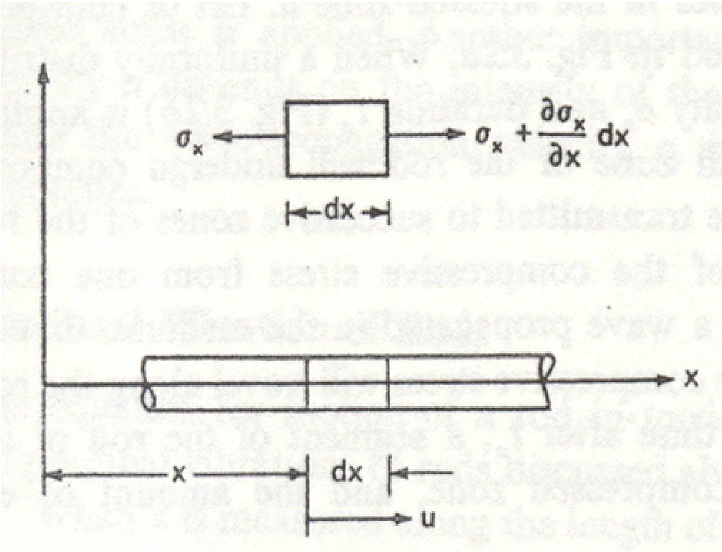
Wave propagation in an Elastic Medium

Longitudinal Vibration of Rods



A : Cross-sectional area

E : Young`s modulus

γ : Unit weight

g : gravitational acceleration

u : displacement function in x direction

$$\sum F_x = -\sigma_x \cdot A + (\sigma_x + \frac{\partial \sigma_x}{\partial x} dx) A$$

applying Newton`s 2nd law,

$$-\sigma_x \cdot A + \sigma_x \cdot A + \frac{\partial \sigma_x}{\partial x} dx \cdot A = \underbrace{\underbrace{dx\ A}_{V}\ \underbrace{\frac{\gamma}{g}}_{\rho}}_{m}\ \underbrace{\frac{\partial^2 u}{\partial t^2}}_{a}$$

i.e. $\frac{\partial \sigma_x}{\partial x} = \frac{\gamma}{g} \frac{\partial^2 u}{\partial t^2}$

and, $\sigma_x = E \frac{\partial u}{\partial x}$, ($\because \varepsilon_x = \frac{\partial u}{\partial x}$)

differentiating w. r. t. x

$$\frac{\partial \sigma_x}{\partial x} = E \frac{\partial^2 u}{\partial x^2}$$

Thus, $\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}$, $\because \frac{\gamma}{g} = \rho$ (mass density)

$$= C_r^2 \frac{\partial^2 u}{\partial x^2}$$

where, $C_r = \sqrt{\frac{E}{\rho}}$: Longitudinal wave velocity of rod

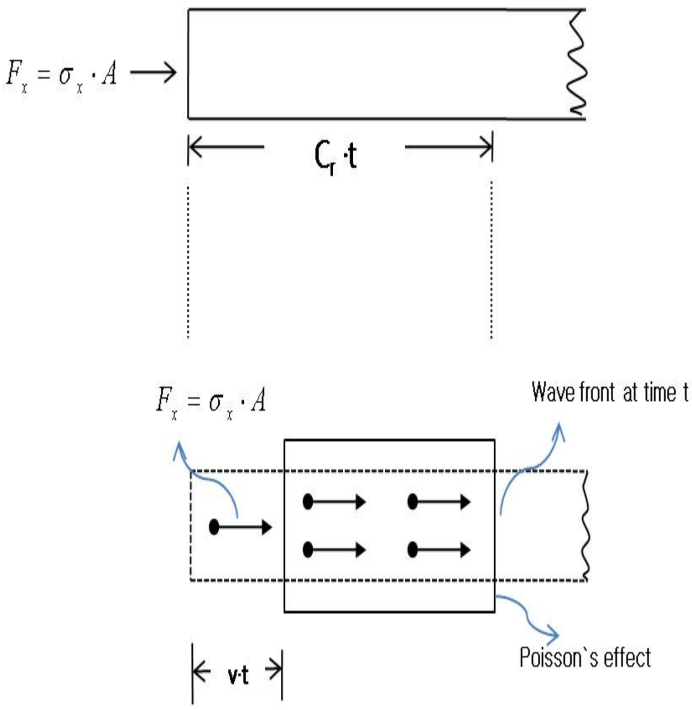
※ $\boxed{\dfrac{E}{\rho} = C^2}$ (refer to ' University Physics' by Sears & Zemansky, pp. 302, pp.115)

- Impulse-momentum principle

The vector impulse of the resultant force on a particle, in any time interval, is equal in magnitude and direction to the vector change in momentum of the particle.

$$\int_{t_1}^{t_2} \vec{F}\, dt = m\vec{v}_2 - m\vec{v}_1$$

- Calculation of longitudinal wave velocity of rod



Longitudinal momentum $= mv$

$$= \underbrace{\rho \cdot C_r \cdot t \cdot A}_{V} \cdot v$$

Longitudinal impulse

$$= \int_0^t F_x dt$$
$$= F_x t$$
$$= \sigma_x \cdot A \cdot t$$
$$= E \cdot \varepsilon_x \cdot A \cdot t$$
$$= E \cdot \frac{v}{C_r} \cdot A \cdot t \left(\because \varepsilon = \frac{\Delta L}{L}\right) \text{...Eq.①}$$

$$\rightarrow \rho \cdot C_r \cdot t \cdot A \cdot v = E \cdot \frac{v}{C_r} \cdot A \cdot t$$

$$\therefore C_r^{\,2} = \frac{E}{\rho}$$

※ Remarks

- When a wave travel in a material substance, it travels in one direction with a certain velocity (C_r), while every particle of the substance oscillates about its equilibrium position(i.e., it vibrates)
- Wave velocities depend upon the elastic properties of the substance through which it travels

Ex. $C_r = \frac{E}{\rho}$, $C_f = \frac{B}{\rho}$

[B : Bulk modulus, C_f : wave velocity of the liquid confined in a tube]

- Particle velocity(v) depends on the intensity of stress or strain induced, while C_r is only a function of the material properties.

From Eq. ① of page 2/13

$$\sigma_x = E \cdot \varepsilon_x = E \cdot \frac{v}{C_r}$$

$\rightarrow$ $v = \frac{\sigma_x \cdot C_r}{E}$ $\rightarrow$ i.e., stress dependent

- When compressive stress applied, both C_r & v are in the same direction (∵compressive σ_x $\rightarrow$ Positive), and for tensile stress, opposite direction.

Solution of Wave Equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}} \quad \ldots①$$

d'Alembert's Solution

- by the chain rule (if u is a function possessing a second derivative)

$$\frac{\partial f(x-ct)}{\partial t} = -cf'(x-ct), \quad \frac{\partial f(x-ct)}{\partial x} = f'(x-ct)$$

$$\frac{\partial^2 f(x-ct)}{\partial t^2} = c^2 f''(x-ct), \quad \frac{\partial^2 f(x-ct)}{\partial x^2} = f''(x-ct)$$

→ thus, $u = f(x-ct)$ satisfies Eq. ①

more generally,

$$u = f(x-ct) + g(x+ct) \quad \ldots②$$

Eq. ② is a complete solution of Eq. ①, i.e., any solution of ① can be expressed in the form ②

Ex. Suppose that the initial displacement of the string(rod, or anything satisfying Eq. ①) at any point x is given by $\phi(x)$, and that the initial velocity by $\theta(x)$, then (i.e., IC given)

$$u(x,0) = \phi(x) = [f(x-ct) + g(x+ct)]_{t=0} = f(x) + g(x) \quad \ldots③$$

$$\left.\frac{\partial u}{\partial t}\right|_{x,0} = \theta(x) = [-cf'(x-ct) + cg(x+ct)]_{t=0} = -cf'(x) + cg'(x) \quad \ldots④$$

Dividing Eq. ④ by c, and then integrating w. r. t. x

$$-f(x)+g(x)=\frac{1}{c}\int_{x_0}^{x}\theta(x)dx$$

Combining this with Eq. ③, [and introducing dummy variable, s]

$$f(x)=\frac{1}{2}[\phi(x)-\frac{1}{c}\int_{x_0}^{x}\theta(s)ds]\,,\quad g(x)=\frac{1}{2}[\phi(x)+\frac{1}{c}\int_{x_0}^{x}\theta(s)ds]$$

Now, $u=u(x,t)=f(x-ct)+g(x+ct)$

$$=\left[\frac{\phi(x-ct)}{2}-\frac{1}{2C}\int_{x_0}^{x-ct}\theta(s)ds\right]+\left[\frac{\phi(x+ct)}{2}+\frac{1}{2c}\int_{x_0}^{x+ct}\theta(s)ds\right]$$

$$=\frac{\phi(x-ct)+\phi(x+ct)}{2}+\frac{1}{2C}\int_{x-ct}^{x+ct}\theta(s)ds$$

Seperation of Variables

[for the undamped torsionally vibrating shaft of finite length]

$$\frac{\partial^2\theta}{\partial t^2}=a^2\frac{\partial^2\theta}{\partial x^2}$$

- Assume that $\theta(x,t)=X(x)T(t)$

then,

$$\frac{\partial^2\theta}{\partial x^2}=X''T\,,\quad \frac{\partial^2\theta}{\partial t^2}=X\ddot{T}$$

$$\rightarrow\ X\ddot{T}=a^2X''T$$

$$\rightarrow\ \frac{\ddot{T}}{T}=a^2\frac{X''}{X}=u\ \text{(constant)}$$

$$\rightarrow\ \ddot{T}=uT\ \text{ and }\ X''=\frac{u}{a^2}X$$

- Consider real values of u

$$u > 0\,,\ \ u = 0\,,\ \ u < 0$$

If $u > 0$, (we can write) $u = \lambda^2$

$$\ddot{T} = \lambda^2 T \ \rightarrow\ T = Ae^{\lambda t} + Be^{-\lambda t}$$

$$X'' = \frac{\lambda^2}{a^2}X \ \rightarrow\ X = Ce^{\lambda x/a} + De^{-\lambda x/a}$$

$$\rightarrow\ \theta(x,t) = X(x)T(t) = (Ce^{\lambda x/a} + De^{-\lambda x/a})(Ae^{\lambda t} + Be^{-\lambda t})$$

(However, this cannot describe the vibrating system because it is not periodic.)

If $u = 0$

$$\ddot{T} = 0 \ \rightarrow\ T = At + B$$

$$X'' = 0 \ \rightarrow\ X = Cx + D$$

$$\rightarrow\ \theta(x,t) = X(x)T(t) = (Cx + D)(At + B)$$

(This Eq. is not periodic either.)

If $u < 0$, we can write $u = -\lambda^2$

$$\ddot{T} = -\lambda^2 T \ \rightarrow\ T = A\cos\lambda t + B\sin\lambda t$$

$$X'' = -\frac{\lambda^2}{a^2}X \ \rightarrow\ X = C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x$$

$$\rightarrow\ \theta(x,t) = X(x)T(t) = (C\cos\frac{\lambda}{a}x + D\sin\frac{\lambda}{a}x)(A\cos\lambda t + B\sin\lambda t)$$ Eq. ①

* Periodic : repeating itself every time t Increases by $\dfrac{2\pi}{\lambda}$

→ period = $\dfrac{2\pi}{\lambda}$, frequency = $\dfrac{\lambda}{2\pi}$

λ : circular(natural) frequency

- Now, find values of λ and the constants A,B,C,D from B.C and/or I.C

These are 3 cases ;

① Both ends fixed

② Both ends free

③ One end fixed, one end free

① Both ends fixed [$\theta(0,t) = \theta(l,t) = 0$ for all t]

$$\theta(0,t) \equiv 0 = C(A\cos\lambda t + B\sin\lambda t)$$

If $A = B = 0$, satisfied, but leads to trivial solution

[$\because \theta(x,t) = 0$ at all times]

→ Let $C = 0$, then from Eq. ①

$$\theta(x,t) = D\sin\frac{\lambda}{a}x(A\cos\lambda t + B\sin\lambda t) \quad \ldots ②$$

$$\theta(l,t) \equiv 0 = D\sin\frac{\lambda}{a}l(A\cos\lambda t + B\sin\lambda t)$$

$A \neq 0, B \neq 0$ (set already)

If D=0 → leads to the trivial case again ($\because C = 0$ already)

$\therefore \sin\frac{\lambda l}{a} = 0$, or $\frac{\lambda l}{a} = n\pi$

→ $\lambda_n = \frac{n\pi a}{l}$, n = 1, 2, 3… [remember that a : wave velocity]

$$\rightarrow \theta_n(x,t) = \sin\frac{\lambda_n}{a}x(A_n\cos\lambda_n t + B_n\sin\lambda_n t)$$

$$= \sin\frac{n\pi x}{a}(A_n\cos\frac{n\pi at}{l} + B_n\sin\frac{n\pi at}{l})$$

$$\rightarrow \theta(x,t) = \sum_{n=1}^{\infty}\theta_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}(A_n\cos\frac{n\pi at}{l} + B_n\sin\frac{n\pi at}{l}) \quad \ldots ③$$

- Initial Conditions : $\theta(x,0) = f(x)$, $\left.\dfrac{\partial \theta}{\partial t}\right|_{x,0} = g(x)$

$\theta(x,0) \equiv f(x) = \sum_{n=1}^{\infty} A_n \sin\dfrac{n\pi x}{l}$ (from Eq. ③ after t=0 substituted)

$A_n = \dfrac{2}{l}\int_0^l f(x)\sin\dfrac{n\pi x}{l}dx$ [Fourier series : Euler coefficients in the half-range sine expansion of $f(x)$ over $(0,l)$]

and, $\dfrac{\partial \theta}{\partial t} = \sum_{n=1}^{\infty} \sin\dfrac{n\pi x}{l}[-A_n \sin\dfrac{n\pi at}{l} + B_n \cos\dfrac{n\pi at}{l}]\dfrac{n\pi a}{l}$

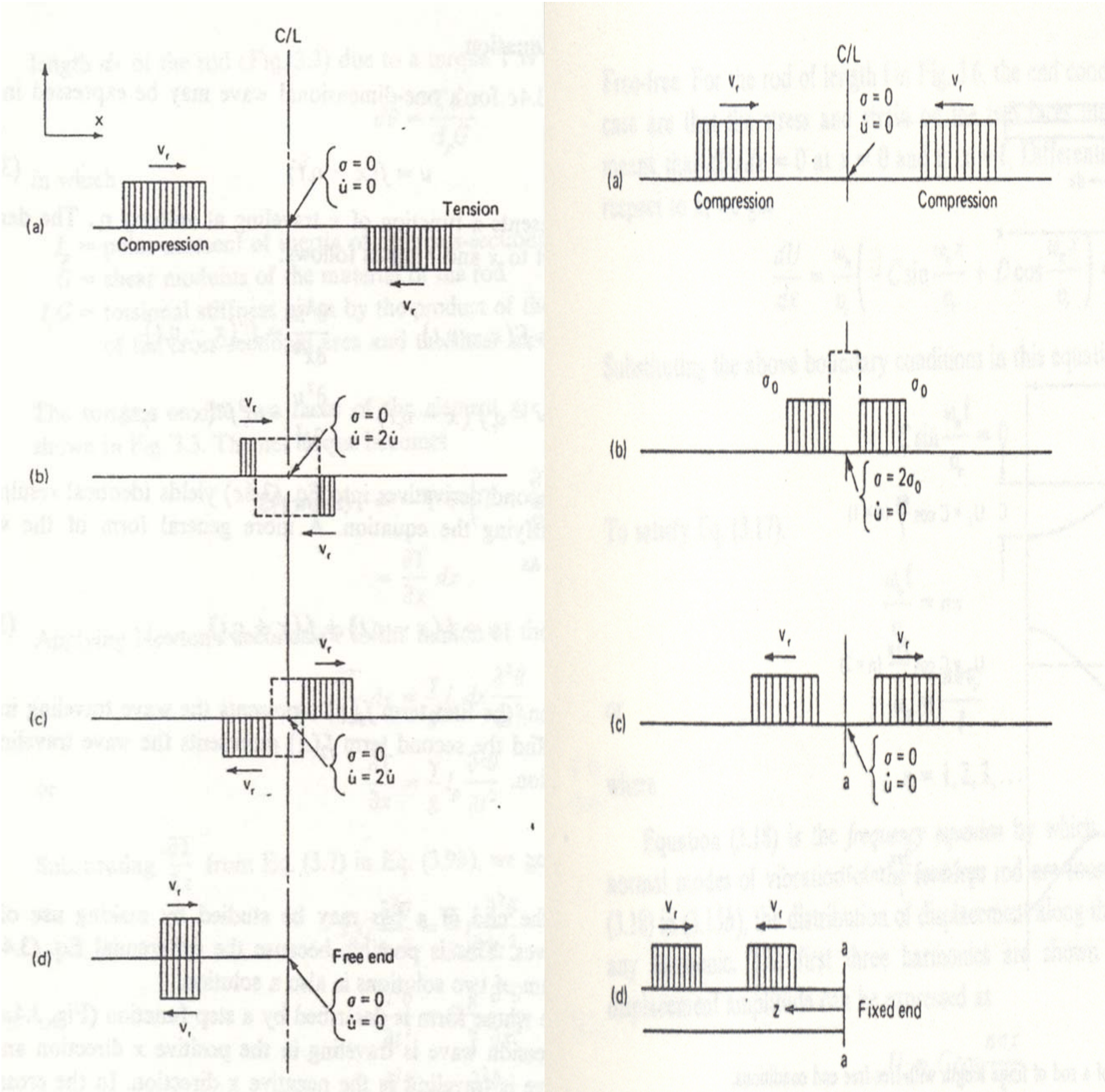
$$\left.\frac{\partial \theta}{\partial t}\right|_{x,0} \equiv g(x) = \sum_{n=1}^{\infty} (\frac{n\pi a}{l} B_n)\sin\frac{n\pi x}{l}$$

$$\rightarrow \frac{n\pi a}{l} B_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx$$

or $B_n = \dfrac{2}{n\pi a}\int_0^l g(x)\sin\dfrac{n\pi x}{l}dx$

- End Conditions for free end & for fixed end

    (linear Eq. → superposition valid)



※ In compression, wave travel & particle velocity

    → same direction

In tension → opposite direction

Experimental Determination of Dynamic Elastic Moduli

- Travel – time method:

measure the time (t_c) for an elastic wave to travel a distance l_0 along a rod.

Since, $C_r^2 = \dfrac{E}{\rho}$

$$E = \rho C_r^2 = \frac{\gamma}{g}\left(\frac{l_0}{t_c}\right)^2 \qquad \left[G = \frac{\gamma}{g}\left(\frac{l_0}{t_s}\right)^2\right]$$

Shear modulus for a torsional wave

- Resonant – column method:

A column of material is excited either longitudinally or torsionally, and the wave velocity is determined from the frequency at resonance and from the dimensions of the specimen.

[End conditions: free – free or fixed – free]

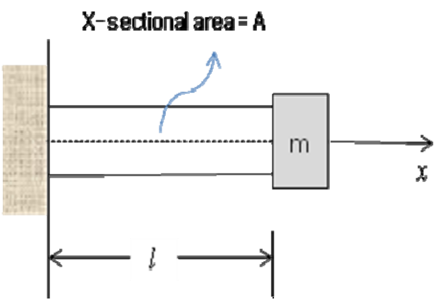
- a free – free column,

$$\omega_n = 2\pi f_n = \frac{\pi C_r}{l} \quad \text{, for n = 1} \qquad \left(\leftarrow \lambda_n = \frac{n\pi a}{l}\right)$$

$$\rightarrow C_r = 2 f_n l$$

$$\rightarrow E = \rho (2 f_n l)^2 = \frac{\gamma}{g}(2 f_n l)^2$$

- a fixed – free with a mass at the free end.



$$C_r = \frac{2\pi f_n l}{\beta} \qquad \text{where, } \beta \tan \beta = \frac{A \cdot l \cdot \gamma}{m \cdot g} = \frac{W_{rod}}{W_{mass}}$$

$$E = \rho \left(\frac{2\pi f_n l}{\beta}\right)^2$$

[refer to Vibrations of Soils & Foundations] by Richart, et. al

Waves in an Elastic –Half Space

1. Compression wave (Primary wave, P wave, dilatational wave, irrotational wave)

$$C_c = \sqrt{\frac{\lambda + 2G}{\rho}} \quad \left(> C_{rod} = \sqrt{\frac{E}{\rho}} \right) \quad \because \text{ Confined laterally}$$

, λ & G : Lame's Constants

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

- if ν=0.5 , $C_c \to \infty$

  In water-saturated soils, C_c is a compression wave velocity of water, not for soil ($\because$ water relatively incompressible)

2. Shear wave (Secondary wave, S wave, distortional wave, equivoluminal wave)

$$C_s = \sqrt{\frac{G}{\rho}} \quad \left(= C_{rod} = \sqrt{\frac{G}{\rho}} \right)$$

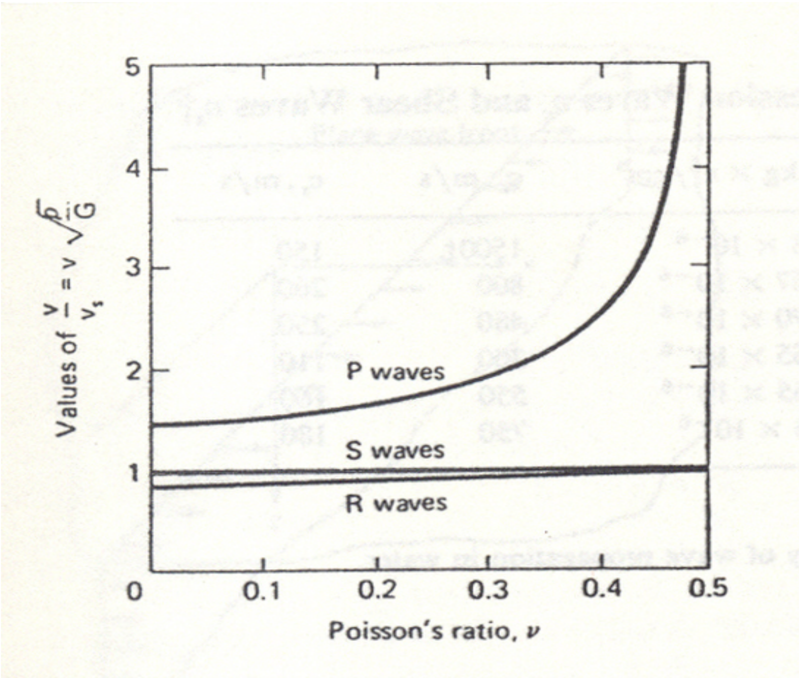
- In water – saturated soils, C_s represents the Soil properties only, since water has no shear strength (i.e. → $G = 0$)

  Thus, in field experiments, shear wave is used in the determination of soil properties.

3. Rayleigh wave (R wave)

C_R : refer to Fig. 3.10 → practically the same with C_S

The elastic wave which is confined to the neighborhood of the surface of a half space



4. Love wave (exists only in layered media)

a horizontally polarized shear wave trapped in a superficial layer and propagated by multiple total deflection (Ref : Kramer pp. 162 ~ 5.3.2)

- Remarks

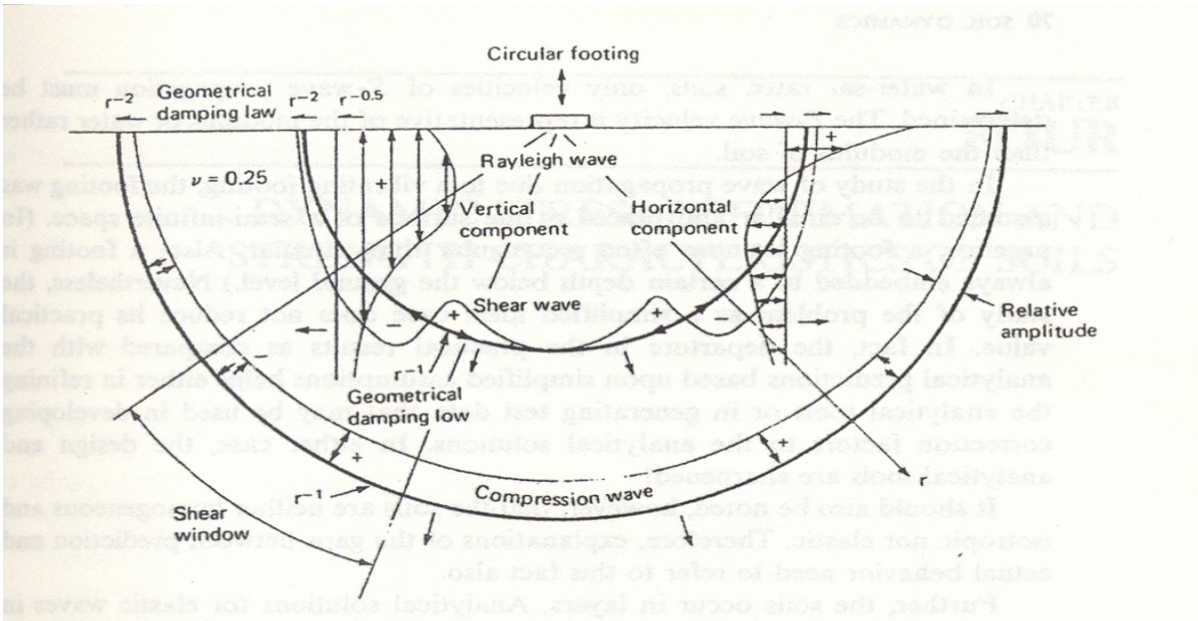


Figure 3.13 Distribution of displacement waves from a circular footing on a homogeneous, isotropic, elastic half space. (*After Woods, 1968.*)

- The distribution of total input energy ;

  R-wave (67%), S-wave (26%), P-wave(7%)

- Geometrical damping : (or Radiation damping)

  All of the waves encounter an increasingly larger volume of material as they travel outward

  → the energy density in each wave decrease with distance from the source

  → this decrease in energy density (i.e., decrease in displacement amplitude) is called geometrical damping

- Attenuation of the waves by geometric damping

  Body waves (P , S) $\propto \frac{1}{r}$

  " Body waves " on the surface $\propto \frac{1}{r^2}$

  R – wave $\propto 1/\sqrt{r}$ $\longrightarrow$ i.e., decay the slowest

  $\therefore$ R – wave is of primary concern for foundations on or near the surface of earth ($\because$ 67% & $1/\sqrt{r}$)