fig. 16.2. Continuous footings may be stepped to allow increased bearing capacity or greater basement area, to go below zones of poor soil, etc., as shown in Fig. 16.2c.

When spread footings for the columns begin to occupy a large percentage of the foundation site or of the area between columns, the designer must weigh formwork costs against the extra footing materials required by using continuous footings or mat (Chapter 17) foundations. These are the considerations of concern for Figs. 16.2c through f. Sometimes, however, alternate footings may be placed and formwork saved by using a spacer board against the already

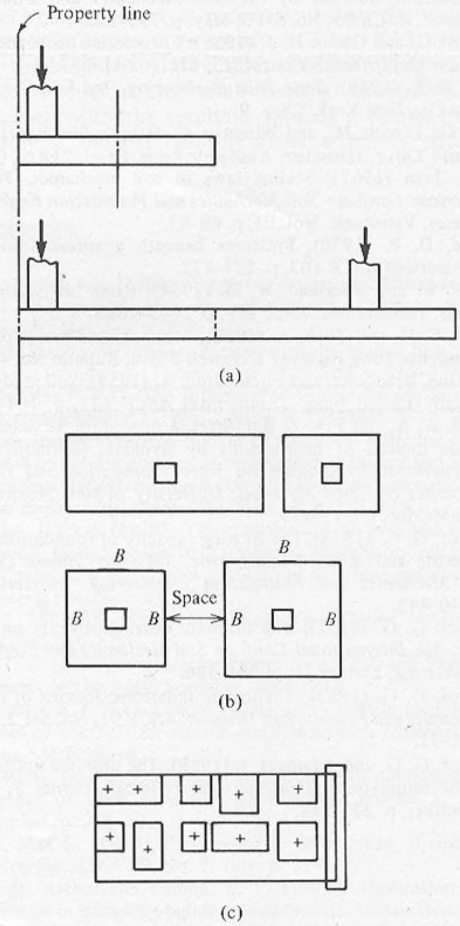


Fig. 16.1 Conditions for use of combined footings. (a) Footings which will be subjected to overturning moment and nonuniform soil pressure can be combined using conventional design practice; (b) footings loaded and placed so that space between footings is small. Generally, if space between footings is less than B it will be more economical to combine or abut two footings; (c) if footings occupy most of foundation space, consider a mat or combining footings.

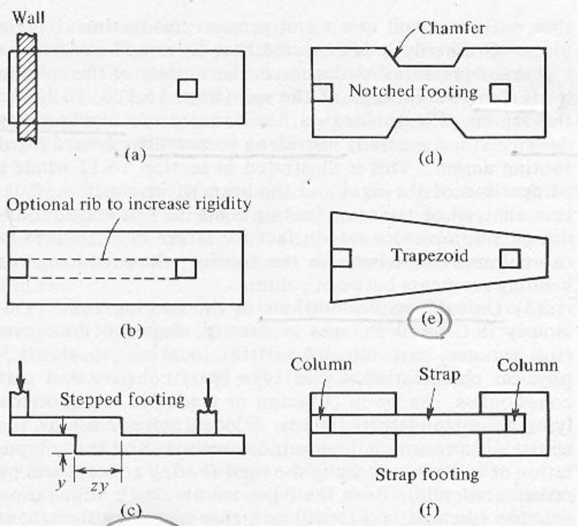


Fig. 16.2 Types of continuous footings.

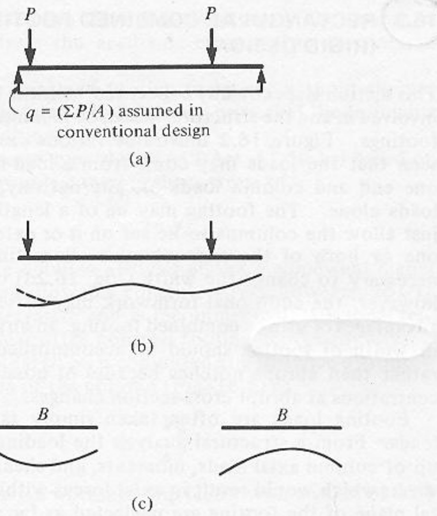


Fig. 16.3 (a) Rigid design assumption of pressure distribution; (b) longitudinal pressure distribution (probable); (c) end view of probable pressure distribution. Edge pressures depend on soil type.

poured footing. This arrangement may save on reinforcing steel (negative) as well as formwork costs.

On the other hand, it is sometimes uneconomical to use a footing of constant width or depth to span the distance between two columns. A possible solution is the *strap* or *cantilever* (sometimes called a *pumphandle*) footing. Here the strap is a shear and moment transferal device and is designed as a beam.

The design considerations of such members, both soil and structural, will be discussed in the following pages.

16.2 ALLOWABLE SOIL BEARING PRESSURES

Design of the footings of this chapter is based on an allowable soil pressure as obtained from the equations presented by Vesić in Chapter 3 or from the SPT penetration number (see Table 15.1). Settlements and differential settlement (Chapter 4) also will be factors to consider and may limit the allowable pressure value. Generally the same factors considered in Chapter 15, sections 2 through 6 are applicable to the footings in this chapter.

Allowable bearing pressure generally will require computation of an ultimate bearing pressure based on the footing width B, which for a rectangular footing will be its least lateral dimension and for a trapezoidal footing its average width. Eccentrically loaded footings should be considered individually, but a bearing pressure value can be obtained using equivalent width B' (section 15.7). The ultimate bearing pressure is divided by an appropriate safety factor F of 2 or 3. A safety factor of $F = 3$ is generally preferred although there is some diverse opinion of what the numerical value of the factor should be as well as how to obtain it (Brinch Hansen, 1967; Johnson and Kavanegh, 1968; Jumikis, 1967*; and Bowles, 1968). At present, however, it seems rational to use an F-value to compute the allowable bearing pressure for the condition of

$$\text{Dead load + Design live load:} \quad F = 3$$

and when part of the loads are temporary or transient

$$\text{Dead load + Design live load} + K_i: \quad F = 2$$

where

K_1 = wind load ($i = 1$)

K_2 = earthquake ($i = 2$)

K_3 = snow load ($i = 3$)

but K_1 not usually taken simultaneously with K_2 or K_3. Additional consideration should be taken of load duration, foundation soil type, and groundwater conditions when applying the safety factor since a transient load on a cohesionless soil will have an immediate effect. On a soil with a low coefficient of permeability part of the transient load may be gone before any foundation displacement can occur.

Actual soil pressures beneath flexible footings will probably be somewhat as shown in Fig. 16.3. In reality it is expected that most footings designed as rigid are not and are probably intermediate between absolutely rigid and flexible.

Actual soil conditions such as stratification, lenses of different soil types, changes in density, etc., as well as footing shape will cause the actual foundation pressure to deviate from the theoretical pressure of Eq. 15.14.

Referring again to Fig. 15.12, if the high edge pressures shown were used in design it could result in larger bending moments at critical sections for bending. Shear values would probably change very little, if at all. Considering, however, that an absolutely rigid footing will not (cannot?) be built and any flexibility or edge differential movement will relieve the high edge stresses, it is recommended that a *planar soil pressure distribution be used for the conventional design of combined footings* (except as noted) in this chapter. There may be an occasional special case in which this method of design could result in an unsafe footing, but it is expected that the designer will recognize the occurrence of this event.

*Jumikis defines

$$F = \frac{\text{Favorable quantities (forces, moments, etc.)}}{\text{Unfavorable quantities}}$$

Bowles (1968), Chapter 7, shows that even this method of evaluating F requires careful interpretation.

16.3 RECTANGULAR COMBINED FOOTINGS (RIGID DESIGN)

This section is specifically concerned with the considerations involved in and the structural design of rectangular combined footings. Figure 16.2 illustrates various cases. It can be seen that the loads may come from a load-bearing wall at one end and column loads or, alternatively, from column loads alone. The footing may be of a length sufficient to just allow the columns to be set on it or extending beyond one or both of the end columns. Sometimes it may be necessary to change the width (Fig. 16.2d) of the footing; however, the additional formwork may offset some of the advantages of using a combined footing. In any case, changes in width of footing should be accomplished by chamfers rather than abrupt notches because of possible stress concentrations at abrupt cross-section changes.

Footing loads are often taken simply as axial column loads. From a structural analysis the loading can be made up of column axial loads, moments, and shears. Ordinarily, shears which would result in axial forces within the horizontal plane of the footing are neglected as far as the member proportioning is concerned. Shears may, however, contribute to a lateral stability problem of a building and/or foundation translation.

Rectangular combined footings should be designed for a uniform soil pressure if foundation space limitations will allow it, to take advantage of computation simplicity. If the conditions shown in Fig. 16.4b obtain, this may not be possible, resulting in somewhat more complicated computations. Still another problem arises when the load on the combined footings consists of more than two column loads and with or without column moments. This latter problem *appears* indeterminate until one realizes that the assumption of a planar soil pressure distribution together with the column loads provides enough information to make a design. An indirect solution to the proportioning of the footing for this latter problem has been presented (Jacoby and Davis, 1941) with the solution consisting of providing a footing centrally loaded with the soil reaction.

There are limitations and assumptions in the design of combined footings generally as follows:

(1) A rigid design is assumed. Unless the footing is extremely thick it is not a rigid member. In fact, some designers make the depth of the footing such that stirrups are required for shear stresses. Except for certain types of slabs, however, the author does not recommend the use of stirrups in rigid footings.*

(2) A uniform or planar soil pressure distribution is obtained beneath the footing. This situation may obtain if the footing is fairly rigid and the underlying soil is soft so that sufficient soil creep and pressure readjustment takes place. Ordinarily it is expected that there will always be a higher soil pressure distribution in the vicinity of the column loads due to "dishing" of the soil (Figs. 16.12b, 16.13) in this region. For this reason, the conventional rigid footing design method generally provides a conservative (overdesign) footing design. This is illustrated in section 16.11 where a comparison of the rigid and the beam on an elastic foundation analysis of the same footing is made. The elastic foundation solution accounts in fact for larger soil pressures in the column zones beneath the footing, thus reducing the bending moments between columns.

(3) Uniform soil conditions in the footing zone. Obviously if the soil changes in density, degree of looseness (for instance, part cut and part fill, local soft spots, etc.), physical characteristics, soil type (part cohesive and part cohesionless, change in cohesion or gradation, etc.), underlying and/or undetected voids, or other soil conditions, the actual soil pressure is highly indeterminate and the computation of soil pressure using the rigid footing concept will be rather academic. Even the beam on an elastic foundation solution (section 16.11) will be highly suspect, although, as will be seen later, a certain amount of adjustment of soil parameters can be made in this solution.

(4) Neglect of contribution of superstructure rigidity. The footing rests on a somewhat elastic media and the amount of deformation at a column is related (but not necessarily linearly) to the column load. As soon as deformation of the foundation soil begins to take place, however, the rigidity of the superstructure comes into play, tending to bridge or decrease certain column loads and concentrating additional loads to other columns. Several writers (Meyerhof, 1953; Chameeki, 1956; Grasshoff, 1957; Somner, 1965; and Lee and Harrison, 1970) have considered the problem with approximate solutions. Analytical solutions to date verify what one would intuitively expect, namely that adding an equivalent amount of moment of inertia to the combined footing to account for the effect of superstructure rigidity on the footing decreases the computed bending moments in the footing. It naturally follows that the stiffer the superstructure the greater the reduction of footing bending moments.

In considering the effect of superstructure on the footing some attention might be given to transmitting column loads to the footing. Certainly a flexible column would react somewhat pin-ended (no moment—only axial load). On the other hand, does it necessarily follow that a stiff column will transmit a footing moment? and if so how much? The amount is going to depend on footing rotation; remember that there will probably be rotation unless the footing is very rigid. Footing moments from the column will also depend on how the column is attached to the foundation.

Fortunately the current state-of-the-art of soil exploration can discover most of the potential soil problems; and soil

*The ACI 318–71 does not recommend not using stirrups, but if they are used they are only 50 percent effective (Art. 11.11.1).

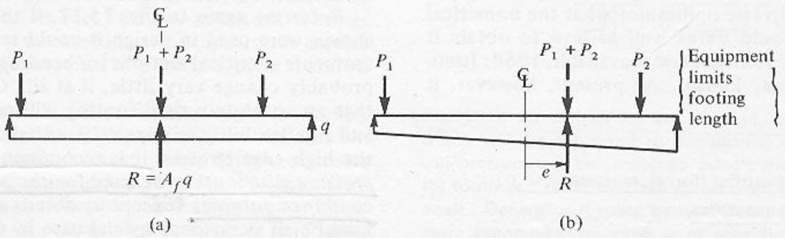


Fig. 16.4 (a) Soil pressure is uniform if sum of loads falls at center of footing area; (b) soil pressure is linear but nonuniform if any eccentricity results between center of resultant and center of footing area.

stabilization science can improve site characteristics to a reasonable degree of prediction. Therefore, the design of combined footings is no more hazardous than the ordinary spread footings of Chapter 15. Because they are usually overdesigned there have been no building failures to date attributed to failures of combined footings (see also Feld, 1962).

Once the type of foundation has been selected and the structural design of members begun, if overdesign of any of the members of the structure is made it is axiomatic that it should be in the foundation members. Reasons are as follows:

(1) There are usually only a few footings relative to other structural members and with a one-to-two-inch increase of concrete depth the increased total volume of concrete is on the order of 5 to 10 yd^3 for all the footings of most structures.

(2) An assumption of 5 to 10 lb of steel extra (average) per footing will amount to considerable overdesign, but the total increase will be less than 1000 lb for most jobs.

(3) The extra reinforcing steel may reduce the requirement for hooks and bends, thus offsetting the material costs.

(4) The extra concrete required to design footings for no shear steel (stirrups) is offset in cost by not requiring stirrups plus the labor costs of placing the steel.

(5) It is difficult and extremely expensive to change the foundation design if members are underdesigned, especially if an appreciable amount of the superstructure is in place.

The design of a combined footing with any number of loads proceeds as follows (refer also to Table 15.3).

(1) Locate the point of application of the resultant of all the loads on the footing (refer to Fig. 16.4).

(2) Compute the area of the footing such that the allowable soil pressure is nowhere exceeded and with:

(a) a uniform soil pressure distribution if the resultant of the loads does not coincide with the center of footing area as

$$q = \frac{\Sigma P}{A}$$

(b) a planar soil pressure distribution if the resultant of the loads does not coincide with the center of footing area as

$$q = \frac{\Sigma P}{A} \pm \frac{\Sigma P e \bar{x}}{I}$$

with terms identified in Fig. 16.4.

(3) Convert the loads to ultimate loads by means of load factors as follows:

$$U = 1.4\,(\text{Dead}) + 1.7\,(\text{Live})$$

$$U = 0.75\,[1.4\,(\text{Dead}) + 1.7\,(\text{Live}) + 1.7\,(\text{Wind or Earthquake})]$$

Then find a fictitious ultimate soil pressure (q_{ult}) per footing length as

$$q_{\text{ult}} = \frac{\Sigma U}{L} \pm \frac{6 \Sigma U e}{L^2}$$

noting that $e = 0$ for case 2a.

(4) Using ultimate loads U and soil pressure q_{ult}, construct the shear and moment diagram taking column loads as *point loads*. This results in incorrect shear and moment diagrams within the column zones, but correct shear and moment values exterior from the column faces which are the design locations. The savings in time offsets the academic error within the column zones.

(5) Find the depth of footing to satisfy without the use of stirrups (generally) the most critical of the following two conditions.

(a) Wide beam type shear (ACI Art. 11.10.1a) based on an allowable concrete shear stress of

$$v_c = 2\phi\sqrt{f_c'}$$

A more detailed analysis allows a somewhat higher value of allowable concrete stress; however, $2\phi\sqrt{f_c'}$ is conservative and is recommended. Diagonal tension shear often controls footing depths anyway. The actual stress is computed at a distance d from the face of the column (Fig. 16.5) as

$$v_u = \frac{V_u}{bd}$$

where ϕ is a workmanship factor with a value of 0.85 for shear.

(b) Diagonal tension (punching shear as shown in Fig. 16.5 based on an allowable concrete shear stress of

$$v_c = 4\phi\sqrt{f_c'} \quad \text{(ACI 318–71 Art. 11.10.3)}$$

and the actual shear stress computed as

$$v_u = \frac{V_u}{b_0 d}$$

(6) With the depth of section established compute the area of steel required to satisfy bending in the longitudinal direction as

$$M_u = \phi A_s f_y\,(d - a/2)$$

and check minimum steel requirements of ACI Art. 7.13. The depth a of rectangular stress block of the concrete is

$$a = \frac{A_s f_y}{0.85\, f_c' b}$$

where b = width of strip being analyzed—generally 12 inches.

Usually the depth d will be the overall depth of concrete D_c less 3 inches of steel cover less one-half a bar diameter:

$$d = D_c - 3 - (\text{Bar diameter})/2$$

(7) Select steel bars to satisfy bond and anchorage (ACI 318–71 Art. 12.5 and 12.6) using Tables 15.5a and b.

(8) Select steel in the short direction considering each column to be supported by a fictitious beam of width

$$W_i = a + d \text{ (Interior columns)}$$

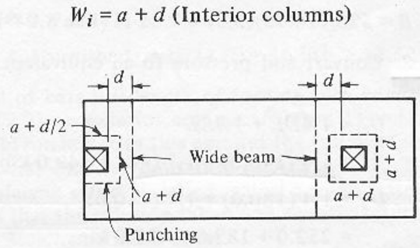


Fig. 16.5 Locations to investigate for shear. Shear ordinarily controls the depth of footing.

or

$$W_e = a + d/2 \text{ (Exterior columns)}$$

Depending on column placement on the footing the distance W_e may increase up to $a + d$ as for interior columns. Use minimum percentage of steel to satisfy shrinkage for the remainder of the footing in the transverse direction. The validity of this placement is based on bending primarily longitudinal with the exception of transversal bending in the immediate vicinity of the columns. Further, most combined footing L/B ratios will be 2 or more; thus there is a similarity to one-way slab design.

EXAMPLE 16.1 Design a combined footing for the loading conditions shown in Fig. E1.1. Use ACI 318–71, strength design (USD), and conventional design procedures.* Note that procedures to obtain loads and soil pressures have been considered in this chapter, Chapter 15, and elsewhere; however, this example uses a value of recommended allowable

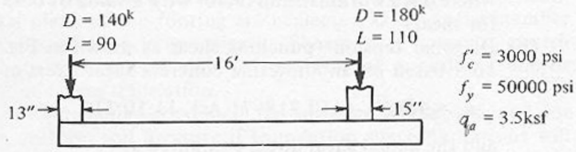


Fig. E1.1

soil pressure as the structural designer usually gets in a report from the soil engineer. It is assumed the structural designer would somehow be able to obtain column loads to apply to the footing as shown. This example does not consider the rigidity effect of superstructure on the foundation.

Step 1. Compute the footing dimensions

$$\Sigma M \text{ Column } 1 = 0$$

$$(230 + 290)\,\bar{x} = 16\,(290)$$

$$= 16\,(290)/520 = 8.92 \text{ ft}$$

The length of the footing is

$$L = 2\,(8.92 + .542) = 18.93 \text{ ft (Use 19.0 ft)}$$

To avoid computational errors the actual computed footing length will be used for all computations. The as built footing will be 19.0 ft. Incidentally, the moment diagram will not close unless the computed length of the footing is used. Any other footing length will introduce an eccentricity of soil reaction for the assumptions used.

The width of the footing is

$$BLq_a = \Sigma P$$

$$B = 520/(18.93)(3.5) = 7.85 \text{ ft (Use 8.0 ft)}$$

Step 2. Convert soil pressure to an equivalent ultimate load value:

$$U_1 = 1.4DL + 1.7LL$$

$$= 1.4\,(140.0) + 1.7\,(90) = 349.0 \text{ kips}$$

$$U_2 = 1.4\,(180.0) + 1.7\,(110.0)$$

$$= 252.0 + 187.0 = 439.0 \text{ kips}$$

*These procedures are essentially the recommendations of ACI committee 436 (*Journal American Concrete Institute*, October 1966).

$$\text{Factor} = (349.0 + 439.0)/520 = 1.515$$

$$q_{\text{ult}} = 3.5\,(1.515) = 5.3 \text{ ksf}$$

Step 3. Draw shear and moment diagrams for the footing. In this example the column loads are treated as distributed loads.* The column loads may be treated as point loads without affecting the structural design since the shear and moment values in the design zones are identical using either approach. Treating the column loads as point loads is generally preferred for hand computation as it is simpler.

$$\text{Load/ft of beam} = \frac{V_u}{L} = \frac{788.0}{18.93} = 41.63 \text{ kips/ft}$$

$$= q_{\text{ult}} \times B = 5.3\,(7.85) = 41.63 \text{ kips/ft}$$

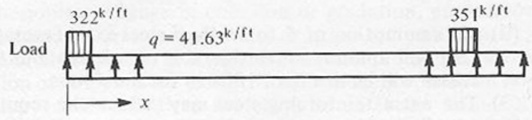


Fig. E1.2

considering the column load as a *distributed* load from $x = 0$ to 1.0833 ft.

$$q = 41.63 - 349/1.0833 = 280.536 \text{ k/ft}$$

$$V = \int q\,dx = 280.536x + C_0 \; (C_0 = 0 \text{ since } V = 0 \text{ at } x = 0)$$

$$V = 280.536\,(1.083) = -303.9 \text{ kips}$$

$$M = \int V\,dx = 280.536x^2/2 + C_1 \; (C_1 = 0 \text{ since } M = 0 \text{ at } x = 0)$$

$$M = \frac{280.536}{2}(1.083)^2 = -164.60 \text{ ft-k}$$

Compute the shear and moment values between columns. Take column loads as point loads at this stage since the resulting values will be identical. For values of $1.083 < x \leq 15.917$ ft

$$q = 41.63 \text{ k/ft}$$

$$V = \int q\,dx = 41.63x - 349.0$$

$$M = \int V\,dx = 41.63\,\frac{x^2}{2} - 349.0\,(x - .542)$$

Evaluating the shear and moment at other selected points the following table can be obtained:

x, ft	*V, kips*	*M, ft-kips*	
0	0	0	
1.083	−303.9	− 164.60	Interior face of column 1
2	−265.75	− 425.59	
6	− 99.24	−1155.56	
8	− 15.99	−1270.79	
8.38	0.0	−1273.86	Maximum moment
10	67.27	−1219.51	
14	233.77	− 617.42	
15.917	313.57	− 92.80	Interior face of column 2
16.542	120.10	+ 50.11	Center of column 2
17.167	− 73.39	+ 64.70	Exterior face of column 2
18.0	38.71	+ 10.62	
18.93	0	0	

*Since this problem had been programmed on the computer.

The maximum moment and its location as shown in the preceding table is obtained from the setting shear to zero:

$$41.63x - 349 = 0$$

$$x = 8.38 \text{ ft}$$

The maximum moment is computed as

$$M_{\max} = 41.63\left(\frac{8.38}{2}\right)^2 - 349\,(8.38 - .542) = -1273.86 \text{ ft-k}$$

The shear and moment diagrams are plotted in Fig. E1.3.

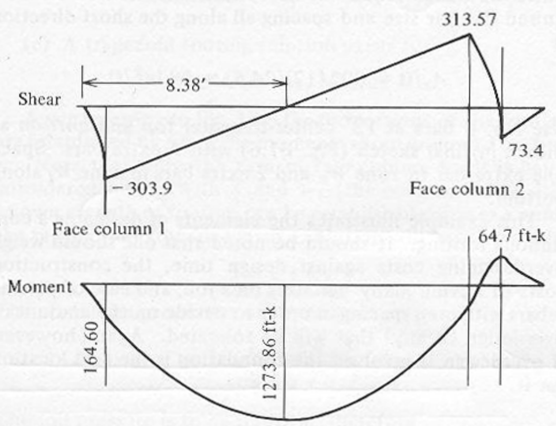


Fig. E1.3

Step 4. Find depth for wide beam shear at the location of largest shear (the interior column in our example as shown in Fig. E1.3); the allowable concrete stress is

$$2\phi\sqrt{f_c'} = 93.1 \text{ psi} = 13.406 \text{ ksf}$$

$$V_{\text{beam}} = 313.57 - 41.63d$$

$$V_{\text{conc}} = Bdv_c = 7.85\,(d)(13.406) = 105.237d$$

Equating

$$41.63d + 105.237d = 313.57$$

$$d = 313.57/146.867 = 2.135' = 25.6''$$

Check diagonal tension

$$v_c = 4\phi\sqrt{f_c'} = 186.2 \text{ psi} = 26.81 \text{ ksf}$$

At column 1 (Fig. E1.4a) the perimeter in shear is

$$p = 2\,(1.083 + 2.135/2) + (1.083 + 2.135) = 7.519 \text{ ft}$$

$$V_{\text{shear}} = pdv_c = 7.519\,(2.135)(26.81) = 430.38 > 349 \text{ kips} \quad (O.K.)$$

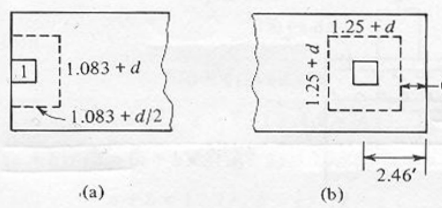


Fig. E1.4

At column 2 (Fig. E1.4b) the perimeter in shear is

$$p = 4\left(1.25 + \frac{2.135}{2}\right) = 5.00 + 4.27 = 9.27 \text{ ft}$$

$$V_{\text{shear}} = 9.27\,(2.135)(26.81) = 530.5 > 439 \text{ kips } (O.K.)$$

Note that strictly the allowable shear value should be compared to the ultimate column loads *less* the soil pressure on on the base as

$$V_{\text{conc shear}} = \text{Col. load} - Aq_{\text{ult}}$$

However, this computation is somewhat academic if the allowable concrete shear is larger than the column load as in these two checks. Check ACI Art. 10.7 to see if this member should be classified as a deep member:

$$D/L = 2.13/16 < 2/5 \text{ or } 4/5$$

One might consider using the span length of column face to column face ($L = 14.83'$) instead of column to column for this check.

Step 5. Compute the area of steel required. Steel will be required in the zone between columns on the top side of the footing (negative steel) and on the base side in the cantilevered portion. Steel will also be required perpendicular to the long axis in zone W_i and W_e, plus shrinkage steel for the remainder of the footing length.

(a) Computing negative steel first:

$$M_{u\max} = -1273.9 \text{ ft-k for width of 8.0 ft}$$

$$\frac{M_u}{\phi f_y} = A_s(d - a/2) \quad \text{Table 15.3)}$$

$$a = \frac{A_s f_y}{.85 f_c' b} = \frac{50\,(A_s)}{.85\,(3)(12)} = 1.632A_s$$

Substituting values,

$$A_s(25.6 - 1.632A_s/2 = +\frac{1273.9\,(12)}{50\,(.9)(8.0)}$$

$$.816A_s^2 - 25.6A_s = -42.5$$

$$A_s^2 - 31.37A_s = -52.0$$

$$A_s = 1.69 \text{ in}^2/\text{ft}$$

Check p for shrinkage (note that minimum $p \geqslant 200/f_y$ does not apply for constant thickness members):

$$p = \frac{1.69}{12\,(25.6)} = .0055 > .002 \quad \text{ACI Art. 7.13.1}$$

Total steel required across footing is

$$1.69\,(8.0) = 13.52 \text{ in}^2$$

Use 16 No. 9 bars at 6″ center-to-center

$$A_s\,(\text{furnished}) = 16\,(1.0) = 16 \text{ in}^2 > 13.52 \text{ in}^2$$

Run all of bars full length of footing with *hooks* (ACI Art. 12.1, 12.2) on exterior column (column 1) end. Note that we could run less than this amount the full distance (i.e., cut some bars) but the limitation of ACI Art. 12.3.3 plus the extra placing effort weighed against a small material savings implies that the solution taken will be satisfactory.

Use 3 inches of clear cover.

Check ACI Art. 10.6.2 (refer to Fig. E1.5)

$$Z = f_s\sqrt[3]{t_b A}$$

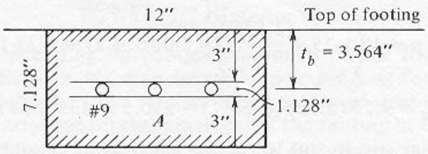


Fig. E1.5

where

$$f_s = .60 f_y = 30 \text{ ksi}$$

$$t_b = 3.564 \text{ in}$$

$$A = 7.128 \times 12 = 85.5 \text{ in}^2$$

$$Z = 30\sqrt{(3.564)(85.5)} = 30\,(8.52) = 166 > 145$$

(b) Compute positive steel in cantilever part:

$$M_u = 64.7 \text{ ft-k}/B = 8.1 \text{ ft-k/ft}$$

$$.816A_s^2 - 25.6A_s = -8.1\,(12)/.9\,(50)$$

$$A_s^2 - 31.37A_s = -2.65$$

$$A_s = .09 \text{ in}^2/\text{ft}$$

$$p = .09/12\,(25.6) = .00029 < .002$$

Shrinkage steel requirements control and the required steel area is

$$.002\,(12)(25.6) = 0.61 \text{ in}^2/\text{ft}$$

Use 8-No. 9 bars at 12" center-to-center.
Check ACI Art. 7.4.3:

$$3t = 3\,(25.6) = 76.8 \text{ in} > 18 \text{ in} > 12 \text{ in spacing.} \quad (O.K.)$$

This allows all the longitudinal rebars of same size at an even spacing.

Run 4 bars ($\frac{1}{2}A_s$) full length of footing. Run 4 bars beyond interior column a distance of 36.5" (Table 15.5a) ACI Art. 12.1.5.

(c) Design transverse steel (short direction). Use actual footing dimensions and place short steel on top of longitudinal steel:

$$q_{\text{ult}} = (349 + 439)/(8)(19) = 5.18 \text{ ksf}$$

(1) For exterior column (Col. 1)

$$a + d/2 = 13 + 25.6/2 = 25.8 = 2.15'$$

d for steel = 25.6 + 0.56 − 1.692 assuming No. 9 bars

$$d = 24.5 \text{ in}$$

The length of cantilevered footing for bending moment is

$$L = (8 - 1.083)/2 = 6.917/2 = 3.46'$$

$$M = qL^2/2 = 5.18\,(3.46)^2/2 = 31.0 \text{ ft-k}$$

From previous computations

$$.816A_s^2 - 24.5A_s = -31\,(12)/.9\,(50)$$

$$A_s^2 - 30A_s = -8.27$$

$$A_s = .29 \text{ in}^2/\text{ft}$$

$$p = .29/12\,(24.5) = .00098 < .002$$

Use

$$A_s = .002\,(12)(24.5) = 0.59 \text{ in}^2/\text{ft}$$

Since shrinkage requirements control the area of steel required the bar size and spacing all along the short direction is

$$A_s/\text{ft} = .002\,(12)(24.5) = .59 \text{ in}^2/\text{ft}$$

Use No. 6 bars at 12" center-to-center *top* and *bottom* as shown in final sketch (Fig. E1.6) with 3 extra bars. Space one extra bar in zone W_e and 2 extra bars in zone W_i along bottom.

This example illustrates the elements of designing a continuous footing. It should be noted that one should weigh overdesigning costs against design time, the construction costs of having many bar sizes on a job, and ease of placing rebars with even spacing in order to decide on the amount of overdesign (if any) that will be tolerated. Again, however, if overdesign is involved the foundation is the best location for it.

16.4 TRAPEZOIDAL SHAPED FOOTINGS

When an exterior column of a building has a larger load than the adjacent interior column and for some reason it is necessary to place the two columns on a single footing, the resulting combined footing will be *trapezoidal* in shape if it is desired to have an assumed uniform distribution of soil pressure beneath the footing. Referring to Fig. 16.6 one can see that the minimum length of footing is out-to-out column faces; any maximum length may be selected. From an inspection of the figure one notes several points of interest:

(a) No solution exists if

$$\bar{x} + w_1 < L/3$$

since this would result in a triangular footing with the result that the interior column would be partly on empty space.

This case, if the column spacing is large or poor proportions (since $a \cong 2b$) are obtained, is considered in section 16.5.

(b) The solution becomes a rectangular combined footing if

$$\bar{x} + w_1 = L/2$$

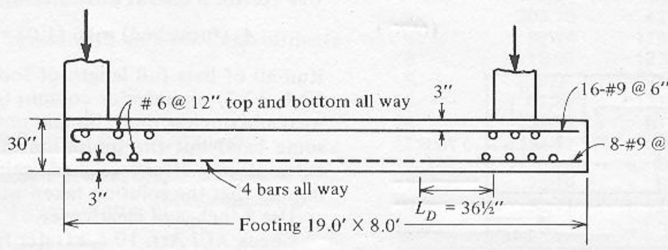


Fig. E1.6

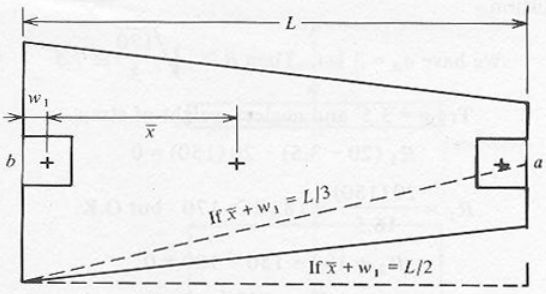


Fig. 16.6 Trapezoidal footing.

(c) A trapezoid footing solution exists for

$$L/3 < (\bar{x} + w_1 < L/2)$$

Again referring to Fig. 16.6 the proportions of the footing are established by summing moments about column 1 to find $\bar{x}$. Note that only the case of two column loads is being considered here. With $\bar{x}$ and w_1 (the column half-width) known a length of footing can be established using as a guide the range of lengths for which a solution will exist. Now let

$$x = \bar{x} + w_1$$

Then from the properties of a trapezoid

$$x = \frac{L}{3}\left(\frac{2b + a}{a + b}\right) \tag{16.1}$$

The soil pressure is to be uniform; therefore

$$qA = \Sigma V$$

and

$$A = \frac{(a + b)}{2} L \tag{16.2}$$

Solving the above three equations one can find the unknown end dimensions a, b.

Next one converts the actual loads to ultimate loads, finds q_{ult} so that shear and moment diagrams can be drawn. The footing is designed in a fashion similar to the rectangular combined footing.

EXAMPLE 16.2. Proportion a trapezoid footing for the conditions shown in Fig. E2.1.

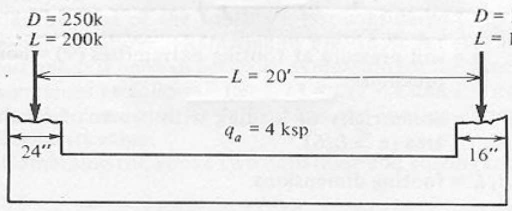


Fig. E2.1

Solution:

$$770\,\bar{x} = 20\,(320)$$

$$\bar{x} = 8.31' \quad x = 8.31 + 1.0 = 9.31'$$

$$A = (a + b)L/2 = (a + b)(21.67/2) = 10.835\,(a + b) = 770/4$$

$$a + b = 17.77 \quad b = 17.77 - a$$

$$x = L/3\left(\frac{2a + b}{a + b}\right) = 9.31$$

$$\frac{2a + b}{17.77} = 9.31\left(\frac{3}{21.67}\right)$$

$$2a + b = 17.77\,(1.29)$$

$$2a + 17.77 - a = 22.92$$

$$a = 5.15' \quad \therefore b = 12.62'$$

$$U_1 = 1.4\,(250) + 1.7\,(200) = 690$$

$$U_2 = 1.4\,(1.80) + 1.7\,(140) = 490$$

$$q_u = 1180/192.5 = 6.13 \text{ ksf}$$

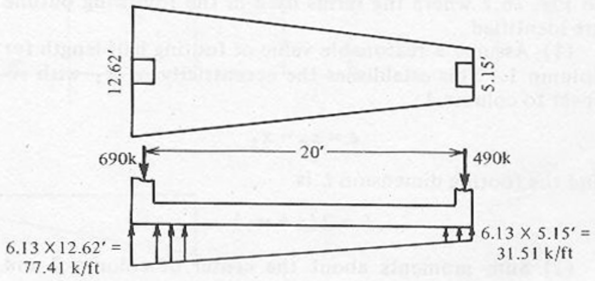


Fig. E2.2

16.5 STRAP OR CANTILEVER FOOTINGS

The previous section indicated situations of column loadings or spacings such that a combined footing could not be made to work (at least with a uniform soil pressure), i.e., when the distance $x < L/3$. A large spacing between two columns can create a situation where a continuous footing is uneconomical due to the use of a large quantity of concrete in the footing and because of the high negative bending moments between columns.

Figure 16.7 illustrates an alternative to the combined footing, namely the strap footing (also termed a *cantilever* and, by some designers, a *pumphandle*). Essentially this type of footing uses spread footings or pads beneath the columns and a rigid beam connecting the two pads to transmit the unbalanced shear and moment from the statically unbalanced footing to the second footing.

Two simplifying assumptions are made to obtain this solution; first, uniform soil pressures are obtained beneath each footing pad and, secondly, the strap or beam connecting the two footings is perfectly rigid. Often the connecting beam is assumed to be weightless—it doesn't matter if the footings are assumed weightless since those weights cancel. The rigid connecting beam is assumed to have no vertical soil reactions. This can be accomplished by loosening the soil

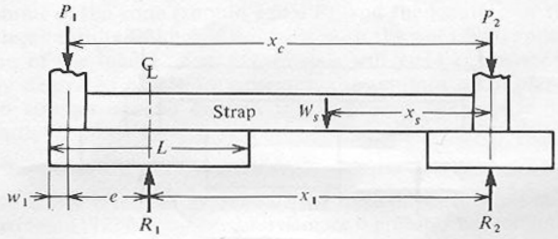


Fig. 16.7 Strap or cantilever footing.

Referring to Fig. 16.10 for signs (use the *left hand* rule: with thumb pointing in direction of arrow, $+M$ direction is indicated by fingers grasping the axis) the following equations of statics can also be written

$$P = \int q\,dA \quad M_x = \int qy\,dA \quad M_y = \int qx\,dA \qquad (16.9)$$

Substituting Eq. 16.9 into Eq. 16.8 and noting that

$$\int_0^A dA = \int_0^A x\,dA = \int_0^A y\,dA = 0$$

with respect to the centroid of the footing we obtain

$$\begin{aligned} P &= cA && \text{(a)} \\ M_x &= aI_{xy} + bI_x && \text{(b)} \qquad (16.10) \\ M_y &= aI_y + bI_{xy} && \text{(c)} \end{aligned}$$

Solving Eq. 16.10a directly the value of the c-coefficient is

$$c = P/A$$

and solving 16.10b and 16.10c simultaneously the a and b coefficients are

$$a = \frac{M_y - M_x\,(I_{xy}/I_x)}{I_y\,(1 - I_{xy}^2/I_xI_y)}$$

and

$$b = \frac{M_x - M_y\,(I_{xy}/I_y)}{I_x\,(1 - I_{xy}^2/I_xI_y)}$$

From which Eq. 16.8 becomes

$$q = \frac{M_y - M_x\,(I_{xy}/I_x)}{I_y\,(1 - I_{xy}^2/I_xI_y)}\,(x) + \frac{M_x - M_y\,(I_{xy}/I_y)}{I_x\,(1 - I_{xy}^2/I_xI_y)}\,(y) + \frac{P}{A} \qquad (16.11)$$

where

I_{xy} = product of inertia and may be (+) or (−)

M_x, M_y = moments defined in Fig. 16.10

x, y = distance from center of area to point where pressure is desired; may be (+) or (−)

q = soil pressure, positive = compression

I_xI_y = moments of inertia with respect to the *centroidal* x and y axes.

Note that a negative value of q indicates soil tension and Eq. 16.11 becomes invalid. For the case of a partially effective footing area the designer should use a trial procedure. However, as this will most likely be an isolated case a redesign so that the footing is totally effective will probably be more economical than the increased cost of the designer's time chargeable to that footing. Strict attention to signs is necessary to arrive at correct answers when using Eq. 16.11.

16.8 MODULUS OF SUBGRADE REACTION

The modulus of subgrade reaction (also soil "spring" constant, or coefficient of subgrade reaction) is expressed as

$$k_s = q/y$$

with units of force/length^3 (FL^{-3}), since q, the intensity of contact pressure is in FL^{-2} and the soil deformation, y, is in units of length.

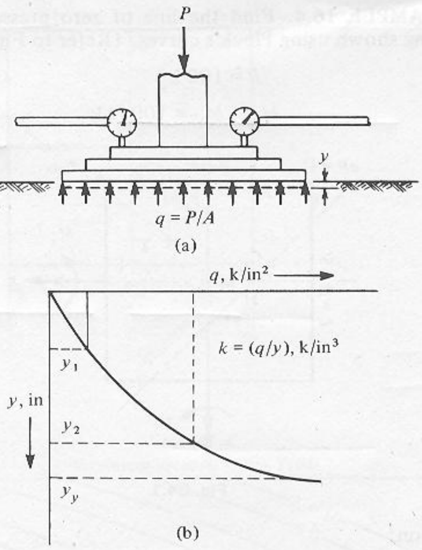


Fig. 16.11 (a) Line details of plate load test with three plates stacked. Dial gauges must be independently attached; (b) presentation of data to evaluate the modulus of subgrade reaction k_s. Note sensitivity of k_s to curve coordinates.

The modulus of subgrade reaction can be obtained by performing a plate load test and plotting a curve of q vs. y as shown in Fig. 16.11. It is difficult to load a plate uniformly (exactly axial and concentrically); in addition, the plate will bend unless it is quite rigid. To avoid the bending problem, a 30-inch-diameter plate may have stacked on it 24- and 18-inch plates. A 12-inch square or round plate may be used, but the zone of influence is so shallow ($\cong 2B$) that unless test pits are dug the results may not be very indicative of actual soil performance. Plate shape influences results also, i.e., a square plate will have a different load settlement curve than a round one.

To extrapolate results of a plate load test to the actual structure is a real problem. The plate load test for a 30-inch-diameter plate tested to 10 ksf will require a dead load apparatus of around 50–60 kips (slight allowance for inefficiency); therefore, only small plates are practical to field load test. Empirically, Terzaghi (1955) proposed the following formula for clay soils when the contact pressures are less than one-half the ultimate bearing capacity

$$k_s/k_p = B_p/B \qquad (16.12)$$

where k_p is the plate load value of the subgrade modulus, using a plate of dimension B_p; k_s is the value to use under the actual footing of width B.

On cohesionless soils it was proposed to use

$$k_s = k_p\left(\frac{B+1}{2B}\right)^2 \qquad (16.13)$$

The work of Bond (1961) and others indicates that this equation may not give reliable values of k_s for medium dense to dense sands (say $D_r > .4$). It appears that the use of this equation for sands may give a value of k_s as much as 100 percent too small over the usual range of B of 5 to 10 feet.

To obtain the modulus of subgrade reaction for a rectangular plate of dimensions B and $L = mB$ using the subgrade modulus of a square plate Terzaghi (1955) proposed

$$k_s = k_p \left(\frac{m + 0.5}{1.5}\right) \qquad (16.14)$$

To attempt to find a value of modulus of subgrade reaction using laboratory tests, several proposals have been made. Vesić (1961, 1961a) proposed using the modulus of elasticity from laboratory triaxial tests. Although Vesić did not say so, it is obvious the confining cell pressure (σ_3) should be representative of the depth of average stress influence zone (about .5B to B). The modulus of elasticity is used as

$$k_{sB} = 0.65 \sqrt[12]{\frac{E_s B^4}{E_b I}} \frac{E_s}{1 - \nu^2} \qquad (16.15)$$

where

$k_{sB} = k_s B\ (FL^{-2})$
B = width of footing
E_b = modulus of elasticity of footing
I = moment of inertia of footing
E_s = modulus of elasticity of the soil
ν = Poisson's ratio

Vesić also considered extrapolating results of a plate load test to footings of width B as follows:

(a) Solve the equation

$$\frac{E_s}{1 - \nu^2} = BI_w q/y$$

The shape factor I_w = 0.82 (see Bowles, 1968 p. 87) for a square plate and $q/y = k$ and if $B = 1$; then

$$\frac{E_s}{1 - \nu^2} = k'$$

(b) Put this value in Eq. 16.15 with $\nu^2 = 0$ and obtain

$$k_s B = .52 \sqrt[12]{\frac{k' B^4}{E_b I}}\, k' \qquad (16.16)$$

Other methods to obtain a value of modulus of subgrade reaction include using consolidation test data (Yong, 1960) as follows:

$$S = \Delta q m_v H$$

or

$$1/m_v H = \Delta q/S = q/y$$

or

$$k_s = 1/m_v H \qquad (16.17)$$

Thus, if a consolidation test is performed and the coefficient of volume compressibility m_v is evaluated, k_s can be computed by using a value of H = 0.5B to B. Tschebotarioff (1951) indicated that this is not a very highly recommended method, especially for silt soils; however, later work by Recordon (1957) indicates that this method may in fact provide reasonable values of subgrade modulus.

Another method to find k_s is to use CBR test data (Nascimento and Simoes, 1957; Recordon, 1957; Black, 1961; Barata, 1967). If we assume that the modulus of elasticity of the soil is approximately

$$E = \frac{\sigma}{2}\left(\frac{B}{y}\right) \qquad (16.18)$$

then using the equation from mechanics of materials $\delta = PL/AE = \sigma L/E$, the average stress in the influence zone of depth B, and terms consistent with this chapter in the above equation, we may assume that the average strain through the stress zone is

$$\epsilon \cong \text{penetration (say 0.05, .1, .2 in.)}/B = y/B$$

The modulus of elasticity of the CBR test can be computed for any CBR penetration value y-inches. The diameter B of the CBR piston is 1.95 inches (A = 3.00 in^2). The piston load is recorded at various penetrations and the load can be converted to the stress in Eq. 16.18 by dividing by the area of the piston. From Eq. 16.18 it is observed that $\sigma/y = k_s$; therefore

$$k_s = 2E/B = 2E/1.95 \cong E$$

If one uses the stress at 0.1-in penetration

$$\boxed{k_s = 10\ \text{CBR}} \qquad (16.19)$$

Barata (1967) shows that the value of k_s computed using the CBR test (which includes a surcharge, incidentally) should be corrected for the fact the failure zone probably interferes with the side of the 6-inch mold. After correction one may take the equivalent value of k_s for a 1-ft plate as one-half the k_s from CBR.

This chapter has presented several methods of obtaining an indication of the modulus of subgrade reaction:

(1) Plate load tests (Palmer, 1948, describes in some detail field methods).
(2) Extrapolate from triaxial tests.
(3) Extrapolate from consolidation tests.
(4) Extrapolate from CBR tests.
(5) Estimate for cohesive soils using E_s from unconfined compression test divided by B.

Major problems associated with the concept of modulus of subgrade reaction are:

(a) Soil is not elastic and results are some sensitive to curve coordinates q and y
(b) Depth and footing size effects
(c) Footing shape factor
(d) Soil stratification or other changes with depth which may not show when testing with a small plate
(e) Duplicating in-situ conditions when laboratory testing.

Values of k_s which one might expect to get range from about 50 k/ft^3 for loose wet sand to, say 2000 k/ft^3 for dense sand. In clays, values might run from 100 k/ft^3 for soft clay to 500 k/ft^3 for hard clay. Generally, one should not use a table of values, as each project becomes an individual matter depending on soil, location of water, size of foundation, etc.

The modulus of subgrade reaction would be expected to increase as a footing is placed at a greater depth in the ground if the soil modulus can be written as

$$E_s = Cz$$

Referring to Fig. 15.4 and taking the average depth of stress influence as B with an approximate average stress intensity within this zone of $q_0/2$, then

$$E_s = \frac{q_{av}}{\epsilon} = \frac{q_0/2}{y/B} = \frac{Bq}{2y}$$

But already it has been shown in Eq. 16.18 that

$$E_s \cong (B/2)\, k_s$$

Now considering a footing located at a depth D in a soil mass for which the modulus of elasticity is proportional to depth, the average modulus occurring at $B/2$ below the base of the footing is

$$E_s = C_z = C(D + B/2)$$

where C is a constant of proportionality. Rearranging, we have

$$E_s = (CB/2)(2D/B + 1)$$

Equating this value of E_s with the previous value of $E_s = (B/2)k_s$ and using k_{sd} to define the modulus of subgrade reaction at the depth of $D + B/2$ the proportionality constant is

$$C = k_{sd}/(2D/B + 1)$$

At the ground surface $D = 0$ and $k_s = k_{ss}$ and C becomes k_{ss}. Equating C-values:

$$k_{sd}/(2D/B + 1) = k_{ss}$$

$$k_{sd} = k_{ss}/(1 + 2D/B) \qquad (16.20)$$

From this it is seen that the modulus of subgrade reaction increases with depth. It should be pointed out that Eq. 16.20 depends heavily on the assumption of stress penetration into the soil and on the assumption of increase in modulus of elasticity with depth. It is doubtful that k_{sd} is much greater than $2k_{ss}$ for any D/B ratios larger than 0.5.

16.9 RIGIDITY OF CONTINUOUS FOOTINGS

The conventional analysis of footings, in general, uses the concept of a rigid footing. Borowicka's curves (Fig. 15.12) show that this situation results in a nonuniform soil pressure distribution against the footing base. Actually to have a uniform soil pressure distribution of $q = P/A$ requires a very flexible footing. Now if one accepts the concept of soil being elastic (modulus of elasticity or coefficient of subgrade reaction) the settlement of a rigid footing would be uniform and for a flexible footing the settlement would be nonuniform—but if this is the case then how can the contact pressure be uniform (recall $q = ky$)? In reality, of course, one has a soil structure interaction problem and there is nonuniform soil pressure and differential settlements within the footing. These concepts are illustrated for a continuous footing with two column loads (the footing of Example 16.1) in Fig. 16.12. From these curves one can see that for

$\lambda L < 0.8$	the member is rigid
$0.8 < \lambda L < 3$	the member is intermediate
$3 < \lambda L$	the member is flexible

where

$$\lambda L = L \sqrt[4]{\frac{k_s B}{4EI}} \qquad (16.21)$$

L = member length

B, E, I = member (footing) properties (width, modulus of elasticity, and moment of inertia) in consistent units

k_s = modulus of subgrade reaction units of FL^{-3}

The above values of λL defining a type of flexible member have also been proposed by Vesić (1961a).

From inspection of the curves of Fig. 16.12 and referring to the deflection curve we see that a *rigid* member is char-

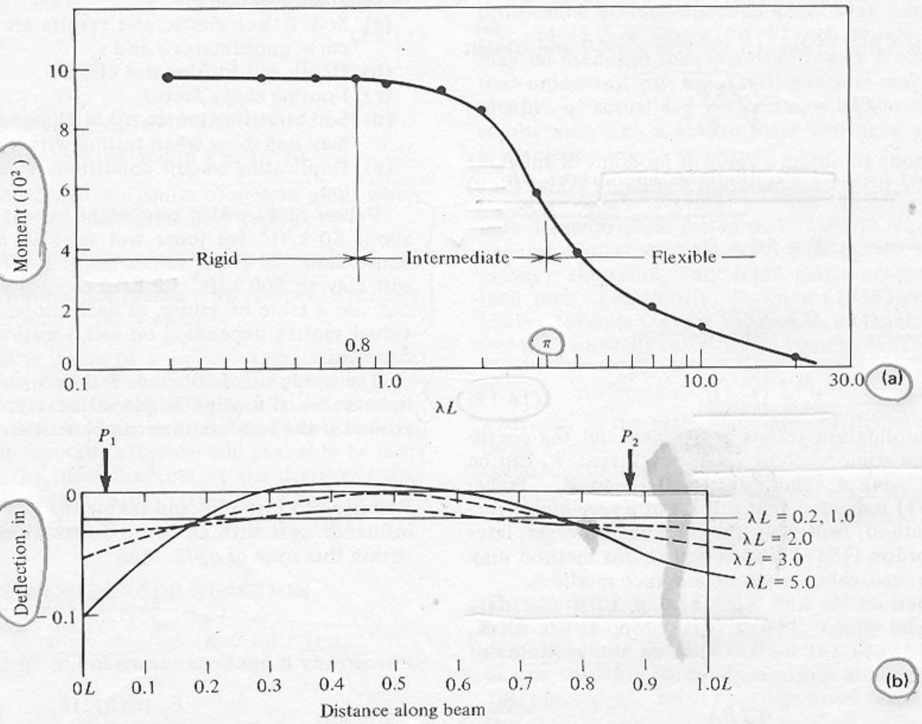


Fig. 16.12 (a) Bending moment as a function of λL (beam of Example 16-1); (b) deflections vs. λL at 0.1 points along beam of Example 16-1.

acterized by high bending moments and relatively small, uniform deflections.

An *intermediate* member, as the term implies, has intermediate bending and deflection values.

The *flexible* member has very large bending moments and deflections in the immediate vicinity of the loads and small values elsewhere.

One may deduce that the assumption of a rigid footing results in designing for assumed bending moments which are larger than the actual bending moments may be. The resulting design is conservative, generally, but may not be economical. This is illustrated in Example 16.5, which reconsiders Example 16.1 as a beam on an elastic medium.

16.10 THE CONTINUOUS FOOTING AS A BEAM ON AN ELASTIC FOUNDATION

The beam on an elastic foundation problem has occupied the attention of many investigators for some time. There seems to have emerged two branches of thought, namely, to treat the soil as a bed of springs (so-called Winkler foundation proposed by E. Winkler, ca. 1867) or as an elastic solid (Biot, 1937).

The majority of the solutions have been of rigorous and semirigorous mathematical solutions: Hetenyi (1946), De Beer (1948), Levinton (1949), Popov (1951), Gazis (1958), Ray (1958), Malter (1960), Vesić (1961), Vesić (1961a), Dodge (1964), Iyengar (1965), Reti (1967), and Szava-Kovats (1967), to cite a few sources. Many of these solutions went directly to the mathematics of the problem and usually attempted to present results in terms of influence values tabulated or charted so that the designer could, in a reasonable length of time, come up with a solution.

Only a few investigators—notably De Beer (1948a) and Vesić and Johnson (1963)—have obtained experimental data to establish any validity of the analytical solution. In general, the experimental results do not appear to be beyond normal magnitudes of error when dealing with soil. A few measured values have not been very good; however, tests run on the surface of sand and/or carried to ultimate load may yield poor results as compared to normal service loads which stress the soil from 1/3 to 1/2 of the ultimate. A look at Fig. 16.11 illustrates that beyond y_y the subgrade modulus k_s has no significance; therefore, in this range of testing one should expect to obtain considerable disagreement between experimental and theoretical results.

The major problem with the beam on an elastic foundation is to establish k_s, or E_s depending on the approach chosen by the designer. Since $E_s = f(k_s)$ one should expect similar results from either solution if the correct elastic value of the soil is used.

The basic solution for the Winkler foundation (preferred by the author because of simplicity) is as follows (refer to Fig. 16.13):

$$EI\frac{d^4y}{dx^4} = -k_s y \quad \text{(Fig. 16.13f)} \tag{16.22}$$

This is a linear fourth-order differential equation whose general solution is

$$y = e^{\lambda x}(A\cos\lambda x + B\sin\lambda x) + e^{-\lambda x}(C\cos\lambda x + D\sin\lambda x) \tag{16.23}$$

where λ is as defined in Eq. 16.21 and is used to account for the width of the footing (or beam) and as a convenience to simplify Eq. 16.22.

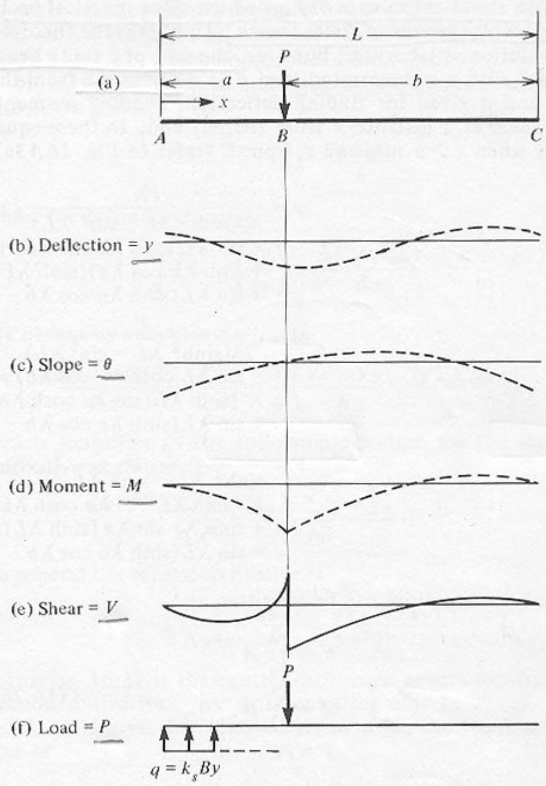


Fig. 16.13 The beam on an elastic foundation.

Successive differentiations of Eq. 16.23 yield

$$dy/dx = \theta = \text{slope}$$
$$d^2y/dx^2 = M = \text{bending moment}$$
$$d^3y/dx^3 = V = \text{shear}$$

These equations contain arbitrary constants of A, B, C, D which must be found. For the specific case shown in Fig. 16.13, boundary conditions are

Shear = 0 at $x = 0, x = L$
Shear to left of a + shear to right of $a = P$
Moment = 0 at $x = 0, x = L$
$M_{\text{left}} = M_{\text{right}}$ at $x = a$
$\theta_{\text{left}} = \theta_{\text{right}}$ at $x = a$
$y_{\text{left}} = y_{\text{right}}$ at $x = a$

The classic solution is the case of Fig. 16.13, where $a = b = \infty$. For this the boundary conditions are:

$$dy/dx = 0 \text{ at } x = 0 \text{ and } \pm\infty,$$
$$d^2y/dx^2 = 0 \text{ at } x = \pm\infty$$
$$d^3y/dx^3 = -P/2 \text{ at } x = 0 = 0 \text{ at } x = \pm\infty$$

Solving, one obtains

$$\begin{aligned} y &= \frac{P}{2k_s'}e^{-\lambda x}(\cos\lambda x + \sin\lambda x) \\ \theta &= \frac{P\lambda^2}{k_s'}e^{-\lambda x}\sin\lambda x \\ M &= \frac{P}{4\lambda}e^{-\lambda x}(\cos\lambda x - \sin\lambda x) \\ V &= -\frac{P}{2}e^{-\lambda x}\cos\lambda x \end{aligned} \tag{16.24}$$

$k_s' = k_sB$ in units of FL^{-2}

The above solution is of limited use since practical problems involve beams of finite length. To obtain the theoretical solution is laborious; however, the case of a finite beam loaded with a concentrated load P at a distance a from the left end is given for finding deflection, bending moment, and shear at a distance x from the left end. In these equations when $x > a$ measure x from C (refer to Fig. 16.13a).

$$y = \frac{P\lambda}{k_s'(\sinh^2 \lambda L - \sin^2 \lambda L)} \{2 \cosh \lambda x \cos \lambda x (\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + (\cosh \lambda x \sin \lambda x + \sinh \lambda x \cos \lambda x)[\sinh \lambda L (\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L (\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\}$$

$$M = \frac{P}{2\lambda(\sinh^2 \lambda L - \sin^2 \lambda L)} \{2 \sinh \lambda x \sin \lambda x (\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + (\cosh \lambda x \sin \lambda x - \sinh \lambda x \cos \lambda x) \times [\sinh \lambda L (\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L (\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\} \qquad (16.25)$$

$$V = \frac{P}{\sinh^2 \lambda L - \sin^2 \lambda L} \{(\cosh \lambda x \sin \lambda x + \sinh \lambda x \cos \lambda x) \times (\sinh \lambda L \cos \lambda a \cosh \lambda b - \sin \lambda L \cosh \lambda a \cos \lambda b) + \sinh \lambda x \sin \lambda x [\sinh \lambda L (\sin \lambda a \cosh \lambda b - \cos \lambda a \sinh \lambda b) + \sin \lambda L (\sinh \lambda a \cos \lambda b - \cosh \lambda a \sin \lambda b)]\}$$

The above equations can be rewritten as

$$y = \frac{P\lambda}{k_s'} A \qquad M = \frac{P}{2\lambda} B \qquad V = PC \qquad (16.26)$$

since coefficients A, B, C correspond only to trigonometric identities in Eq. 16.25. The only practical solution to a problem is to program these equations on the computer. Example 16.1 has been reworked this way with data shown in Table E5.1 so that the user may prepare and debug his computer program. Note that Table E5.1 accounts for shear to left and right of column; deflections are in feet. The shear and moment values would be multiplied by the ratio of the sum of the ultimate loads divided by the actual loads of 520 kips to convert values to M_u and V_u for strength design. Table 16.1 is a partial computer output of Eq. 16.25 for additional user assistance.

There are two factors (besides what to use for k_s') to consider:

(a) This solution implies that the soil can take tension as well as compression.

(b) The solution cannot easily adapt to a change in k_s', cavities in the base, or changes in moment of inertia I of the beam. Note: $k_s' = k_s B$.

Because of these shortcomings the element method of the next section is preferred, since all sorts of contingencies may be accounted for.

EXAMPLE 16.5. Analyze the continuous footing of Example 16.1 as a beam on an elastic foundation via Eq. 16.25. Use superposition, program on the computer for moments at the 0.1 points, and arbitrarily place the column loads as point loads at the nearest 0.1 points.

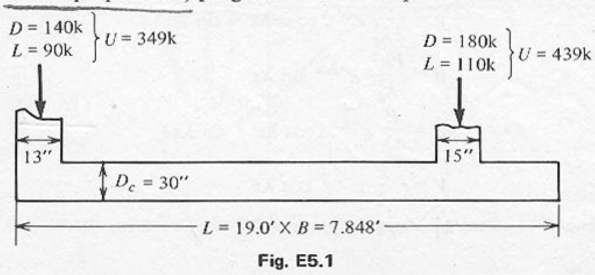


Fig. E5.1

Note: To convert to USD moments and shears multiply output values by the factor

$$\text{Factor} = \frac{349 + 430}{230 + 290} = 1.515$$

TABLE 16.1 TYPICAL COMPUTER OUTPUT DATA FROM PROGRAMMING THE TRIGONOMETRIC QUANTITIES OF EQS. 16.25. NOTE OUTPUT IS FOR 1/10 POINTS AND POINTS JUST TO LEFT AND RIGHT OF LOAD POINT ARE CONSIDERED. ONLY SUFFICIENT DATA FOR DEBUGGING A COMPUTER PROGRAM IS GIVEN.

λL = 1.0

Load at 0.0L				*Load at .5L*			
Dist	*Def*	*Mom*	*Shear*	*Dist*	*Def*	*Mom*	*Shear*
0.0	4.0378	0.0000	0.0000	0.0	.9814	0.0000	0.0000
0.0	4.0378	0.0000	−1.0000	.1	.9896	.0098	.0985
.1	3.4176	−.1616	−.6272	.2	.9976	.0394	.1979
.2	2.8004	−.2550	−.3163	.3	1.0048	.0890	.2980
.3	2.1883	−.2923	−.0669	.4	1.0102	.1587	.3988
.4	1.5820	−.2858	.1214	.5	1.0124	.2486	.4999
.5	.9814	−.2477	.2496	.5	1.0124	.2486	−.4999
.6	.3856	−.1900	.3179	.6	1.0102	.1587	−.3988
.7	−.2062	−.1245	.3268	.7	1.0048	.0890	−.2980
.8	−.7957	−.0631	.2767	.8	.9976	.0394	−.1979
.9	−1.3839	−.0177	.1677	.9	.9896	.0098	−.0985
1.0	−1.9716	0.0000	0.0000	1.0	.9814	0.0000	0.0000

λL = 3.0

Load at .2L				*Load at .5L*			
Dist	*Def*	*Mom*	*Shear*	*Dist*	*Def*	*Mom*	*Shear*
0.0	.9182	0.0000	0.0000	0.0	.0327	0.0000	0.0000
.1	.8336	.0801	.2629	.1	.1623	.0068	.0292
.2	.7324	.3101	.4986	.2	.2903	.0428	.0972
.2	.7324	.3101	−.5013	.3	.4097	.1310	.2026
.3	.5913	.0712	−.3019	.4	.5045	.2926	.3407
.4	.4359	−.0614	−.1479	.5	.5452	.5442	.4999
.5	.2903	−.1154	−.0394	.5	.5452	.5442	−.4999
.6	.1647	−.1168	.0283	.6	.5045	.2926	−.3407
.7	.0596	−.0883	.0614	.7	.4097	.1310	−.2026
.8	−.0295	−.0488	.0657	.8	.2903	.0428	−.0972
.9	−.1099	−.0145	.0446	.9	.1623	.0068	−.0292
1.0	−.1874	0.0000	0.0000	1.0	.0327	0.0000	0.0000

although each column load factor should be considered separately if the individual factors differ appreciably from the average (1.517 and 1.514 in this example for negligible difference).

The output is shown in Table E5.1. From the table it can be seen that the maximum moment is about 12 percent less than the "rigid" design value.

Input Data:

$$P_1 = 230 \text{ kips}; P_2 = 290 \text{ kips};$$
$$k_sB = 152(7.848) = 1192.96 \text{ ksf} = k_s'$$
$$D_c = 30 \text{ in.}; E_c = 3000 \text{ ksi};$$
$$I = Bt^3/12 = 10.219 \text{ ft}^4; \lambda L = 2.48.$$

TABLE E5.1 THE TOTAL DEFLECTION, MOMENT AND SHEAR AT THE 0.1 POINTS

Distance, ft	*Deflection, ft*	*Moment, ft-k*	*Shear, kips*
0.0	0.0403	0.0	0.0
0.0	0.0403	0.0	−348.4495
1.9000	0.0313	−540.4719	−225.6334
3.8000	0.0235	−875.9280	−131.9437
5.7000	0.0177	−1056.7764	−61.7413
7.6000	0.0144	−1120.5657	−7.3309
9.5000	0.0136	−1089.1458	−39.9574
11.4000	0.0153	−967.6533	88.9210
13.3000	0.0193	−744.9695	147.7496
15.2000	0.0249	−395.5894	223.2625
17.1000	0.0315	116.9417	320.0002
17.1000	0.0315	116.9418	−119.3491
19.0000	0.0380	−0.0002	0.0001

16.11 SOLUTION OF A BEAM ON AN ELASTIC FOUNDATION USING FINITE DIFFERENCES

The preceding section considered the rigorous solution of a beam on an elastic foundation. This section will consider in some detail a solution as proposed by Malter (1960) using finite differences. This method requires the solution of 10 or more simultaneous equations, thus necessitating the use of a computer. The solution can be adapted to continuous footings with any number of column loads which may include both axial loads as well as moments.

Referring to Fig. 16.14, the average slope of the elastic curve at station 3 is

$$\left(\frac{dy}{dx}\right)_3 \Rightarrow \left(\frac{\Delta y}{\Delta x}\right)_3 = \left(\frac{y_4 - y_3}{h}\right)_3$$

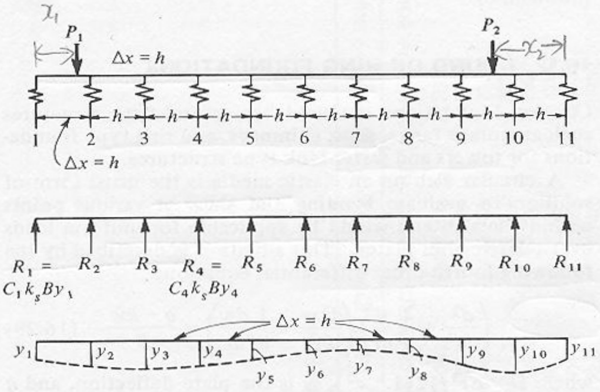


Fig. 16.14 Mathematical model for the finite difference solution for a beam on an elastic foundation. (a) Assumed loading (Winkler foundation); (b) equivalent loading; (c) deflection.

The above expression is termed a finite difference expression as Δy, Δx will assume finite values as indicated as opposed to dy/dx, which approaches a limit as $dx \to 0$.

Now, taking forward and back differences at stations 4 and 3, we obtain

$$\frac{y_4 - y_3}{h} \quad \text{and} \quad \frac{y_3 - y_2}{h}$$

Since the second derivative is

$$\frac{d^2y}{dx^2} = d\left(\frac{dy}{dx}\right) = \frac{\Delta(\Delta y/\Delta x)}{\Delta x}$$

We obtain by substitution

$$\frac{\Delta(\Delta y/\Delta x)}{\Delta x} = \frac{\Delta^2 y}{\Delta x^2} = \frac{1}{h}\left(\frac{y_4 - y_3}{h} - \frac{(y_3 - y_2)}{h}\right)$$

which simplifies to the following equation for the second derivative at station 3:

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{y_4 - 2y_3 + y_2}{h^2}$$

In general the second derivative is

$$y'' = \frac{\Delta^2 y}{\Delta x^2} = \frac{y_{n+1} - 2y_n + y_{n-1}}{(\Delta x^2)} \qquad (16.27)$$

Equation 16.27 is the *central difference* expression for the second derivative. By differentiating Eq. 16.27 one obtains the central difference expression for the third derivative as

$$y''' = \frac{y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}}{2(\Delta x)^3} \qquad (16.28)$$

These two expressions are sufficient to solve the beam on an elastic foundation problem by finite differences. There are many other forms of difference expressions, e.g., first central differences, second central differences, first forward, second forward, first backward, and second backward (see Bowles, 1968, p. 248).

From mechanics of materials one can write

$$EIy'' = M \text{ (moment)}$$
$$EIy''' = V \text{ (shear)}$$

and using finite differences and h for Δx (Fig. 16.15):

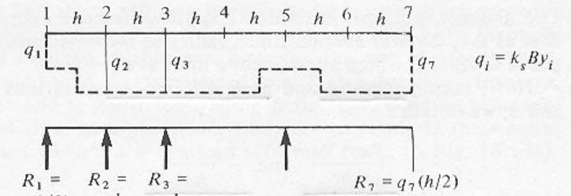
$$\frac{EI}{h^2}(y_{n+1} - 2y_n + y_{n-1}) = M_n$$

$$\frac{EI}{h^3}(y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}) = V_n$$

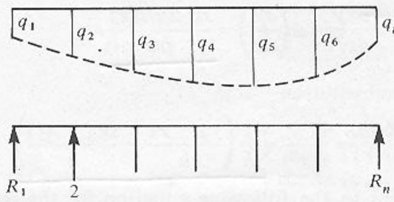
From the concept of the Winkler foundation consisting of a series of springs one can replace the foundation with a series of concentrated springs on the base of the footing as shown in Fig. 16.14b. It is recommended to use 10 divisions of $\Delta x = h$ for the beam elements. Fewer than 10 may not give good results; more divisions increases work with insignificant increase in precision. For computational simplicity make h = constant.

One may use any type of pressure distribution of soil to footing; however, the author recommends a stepped pressure distribution (see Fig. 16.15). Using the stepped pressure distribution the reactions against the beam of Fig. 16.14b become

$$R_1 = \tfrac{1}{2} k_s B h y_1$$
$$R_2 \text{ to } R_{10} = k_s B h y_i$$
$$R_{11} = \tfrac{1}{2} k_s B h y_{11}$$

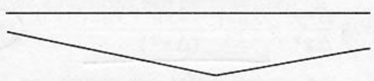


(a)



$$R_1 = h/24(7q_1 + 6q_2 - q_3)$$
$$R_2 - R_{n-1} = h/12(q_{n-1} + 10q_n + q_{n+1})$$
$$R_n = h/24(7q_n + 6q_{n-1} - q_{n-2})$$

(b)



$$R_1 = h/6(2q_1 + q_2)$$
$$R_2 - R_{n-1} = h/6(q_{n-1} + 4q_n + q_{n+1})$$
$$R_n = h/6(2q_n + q_{n-1})$$

(c)

Fig. 16.15 Various assumed soil pressure distributions and corresponding equations to compute the equivalent soil reactions on beam for finite difference solution for beam on an elastic foundation. (a) Stepped pressure distribution; (b) parabolic pressure distribution; (c) linear pressure distribution.

The bending moment at station 2 is

$$\frac{EI}{h^2}(y_1 - 2y_2 + y_3) = R_1 h - P_1(h - x_l)$$

Let

$$\frac{EI}{h^2} = C$$

Arranging for a computer solution, we have

$$Cy_1 - 2Cy_2 + Cy_3 - \tfrac{1}{2}k_sBhy_1 = P_1(h - x_l)$$
$$y_1(C - \tfrac{1}{2}k_sBh) - 2Cy_2 + Cy_3 = -P_1(h - x_l)$$

and at station 3

$$C(y_2 - 2y_3 + y_4) = R_1(2h) + R_2h - P_1(2h - x_l)$$

In like manner one proceeds through each station including 10, yielding 9 equations. Next one may sum moments at either end; thus at right end

$$R_1(10h) + R_2(9h) + R_3(8h) + R_4(7h) + R_5(6h) + R_6(5h) + R_7(4h) + R_8(3h) + R_9(2h) + R_{10}h - P_1(10h - x_l) - P_2(2h - x_2) = 0$$

Summing forces in the vertical direction gives

$$\sum_{i=1}^{i=11} R_i - P_1 - P_2 = 0$$

We have now obtained 11 equations in 11 unknown values of y. The moment and shear at each station is obtained by back substitution of the beam deflections (y's) into Eqs. 16.27 and 16.28 at each station.

EXAMPLE 16.6. Beam on an elastic foundation by finite difference: Resolve Example 16.1 and compare results.

Solution: Actual loads will be used and the results compared by dividing the output of Example 16.1 by the ratio of $\Sigma U/\Sigma P = (349 + 439)/520 = 1.515$.

Note: Actual B-dimension used for comparison purposes. Take

$$E_c = 3000.0 \text{ ksi}$$
$$I = 7.848(2.5)^3/12 = 10.219$$
$$h = 19/10 = 1.90 \text{ ft}$$
$$C = EI/h^2 = 1222883.1$$
$$k_s = 152 \ k/\text{ft}^3$$
$$k_sB = 1192.96 \ k/\text{ft}^2$$

Assume stepped pressure distribution (Fig. 16.15a).

Set up coefficient matrix:

First equation: $\Sigma M_2 = 0$

$$C(y_1 - 2y_2 + y_3) + \tfrac{1}{2}k_sBhy_1(h) - P_1(h - .542) = 0$$
$$1222883y_1 - 2445766y_2 + 1222883y_3 + 2153y_1 = 312$$
$$1225036y_1 - 2445766y_2 + 1222883y_3 = 312 \quad (1)$$

Second equation: $\Sigma M_3 = 0$

$$C(y_2 - 2y_3 + y_4) + \tfrac{1}{2}k_sBhy_1(2h) + k_sBhy_2h - P_1(2h - .542) = 0$$
$$C(y_2 - 2y_3 + y_4) + 4307y_1 + 4307y_2 - 750 = 0$$
$$4307y_1 + 1227190y_2 - 2445766y_3 + 122883y_4 = 750 \quad (2)$$

In similar manner the remaining seven equations are written. Next, sum moments about either end and set them equal to zero. Example uses *left* end for Eq. 11. For Eq. 10 $\Sigma F_v = 0$ gives $\Sigma_1^{11} R_i - \Sigma P = 0$ or

$$1133y_1 + 2267y_2 + 2267y_4 + \cdots + 2267y_{10} + 1133y_{11} = 520 \quad (10)$$

The recommended procedure is to have the coefficient matrix generated on the computer and back substitution for final output as shown (Fig. E6.1). Note there are minor differences in y-coefficients by hand and computer, but they do not affect final output. Output data is plotted in Fig. E6.2 and compared to earlier solutions of same problem.

16.12 ROUND OR RING FOUNDATIONS

Circular foundations are used for several type structures such as storage tanks, silos, chimneys, and ring type foundations for towers and water tank type structures.

A circular slab on an elastic media is the usual form of solution to evaluate bending and shear at various points within the slab and would be applicable for uniform loads over relative thin plates. This situation is described by the following fourth-order differential equation:

$$\left(\frac{d^2}{dr^2} + \frac{1}{\nu}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2} + \frac{1}{\nu}\frac{dw}{dr}\right) = \frac{q - kw}{D} \quad (16.29)$$

where $D = Et^3/12(1 - \nu^2)$; w is the plate deflection, and q is the uniform load on the plate. Timoshenko (1959) solves