B-spline curve

Properties of B-spline curves

- B-spline curve:

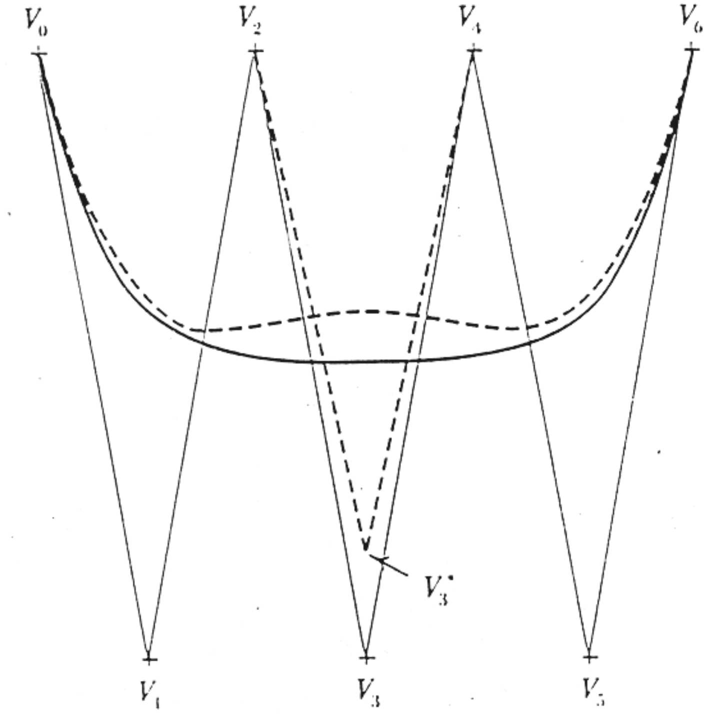
  Degree of curve is independent of number of control points

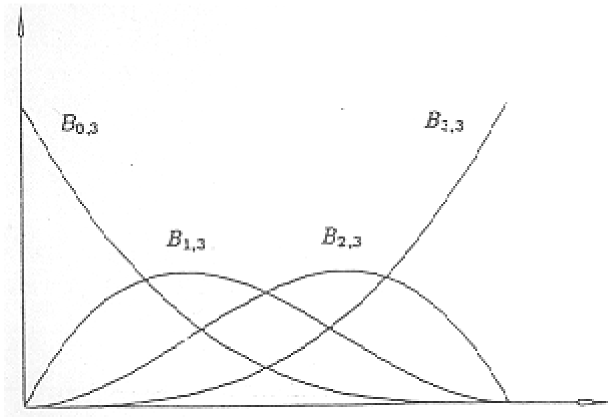
- Bezier curve: global modification

  Modification of any one control point changes the curve shape everywhere

  All the blending functions have non-zero value in the whole interval $0 \leq u \leq 1$

Bezier curve: global modification



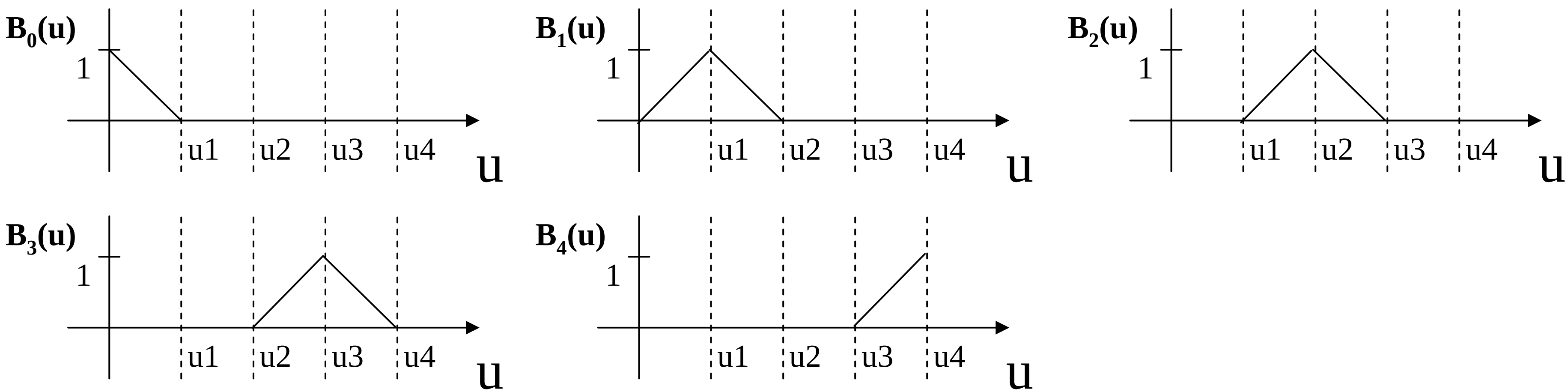


Bezier curve of degree 3

Desired Blending Function $B_i(u)$

$$P(u) = \sum_{i=0}^{n} P_i B_i(u)$$

Consider degree 1 blending functions, and n=4



P_0	has an effect only for	$0 \le u \le u_1$
P_1	has an effect only for	$0 \le u \le u_2$
P_2	has an effect only for	$u_1 \le u \le u_3$
P_3	has an effect only for	$u_2 \le u \le u_4$
P_4	has an effect only for	$u_3 \le u \le u_4$

Resulting Curve

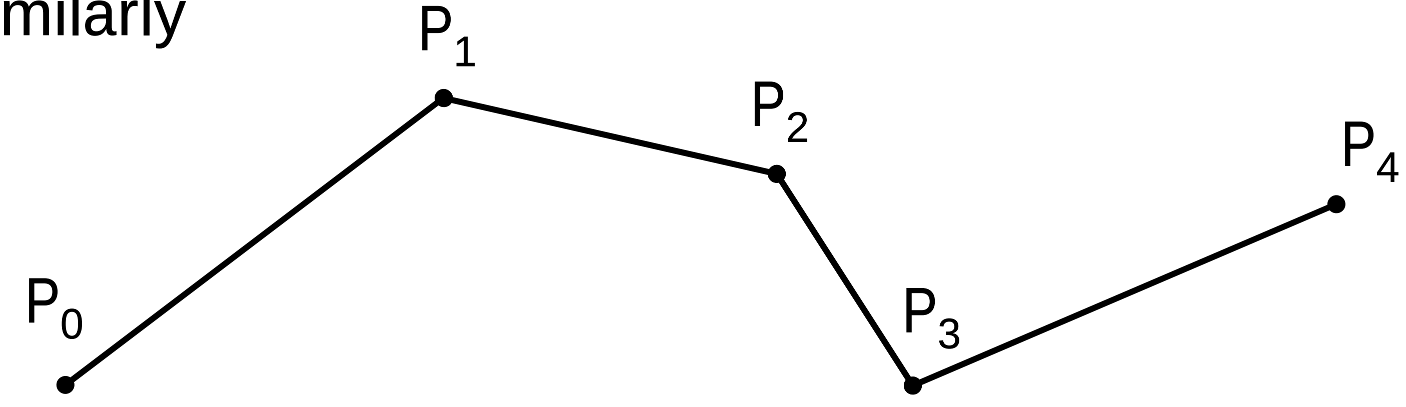
For $0 \le u \le u_1$ $\quad P(u) = P_0 B_0(u) + P_1 B_1(u) = P_0(1-u) + P_1 u$

… straight line from P_0 to P_1

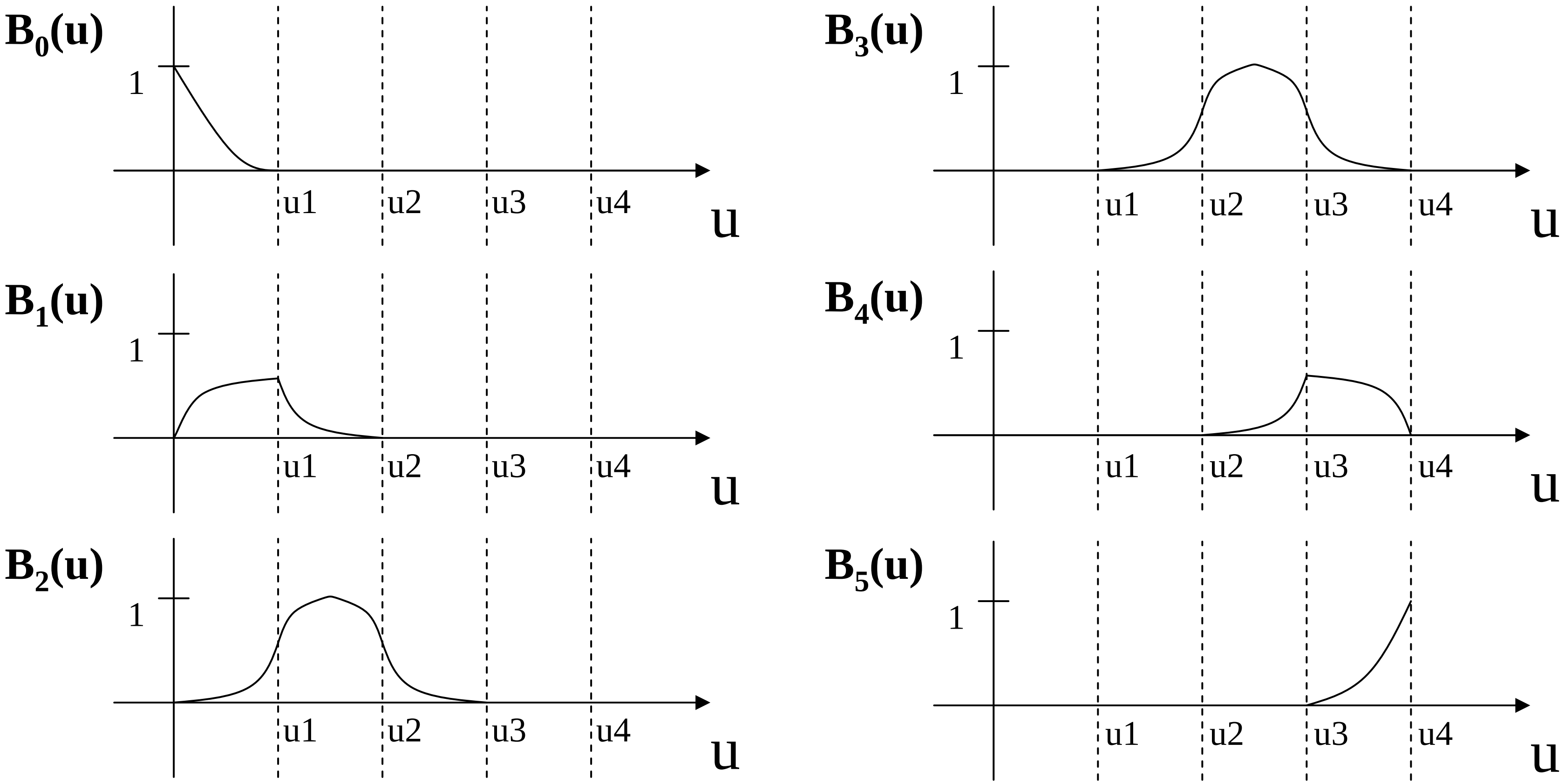
For $u_1 \le u \le u_2$ $\quad P(u) = P_1 B_1(u) + P_2 B_2(u) = P_1(2-u) + P_1(u-1)$

… straight line from P_1 to P_2

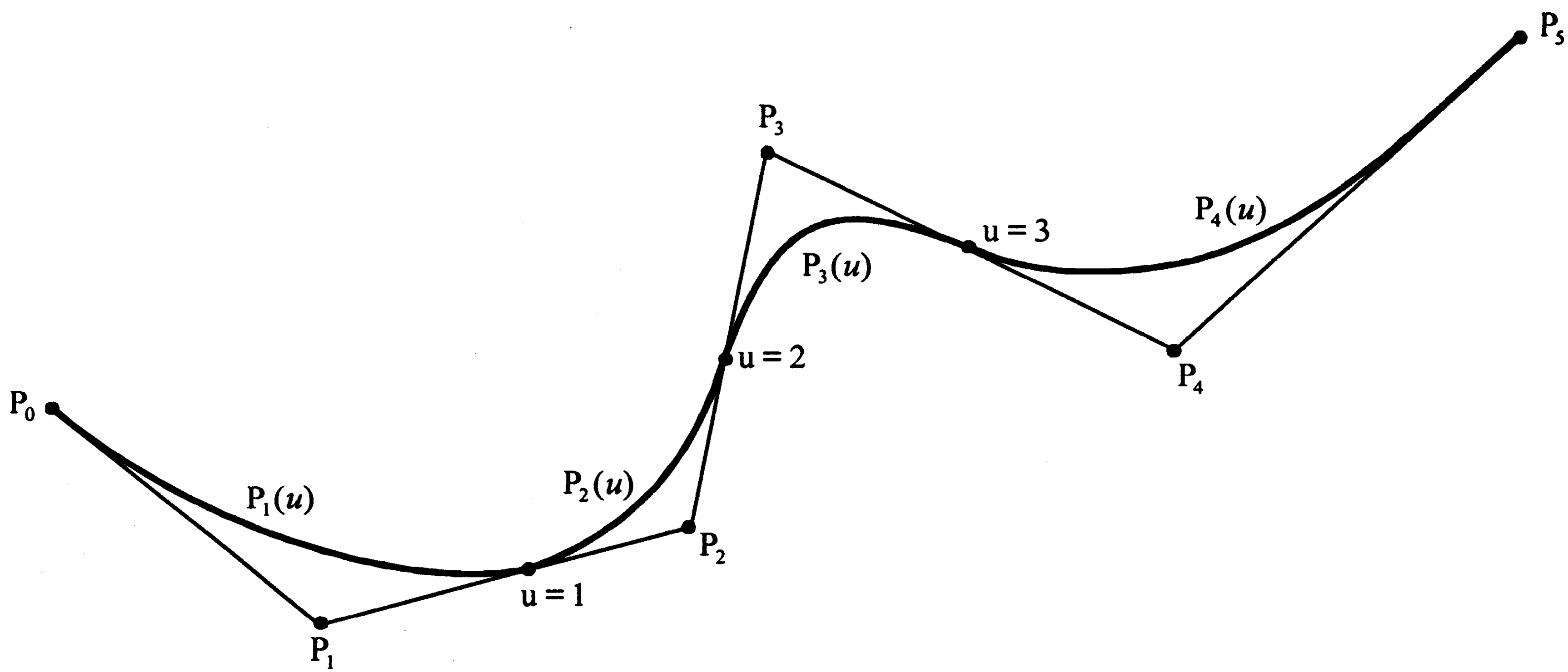
Similarly



Consider degree 2 blending functions, and n=5



P_0	has an effect only for	$0 \leq u \leq u_1$
P_1	has an effect only for	$0 \leq u \leq u_2$
P_2	has an effect only for	$0 \leq u \leq u_3$
P_3	has an effect only for	$u_1 \leq u \leq u_4$
P_4	has an effect only for	$u_2 \leq u \leq u_4$
P_5	has an effect only for	$u_3 \leq u \leq u_4$



P_0
P_1
P_2
P_3
P_4
P_5
$P_1(u)$
$P_2(u)$
$P_3(u)$
$P_4(u)$
u = 1
u = 2
u = 3

B-spline curve equation

$$P(u) = \sum_{i=0}^{n} P_i N_{i,k}(u) \qquad t_{k-1} \le u \le t_{n+1} \qquad \text{(a)}$$

$$N_{i,k}(u) = \frac{(u - t_i)N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u)N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}} \qquad \left(\frac{0}{0} = 0\right) \qquad \text{(b)}$$

$$N_{i,1}(u) = \begin{cases} 1 & t_i \le u \le t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(c)}$$

→ Art any value of u, there should be only one non-zero Ni,1(u)

$N_{i,k}$: degree (k-1) of u, <u>k : order</u> (independent of number of control points n)

t_i: knot values, boundary of non-zero range of each blending function

t_0 (for i=0)to t_{n+k} (for i=n) are needed (<u>n+k+1 values</u>)

B-spline curve equation – cont’

- Only the differences in t_i ($i=0, \cdots, n+k$) is important in (b)
- Can be shifted as a whole, parameter range should be shifted together
- A portion of B-spline curve is affected by a limited number of control points
- For u in $[t_l, t_{l+1}]$

  Control points associated with blending functions that are non-zero in $[t_l, t_{l+1}]$

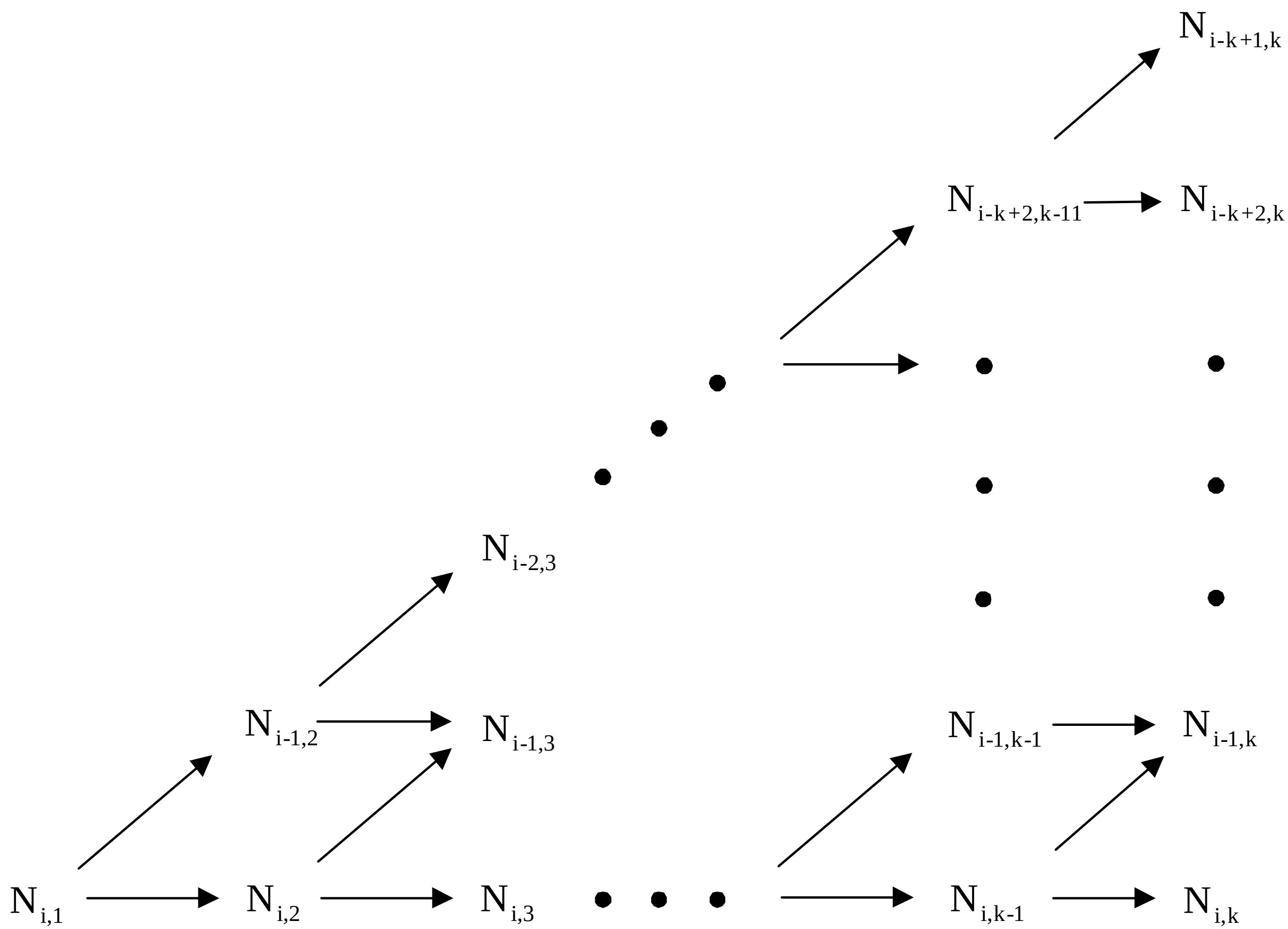
  $N_{l,1}(u)$ is nonzero in $[t_l, t_{l+1}]$ among $N_{i,1}(u)$

  Substitute $N_{l,1}(u)$ into the right-hand side of (b)

  $N_{l,2}(u)$, $N_{l-1,2}(u)$ can be non-zero

  Apply Recursive

  From $N_{l,2}(u)$, $N_{l,3}(u)$, $N_{l-1,3}(u)$; from $N_{l-1,2}(u)$ $N_{l-1,3}(u)$, $N_{l-2,3}(u)$can be non-zero



N i-k+1,k
N i-k+2,k-11
N i-k+2,k
N i-2,3
N i-1,2
N i-1,3
N i-1,k-1
N i-1,k
N i,1
N i,2
N i,3
N i,k-1
N i,k

B-spline curve equation – cont'

- Control points that have influence in the region $[t_l, t_{l+1}]$ are

$P_{l-k+1}, P_{l-k+2}, \cdots, P_l$ k control points

Control points modify: Example

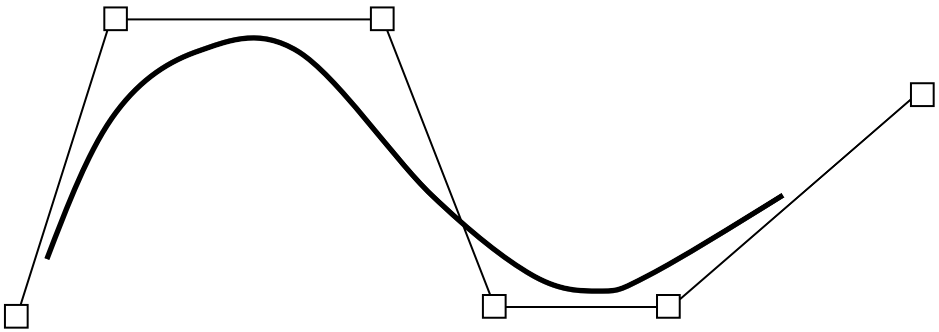
B-spline curve - Knot

- Knot: $t_0, t_1, \cdots, t_{n+k}$
  - parameter range is determined by knots
  - Periodic knots

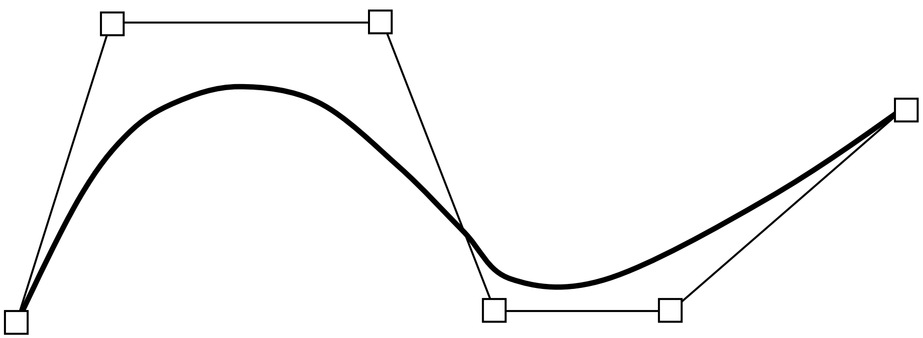
    $t_i = i - k \qquad 0 \le i \le n+k$
  - Non-periodic knots

$$t_i = \begin{cases} 0 & 0 \le i < k & \text{duplicates k times} \\ i - k + 1 & k \le i \le n & \\ n - k + 2 & n < i \le n + k & \text{duplicates k times} \end{cases} \qquad \text{(d)}$$

Periodic VS non-periodic



Periodic knot



Non-periodic knot

B-spline curve - Knot

- By duplicating knot k times at the ends
  - Curve passes through first control point and last control point
- Periodic knot

  First control point and last control point do not pass through curve as other control points

  ⇒non-periodic knots are used in most CAD systems
- knot interval is uniform in (d)

  uniform B-spline (vs. non-uniform B-spline)
- During manipulation of curve shape, knots are added or removed

  non-uniform knot ⇒ non-uniform B-spline curve

Example program

- Knot Insertion

  Example

Expansion of curve equation

- Ex)

  K=3, $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ non-periodic uniform B-spline

  $t_0=0,\ t_1=0,\ t_2=0,\ t_3=1,\ t_4=1,\ t_5=1$

  $$0 \le u \le 1$$

  $$\uparrow \qquad\qquad \uparrow$$

  $$(=t_2) \qquad (=t_3)$$

$$N_{0,1}(u) = \begin{cases} 1 & t_0 \le u \le t_1 \qquad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{1,1}(u) = \begin{cases} 1 & t_1 \le u \le t_2 \qquad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{2,1}(u) = \begin{cases} 1 & t_2 \le u \le t_3 \qquad (0 \le u \le 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & t_3 \le u \le t_4 \qquad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & t_4 \le u \le t_5 \qquad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

Expansion of curve equation – cont'

- At u=0, select $N_{2,1}(u)$ to be non-zero among $N_{0,1}(0)$, $N_{1,1}(0)$, $N_{2,1}(0)$
- Selection of any one is O.K.
- At u=1, select $N_{2,1}(u)$ similarly

=> Only $N_{2,1}(u)$ needs to be considered among blending functions of order 1

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = \frac{(1-u)N_{2,1}}{1} = (1-u)$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = \frac{uN_{2,1}}{1} = u$$

$$N_{0,3}(u) = \frac{(u - t_0)N_{0,2}}{t_2 - t_0} + \frac{(t_3 - u)N_{1,2}}{t_3 - t_1} = \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3}(u) = \frac{(u - t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4 - u)N_{2,2}}{t_4 - t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3}(u) = \frac{(u - t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5 - u)N_{3,2}}{t_5 - t_3} = u^2$$

$$\therefore P(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$$

Expansion of curve equation – cont’

- Consider Bezier curve defined by P_0, P_1, P_2

$$\mathbf{P}(u) = \binom{2}{0} u^0 (1-u)^2 \mathbf{P}_0 + \binom{2}{1} u^1 (1-u)^1 \mathbf{P}_1 + \binom{2}{2} u^2 (1-u)^0 \mathbf{P}_2$$

- Non-periodic B-spline curve having k (order) control points ends in Bezier curve
- Bezier curve is a special case of B-spline curve

Example

- Ex)

K=3, $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4, \mathbf{P}_5$

$t_0=0,\ t_1=0,\ t_2=0,\ t_3=1,\ t_4=2,\ t_5=3,$
$t_6=4,\ t_7=4,\ t_8=4$

$0 \leq u \leq 4$

$$N_{2,1}(u) = \begin{cases} 1 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & 1 \leq u \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & 2 \leq u \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{5,1}(u) = \begin{cases} 1 & 3 \leq u \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = (1 - u)N_{2,1}$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = u \quad N_{2,1} + (2 - u)N_{3,1}$$

$$N_{3,2}(u) = \frac{(u - t_3)N_{3,1}}{t_4 - t_3} + \frac{(t_5 - u)N_{4,1}}{t_5 - t_4} = (u - 1)N_{3,1} + (3 - u)N_{4.1}$$

$$N_{4,2}(u) = \frac{(u - t_4)N_{4,1}}{t_5 - t_4} + \frac{(t_6 - u)N_{5,1}}{t_6 - t_5} = (u - 2)N_{4,1} + (4 - u)N_{5.1}$$

$$N_{5,2}(u) = \frac{(u - t_5)N_{5,1}}{t_6 - t_5} + \frac{(t_7 - u)N_{6,1}}{t_7 - t_6} = (u - 3)N_{5,1}$$

$$N_{0,3}(u) = \frac{(u - t_o)N_{0,2}}{t_2 - t_0} + \frac{(t_3 - u)N_{1,2}}{t_3 - t_1} = (1-u)N_{1,2} = (1-u)^2 N_{2,1}$$

$$N_{1,3}(u) = \frac{(u - t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4 - u)N_{2,2}}{t_4 - t_2} = u \quad N_{1,2} + \frac{2-u}{2} N_{2,2}$$

$$= \left[u(1-u) + \frac{(2-u)u}{2} \right] N_{2,1} + \frac{(2-u)^2}{2} N_{3.1}$$

$$N_{2,3}(u) = \frac{(u - t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5 - u)N_{3,2}}{t_5 - t_3} = \frac{u}{2} N_{2,2} + \frac{3-u}{2} N_{3,2}$$

$$= \frac{u^2}{2} N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2} \right] N_{3,1} + \frac{(3-u)^2}{2} N_{4,1}$$

$$N_{3,3}(u) = \frac{(u - t_3)N_{3,2}}{t_5 - t_3} + \frac{(t_6 - u)N_{4,2}}{t_6 - t_4} = \frac{u-1}{2} N_{3,2} + \frac{4-u}{2} N_{4,2}$$

$$= \frac{(u-1)^2}{2} N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2} \right] N_{4,1} + \frac{(4-u)^2}{2} N_{5,1}$$

$$N_{4,3}(u) = \frac{(u - t_4)N_{4,2}}{t_6 - t_4} + \frac{(t_7 - u)N_{5,2}}{t_7 - t_5} = \frac{u - 2}{2} N_{4,2} + (4 - u)N_{5,2}$$

$$= \frac{(u-2)^2}{2} N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3) \right] N_{5,1}$$

$$N_{5,3}(u) = \frac{(u - t_5)N_{5,2}}{t_7 - t_5} + \frac{(t_8 - u)N_{6,2}}{t_8 - t_6} = (u - 3)N_{5,2} = (u - 3)^2 N_{5,1}$$

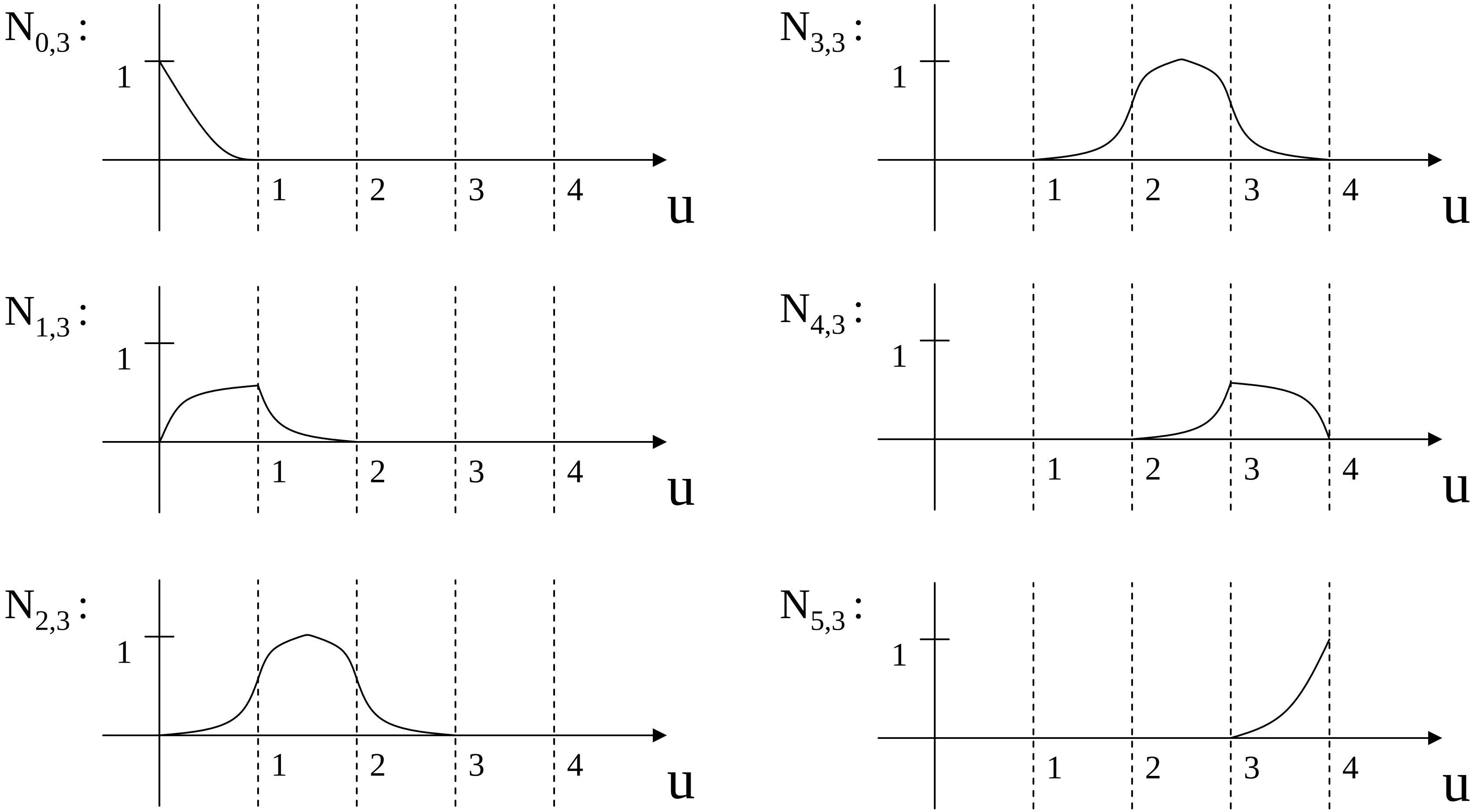
$$\therefore \mathbf{P}(u) = (1-u)^2 N_{2,1}\mathbf{P}_0 + \left\{\left[u(1-u) + \frac{(2-u)u}{2}\right]N_{2,1} + \frac{(2-u)^2}{2} N_{3,1}\right\}\mathbf{P}_1$$

$$+\left\{\frac{u^2}{2} N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]N_{3,1} + \frac{(3-u)^2}{2} N_{4,1}\right\}\mathbf{P}_2$$

$$+\left\{\frac{(u-1)^2}{2} N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2}\right]N_{4,1} + \frac{(4-u)^2}{2} N_{5,1}\right\}\mathbf{P}_3$$

$$+\left\{\frac{(u-2)^2}{2} N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3)\right]N_{5,1}\right\}\mathbf{P}_4$$

$$+ (u-3)^2 N_{5,1} \quad \mathbf{P}_5$$


N0,3 :
N1,3 :
N2,3 :
N3,3 :
N4,3 :
N5,3 :
1
1 2 3 4
u

Example – cont’

- For each knot interval, coefficients of certain control points = 0
  - -> only subset of control points has influence
- For $0 \le u \le 1$, all $N_{i,1}$ except $N_{2,1}$ are 0

$$\therefore \mathbf{P}_1(u) = (1-u)^2 \mathbf{P}_0 + \left[u(1-u) + \frac{(2-u)u}{2} \right] \mathbf{P}_1 + \frac{u^2}{2} \mathbf{P}_2$$

Example – cont’

- Similarly

$1 \le u \le 2$

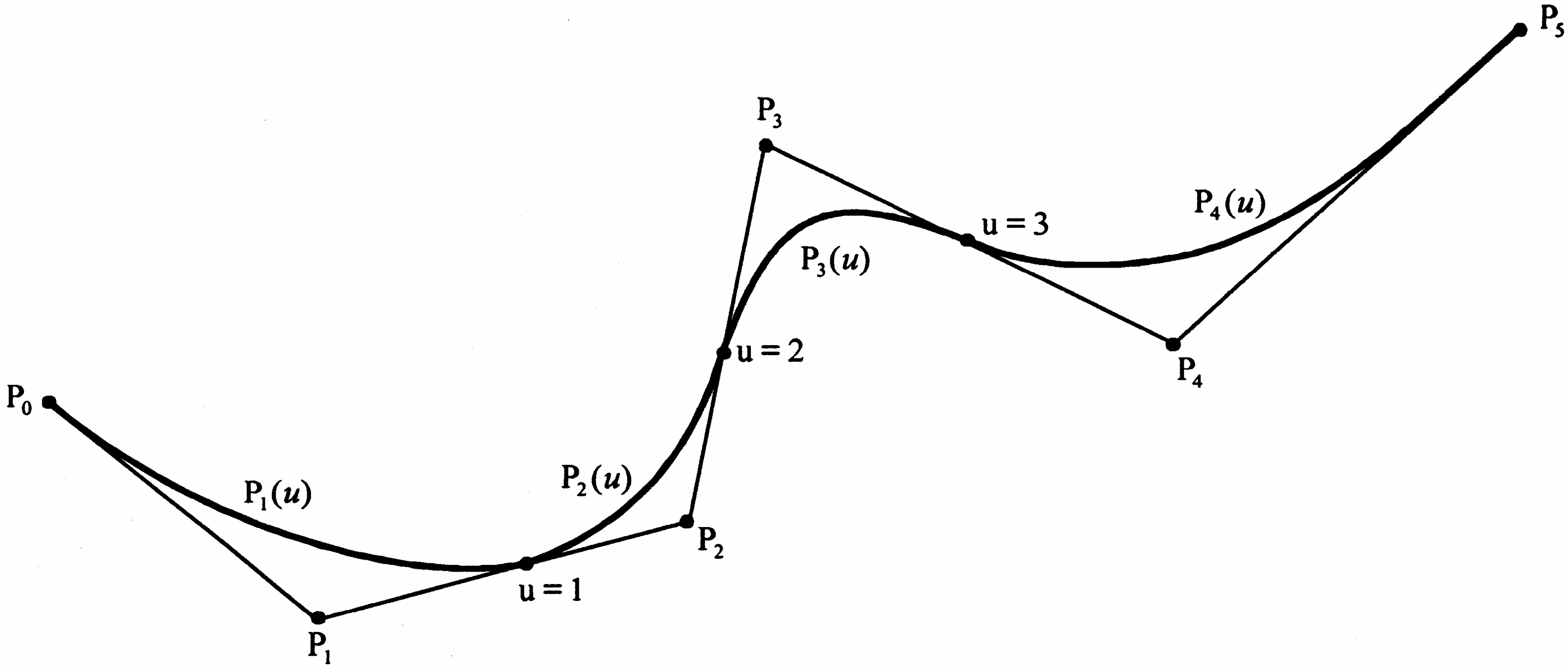
$$\mathbf{P}_2(u) = \frac{(2-u)^2}{2}\mathbf{P}_1 + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]\mathbf{P}_2 + \frac{(u-1)^2}{2}\mathbf{P}_3$$

$2 \le u \le 3$

$$\mathbf{P}_3(u) = \frac{(3-u)^2}{2}\mathbf{P}_2 + \frac{1}{2}(-2u^2 + 10u - 11)\mathbf{P}_3 + \frac{(u-2)^2}{2}\mathbf{P}_4$$

$3 \le u \le 4$

$$\mathbf{P}_4(u) = \frac{(4-u)^2}{2}\mathbf{P}_3 + \frac{1}{2}(-3u^2 + 20u - 32)\mathbf{P}_4 + (u-3)^2\mathbf{P}_5$$


P5
P3
P4(u)
u = 3
P3(u)
u = 2
P4
P0
P2(u)
P1(u)
P2
u = 1
P1

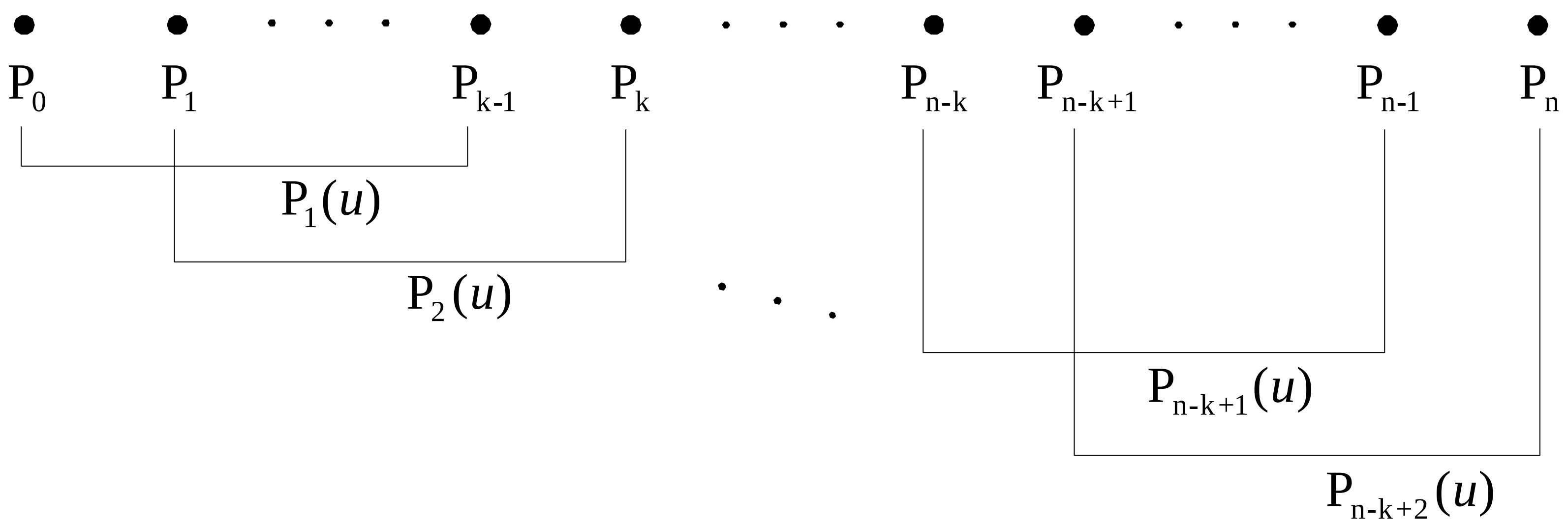
Example – cont'

1. $\mathbf{P}'_1(1) = \mathbf{P}'_2(1), \quad \mathbf{P}'_2(2) = \mathbf{P}'_3(2), \quad \mathbf{P}'_3(3) = \mathbf{P}'_4(3) \quad C^1\text{continuiy}$

   C^2 continuity is not satisfied. (∵ k=3, degree 2)

   - For curve of order k, neighboring curves have same derivatives up to (k−2)−th derivative at the common knot
2. Each curve segment is defined by k control points.
3. Any one control point can influence up to maximum k curve segments.

count curve segment including $\mathbf{P}_{k-1}$



Intersection between curves

- $\mathbf{P}(u) - \mathbf{Q}(v) = 0$
- 3 scalar equations, two unknowns

  $P_x(u) - Q_x(v) = 0$

  $P_y(u) - Q_y(v) = 0$
- Use Newton Raphson method
  - Derivative of P_x, Q_x, P_y, Q_y need to be calculated

  $$f_1(x_1,\ldots,x_n) = 0$$
  $$f_2(x_1,\ldots,x_n) = 0$$
  $$\vdots$$
  $$f_n(x_1,\ldots,x_n) = 0$$

$$f_1(x_1 + \quad x_1, x_2 + \quad x_2, \cdots, x_n + \quad x_n) = f_1(x_1, \cdots, x_n) + \frac{\partial f_1}{\partial x_1} \quad x_1 + \cdots + \frac{\partial f_1}{\partial x_n} \quad x_n$$

$$\vdots$$

$$f_n(x_1 + \quad x_1, x_2 + \quad x_2, \cdots, x_n + \quad x_n) = f_n(x_1, \cdots, x_n) + \frac{\partial f_n}{\partial x_1} \quad x_1 + \cdots + \frac{\partial f_n}{\partial x_n} \quad x_n$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} \cdots \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \quad x_1 \\ \quad x_2 \\ \quad x_n \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ -f_n \end{bmatrix}$$

Intersection between curves

- If initial values of u, v are too far from real solution, the iteration diverges.
- Hard to find all the intersection points.
- Cannot handle the case of overlapping curves.
- Two curves are regarded to intersect each other if they lie within numerical tolerance.
- Control polygons are approximated to the curve by subdivision and initial values of u,v can be provided closely by intersecting control polygons
- Better to detect special situation in advance before resorting to numerical solution.
- Tuning tolerance values is necessary

Straight line vs. curve

- $\mathbf{P}(u) = \mathbf{P}_0 + u(\mathbf{P}_1 - \mathbf{P}_0)$
- $\mathbf{Q}(v) = \mathbf{P}_0 + u(\mathbf{P}_1 - \mathbf{P}_0)$ (a)
- Apply dot product $(\mathbf{P}_0 \times \mathbf{P}_1)$ to both sides of eq(a) gives

  $(\mathbf{P}_0 \times \mathbf{P}_1) \cdot \mathbf{Q}(v) = 0$

  non-linear equation of v

Non-uniform Rational B-spline (NURBS) curve

- Use same Blending functions as B-spline
- Control points are given in homogeneous coordinates $(x_i,\ y_i,\ z_i) \Rightarrow (x_i \bullet h_i,\ y_i \bullet h_i,\ z_i \bullet h_i,\ h_i)$

$$x \cdot h = \sum_{i=0}^{n} (h_i \cdot x_i) \quad N_{i,k}(u)$$

$$y \cdot h = \sum_{i=0}^{n} (h_i \cdot y_i) \quad N_{i,k}(u)$$

$$z \cdot h = \sum_{i=0}^{n} (h_i \cdot z_i) \quad N_{i,k}(u)$$

$$h = \sum_{i=0}^{n} h_i \quad N_{i,k}(u)$$

Non-uniform Rational B-spline (NURBS) curve – cont'

$$\mathrm{P}(u) = \frac{\sum_{i=0}^{n} h_i \mathrm{P}_i N_{i,k}(u)}{\sum_{i=0}^{n} h_i \quad N_{i,k}(u)}$$

Passes through the 1st and the last control points

(When non-periodic knots are used)

Numerator is B-spline with $h_i\mathrm{P}_i$ as contorl points

$\Rightarrow$ $h_0\mathrm{P}_0$, $h_n\mathrm{P}_n$ at parameter boundary values

Similarly denominator has values of h_0, h_n at parameter boundary values

Non-uniform Rational B-spline (NURBS) curve – cont'

Directions of tangent vectors are P_1-P_0, P_n-P_{n-1} at starting and ending points

$$h_i = 1 \quad \sum_{i=0}^{n} N_{i,k}(u) = 1 \quad \Rightarrow \quad B - spline$$

B-spline curve is a special case of NURBS

Non-uniform Rational B-spline (NURBS) curve – cont'

- Curve shape can be changed by changing weight(h_i)

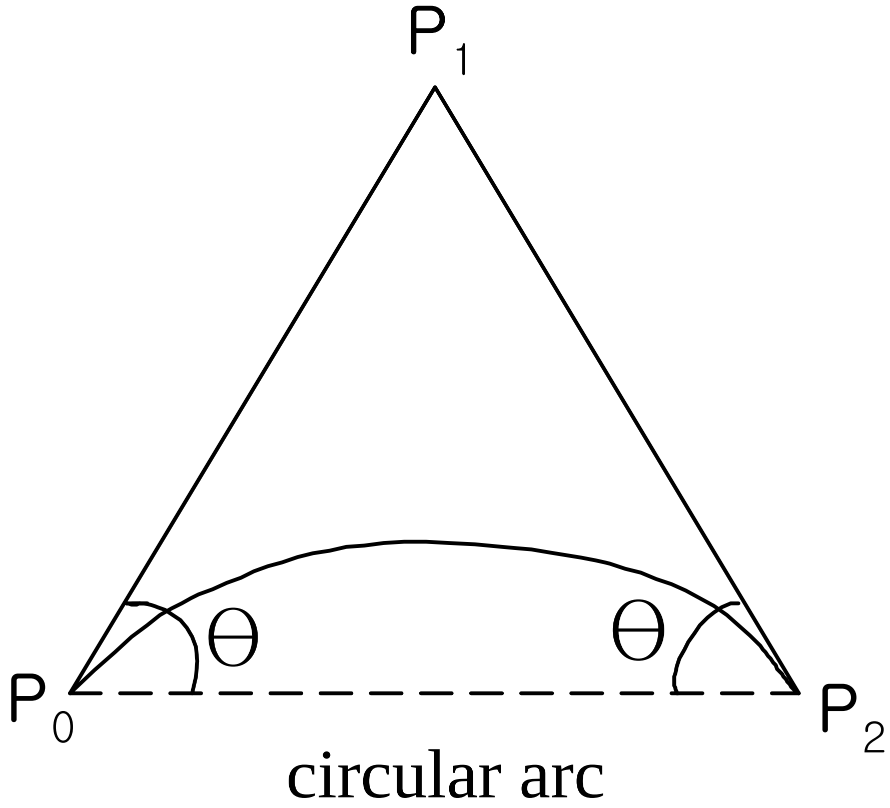
  Increasing weight has an effect of pulling curve toward associated control point

  Example program

- Conic curve can be represented exactly

  Reducing program coding effort

Control points of NURBS curve equivalent to a circular arc



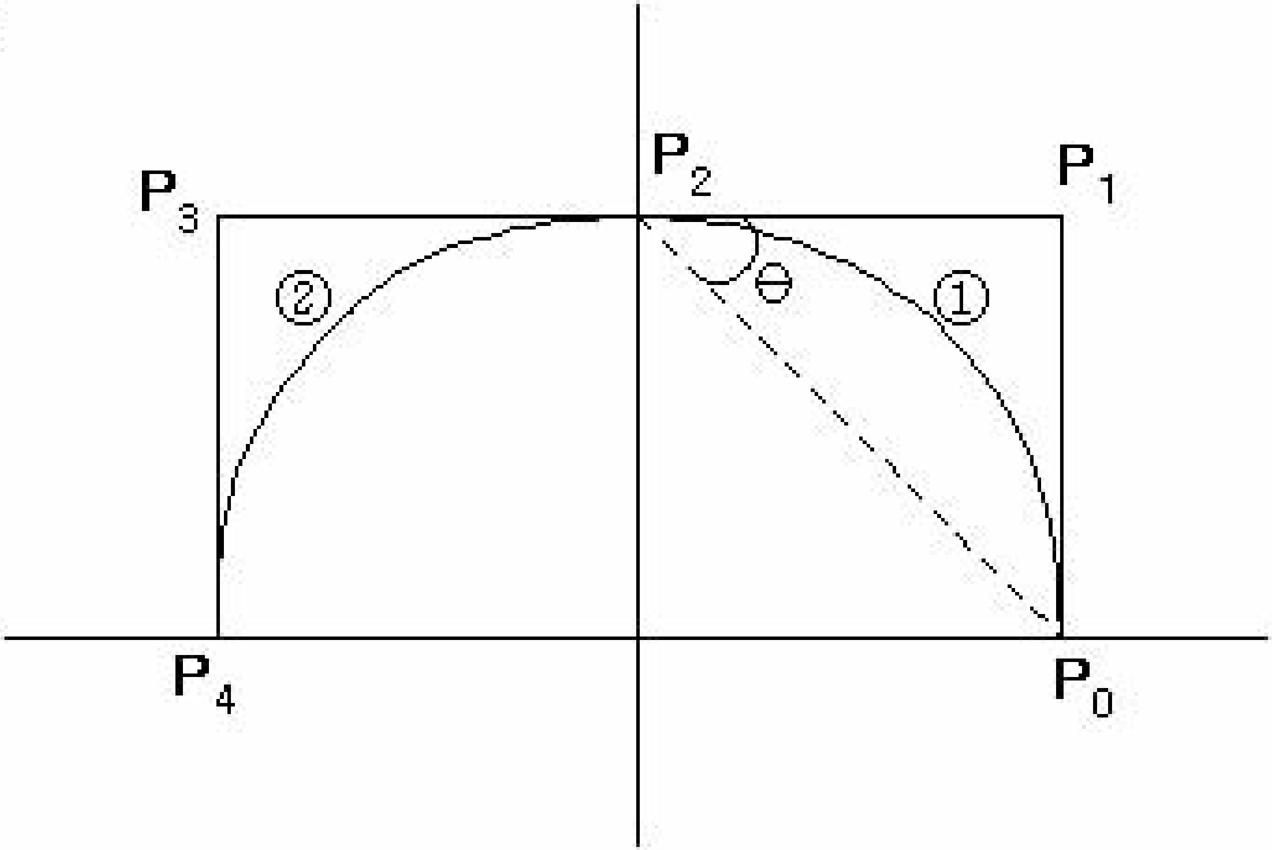
$$h_0 = h_2 = 1$$
$$h_1 = \cos\theta$$

Can be used when center angle is less than 180º.

Arc with a center angle bigger than 180º is split into two and combined later

Example

Ex.



Example – cont'

$$P_0 = (1, \ 0), \qquad P_1 = (1, \ 1), \qquad P_2 = (0, \ 1)$$

$$h_0 = 0 \qquad h_1 = \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$knot \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad (n = 2, \quad k = 3)$$

Similarly $P_2 = (0, \ 1), \quad P_3 = (-1, \ 1), \quad P_4 = (-1, \ 0)$

$$h_2 = 1, \qquad h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad \Rightarrow \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2$$

Example – cont'

Composition

$$P_0 = (1,\ \ 0),\quad P_1 = (1,\ \ 1),\quad P_2 = (0,\ \ 1),$$

$$P_3 = (-1,\ \ 1),\quad P_4 = (-1,\ \ 0)$$

$$h_0 = 1 \qquad h_1 = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2$$