Theory of Vibrations

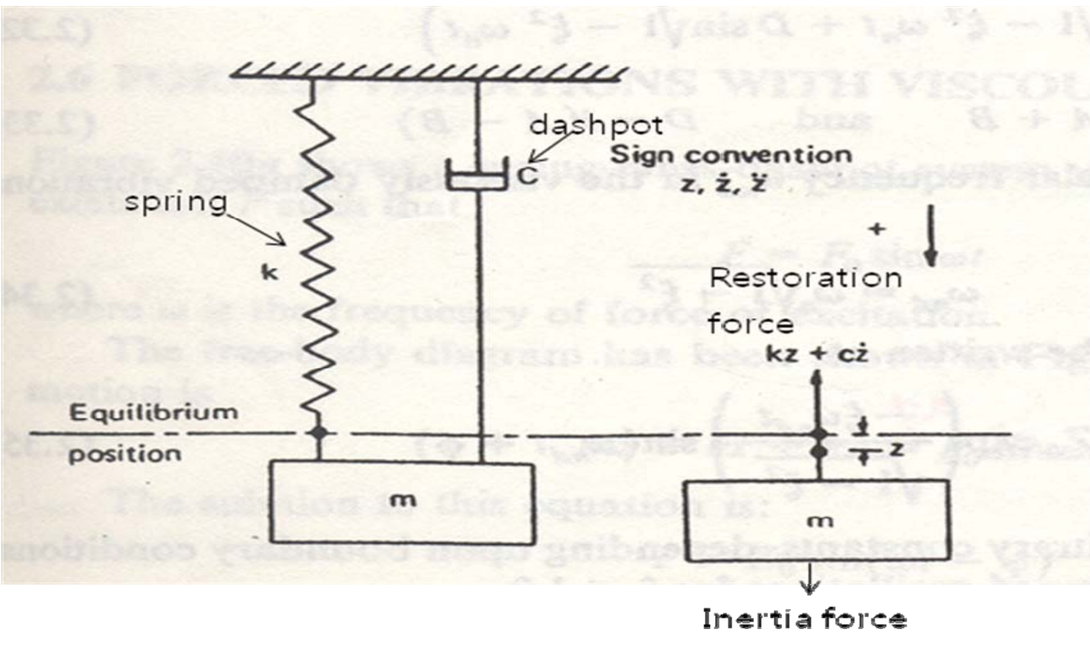
Free Vibrations w/ viscous Damping

Damping : 減 幅 ← comes from friction

= may be a function of $\dot{x}$, x or both, or $\ddot{x}$ or constant

- If the force of damping(F_d) is proportional to velocity
  → termed "viscous damping" (cf. constant coulomb friction)

$F_d = -c\dot{z}$, c : coefficient of viscous damping $[FL^{-1}T]$



- Governing equation :

$$m\ddot{z} + c\dot{z} + kz = 0 \quad \ldots\ ①$$

- Solution : try $z = e^{st}$ backsubstitute this into Eq. ①

$$(s^2 + \frac{c}{m}s + \frac{k}{m})e^{st} = 0$$

$$\rightarrow \; s_{1,2} = -\frac{c}{2m} \pm \sqrt{(\frac{c}{2m})^2 - \frac{k}{m}} \qquad \ldots. ②$$

→ the general solution is

$$z = Ae^{s_1 t} + Be^{s_2 t} \quad \ldots.③$$

- If the radical in Eq.② is zero,

i.e. $(\frac{c}{2m})^2 = \frac{k}{m} = w_n{}^2$

$\rightarrow c_c = 2mw_n$ (termed : "critical damping")

- Damping Factor (ξ) – [damping ratio, (cf. attenuation factor)]

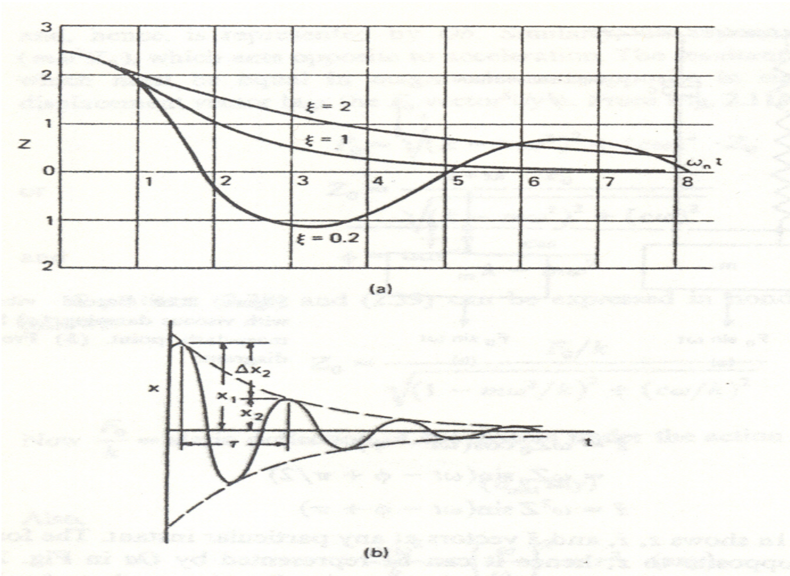
$$\xi = \frac{actual \quad damping}{critical \quad damping} = \frac{c}{c_c}$$

- modification of Eq.② in terms of ξ

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \xi w_n$$

$$\rightarrow \; s_{1,2} = (-\xi \pm \sqrt{\xi^2 - 1})w_n$$

- refer to Fig. 2.9 in text



If ξ ≥ 1, the motion is aperiodic & no oscillation

and when ξ = 1, the system returns to its original position in the minimum time

Practically, ξ < 1

→ $s_{1,2} = (-\xi \pm i\sqrt{1-\xi^2})w_n$

- the general solution becomes

$$z = \exp(-\xi w_n t)(C\cos\sqrt{1-\xi^2}w_n t + D\sin\sqrt{1-\xi^2}w_n t) \quad \ldots. \text{ ④}$$

where, C=A+B, D=i (A-B), A & B in Eq.③

- Circular natural frequency w_{nd} in the viscously damped system is

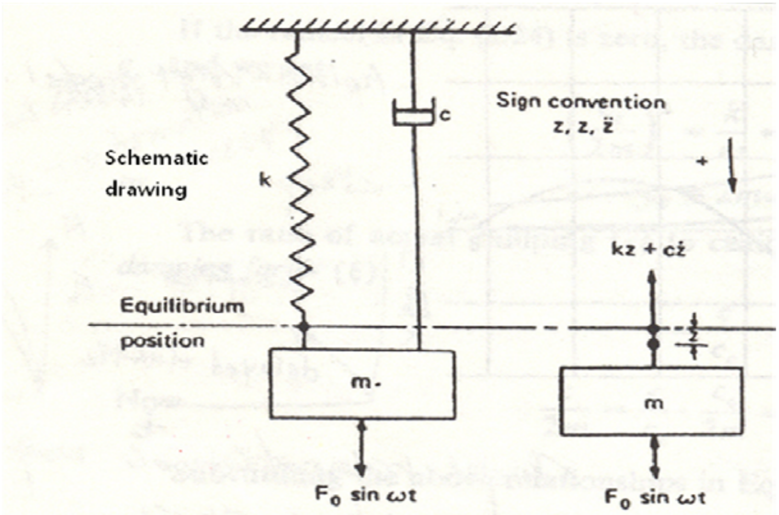
$w_{nd} = w_n\sqrt{1-\xi^2}$ [practically w_{nd} < w_n ∵ ξ < 1]

- Eq. ④ may be expressed as

$$\xi = z_0 \exp(\frac{-\xi w_{nd} t}{\sqrt{1-\xi^2}})\sin(w_{nd}t + \phi)$$

where Z_0, ϕ are arbitrary constants, which are determined from B.C (or I.C)

Forced Vibrations w/ Viscous Damping



- the force of excitation

  $F = F_0 \sin wt$

- the Governing Eq.

  $m\ddot{z} + c\dot{z} + kz = F_0 \sin wt$ …. ①

→ the particular integral to Eq. ①

Try $z = Z_0 \sin(wt - \phi)$

Then $\dot{z} = wZ_0 \cos(wt - \phi)$

$\ddot{z} = -w^2 Z_0 \sin(wt - \phi)$

- Backsubstitute these integrals into Eq. ①

  $-mw^2 Z_0 \sin(wt - \phi) + cwZ_0 \cos(wt - \phi) + kZ_0 \sin(wt - \phi)$

  $= F_0 \sin wt$ …. ②

  L.H.S of Eq. ② becomes

  $-mw^2 Z_0 (\sin wt \cos\phi - \cos wt \sin\phi)$

  $+ cwZ_0 (\cos wt \cos\phi + \sin wt \sin\phi)$

  $+ kZ_0 (\sin wt \cos\phi - \cos wt \sin\phi)$

  $= \sin wt[-mw^2 Z_0 \cos\phi + cwZ_0 \sin\phi + kZ_0 \cos\phi]$

  $+ \cos wt[mw^2 Z_0 \sin\phi + cwZ_0 \cos\phi - kZ_0 \sin\phi]$

- To satisfy the above Eq. always, we should have,

$$-mw^2\cos\phi + cw\sin\phi + k\cos\phi = \frac{F_0}{Z_0}$$

and $$mw^2\sin\phi + cw\cos\phi - k\sin\phi = \frac{0}{Z_0} = 0 \quad [\because Z_0 \neq 0]$$

→ Solve for Z_0 & ϕ

Divide the 2nd Eq. by $\cos\phi$ (i.e. $\cos\phi = 0$, i.e. $\phi = \frac{\pi}{2}$ → $Z_0 = \frac{F_0}{cw\sin\phi}$)

→ $mw^2\tan\phi + cw - k\tan\phi = 0$

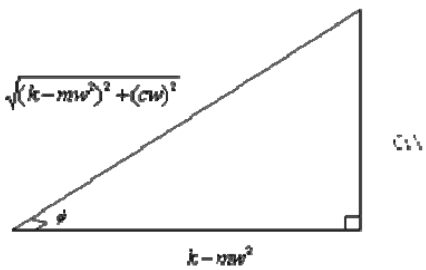
$$\tan\phi = \frac{cw}{k-mw^2}$$

$$\boxed{\therefore \phi = \tan^{-1}\frac{cw}{k-mw^2}}$$

Sub. these into the 1st Eq. i.e.

$$Z_0 = \frac{F_0}{(-mw^2\cos\phi + cw\sin\phi + k\cos\phi)}$$

→ Let`s manipulate the followings



$$\tan\phi = \frac{cw}{k-mw^2}$$

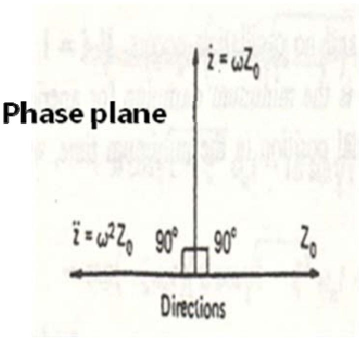
$$\sin\phi = \frac{k-mw^2}{\sqrt{(k-mw^2)+(cw)^2}}$$

$$\cos\phi = \frac{cw}{\sqrt{(k-mw^2)+(cw)^2}}$$

Sub. these into the above Eq. to obtain

$$\boxed{Z_0 = \frac{F_0}{\sqrt{(k-mw^2)^2+(cw)^2}}}$$

" Vectorial Solution of forced vibration with viscous damping"



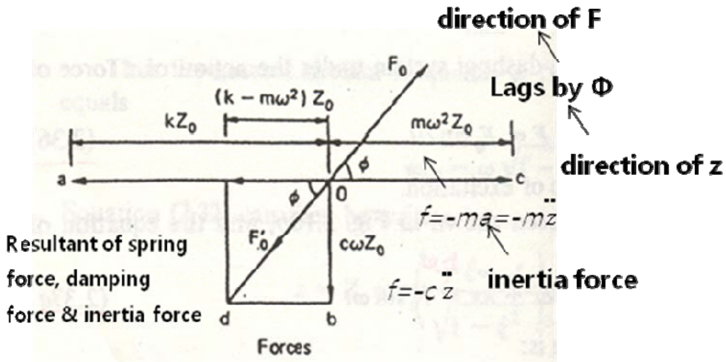


Fig.1 (a) Directions **(b) Forces**

From Fig.1(b)

$$F_0 = \sqrt{(k - mw^2)^2 + (cw)^2} \cdot Z_0$$

$$Z_0 = \frac{F_0}{\sqrt{(k - mw^2)^2 + (cw)^2}}$$

$$\phi = \tan^{-1} \frac{cw}{k - mw^2}$$

$$z = Z_0 \sin(wt - \phi) \quad (\leftarrow \ F = F_0 \sin wt \ )$$

Z lags F by ϕ (Fig 1.(b)) and the magnitude of ϕ

- The non-dimensional terms,

$$Z_0 = \frac{F_0 / k}{\sqrt{(1 - mw^2/k)^2 + (cw/k)^2}}$$

here, F_0 / k = static deflection under the ($= \delta_{st}$)
action of F_0 (at t=0, $F = F_0 \sin wt = F_0$)

→ and $\frac{mw^2}{k} = (\frac{w}{w_n})^2 = r^2$, r = frequency ratio

and $(\frac{cw}{k})^2 = (\frac{c}{c_c} \ \frac{c_c w}{m{w_n}^2})^2 = (\frac{c}{c_c} \ \frac{2mw_n w}{m{w_n}^2})^2 = (2\xi r)^2$

($k = m{w_n}^2$, $c_c = 2mw_n$, $\xi = \frac{c}{c_c}$)

$$\rightarrow \frac{Z_0}{\delta_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

Similarly, $\phi = \tan^{-1} \frac{2\xi r}{1 - r^2}$

- If undamped

$\frac{Z_0}{\delta_{st}} = \frac{1}{1 - r^2}$ = N (magnification factor)

If r (w/w_n) =1, resonance occurs

& w_n = resonance frequency (← natural frequency)

- magnification factor, N depends on ;

  1) the frequency of excitation force
  ($w \rightarrow w_n\sqrt{1-\xi^2}\,(= w_{nd})$, resonance occur)

  2) the damping factor (refer to Fig 2.12 in text)

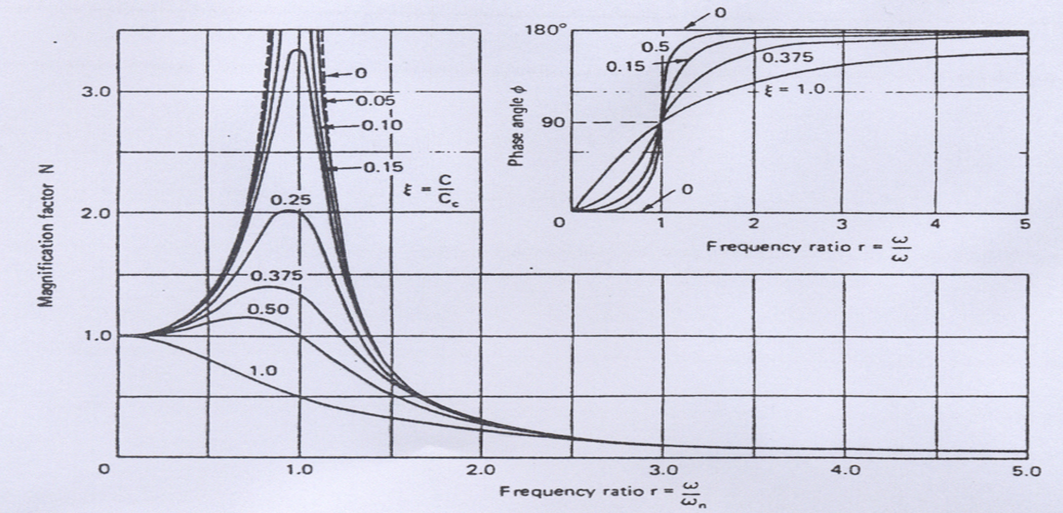
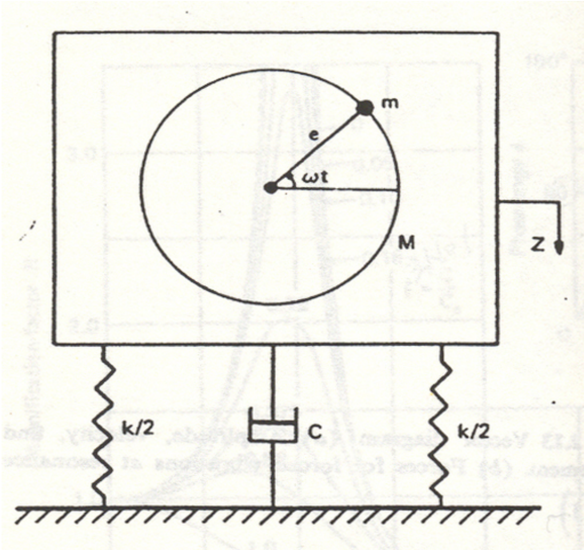


Figure 2.12 Magnification factor and phase angle in forced vibrations. (Reproduced by permission of Prentice-Hall Inc., Englewood Cliffs, N.J., from *Theory of Vibrations with Applications* by W. T. Thomson.)

* For transient vibrations, use free vibration solution

Frequency- dependent Exciting force



- the unbalanced force

  $F = m \cdot e \cdot w^2 \sin wt \quad [(m)(w^2 \cdot e)\sin wt]$

  (cf. $F_0 = \sin wt$)

  → the governing Eq. is

  $m\ddot{z} + c\dot{z} + kz = m \cdot e \cdot w^2 \sin wt$

- solution

$$z = \frac{mew^2}{\sqrt{(k - Mw^2)^2 + (cw)^2}} \sin(wt - \phi)$$

$$Z_0 = \frac{mew^2}{\sqrt{(k - Mw^2)^2 + (cw)^2}}$$

$$\tan\phi = \frac{cw}{k - Mw^2}$$

- in non-dimensional form

$$\frac{Z_0}{me/M} = \frac{r^2}{\sqrt{(1 - r^2)^2 + 4\xi^2 r^2}}$$

$$\tan\phi = \frac{2\xi r}{1 - r^2}$$

- peak amplitudes occur at $w = \dfrac{w_n}{\sqrt{1-2\xi^2}}$, & (refer to Fig 2.15(a))

  For r=1, $\dfrac{Z_0}{(me/M)} = \dfrac{1}{2\xi}$.

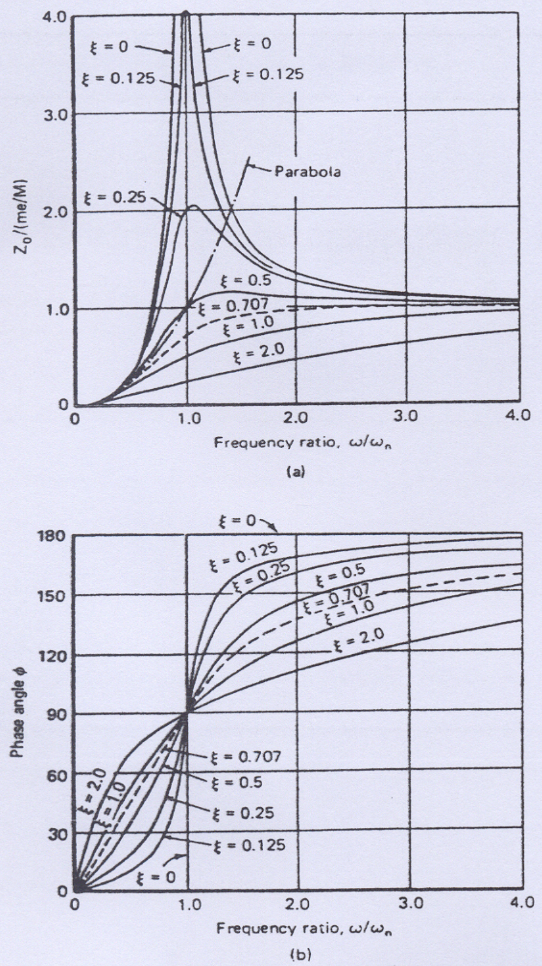


Figure 2.15 Response of system with rotating unbalances. (a) $\dfrac{Z_0}{me/M}$ vs. frequency ratio. (b) phase angle ϕ vs. frequency ratio $\dfrac{\omega}{\omega_n}$.

Transmissibility

- Forces transmitted to the foundation through the spring & the dashpot

(kZ_0) (cwZ_0)

which are out of phase at 90° (Fig 2.11)

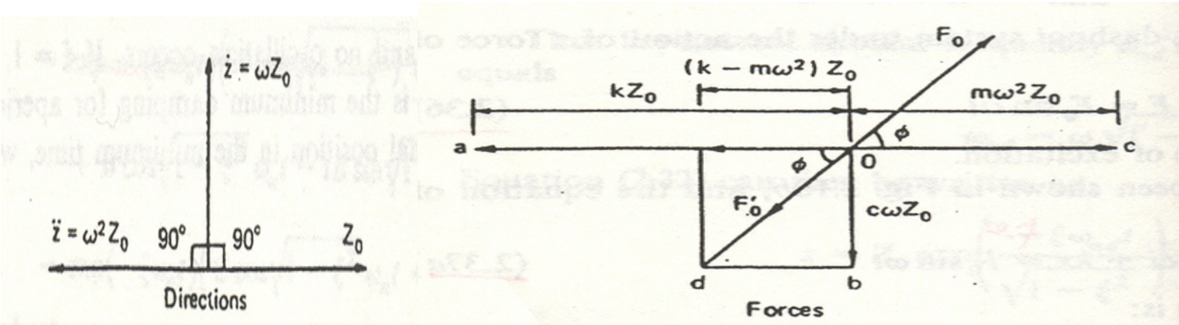


Fig 2. 11 (a) Directions (b) Forces

$$\rightarrow \; F_t = \sqrt{(kZ_0)^2 + (cwZ_0)^2}$$
$$= kZ_0 \sqrt{1 + (\frac{cw}{k})^2}$$
$$\left(= k \frac{mew^2}{\sqrt{(k - Mw^2)^2}} \sqrt{1 + (\frac{cw}{k})^2} \right)$$

- The exciting forces from machine operation

$$F_e = mew^2$$

- Transmissibility(T_r)

$$T_r = \frac{F_t}{mew^2} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

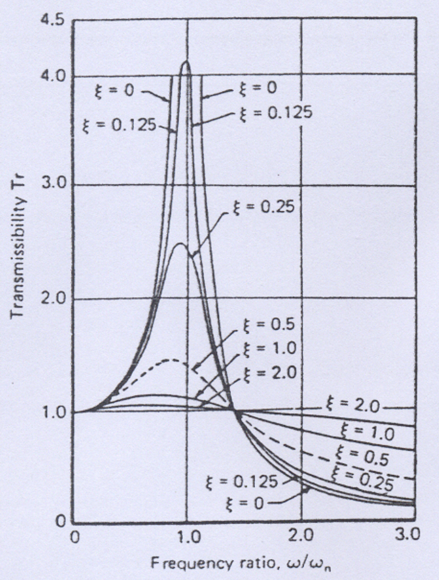


Figure 2.21 Transmissibility (T_r) vs. frequency ratio (r).

- Fig 2.21

① all curves pass through $r = \sqrt{2}$ at $T_r = 1$

② For $r > \sqrt{2}$, all approach the r axis, i.e. $T_r = 0$

→ the higher the ratio, the better the isolation

But may experience the excessive amplitude when starting & stopping