SNU MAE INTRO. ENG. PROB note 10.

Conditional Expectation

Recall X, Y : jointly distributed r.vs.
the conditional prob mass function of X, given that $Y=y$, is (defined for all y s.t. $P\{Y=y\}>0$)

$$P_{X|Y}(x|y) = P\{X=x \mid Y=y\} = \frac{p(x,y)}{P_Y(y)}$$

Def The conditional expectation of X, given that $Y=y$, is

$$E[X|Y=y] = \sum_x x P\{X=x \mid Y=y\}$$
$$= \sum_x x P_{X|Y}(x|y)$$

(defined for all values of y s.t. $P_Y(y)>0$)

Def X, Y : jointly continuous with density $f(x,y)$.
Then

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$

(if $f_Y(y)>0$)

Ex $f(x,y) = \dfrac{e^{-x/y} e^{-y}}{y}$ $\quad 0<x<\infty, \quad 0<y<\infty$

$E[X|Y=y] = ?$

sol. $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = \dfrac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} = \dfrac{\frac{1}{y} e^{-x/y} \cancel{e^{-y}}}{\int_{0}^{\infty} \frac{1}{y} e^{-x/y} \cancel{e^{-y}}\,dx}$

$= \frac{1}{y} e^{-x/y}$.

$E[X|Y=y] = \int_0^\infty x \cdot \frac{1}{y} e^{-x/y}\,dx = y.$

Rmk Conditional expectations satisfy the properties of ordinary expectations.

E.g. $E[g(X)|Y=y] = \begin{cases} \sum_x g(x) P_{X|Y}(x|y) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx & \text{conti} \end{cases}$

$$E\left[\sum_{i=1}^n X_i \middle| Y=y\right] = \sum_{i=1}^n E[X_i|Y=y]$$

Conditional expectation given $Y=y$ can be thought of as an ordinary expectation on a reduced sample space consisting only of outcomes for which $Y=y$.

Notation $E[X|Y]$: ftn of the r.v. Y whose value at $Y=y$ is $E[X|Y=y]$.

a r.v.

Prop $\boxed{E[X] = E[E[X|Y]]}$ $(*)$

$$E[X] = \begin{cases} \sum_y E[X|Y=y]\, P\{Y=y\} & Y \text{ discrete} \\ \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\,dy & Y \text{ conti} \end{cases}$$

<u>Pf</u> of (*) (when X & Y are discrete)

$$\sum_y E[X|Y=y]\,P\{Y=y\} = \sum_y \sum_x x\,P\{X=x|Y=y\}\,P\{Y=y\}$$

$$= \sum_y \sum_x x \frac{P\{X=x, Y=y\}}{\cancel{P\{Y=y\}}} \cancel{P\{Y=y\}}$$

$$= \sum_x x \sum_y P\{X=x, Y=y\}$$

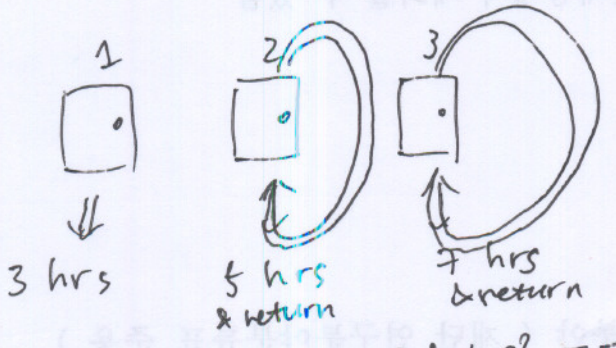
$$= \sum_x x\,P\{X=x\}$$

$$= E[X] \quad //$$

<u>Interpretation of (*)</u>

To calculate $E[X]$, take a weighted avg of the conditional expectation of X, given that $Y=y$, each of the terms $E[X|Y=y]$ being weighted by the probability of the event on which it is conditioned.

<u>Ex 4C</u>



X = hrs

Y = door

$$P\{Y=1\} = P\{Y=2\} = P\{Y=3\} = \frac{1}{3}.$$

$$E[X] = E[X|Y=1]\,P\{Y=1\} + E[X|Y=2]\,P\{Y=2\}$$
$$+ E[X|Y=3]\,P\{Y=3\}$$

$E[X|Y=1]=3$

$E[X|Y=2]=\,?\quad = 5+E[X]$

$E[X|Y=3]=\,?\quad = 7+E[X]$

$\therefore\ E[X] = \frac{1}{3}\{3+5+7+2E[X]\}$

$\therefore\ E[X]=15.$ //

Ex 4e The game of craps.

Roll a pair of dice.

If the sum $=2, 3,$ or 12, lose.

$= 7$ or 11, win

$=$ any other number i, continue to roll until 7 or i

$\begin{cases} \text{if } 7, & \text{lose} \\ \text{if } i, & \text{win} \end{cases}$

R : # of rolls of the dice in a game.

$E[R]$, $E[R|\text{win}]$, $E[R|\text{lose}]$?

sol

$$\left.\begin{array}{l} P\{\text{sum}=2\} = \frac{1}{36} \\ \vdots \\ P\{\text{sum}=7\} = \frac{6}{36} \\ P\{\text{sum}=8\} = \frac{5}{36} \\ \vdots \\ P\{\text{sum}=12\} = \frac{1}{36} \end{array}\right\} \Rightarrow \begin{array}{l} P_i = \text{the prob that sum}=i \\ \quad = \frac{i-1}{36} \quad i=2,\dots,7 \\ \quad = P_{14-i}. \end{array}$$

$$E[R] = \sum_{i=2}^{12} E[R|S=i]\,P_i$$

$$E[R \mid S=i] = \begin{cases} 1 & \text{if } i=2,3,7,11,12 \\ 1+\dfrac{1}{P_i+P_7} & \text{else} \end{cases}$$

$\because$ the number of rolls in this case is a geometric r.v. with parameter P_i+P_7.

$$\therefore\ E[R] = 1+\sum_{i=4}^{6}\frac{P_i}{P_i+P_7}+\sum_{i=8}^{10}\frac{P_i}{P_i+P_7}$$

$$= 1+2\left(\frac{3}{9}+\frac{4}{10}+\frac{5}{11}\right) \quad //$$

To compute $\cancel{P}$ $E[R \mid win]$, need p = prob that we win.

$$p = \sum_{i=2}^{12} P\{win \mid S=i\}\, P_i$$

$$= 1\cdot P_7 + 1\cdot P_{11} + \sum_{i=4}^{6}\frac{P_i}{P_i+P_7}P_i + \sum_{i=8}^{10}\frac{P_i}{P_i+P_7}P_i$$

$$= 0.493$$

prob of getting sum $= i$ before 7.

$$P\{S=i \mid win\} = Q_i$$

$$= \begin{cases} 0 & \text{if } i=2,3,12 \\ P_i/p & \text{if } i=7,11 \\ = \dfrac{P\{S=i,\ win\}}{P\{win\}} = \dfrac{P_i\, P\{win \mid S=i\}}{p} = \dfrac{P_i}{p}\cdot\dfrac{P_i}{P_i+P_7} & \text{if } i=4,5,6,8,9,10. \end{cases}$$

$$E[R|\text{win}] = \sum_i E[R|\text{win}, S=i]\,Q_i$$

given that the initial sum $=i$, # of additional rolls needed and the outcome are indept.

$$= \sum_i E[R|S=i]\,Q_i$$

$$= 1 + \sum_{i=4}^{6} \frac{Q_i}{P_i + P_7} + \sum_{i=8}^{10} \frac{Q_i}{P_i+P_7}$$

$$= 2.983$$

$E[R|\text{lose}]$ can be determined in the same way.

Or,

$$E[R] = E[R|\text{win}]\,p + E[R|\text{lose}]\,(1-p)$$

$$\Rightarrow \quad E[R|\text{lose}] = 3.801 \quad //$$

Ex 4h $U_1, U_2, \ldots$ a sequence of $U(0,1)$ rvs.

$$N = \min\{n : \sum_{i=1}^{n} U_i > 1\}.$$

$$E[N] = ?$$

Sol For $x \in [0,1]$, let

$$N(x) = \min\{n : \sum_{i=1}^{n} U_i > x\}.$$

$\leftarrow$ # of rvs we need until their sum exceeds x.

$$m(x) = E[N(x)]$$

Then

$$m(x) = \int_0^1 E[N(x)|U_1=y]\,dy$$

$$E[N(x)|U_1=y] = \begin{cases} 1 & \text{if } y > x \\ 1+m(x-y) & \text{if } y \le x \end{cases}$$

$$\therefore\ m(x) = \int_0^x 1+m(x-y)\,dy + \int_x^1 1\,dy$$

$$= 1 + \int_0^x m(u)\,du \quad (u = x-y)$$

differentiate

$\Downarrow$

$$m'(x) = m(x)$$

$\Downarrow$

$$\ln[m(x)] = x + C$$

$\Downarrow$

$$\left.\begin{array}{l} m(x) = k e^{x} \\ m(0) = 1 \end{array}\right\} \Rightarrow m(x) = e^{x}.$$

$$\therefore \quad E[N] = m(1) = e \; //$$

Computing Probabilities by Conditioning

E : an arbitrary event

$$X = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{else} \end{cases} \quad \text{indicator variable.}$$

$$\Rightarrow E[X] = P(E)$$

$$E[X \mid Y=y] = P(E \mid Y=y) \quad \text{for any r.v. } Y.$$

$$P(E) = \begin{cases} \sum_y P(E \mid Y=y) \, P(Y=y) & (Y \text{ discrete}) \\ \int_{-\infty}^{\infty} P(E \mid Y=y) \, f_Y(y) \, dy & (Y \text{ conti}) \end{cases} \; //$$

Ex 4i.

Ex 4k.