3 Random Variables

Example .

Tossing Two Dice - we may be interested in the sum of the two equal to Seven rather than (1, 6), (2, 5), ..., (6, 1).

Example .

Coin Flipping - We may be interested in the total # of Heads (H), rather than the actual Head and Tail sequence.

These real valued functions defined on the sample space are called Random Variables.

Example .

Tossing 3 fair coins.

Y = # of Heads: Random Variable taking one of the values 0, 1, 2, 3.

$$P\{Y=0\} = P\{(TTT)\} = \frac{1}{8}$$

$$P\{Y=1\} = P\{(TTH)(THT)(HTT)\} = \frac{3}{8}$$

$$P\{Y=2\} = \frac{3}{8}$$

$$P\{Y=3\} = \frac{1}{8}$$

$$P\left(\bigcup_{i=0}^{3}\{Y=i\}\right) = \sum_{i=1}^{3} P\{Y=i\} = 1$$

Example .

Choose 3 balls from an urn containing 3 white, 3 red, 5 black.

We win \$1 for each white, and lose \$1 for each red.

X = our total wins; random variable taking on 0, ±1, ±2, ±3.

$$P\{X=0\} = \frac{\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1}}{\binom{11}{3}} = \frac{55}{165} \qquad \text{[Remarks: BBB WRB]}$$

$$P\{X=1\}=P\{X=-1\}=\frac{\binom{5}{2}\binom{3}{1}+\binom{3}{2}\binom{3}{1}}{\binom{11}{3}}=\frac{39}{165}$$ [Remarks: BBW WWR]

$$P\{X=2\}=P\{X=-2\}=\frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}}=\frac{15}{165}$$ [Remarks: WWB]

$$P\{X=3\}=P\{X=-3\}=\frac{\binom{3}{3}}{\binom{11}{3}}=\frac{1}{165}$$ [Remarks: WWW]

The Probability that we win money

$$= \mathrm{P}\{\mathrm{X}>0\}=\sum_{i=1}^{3}P\{X=i\}=\frac{55}{165}=\frac{1}{3}$$

Definition:

For a random variable X, the function F defined by

$$F(x) \;= P\{X \leq x\} \qquad\qquad -\infty < x < \infty$$

is called the (cumulative) distribution funtion of X. The probability that the random variable is less than or equal to x.

- If a $\leq$ b, then $\{\mathrm{X} \leq \mathrm{a}\} \subset \{\mathrm{X} \leq \mathrm{b}\}$

$\Rightarrow$ F(a) $\leq$ F(b).

$\therefore F(x)$ is a non-decreasing funtion of x.

• Discrete Random Variable: A random variable that can take at most a countable number of possible values.

- For a discrete random variable X, we define the probability mass funtion, $p(a)$ of X by

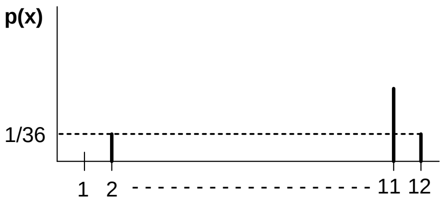
$p(a)$ = P{X = a}

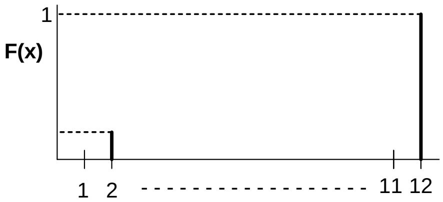
- If X assumes one of the values x_1, x_2, x_3, ..., then

$P(x_i) \geq 0 \qquad i$ = 1, 2, ...

$P(x) = 0 \qquad$ all other values of x

$$\text{and,} \quad \sum_{i=1}^{\infty} \mathrm{P}\{\mathrm{X} = \mathrm{i}\} = 1.$$





Definition:

If X is a discrete random variable with a probability mass funtion $p(x)$, the expectation or expected value of X is denoted by E[X] and is define by,

$$\mathrm{E}[\mathrm{X}] = \sum_{x:p(x)>0} xp(x)$$

$*$ weighted average of the possible values, each value being weighted by the probability.

Example .

Indicator variable for the event A

$$\mathrm{I} = \begin{cases} 1 & \text{if A occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

E(I) = ?

Solution.

$p(1)$ = P(A), $\quad p(0)$ = 1 - P(A).

E[I] = 0 . (1 - P(A)) + 1 . P(A) = P(A).

. Expected value of some funtion of X, say $g(X)$, $\quad \mathrm{E}[g(x)] = ?$

Example .

X : a random variable such that

P{X = -1} = .2, P{X = 0} = .5, P{X = 1} = .3

E[X^2] = ?

Solution.

Let Y = X^2. then

P{Y = 1} = P{X = -1} + P{X = 1} = .5

P{Y = 0} = P{X = 0} = .5

E[X^2] = E[Y] = 0 ∗ .5 + 1 ∗ .5 = .5

note: E[X^2] $\neq (E[X])^2$

Proposition: If X is a discrete random variable that takes on x_i, $i \geq 1$, with respective probability $p(x_i)$, then for any real valued funtion g,

$$\mathrm{E}[g(x)] = \sum_i g(x_i)p(x_i).$$

Proof.

$$\begin{aligned}\sum_i g(x_i)p(x_i). &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p(x_i) \\ &= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\ &= \sum_j y_j \mathrm{P}\{\mathrm{g(X)} = y_j\} \\ &= \mathrm{E[g(x)]}\end{aligned}$$

. E[aX + b] = aE[x] + b for constants a and b.

$$. \ \mathrm{E}[X^n] = \sum_{x:p(x)>0} x^n p(x)$$

Definition:

If X is a random variable with mean μ, then the variance of X is

Var(X) = E[$(X-\mu)^2$] .

$$\cdot\ \mathrm{E}[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x)$$

$$= \sum_x (x)^2 p(x) \text{ - } 2\mu \sum_x xp(x) \text{ - } \mu^2 \sum_x xp(x)$$

$$= \mathrm{E}[X^2] \text{ - } 2\mu.\mu + \mu^2$$

$$= \mathrm{E}[X^2] \text{ - } \mu^2$$

$$\therefore \mathrm{Var(X)} = \mathrm{E}[X^2] \text{ - } [E[X]^2]$$

Example . (Rolling a fair die)

X : Outcome

$$\mathrm{E[X]} = \frac{7}{2}$$

$$\mathrm{E}[X^2] = 1^2.\frac{1}{6} + 2^2.\frac{1}{6} + \text{..........} + 6^2.\frac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var(X)} = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$\cdot$ Var(aX + b) = E$[(aX+b-a\mu-b)^2]$

$= a^2$ E$[(X-\mu)^2]$

$= a^2$ Var(X)

$\cdot$ SD(X) = $\sqrt{Var(X)}$: Standard Deviation of X.

Remark.

Mean $\sim$ Center of Gravity

Variance $\sim$ Moment of Inertia

3.1 Bernoulli & Binomial Random Variables

Example .

$p(0)$ = P{X = 0} = 1 - p failure

$p(1)$ = P{X = 1} = p success

Such a random variable with $p \; \epsilon \; (0, 1)$ is called a <u>Bernoulli random variable</u>.

. If X represents the number of successes that occur in the n independent trials, each with probability $\frac{p}{1-p}$, then X is called a binomial random variable w/ (n, p).

. Bernoulli random variable = Binomial random variable w/ (1, p)

. The probability mass function of a binomial random variable w/ (n, p)

$$p(i) = \left(\begin{array}{c} n \\ i \end{array} \right) p(i)(1-p)^{n-i} \qquad i = 0,\, 1,\, 2,\, ...,\, \text{n}.$$

Note that $\displaystyle\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) p(i)(1-p)^{n-i} = [p + (1-p)]^n = 1$ [Binomial Theorem]

Example .

Flipping five fair coins. outcomes are independent.

X = # of H (success)

: binomial random variable with (n = 5, $p = \frac{1}{2}$)

$$\text{P}\{\text{X} = 0\} = \left(\begin{array}{c} 5 \\ 0 \end{array} \right) \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^0 \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^5$$

$$\text{P}\{\text{X} = 1\} = \left(\begin{array}{c} 5 \\ 1 \end{array} \right) \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^1 \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^4$$

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$$\text{P}\{\text{X} = 5\} = \left(\begin{array}{c} 5 \\ 1 \end{array} \right) \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^5 \left(\begin{array}{c} 1 \\ 2 \end{array} \right)^0$$

⇒ Sums up to 1.

3.1.1 Properties of Binomial Random Variables

X : Binomial with (n, p)

$$E[X^k] = \sum_{i=0}^{n} i^k \binom{n}{i} p(i)(1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p(i)(1-p)^{n-i}$$

$$\text{where as,} \quad i\binom{n}{i} = n\binom{n-1}{i-1} \quad \therefore$$

$$= np\sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1}(1-p)^{n-i}$$

$$\text{where as,} \quad j = i-1$$

$$= np\sum_{j=o}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^{j}(1-p)^{n-1-j}$$

$$= np\ E[(Y+1)^{k-1}]$$

Where Y is binomial with $(n-1, p)$

<u>K = 1</u> : E[X] = np

<u>K = 2</u> : E[X^2] = np E[Y + 1] = $np[(n - 1)p + 1]$

Var (X) = E[X^2] - $(E[X])^2$

= $np((n-1)p+1)$ - $(np)^2 = np(1-p)$.

. Binomial Probability mass funtion first increases and then decreases.

P $\sim$ B(n, p) $\qquad 0 < p < 1$.

$$\frac{P\{X=k\}}{P\{X=k-1\}} = \frac{\frac{n!}{(n-k)!}P^k(1-P)^{n-k}}{\frac{n!}{(n-k+1)!}P^{k-1}(1-P)^{n-k+1}} = \frac{(n-k+1)P}{k(1-P)}$$

P{X = k} $\geq$ P{X = k - 1} <u>If</u> (n - k + 1)P $\geq$ k(1 - P)

i.e. k $\leq$ (n + 1) P.

$\therefore$ As k goes from 0 to n, P{X = k} first increases monotonically and then decrease monotonically, reaching its largest value when k is the larges integer less than or equal to $(n+1)$P.

3.2 Poisson Random Variable

- A random variable X, taking on values 0, 1, 2, ... is said to be a poison random variable with parameter λ if for some $\lambda > 0$,

  $p(i) = \mathrm{P}\{\mathrm{X} = i\} = e^{-\lambda}\, \frac{\lambda^i}{i!} \qquad i = 0,\, 1,\, 2,\, ...$

  - This equation is a probability mass funtion, because

$$\sum_{i=0}^{\infty} \mathrm{P}(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

  - Let $\mathrm{X} \sim \mathrm{B}(np)$, and $\lambda = np$. then

$$\mathrm{P}\{\mathrm{X} = i\} = \frac{n!}{(n-i)i!} P^i (1-P)^{n-i}$$

$$= \frac{n!}{(n-i)i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)......(n-i+1)}{n^i!} \frac{\lambda^i}{i!} \frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^i}$$

Where as, $\dfrac{n(n-1)......(n-i+1)}{n^i!} \approx 1,\ \left(1 - \dfrac{\lambda}{n}\right)^n e^{-\lambda}$ and $\left(1 - \dfrac{\lambda}{n}\right)^i \approx 1$

$$\therefore \mathrm{P}\{\mathrm{X} = i\} \approx\ e^{-\lambda}\ \frac{\lambda^i}{i!}$$

(approximately a poisson random variable with $\lambda = np$.)

► A lot of applications (please see page 150. text)

$$\mathrm{E}[\mathrm{X}] = \sum_{i=0}^{\infty} \frac{i e^{-\lambda} \lambda^i}{i!} = \lambda \sum_{i=1}^{\infty} \frac{e^{-\lambda} \lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda \qquad \text{where as, } \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$$

$$\mathrm{E}[X^2] = \sum_{i=0}^{\infty} \frac{i^2 e^{-\lambda} \lambda^i}{i!} = \lambda \sum_{i=1}^{\infty} \frac{i e^{-\lambda} \lambda^{i-1}}{(i-1)!}$$

$$= \lambda \sum_{j=0}^{\infty} \frac{(j+1) e^{-\lambda} \lambda^j}{j!}$$

$$= \lambda \left[\sum_{j=0}^{\infty} \frac{je^{-\lambda}\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \right]$$

$$= \lambda(\lambda + 1)$$

Var (X) = E[X^2] - $(E(X))^2 = \lambda$.

. If X is a Poisson with parameter λ, then

$$\frac{P\{X = i+1\}}{P\{X = i\}} = \frac{e^{-\lambda} \frac{\lambda^{i+1}}{(i+1)!}}{e^{-\lambda} \frac{\lambda^i}{i!}} = \frac{\lambda}{i+1}$$

$$\mathrm{P}\{\mathrm{X} = i\} = e^{-\lambda}$$

$$\mathrm{P}\{\mathrm{X} = 1\} = \lambda \ \mathrm{P}\{\mathrm{X} = 0\}$$

$$\mathrm{P}\{\mathrm{X} = 2\} = \frac{\lambda}{2} \ \mathrm{P}\{\mathrm{X} = 1\}$$

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3.2.1 Properties of the Cumulative Distribution Function

F(b) = the prob. that the random variable X takes on a value that is less than or equal to b.

(1) F is a non-decreasing funtion. i.e. if a < b then F(a) $\leq$ F(b)

$$(2) \lim_{b \to \infty} F(b) = 1$$

$$(3) \lim_{b \to -\infty} F(b) = 0$$

(4) F is right continous i.e. for any b and any decreasing sequence b_n, $n \geq 1$, that converges to b,

$$\lim_{n \to \infty} F(b_n) = F(b).$$