4 Continuous Random Variables

There are random variables whose set of possible values are uncountable.

Example . The lifetime of a light bulb.

Definition:

X is a continuous random variable if there exist a non-negative funtion f, defined for all real $x \, \epsilon \, (-\infty, \infty)$, having the property that for any set B of real numbers.

$$\mathrm{P}\{\mathrm{X} \, \epsilon \, \mathrm{B}\} = \int_B f(x)dx$$

The function f is called the probability density funtion of the random variable X.

- f must satisfy

$$1 = \mathrm{P}\{\mathrm{X} \, \epsilon \, (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x)dx.$$

$$\mathrm{P}\{a \leq \mathrm{X} \leq b\} = \int_a^b f(x)dx$$

$$\mathrm{P}\{\mathrm{X} = a\} = \int_a^a f(x)dx = 0$$

This shows that the probability of a continues random variable will assume any fixed value is 0.

$$\mathrm{P}\{\mathrm{X} < a\} = \mathrm{P}\{\mathrm{X} \leq a\} = \mathrm{F}(a) = \int_{-\infty}^{a} f(x)dx.$$

Example .

The amount of time, in hours that a computer funtions before breaking down:

a continuous random variable with probability density funtion

$$\mathrm{f(x)} = \begin{cases} \lambda e^{-\frac{x}{100}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Probability that a computer will funtion between 50 and 150 hours before breaking down ?

$$1 = \int_{-\infty}^{\infty} f(x)dx = \lambda \int_{0}^{\infty} e^{-\frac{x}{100}} dx = 100\ \lambda$$

$$\Rightarrow \lambda = \frac{1}{100}$$

$$\text{P}\{50 < \text{X} < 150\} = \int_{50}^{150} \frac{1}{100} e^{-\frac{x}{100}} dx$$

$$= e^{-\frac{1}{2}} \text{ - } e^{-\frac{3}{2}} \approx .384$$

(b) Probability that it will function less than 100 hours ?

$$\text{P}\{\text{X} < 100\} = \int_{0}^{100} \frac{1}{100} e^{-\frac{x}{100}} dx = 1 \text{ - } e^{4} \approx .633$$

- $F(a) = \text{P}\{\text{X} \ \epsilon \ (-\infty, a) = \int_{-\infty}^{a} f(x)dx$

- $F(a) = \text{P}\{\text{X} \ \epsilon \ (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$

$$\Rightarrow \frac{d}{da} \text{ F}(a) = f(a).$$

- For very small ε, and when f(.) is continuous at a,

$$\text{P}\{\ a \text{ - } \frac{\varepsilon}{2} \leq \ \text{X} \ \leq \ a + \frac{\varepsilon}{2}\} + \int_{a+\frac{\varepsilon}{2}}^{a-\frac{\varepsilon}{2}} f(x)dx \approx \ \varepsilon f(a).$$

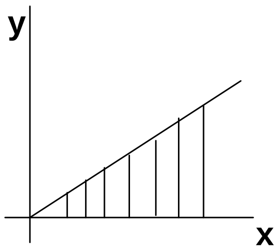
- X : A continuous random variable with probability density funtion $f(x)$, then $f(x)dx \approx \text{P}\{\ x \leq \text{X} \leq x + dx\}$ for small dx

$$\text{E[X]} = \int_{-\infty}^{\infty} xf(x)dx$$: the expected value of X for any real - valued funtion g,

$$\text{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

<u>Lemma</u> For a non-negative random variable Y,

$$\mathrm{E[Y]} = \int_0^\infty \mathrm{P}\{\mathrm{Y} > y\}\,\mathrm{dy}$$



Proof.

$$\mathrm{RHS} = \int_0^\infty \int_y^\infty f_Y(x)\mathrm{dx}\,\mathrm{dy}$$

(By interchanging the order of integration)

$$= \int_0^\infty \left(\int_0^x \mathrm{dy} \right) f_Y(x)\mathrm{dx}$$

$$= \int_0^\infty x f_Y(x)\mathrm{dx}$$

$$= \mathrm{E[Y]}$$

- If a & b are constants, then

$\mathrm{E}[aX + b] = a\mathrm{E[X]} + b$

- $\mathrm{Var(X)} = \mathrm{E}[(X - \mu)^2] = \mathrm{E}[X^2]$ - $(E[X])^2$
- $\mathrm{Var}(aX + b) = a^2\ \mathrm{Var(X)}$.

4.1 Uniform Random Variable

- A random variable is said to be uniformly distributed over the interval (0, 1) if its probability density function is

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathrm{P}\{a \leq X \leq b\} = \int_a^b f(x)\mathrm{dx} = b - a$$

for any $0 < a < b < 1$.

• X $\sim$ U$[\alpha, \beta]$

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \mathrm{F}(a) = \begin{cases} 0 & a \le \alpha \\ \\ \frac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ \\ 1 & a \ge \beta \end{cases}$$

$$\mathrm{E[X]} = \int_{-\infty}^{\infty} x f(x) dx = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx = \frac{\alpha + \beta}{2}$$

$$\mathrm{E}[X^2] = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha}\ x^2 dx = \frac{1}{3}(\alpha^2 + \alpha\beta + \beta^2)$$

$$\mathrm{Var[X]} = \frac{\alpha^2 + \alpha\beta + \beta^2}{3} \text{ - } \left(\frac{\alpha + \beta}{2}\right)^2 = \frac{(\beta - \alpha)^2}{12}$$

4.2 Normal Random Variables

• X is normally distributed with parameters μ and σ^2 if the density of X is given by

$$\mathrm{f(x)} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty \qquad (*)$$

• Many random phenomena obey, at least approximately, a normal probability distribution.

• Is $(*)$ indeed a probability funtion ?

$$\text{Let y} = \frac{x - \mu}{\sigma}\ ,\ \text{ then}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}\ e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ \mathrm{dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \mathrm{dy} \qquad (**)$$

$$\text{Let} \quad \mathrm{I} = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \mathrm{dy} \quad \text{ then}$$

$$\mathrm{I}^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \mathrm{dy}\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \mathrm{dx}\right)$$

$$\text{where as,} \qquad x = r\cos\theta, \quad y = r\sin\theta$$

$$\therefore \qquad = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}}\ r d\theta dr$$

$$= 2\pi \int_0^{\infty} \mathrm{r}e^{-\frac{r^2}{2}}\ dr$$

$$= \ 2\pi$$

$$\therefore \qquad (**) = \ 1$$

$$\mathrm{E[X]} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ \mathrm{dx} + \frac{\mu}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ \mathrm{dx}$$

$$\downarrow \quad x - \mu = y$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} ye^{\frac{-(y)^2}{2\sigma^2}}\ \mathrm{dy} + \ \mu\int_{-\infty}^{\infty} f(x)\ dx$$

$$\left[\text{Where as, } \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} ye^{\frac{-(y)^2}{2\sigma^2}}\ \mathrm{dy} = 0 \qquad \text{and} \qquad \mu\int_{-\infty}^{\infty} f(x)\ dx = 1\right]$$

$$= \ \mu$$

- $\mathrm{Var(X)} = \mathrm{E[(X\ \text{-}\ \mu)^2]}$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\ e^{\frac{-(x-\mu)^2}{2\sigma^2}}\ dx$$

$$\downarrow \quad y = \frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2\ e^{\frac{-(y)^2}{2}}\ \mathrm{dy}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\left[\text{-}\ y\ e^{\frac{-(y)^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-(y)^2}{2}}\ \mathrm{dy}\right]$$

$$\downarrow \ \text{Integration by parts}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\ .\sqrt{2\pi}$$

$$= \sigma^2$$

- If $\quad$ X $\sim$ N(μ, σ^2) , then

Y $= \alpha$X $+ \beta \sim$ N$(\alpha\mu + \beta,\ \ \alpha^2\sigma^2)$

Proof.

(Suppose that $\alpha > 0$).

The cumulative distribution funtion of Y

$$F_Y(a) = \mathrm{P}\{\alpha \mathrm{X} + \beta \leq \mathrm{a}\} = \mathrm{P}\{\mathrm{X} \leq \frac{a-\beta}{\alpha}\} = F_X\left(\frac{a-\beta}{\alpha}\right).$$

$$\downarrow \ \text{differentiate}$$

$$f_Y(a) = \frac{1}{\alpha} f_X\left(\frac{a-\beta}{\alpha}\right) = \frac{1}{\sqrt{2\pi}\alpha\sigma}\ e\left[-\frac{(a-\beta-\alpha\mu)^2}{2(\alpha\sigma)^2}\right]$$

- X $\sim$ N(μ, σ^2) , then

Z $= \frac{X-\mu}{\sigma} \sim$ N(0, 1) $\qquad$: Standard unit normal distribution.

Notation: The cumulative distribution funtion of a standard normal random variable

$$= \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\ \mathrm{dy}$$

Regarding $\Phi(x)$, we can look up the table for $x \geq 0$

For non-negative x,

$\Phi(-x) = 1$ - $\Phi(x) \qquad\qquad -\infty < x < \infty$

($\because$ symmetry of the density funtion)

Distribution funtion of X $\sim$ N(μ, σ^2) ?

$$F_x(a) = \mathrm{P}\{\mathrm{X} \leq a\}$$

$$= \mathrm{P}\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right)$$

$$= \ \Phi\left(\frac{a-\mu}{\sigma}\right).$$

4.3 Binomial Distribution $\rightsquigarrow$ Normal Distribution

De Moivre - Laplace Limit Theorem

S_n = # of successes that occur when n independent trials, each with success probability p, are performed.

Then, for any $a < b$,

$$\mathrm{P}\left\{a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right\} \quad \rightarrow \quad \Phi(b) \text{ - } \Phi(b)$$

(Where as, np is mean and np(1 - p) is Standard deviation)

Remarks. This is the special case of the central limit theorem, to be presented in Chapter-8.

Example 4.h

100 people are put on the special diet. it will be endorsed if $\geq 65\%$ of the people have lower cholesterol after the diet.

Assume that if the diet has no effect on the cholesterol, then , strictly by chance, each persons cholesterol level will be lower than before with probability 1/2.
Probability that the diet will be endorsed when it actually has no effect

$$= \sum_{i=65}^{100} \binom{100}{i} \left(\frac{1}{2}\right)^{100} \quad = \quad \mathrm{P}\{\mathrm{X} \geq 64.5\}$$

$$= \quad \mathrm{P}\left\{\frac{X - 100.\frac{1}{2}}{\sqrt{100.\frac{1}{2}.\frac{1}{2}}} \geq 2.9\right\}$$

$$\approx \quad 1 \text{ - } \quad \Phi(2.9) \quad \approx \quad 0.0019$$

4.4 Exponential Random Variables

- An exponential random variable with parameter λ

  : probability density funtion

$$f(x) = \begin{cases} \lambda \, e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\Rightarrow$ Cumulative distribution function

$$F(a) = P\{X \leq a\} = \int_0^a \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda a} \qquad a \geq 0$$

$$\bullet\ E[X] = \int_0^\infty x\lambda e^{-\lambda x}\, dx$$

$\downarrow$ Integration by parts

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

$$\bullet\ E[X^2] = \int_0^\infty x\lambda e^{-\lambda x}\, dx$$

$\downarrow$ Integration by parts

$$= -x^2e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx$$

$$= 0 + \frac{2}{\lambda}E[X]$$

$$= \frac{2}{\lambda^2}$$

$$\therefore \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

Example .

Amount of time until an Earthquake.

Amount of time until a new war. etc.

Definition:

A non-negative random variable is <u>memoryless</u>

If ,

$\mathrm{P}\{\mathrm{X} > s+t \mid \mathrm{X} > t\ \} = \mathrm{P}\{\mathrm{X} > s\}$ for all $s, t \geq 0$.

If the instrument is alive at age t, the distribution of the remaining amount of time that it survives is the same as the original lifetime distribution.

(It does not remember that it has already been used for a time t)

$$\Leftrightarrow \quad \frac{P\{X > s+t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

$$\Leftrightarrow \quad \mathrm{P}\{\mathrm{X} > s+t\} = \mathrm{P}\{\mathrm{X} > s\}\ \ \mathrm{P}\{\mathrm{X} > t\}.$$

Example .

$\mathrm{e}^{-\lambda(s+t)} = \mathrm{e}^{-\lambda s}\ \mathrm{e}^{-\lambda t}$

$\therefore$ Exponential distributed random variables are memoryless.

Claim:

The exponential distribution is the unique distribution possessing the memoryless property!

– Suppose X is memoryless and let $\overline{F}(x) = \mathrm{P}\{\mathrm{X} > x\}$.

Then

$\overline{F}(s+t) = \overline{F}(s)\overline{F}(t)$

i.e. $\overline{F}(.)$ satisfies $g(s+t) = g(s)\ g(t)$.

The only right continuous solution of this is

$$g(x) = \mathrm{e}^{-\lambda x}.$$

$$\Rightarrow \ \overline{F}(x) = \mathrm{e}^{-\lambda x}.$$

Why ?

$$g\left(\tfrac{z}{n}\right) = g\left(\frac{1}{n} + \frac{1}{n}\right) = g^2\left(\tfrac{1}{n}\right)$$

$$g(1) = g\left(\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}\right) = g^n\left(\tfrac{1}{n}\right) \quad \Rightarrow \quad g\left(\tfrac{1}{n}\right) = (g(1))^{\frac{1}{n}}$$

$$g\left(\tfrac{m}{n}\right) = g^m\left(\tfrac{1}{n}\right) = (g(1))^{\frac{m}{n}}$$

$$\Rightarrow \quad g(x) = (g(1))^x \qquad \text{Since } g \text{ is right continuous.}$$

$$g(1) = (g(\tfrac{1}{2}))^2 \geq 0$$

$$\Rightarrow \quad g(x) = \mathrm{e}^{-\lambda x}$$

$$\lambda = -\log g(1).$$

4.5 Laplace Distribution (Double Exponential)

Density Function $f(x) = \frac{1}{2}\ \lambda\ \mathrm{e}^{-\lambda|x|} \qquad -\infty < x < \infty$

Distribution Funtion

$$f(x) = \begin{cases} \frac{1}{2}\int_{-\infty}^{x} \lambda \mathrm{e}^{\lambda x} dx & x < 0 \\ \int_{-\infty}^{0} \lambda \mathrm{e}^{\lambda x} dx + \frac{1}{2}\int_{0}^{x} \lambda \mathrm{e}^{-\lambda x} dx & x > 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\ \mathrm{e}^{\lambda x} \\ 1 - \frac{1}{2}\ \mathrm{e}^{-\lambda x} \end{cases}$$

• A positive continuous random variable X $\sim$ lifetime of some item, with distribution funtion F and density ρ

Hazard rate (failure rate)

$\lambda(t)$ of F

$$\lambda(t) = \frac{f(t)}{\overline{F}(t)} \qquad \overline{F}(t) = 1 - \mathrm{F}$$

Suppose that the item has survived for t and we want to know the probability that it will not survive for additional dt.

$$\mathrm{P}\{\mathrm{X}\ \epsilon\ (t, t+dt) \mid \mathrm{X} > t\} = \frac{P\{X\epsilon(t, t+dt), X > t\}}{P\{X > t\}}$$

$$= \frac{P\{X\epsilon(t, t+dt)\}}{P\{X > t\}}$$

$$\approx \frac{f(t)}{\overline{F}(t)}\ dt$$

: The conditional probability intensity that a t-unit-old item will fail.

Example . Exponential Distribution

$\Rightarrow$ memoryless $\Rightarrow$ distribution of remaining life for a t-year-old item is the same as for a new item.

$\Rightarrow$ $\lambda(t)$ is constant

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.$$

- The failure rate function $\lambda(t)$ uniquely determines the distribution F.

$$\lambda(t) = \frac{\frac{d}{dt}F(t)}{1 - F(t)}$$

$$\Rightarrow\ -\int_0^t \lambda(t)dt\ +\ k\ =\ \log(1 - F(t))$$

$$\Rightarrow\ 1 \text{ - } F(t) =\ e^k\ \ exp\{-\textstyle\int_0^t \lambda(t)dt\}$$

$$t = 0\ \Rightarrow\ k = 0. \qquad \therefore\ F(t) =\ 1 \text{ - }\ exp\{-\textstyle\int_0^t \lambda(t)dt\}$$

4.6 Gamma Distribution)

A random variable is said to have a gamma distribution with parameters (α, λ)

$\lambda > 0$, $\alpha > 0$
it its density funtion is given by,

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \lambda \mathrm{e}^{\lambda x} dx & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\text{where ,} \quad \Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy \qquad : \text{gamma funtion}$$

Note:

$$\Gamma(\alpha) = \text{ - } \mathrm{e}^{-y}\ y^{\alpha-1}\Big|_0^\infty\ +\ \int_0^\infty e^{-y}(\alpha - 1)y^{\alpha-2}\, dy$$

$$= (\alpha - 1)\Gamma(\alpha - 1).$$

$$\Gamma(1) = \int_0^\infty e^{-y} dy \ = \ 1$$

$$\therefore \quad \Gamma(n) = \ (n \text{ - } 1) \ \Gamma \ (n \text{ - } 1) = (n \text{ - } 1)(n \text{ - } 2)\Gamma \ (n \text{ - } 2) = \ .... \ = (n \text{ - } 1)!$$

- T_A : The time at which the n^{th} event occurs.

N(t) : # of events in [0, t]

$$\text{P}\{T_n \leq t\} = \text{P}\{N(t) \geq n\} = \sum_{j=n}^{\infty} \text{P}\{N(t) = j\}$$

(# of events in [0, t] has a poissons distribution with parameter λt)

$\downarrow$ Differentiate

$$f(t) = \sum_{j=n}^{\infty} \frac{e^{-\lambda t} j(\lambda t)^{j-1}\lambda)}{j!} \text{ - } \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!}$$

$$= \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{j-1}}{(j-1)!} \text{ - } \sum_{j=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^j}{j!}$$

$$= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)1}$$

[gamma distribution with (n, λ)]

- X $\sim$ gamma(α, λ).

$$\text{E[X]} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} dx$$

$$= \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha} dx$$

$$= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}.$$

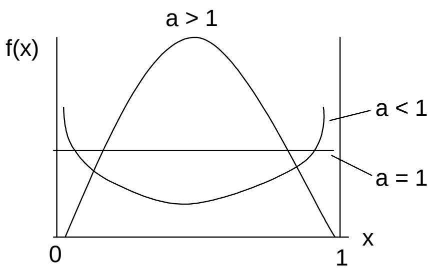
$$\text{Var [X]} = \frac{\alpha^2}{\lambda}.$$

4.7 Beta Distribution

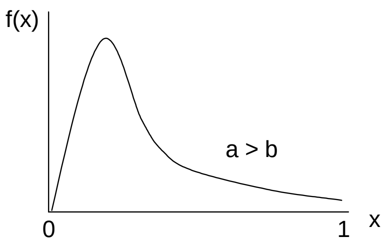
$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{B}\ (a, b) = \int_0^1 x^{a-1}\ (1-x)^{b-1}\ dx$$

– used to model a random phenomenon whose se of possible values are some finite interval $[c, d]$.



(a = b) beta density is symmetric about 1/2



<u>Note:</u>

$$\mathrm{B}\ (a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\mathrm{E[X]} = \frac{a}{a+b}$$

$$\mathrm{Var\ (X)} = \frac{ab}{(a+b)^2(a+b+1)}$$

4.8 Distribution of a funtion of a random variable

Example .

X : a continuous random variable with probability density function f_x.

$$Y = X^2.$$

$$F_y(y) = P\ \{Y \leq y\} = P\{X^2 \leq y\}$$

$$= P\ \{-\sqrt{y} \leq X \leq \sqrt{y}\}$$

$$= F_x(\sqrt{y})\ -\ F_x(-\sqrt{y})$$

$\downarrow$ Differentiate

$$f_Y(y) = \frac{1}{2\sqrt{y}}\ [\ f_x(\sqrt{y}) + f_x(-\sqrt{y})]$$

Theorem

X : a continuous random variable with probability density f_x.

$g(x)$: a strictly monotone (increasing or decreasing), differentiable (and thus continuous) funtion of x.

Then the random variable $Y = g(x)$ has a probability density funtion

$$f_Y(y) = \begin{cases} f_x[g^-1(y)]\ |\frac{d}{dy}g^{-1}(y)| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

Where $g^{-1}(y)$ is define to equal that value of x such that $g(x) = y$

Proof.

Let $y = g(x)$ for some x. Then

$$F_Y(y) = P\ \{g(x) \leq y\} = P\{X \leq g^{-1}(y)\}$$

$$= F_x(g^{-1}(y)).$$

$\downarrow$ differentiating

$$f_Y(y) = f_x(g^{-1}(y))\,\frac{d}{dy}g^{-1}(y)$$

When $y \neq g(x)$ for any x, then $\mathrm{F}_Y(y) = 0$ or 1. $\quad \Rightarrow f_y(y) = 0$

Example .

X : a continuous non-negative random variable with density f.

$\mathrm{Y} = \mathrm{X}^n, \quad f_Y = ?$

Solution.

Let $g(x) = x^n \quad \Rightarrow \quad g^{-1}(y) = y^{\frac{1}{n}}$

$$\frac{d}{dy}\,(g^{-1}(y)) = \frac{1}{n}\,y^{\frac{1}{n}-1}$$

$$\therefore \quad f_Y(y) = \frac{1}{n}\;y^{\frac{1}{n}-1}\,f(y^{\frac{1}{n}})$$

$$\underline{n=2}, \qquad f_Y(y) = \frac{1}{2\sqrt{y}}\;f(\sqrt{y}).$$

(Compare with the previous example)