8 Joint Probability Distribution of Function of Random Variables

X_1, X_2 : jointly continuous random variables with joint density $f_{X_1 X_2}$

Suppose that $\quad Y_1 = g_1\ (X_1, X_2)\ ,\ Y_2 = g_2(X_1, X_2) \quad$ for some funtions g_1, g_2

Assume that,
1. $y_1 = g_1(x_1, x_2),\ \ y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of $y_1 \& y_2$, say, $x_1 = h_1(y_1, y_2),\ \ x_2 = h_2(y_1, y_2)$.

2. $g_1 \& g_2$ have continuous partial derivatives at all points (x_1, x_2) and are such that at all (x_1, x_2),

$$\mathrm{J}(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \equiv \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1} \neq 0.$$

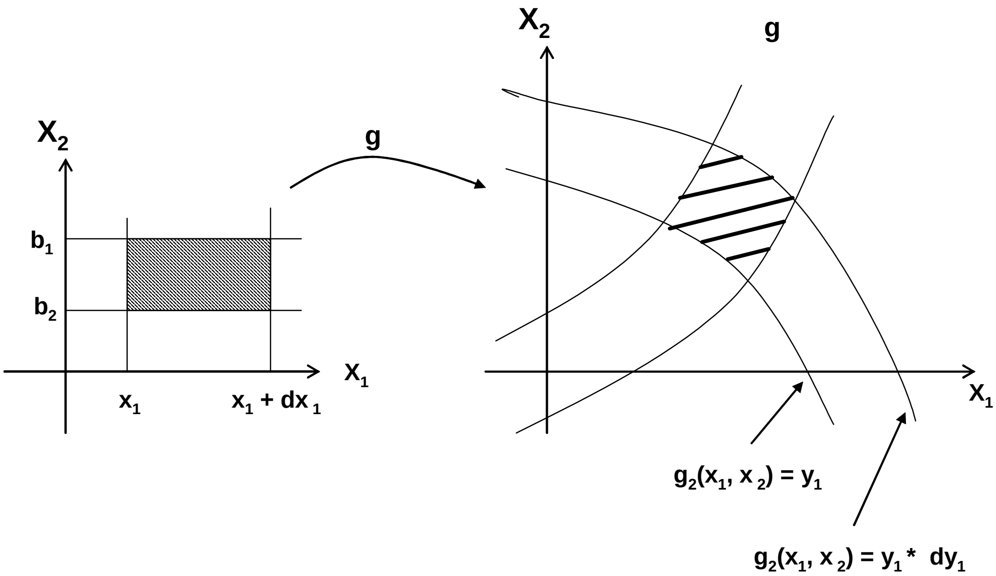
Then, $Y_1 \& Y_2$ are jointly continuous with joint density function,

$$\mathrm{f}_{Y_1,Y_2}(y_1, y_2) = \mathrm{f}_{X_1,X_2}(x_1, x_2)\ |J(x_1, x_2)|^{-1}$$

Proof.

$$\mathrm{P}\{Y_1 \leq y_1, Y_2 \leq y_2\} = \int\int_{(x_1,x_2):\ g_1(x_1,x_2)\leq y_1,\ g_2(x_1,x_2)\leq y_2} f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

differentiate w.r.t. $y_1 \& y_2$



$$f_{X_1,X_2}(x_1,x_2)dx_1dx_2.\ |J|. \approx f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2$$

Example .

(X, Y) : random point in the plane
X, Y : independent unit normal random variables
Joint distribution of R, Θ ?

Solution.

$$\text{r} = g_1(x,y) = \sqrt{x^2+y^2}$$

$$\theta = g_2(x,y) = \tan^{-1}\tfrac{x}{y}$$

$$\text{J} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$$f(x,y) = \frac{1}{2\pi}\, e^{-\frac{1}{2}(x^2+y^2)}$$

$$f(r,\theta) = \frac{1}{2\pi} r\, e^{-\frac{1}{2}r^2} \qquad 0<\theta<2\pi,\ 0<r<\infty$$

Θ : uniform over $(0,\ 2\pi)$.

R : Rayleigh distribution $\quad f(r) = re^{-\frac{r^2}{2}}$
example) X, Y: horizontal/vertical miss distance
absolute value of the error $\sim$ Rayleigh.

Example .

(X, Y) : random point in the plane
X, Y : independent unit normal random va
Joint distribution of R^2, Θ ?

Solution.

$$g_1(x,y) = x^2 + y^2$$

$$\text{J} = \begin{vmatrix} 2x & 2y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = 2$$

$$f(d,\theta) = \frac{1}{4\pi}e^{-\frac{d}{2}}$$

$R^2 \sim$ exponential distribution with parameter $\frac{1}{2}$.
$R^2 = X^2 + Y^2 \quad \Rightarrow \quad R^2 \sim \chi^2$ distribution with 2 d.o.f. $\therefore$ The exp distribution with parameter $\frac{1}{2}$ = χ^2 distribution with 2 d.o.f.

• We can simulate (generate) normal random variables by making a suitable transformation on uniform random variables.

U_1, U_2 : independent random variables each uniform over (0, 1)

X_1, X_2 : two independent unit normal random variables.

$R^2 = X_1^2 + X_2^2,\ \Theta \quad \Rightarrow \quad$ exponential distribution with $\lambda = \frac{1}{2}$

P$\{-2\log U_1 < x\}$ = P$\{\log U_1 > -\frac{x}{2}\}$ = P$\{U_1 > e^{-\frac{x}{2}}\}$ = 1 - $e^{-\frac{x}{2}}$

$\therefore$ -2log U_1 as an exp distribution with $\lambda = \frac{1}{2}$.

$2\pi U_2$: uniform over (0, 2 π), use it to generate Θ

$\therefore \quad X_1 = \sqrt{-2\log U_1}\ cos(2\pi U_2)$, $X_2 = \sqrt{-2\log U_1}\ sin(2\pi U_2)$, independent standard normal.

• When the joint density function of the n random variables $X_1, X_2, ..., X_n$ is given and we want to compute the joint density function of $Y_1, ..., Y_n$, where

$$Y_1 = g_1(X_1, ..., X_n)$$

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$$Y_n = g_n(X_1, ..., X_n)$$

Assuming $g_i's$ have continuous partial derivatives and that, at all points $(x_1, ..., x_n)$

$$\mathrm{J}(x_1, x_2) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots\cdots \frac{\partial g_1}{\partial x_n} \\ . & \\ . & \\ . & \\ \frac{\partial g_n}{\partial x_1} & \cdots\cdots \frac{\partial g_n}{\partial x_n} \end{vmatrix} \neq 0.$$

Suppose that $y_1 = g_1(x_1, ..., x_n), \quad y_n = g_n(x_1, ..., x_n)$ have a unique solution, say $x_1 = h_1(y_1, ..., y_n), \quad x_n = h_n(y_1, ..., y_n)$, then

$f_{Y_1,\cdots,Y_n}\ (y_1, ..., y_n) = f_{X_1,\cdots,X_n}\ (x_1, ..., x_n)\ |J(x_1, ...x_n)|^{-1}$

where $\quad x_i = h_i(y_1, ..., y_n), \quad i = 1, ..., n.$

Example .

X_1, X_2, X_3: independent unit normal random variables.
$Y_1 = X_1 + X_2 + X_3$
$Y_2 = X_1 - X_2$
$Y_3 = X_1 - X_3$
$\Downarrow$
$X_1 = \frac{1}{3}(Y_1 + Y_2 + Y_3)$
$X_2 = \frac{1}{3}(Y_1 - 2Y_2 + Y_3)$
$X_3 = \frac{1}{3}(Y_1 + Y_2 - 2Y_3)$

$$\mathrm{J} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3$$

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}}\ e^{-\sum_{i=1}^{3} \frac{{x_i}^2}{2}}$$

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \frac{1}{3}\frac{1}{(2\pi)^{\frac{3}{2}}}\ exp\left[-\frac{1}{2}\left[\left(\frac{y_1 + y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - 2y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 + y_2 - 2y_3}{3}\right)^2\right]\right]$$