SNU MAE Intro. Eng. Prob note 13

Q. An upper bound for $P\{X-\mu \geq a\}$?

One answer: $P\{X-\mu \geq a\} \leq P\{|X-\mu| \geq a\} \leq \frac{\sigma^2}{a^2} \quad (a>0)$

↑ Chebyshev.

But we can do better!

<u>Prop</u> [One-sided Chebsehv Inequality]

X : a rv with mean 0, finite variance σ^2.

Then for any $a>0$,

$$P\{X \geq a\} \leq \frac{\sigma^2}{\sigma^2+a^2}.$$

<u>Pf</u> Let $b>0$, then $X \geq a \iff X+b \geq a+b$.

$$\begin{aligned} \therefore \quad P\{X \geq a\} &= P\{X+b \geq a+b\} \\ &\leq P\{(X+b)^2 \geq (a+b)^2\} \\ &\leq \frac{E[(X+b)^2]}{(a+b)^2} = \frac{\sigma^2+b^2}{(a+b)^2} \quad \text{Markov} \end{aligned}$$

Let $b = \frac{\sigma^2}{a}$. (the value of b that minimizes $\frac{\sigma^2+b^2}{(a+b)^2}$) //

<u>Cor</u> If $E[X]=\mu$, $Var[X]=\sigma^2$, then for $a>0$,

$$P\{X \geq \mu+a\} \leq \frac{\sigma^2}{\sigma^2+a^2}$$

$$P\{X \leq \mu-a\} \leq \frac{\sigma^2}{\sigma^2+a^2}.$$

<u>Pf</u> $X-\mu$ & $\mu-X$: mean 0, variance σ^2.

apply one-sided Chebyshev. //

Ex 200 people (100 M & 100 W)

→ randomly divided into 100 pairs of 2 each.

Upper bound for

P{ at most 30 pairs will consist of a M & a W } ?

sol

$$X_i = \begin{cases} 1 & \text{if man } i \text{ is paired with a W} \\ 0 & \text{else} \end{cases}$$

Then $X = \sum_{i=1}^{100} X_i$: # of M-W pairs.

$$E[X_i] = P\{X_i = 1\} = \frac{100 \leftarrow \text{women}}{199 \leftarrow \text{the other people}}$$

for $i \neq j$,

$$E[X_i X_j] = P\{X_i = 1, X_j = 1\}$$

$$= P\{X_i = 1\} P\{X_j = 1 \mid X_i = 1\} = \frac{100}{199} \frac{99 \leftarrow \text{the other women remaining}}{197 \leftarrow \text{the other people remaining}}$$

$$\therefore \ E[X] = \sum_{i=1}^{100} E[X_i] = 100 \times \frac{100}{199} \approx 50.25$$

$$Var(X) = \sum_{i=1}^{100} Var(X_i) + 2 \sum_{i<j}\sum Cov(X_i, X_j)$$

$$= 100 \times \left(\frac{100}{199} \times \frac{99}{199} \right) + 2 \binom{100}{2} \left[\frac{100}{199} \frac{99}{197} - \left(\frac{100}{199} \right)^2 \right]$$

$$\approx 25.126$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2$$

$$= P\{X_i^2 = 1\} - \text{"}$$

$$= \frac{100}{199} - \left(\frac{100}{199} \right)^2.$$

Chebyshev : $P\{X \leq 30\} \leq P\{|X - 50.25| \geq 20.25\} \leq \frac{25.126}{(20.25)^2}$

one-sided Chebyshev : $P\{X \leq 30\} = P\{X \leq 50.25 - 20.25\}$ improved.

$$\leq \frac{25.126}{25.126 + (20.25)^2}$$

- More effective bounds when the moment gen ftn is known?

Let $M(t) = E[e^{tX}]$.

Then for $t > 0$,

$$P\{X \geq a\} = P\{e^{tX} \geq e^{ta}\} \underset{\uparrow \text{ Markov.}}{\leq} E[e^{tX}]\, e^{-ta}$$

For $t < 0$,

$$P\{X \leq a\} = P\{e^{tX} \geq e^{ta}\} \leq \quad \text{"}$$

<u>Prop</u> [Chernoff bounds]

$$\boxed{\begin{aligned} P\{X \geq a\} &\leq e^{-ta} M(t) \quad \text{for all } t > 0 \\ P\{X \leq a\} &\leq e^{-ta} M(t) \quad \text{"} \quad t < 0 \end{aligned}}$$

<u>Rmk</u> Obtain the best bound by using t that minimizes $e^{-ta} M(t)$.

<u>Ex</u> Z : a standard normal r.v.

$\Rightarrow M(t) = e^{t^2/2}$

$$P\{Z \geq a\} \leq e^{-ta} e^{\frac{t^2}{2}} \quad \text{for all } t > 0$$

minimize $\frac{t^2}{2} - ta \Rightarrow t = a$.

$$\therefore P\{Z \geq a\} \leq e^{-\frac{a^2}{2}}$$

Similarly, $P\{Z \leq a\} \leq e^{-\frac{a^2}{2}}$. //

Def A twice-differentiable real-valued ftn $f(x)$ is convex if $f''(x) \geq 0$ for all x;
concave ≤ 0

Ex $f(x) = x^2, e^{ax}, x^{\frac{1}{n}}$ for $x \geq 0$: convex
$g(x) = -f(x)$ (f convex) : concave.

Prop [Jensen's inequality]

$f(x)$: convex. then

$$E[f(X)] \geq f(E[X]),$$

provided that E's exist and finite.

Pf Taylor's series expansion

$$f(x) = f(\mu) + f'(\mu)(x-\mu) + \frac{\overbrace{f''(\xi)}^{\geq 0}(x-\mu)^2}{2}$$

ξ : some value btwn x & μ.

$\Rightarrow \quad f(x) \geq f(\mu) + f'(\mu)(x-\mu)$

$\Rightarrow \quad f(X) \geq f(\mu) + f'(\mu)(X-\mu)$

$\Rightarrow \quad E[f(X)] \geq f(\mu) + f'(\mu) E[X-\mu] = f(\mu)$. //

Ex Investment choice : invest all the money in
Ⓐ a risky proposition → random variable X, mean m
Ⓑ a risk-free venture → return of m with prob 1.

Make a decision to maximize $E[u(R)]$

u ← utility ftn, R ← return.

- If u is convex, then
  $E[u(X)] \geq u(m)$: risky option is better.

- If u is concave, then
  $E[u(X)] \leq u(m)$: risk-free option is better.