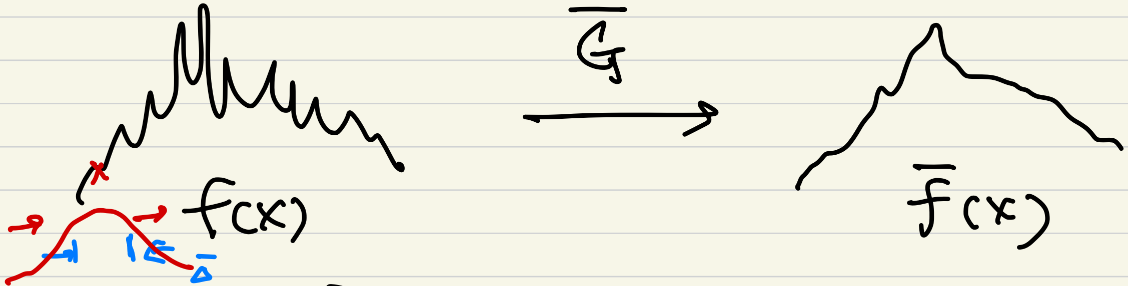
Large eddy simulation

① Compromise between DNS and RANS

② Filtering

→ is an operation which damps out all spatial fluctuations of flow variables smaller than a prescribed length scale

filter width



$\overline{(\cdot)}$: filtered variable

- $\bar{f}(\underline{x}) = \int f(\underline{x}') \underbrace{\bar{G}(\underline{x}, \underline{x}')}_{\text{filter}} d\underline{x}'$: convolution integral

- Gaussian filter : $\bar{G}(x_i - x_i') = \left(\dfrac{6}{\pi \bar{\Delta}}\right)^{\frac{1}{2}} \exp\left[-6 \dfrac{(x_i - x_i')^2}{\bar{\Delta}^2}\right]$

$\hat{G}(k) = \exp\left[-\dfrac{k^2 \bar{\Delta}^2}{24}\right]$

↑ wavenumber

$\bar{\Delta}$: filter width

(sharp) cut-off filter : $\bar{G}(x_i - x_i') = \dfrac{\sin[\pi(x_i - x_i')/\bar{\Delta}]}{\pi(x_i - x_i')}$

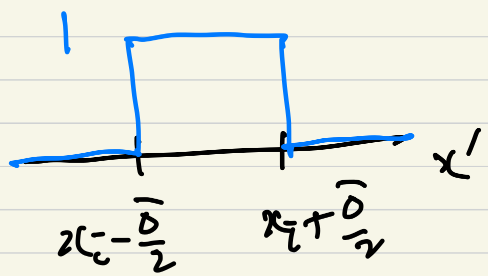
$\bar{G}(k) = \begin{cases} 1 & \text{for } k \le k_c \; (= \frac{\pi}{\bar{\Delta}}) \\ 0 & \text{for } k > k_c \end{cases}$

cut-off wave-number

→ removes modes w/ wavenumber $k > \frac{\pi}{\bar{\Delta}}$

appropriate for spectral method.

Box filter : $\bar{G}(x_i - x_i') = \begin{cases} 1 & \text{for } x_i - \frac{\bar{\Delta}}{2} < x_i' < x_i + \frac{\bar{\Delta}}{2} \\ 0 & \text{otherwise} \end{cases}$



$\bar{G}(k) = \dfrac{\sin(\frac{1}{2}k\bar{\Delta})}{\frac{1}{2}k\bar{\Delta}}$

appropriate for FDM, FVM, FEM

commonly used

③ Filtering the Navier-Stokes eqs.

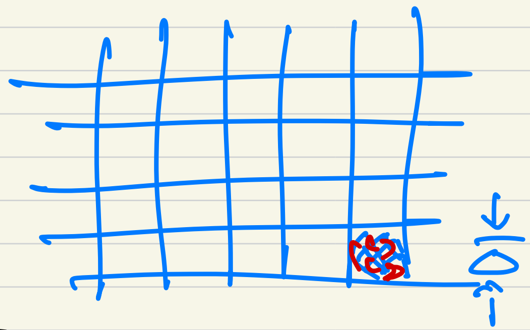
Apply a filter to N-S eqs.

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \bar{u}}{\partial x}$$

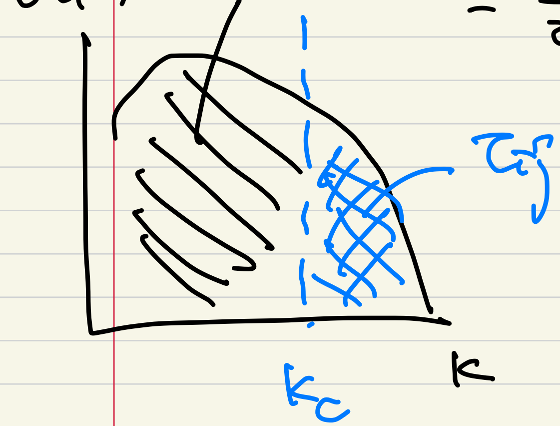
$$\begin{cases} \dfrac{\partial \bar{u}_i}{\partial x_i} = 0 \\ \rho \dfrac{\partial \bar{u}_i}{\partial t} + \rho \dfrac{\partial}{\partial x_j} \overline{u_i u_j} = -\dfrac{\partial \bar{p}}{\partial x_i} + \mu \nabla^2 \bar{u}_i \end{cases}$$

$\bar{u}_i$: filtered velocity

$\bar{p}$: filtered pressure



$$= \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) + \frac{\partial}{\partial x_j}(\overline{u_i u_j} - \bar{u}_i \bar{u}_j)$$



$$\tau_{ij} \equiv \overline{u_i u_j} - \bar{u}_i \bar{u}_j$$

should be modeled

← residual stress tensor

(subgrid scale stress tensor

$$\Rightarrow \rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \underbrace{\mu \nabla^2 \bar{u}_i}_{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{u}_i}{\partial x_j}\right)} - \rho \frac{\partial \tau_{ij}}{\partial x_j}$$

④ $\boxed{\text{Smagorinsky model}}$ - eddy viscosity model

$$\tau = 2\mu S_{ij}$$

$$\boxed{\tau_{ij} = -2\nu_t \bar{S}_{ij}}$$

eddy viscosity of residual motions
subgrid-scale

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \qquad \overline{(\cdot)} = \overline{(\cdot)}(\underline{x}, t)$$

mixing-length hypothesis

$$\nu_t = l_s^2 \bar{S} \qquad \bar{S} = \sqrt{2\bar{S}_{ij}\bar{S}_{ij}} \text{ : filtered rate of strain}$$

$$= (C_s \bar{\Delta})^2 \bar{S} \geq 0 \qquad \bar{\Delta} \text{ : filter width}$$

Smagorinsky constant

ex) grid size

l_s : Smagorinsky length scale
(similar to mixing length)

the rate of transfer of energy to the residual motions

$$P_r = -\tau_{ij}\bar{S}_{ij} = 2\nu_t \bar{S}_{ij}\bar{S}_{ij} = \nu_t \bar{S}^2 \geq 0$$

$\Rightarrow$ no backscatter

energy transfer from small scales to large scales.

with $\bar{\Delta}$ in the inertial subrange

(i.e., $l_{DI} < \bar{\Delta} < l_{EI}$)

P_t is balanced by the dissipation ε

$$\rightarrow \varepsilon = \langle P_t \rangle = \langle \nu_t \bar{S}^2 \rangle = l_s^2 \langle \bar{S}^3 \rangle$$

Using the Kolmogorov spectrum ($E \sim k^{-5/3}$)

isotropic turb.

and $\langle \bar{S}^3 \rangle \doteq \langle \bar{S}^2 \rangle^{3/2}$,

$$C_s = l_s / \bar{\Delta} \simeq 0.17$$

$\bar{\Delta} = \pi / k$

Damping ft. is required near the wall for C_s (Moin & Kim 1982)

$$\boxed{C_s \bar{\Delta} \rightarrow C_s \bar{\Delta} \left[1 - \exp(y^+ / A^+) \right]}$$

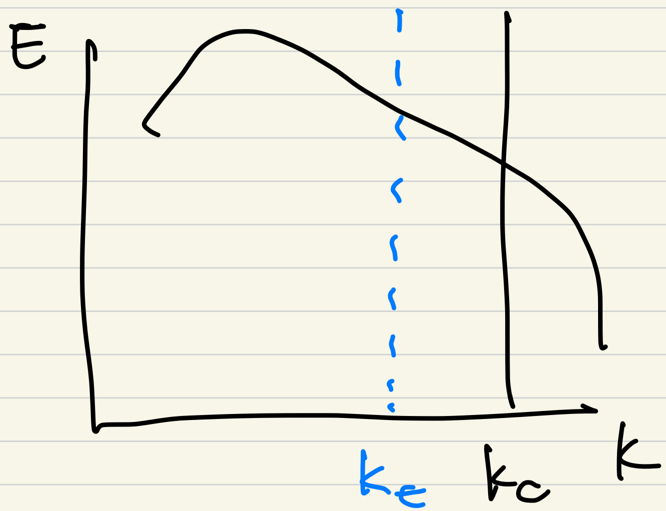
van Driest damping ft.

Problems : ① we don't know C_s

② $\tau_{ij} \neq 0$ for laminar flow.

⑤ Dynamic procedure of obtaining C_s

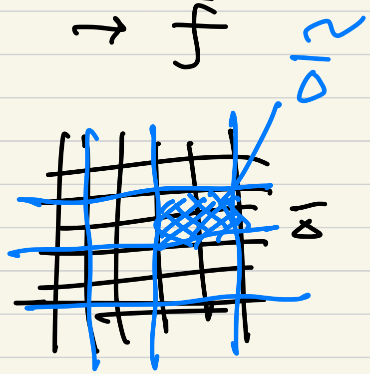
Dynamic subgrid-scale model (Germano et al. 1991)



$\pi/\bar{\Delta} = k_c$: cut-off filter $\rightarrow \bar{f}$
grid "

$\pi/\tilde{\bar{\Delta}} = k_t$: test " $\rightarrow \tilde{\bar{f}}$

$k_t < k_c \rightarrow \tilde{\bar{\Delta}} > \bar{\Delta}$



Apply two filters to N-S eqs.

$$\frac{\partial \tilde{\bar{u}}_i}{\partial x_i} = 0$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j} \tilde{\bar{u}}_i \tilde{\bar{u}}_j = -\frac{\partial \tilde{\bar{p}}}{\partial x_i} + \mu \nabla^2 \tilde{\bar{u}}_i - \frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} = \widetilde{\overline{u_i u_j}} - \widetilde{\overline{u_i}}\,\widetilde{\overline{u_j}}$$

Let $L_{ij} \equiv \widetilde{\overline{u_i}\,\overline{u_j}} - \widetilde{\overline{u_i}}\,\widetilde{\overline{u_j}}$: computable quantity

$$\tau_{ij} = \overline{u_i u_j} - \overline{u_i}\,\overline{u_j}$$

$$\rightarrow \boxed{L_{ij} = T_{ij} - \widetilde{\tau_{ij}}} \quad ①$$ Germano identity

② $(C_s \overline{\Delta})^2$

modeling : $\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2C\overline{\Delta}^2 |\overline{S}|\,\overline{S}_{ij}$ $\quad C = C_s^2$

$$T_{ij} - \frac{1}{3}\delta_{ij}T_{kk} = -2C\widetilde{\overline{\Delta}}^2 |\widetilde{\overline{S}}|\,\widetilde{\overline{S}}_{ij} \quad — ③$$

② & ③ $\longrightarrow$ ① :

$$L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} = -2C\underbrace{\left[\widetilde{\overline{\Delta}}^2 |\widetilde{\overline{S}}|\,\widetilde{\overline{S}}_{ij} - \widetilde{\overline{\Delta}^2 |\overline{S}|\,\overline{S}_{ij}}\right]}_{=M_{ij}}$$

$$\rightarrow C(\underline{x}, t) = \frac{L_{ij} - \frac{1}{3}\delta_{ij}L_{kk}}{M_{ij}} \quad ?$$

least square method : $Q = \left(L_{ij} - \frac{1}{3}\delta_{ij}L_{kk} + 2cM_{ij}\right)^2$

$\frac{\partial Q}{\partial c} = 0 \rightarrow \boxed{c = -\frac{1}{2}\frac{L_{ij}M_{ij}}{M_{ij}M_{ij}}}$ dynamic Smagorinsky constant

$c(x,t)$

Lilly (1992, PoF)

very large fluctuations of c

including negative values → unstable

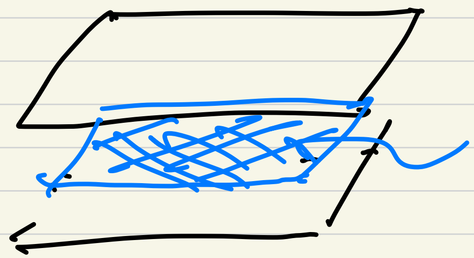
$$\Rightarrow \quad c = -\frac{1}{2}\frac{\langle L_{ij}M_{ij}\rangle_h}{\langle M_{ij}M_{ij}\rangle_h}$$

h: homogeneous directions

averaging

clipping $c<0 \rightarrow c=0$

or $c<0$ but $\nu_t + \nu > 0$



$h \Rightarrow c(y,t)$

$\cancel{c(x,y,z,t)}$

- Vreman model (2004, PoF)

$$\tau_{ij} = -2\nu_t \overline{S}_{ij}$$

$$\nu_t = C_v \sqrt{\Pi_\beta / \overline{\alpha}_{ij}\overline{\alpha}_{ij}}$$

zero for 13 laminar shear flows among 320 possible velocity derivative matrices

$$\overline{\alpha}_{ij} = \partial \overline{u}_j / \partial x_i \,, \quad \Pi_\beta = \beta_{11}\beta_{22} - \beta_{12}^2 + \beta_{11}\beta_{33} - \beta_{13}^2 + \beta_{22}\beta_{33} - \beta_{23}^2 \,,$$

$$\beta_{ij} = \sum_{m=1}^{3} \overline{\Delta}_m^2 \overline{\alpha}_{mi} \overline{\alpha}_{mj}$$

$$\overline{\alpha}_{ij} = \begin{pmatrix} \overline{\alpha}_{11} & \overline{\alpha}_{12} & \overline{\alpha}_{13} \\ \overline{\alpha}_{21} & \overline{\alpha}_{22} & \overline{\alpha}_{23} \\ \overline{\alpha}_{31} & \overline{\alpha}_{32} & \overline{\alpha}_{33} \end{pmatrix}$$

$C_v = 0.07$ for isotropic turbulence

is not universal

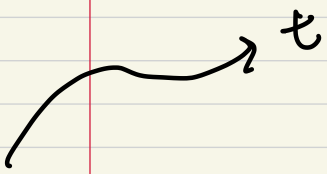
→ dynamic procedure to determine C_v is required as done for Smagorinsky model

Park et al. (PoF 2006), Lee et al. (PoF 2010)

$C_s(\underline{x}, t) \to$ unstable $\to \langle C_s \rangle_{homo}$ $\Leftarrow$ cannot be used for complex geometry because there is no homo. direction

To overcome this problem,

dynamic localization model (Ghosal et al. 1995)



Lagrangian dynamic model (Meneveau et al. 1996)

Vreman model $\to C_v \to$ dynamic $C_v \to \langle C_v \rangle_{volume}$

$C_v(t)$

dynamic global model $\leftarrow$ stable sol.

many subgrid-scale models are available!