Theory of Vibrations

- Definitions :

Period : The time elapsed in repeating a periodic motion once.

Cycle : Motion completed during a period is referred to as a cycle.

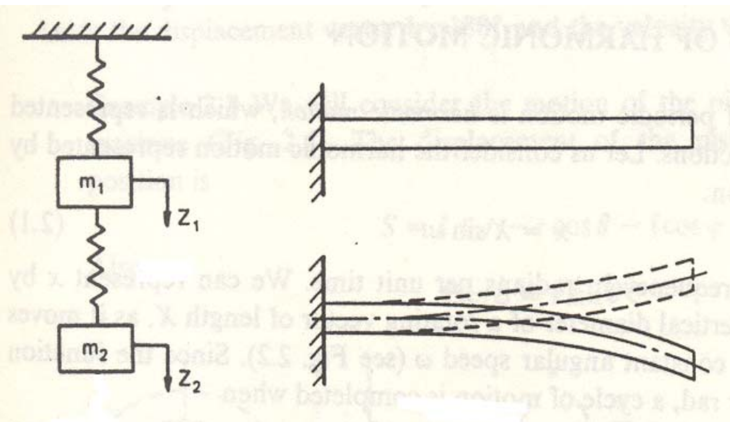
Frequency : The number of cycles of motion in a unit of time. (cf. Hz : cycle/sec)

Natural frequency : The frequency with which an elastic system vibrates under the action of forces inherent in the system.

Forced vibrations : Vibrations that occur under the excitation of external forces. Forced vibrations occur at the frequency of the exciting force.

Degrees of freedom : The number of independent coordinates necessary to describe the motion of a system.

Ex)

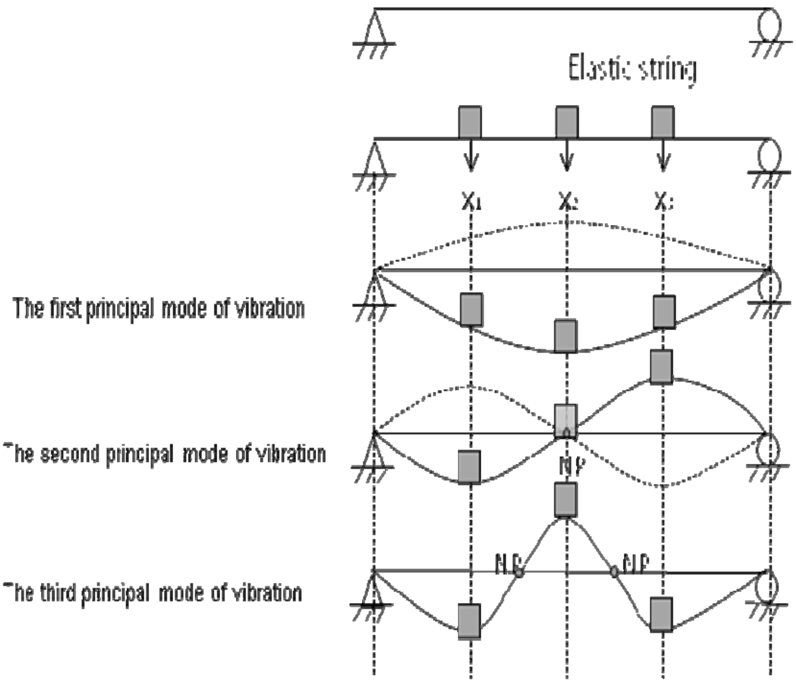


n=2 n=∞

Rosonance : If the frequency of excitation coincides with any one of the natural frequencies of the system, resonance is said to occur.

Principal modes of vibration : In a principal mode, each point in the system vibrates with the same frequency. The vibration of a multidegree freedom system can always be represented by the superposition of principal modes.

Ex)



distributed mass system
($n \rightarrow \infty$)

Concentrated mass system
($n = 3$)

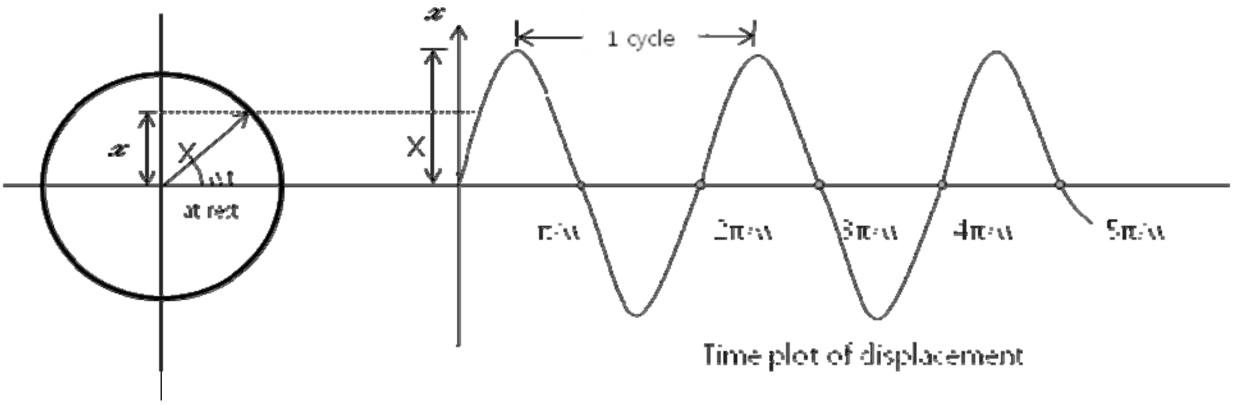
N.P = nodal point
: a point of zero deflection

Fundamental frequency : the lowest frequency among the principal
(=the frequency of the first mode) modes of the vibrating system

Normal mode of vibrations : when the amplitude of some point of the system vibrating in one of the principal modes is made equal to unity(i.e. normalized), the motion is called the normal mode of vibration.

Harmonic motion

The simplest form of periodic motion, occurring under the influence of elastic restoring force in the absence of all friction, which is represented by sine or cosine functions



(Phase plane representation of a harmonic motion)

- The harmonic motion

The displacement, $x = X \sin wt$ … ①

w : circular (angular velocity) frequency in rads / unit time

- One cycle is completed when

$$wt = 2\pi$$

t_{cy} = T = $\frac{2\pi}{w}$ T : time period of motion

then, $f = \frac{1}{T} = \frac{w}{2\pi}$ f : frequency

(reciprocal)

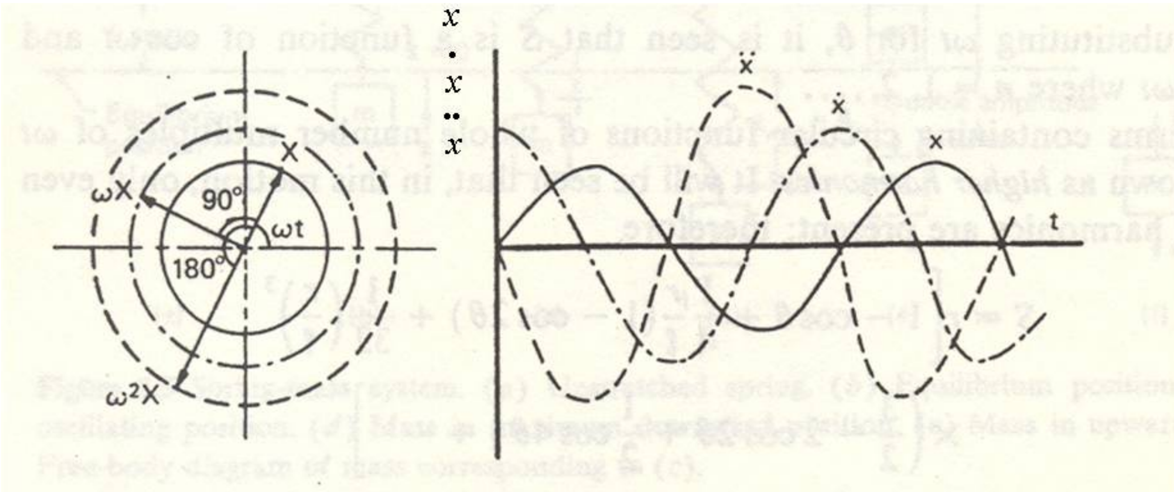
- Velocity of motion $= \dfrac{dx}{dt} = \dot{x} = wX\cos wt$

  $= wX\sin(wt + \dfrac{\pi}{2})$

- Acceleration $= \dfrac{d^2x}{dt^2} = \ddot{x} = -w^2 X\sin wt$

  $= w^2 X\sin(wt + \pi)$

  $= -w^2 x$



Displacement(x), Velocity($\dot{x}$), Acceleration($\ddot{x}$)

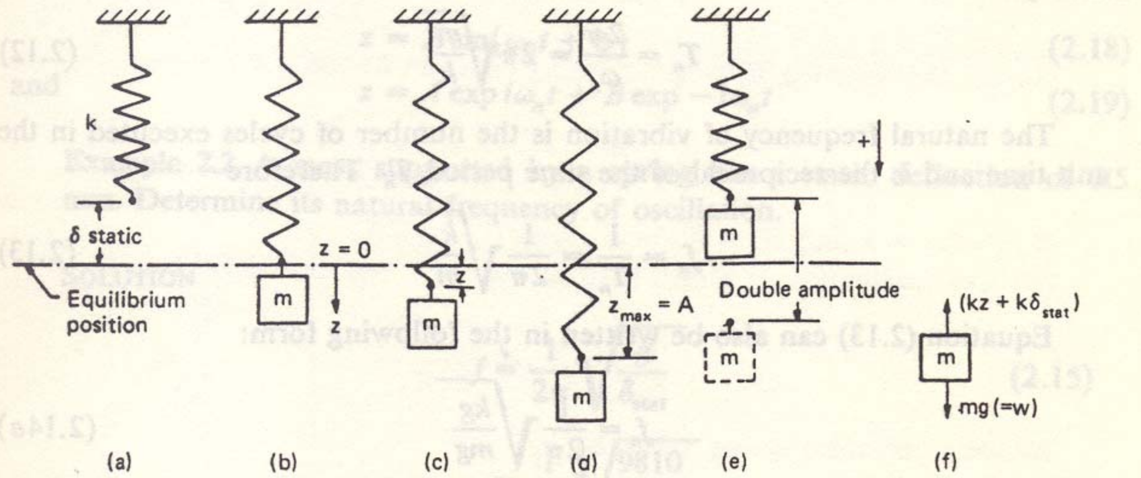
- Thus, velocity and acceleration vectors lead the displacement vector by $\dfrac{\pi}{2}$ and π rads. (these angles -> phase angles)

$z_1 = A_1\sin wt$

$z_i = A_i\sin(wt - \phi_i)$

$\begin{cases} \text{Phase lag (if } \phi > 0) \\ \text{Phase lead (if } \phi < 0) \end{cases}$

Free vibrations of a spring-mass system



Spring-mass system. (*a*) Unstretched spring. (*b*) Equilibrium position. (*c*) Mass in oscillating position. (*d*) Mass in maximum downward position. (*e*) Mass in upward position. (*f*) Free-body diagram of mass corresponding to (*c*).

- $\delta_{st} = \dfrac{w}{k}$, k : the spring constant (force per unit deflection)

- the equation of motion (according to Newton`s 2nd law, F=ma)

F=ma $\rightarrow \sum F = m\ddot{z}$

$$-(\cancel{k\delta_{st}} + kz) + \cancel{W} = m\ddot{z}$$

$\therefore$ $\boxed{m\ddot{z} + kz = 0}$ (since, $k\delta_{st}$=W) … ①

Governing Eq. (Linear 2nd order differential eq.)

- the general solution (cf. particular solution)

$$z = A\sin w_n t + B\cos w_n$$

where A, B = arbitrary constants, w_n = circular natural frequency

- Bask substitute the solution into the governing eq.

$$(-w_n^2 + \frac{k}{m})z = 0$$

$$\rightarrow w_n = \sqrt{\frac{k}{m}} \qquad (\because z \text{ is not always zero})$$

- for one cycle of motion to be completed.

$$w_n t = 2\pi \quad \rightarrow \quad t(=T_n) = \frac{2\pi}{w_n} = 2\pi\sqrt{\frac{m}{k}}$$

where, T_n = natural period

- thus, natural frequency

$$f_n = \frac{1}{T_n} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\rightarrow \text{ or, } f_n = \frac{1}{2\pi}\sqrt{\frac{k\cdot g}{m\cdot g}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \quad (\because \frac{mg}{k} = \frac{W}{k} = \delta_{st})$$

* If you know the quantity of mass(m) & the spring constant(k)

or if you know the static deflection(δ_{st}) $\rightarrow$ f_n (T_n, w_n) determined

- for particular solutions, substitute the initial conditions.

$$\text{at} \quad t=0, \quad z=z_0, \quad \dot{z}=v_0$$

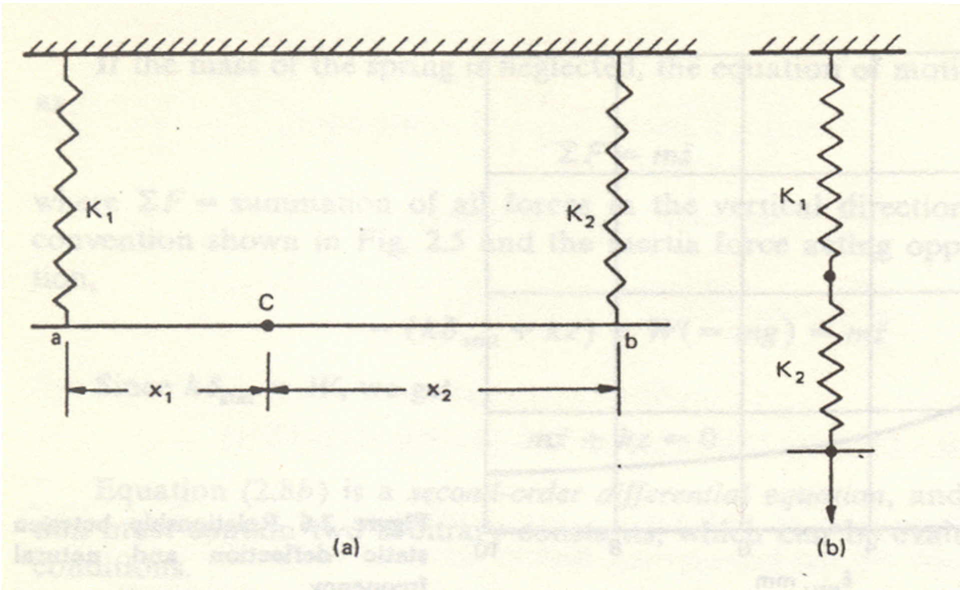
$$\rightarrow z = \frac{v_0}{w_n}\sin w_n t + z_0 \cos w_n t$$

- Other types of expressions

$$z = Z\sin(w_n t + \phi)$$

$$z = Ae^{iw_n t} + Be^{-iw_n t}$$

Ex) Equivalent Spring Constant



Spring attached in parallel in series

(a) load ratio at a and b = $x_2/(x_1+x_2)$: $x_1/(x_1+x_2)$

& deflection at a and b = $1\times x_2/(x_1+x_2)/K_1$: $1\times x_1/(x_1+x_2)/K_2$

for a unit load at c

- then, the deflection of pt.c is

$$\underbrace{\frac{x_1}{x_1+x_2}\cdot\frac{1}{K_2}}_{\text{Deflection of pt.b}} + \underbrace{\left(\frac{x_2}{x_1+x_2}\cdot\frac{1}{k_1} - \frac{x_1}{x_1+x_2}\cdot\frac{1}{k_2}\right)}_{a_{def} - b_{def}}\frac{x_2}{x_1+x_2} = \frac{1}{(x_1+x_2)^2}\left(\frac{x_1^2}{K_2}+\frac{x_2^2}{K_1}\right)$$

- from the definition $K = \dfrac{load}{deflection}$

$$K_{eq} = \frac{1(\because unit-load)}{\frac{1}{(x_1+x_2)^2}(\frac{x_1^2}{K_2}+\frac{x_2^2}{K_1})}$$

$$= \frac{(x_1+x_2)^2}{x_1^2/K_2 + x_2^2/K_1}$$

\- if $x_1 = x_2 = x$ & $K_1 = K_2 = K$

$$\rightarrow K_{eq.} = 2K$$