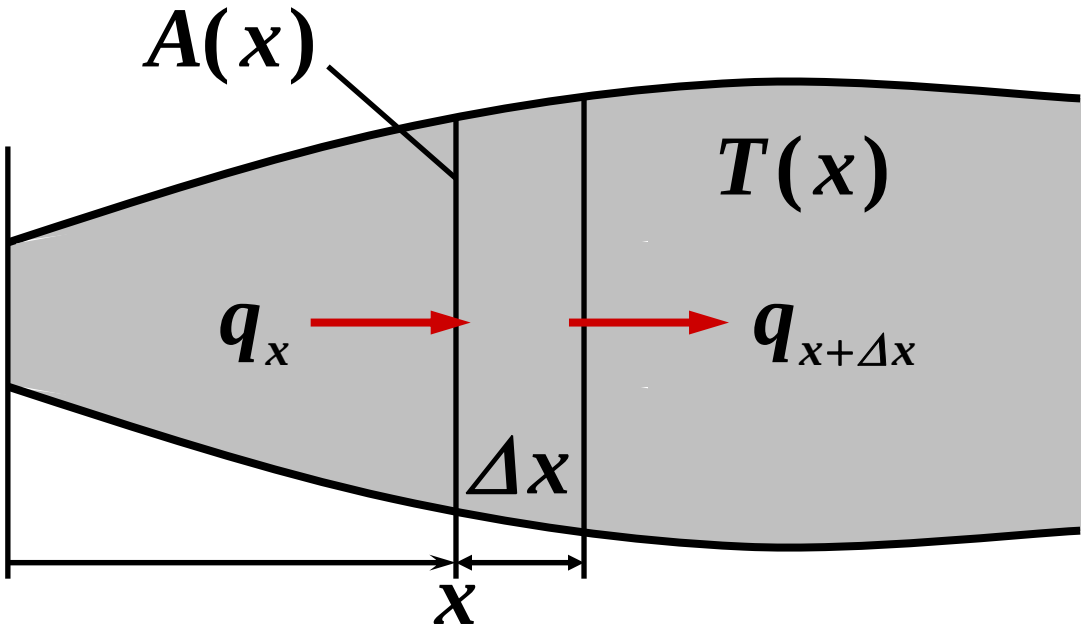
ONE-DIMENSIOANL, STEADY-STATE CONDUCTION

- 1-D Conduction with Varying Cross-Section
- Plane Wall
- Alternate Conduction Analysis
- Electric Network Analogy
- Radial System
- Conduction with Heat Generation
- Heat Transfer from Extended Surfaces

1-D Conduction: Varying Cross-section



steady state: $q_x = q_{x+\Delta x}$, $q''_x(x)A(x) =$ **constant**

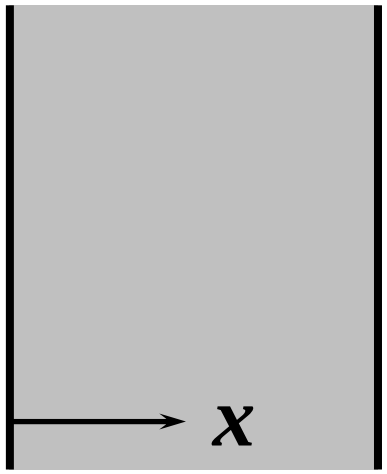
$$q_x = -kA(x)\frac{dT}{dx}$$

$$q_{x+\Delta x} = -kA(x)\frac{dT}{dx} + \frac{d}{dx}\left[-kA(x)\frac{dT}{dx}\right]\Delta x$$

$$\frac{d}{dx}\left[kA(x)\frac{dT}{dx}\right]=0$$

When k = const., $$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right]=0$$

Plane Wall

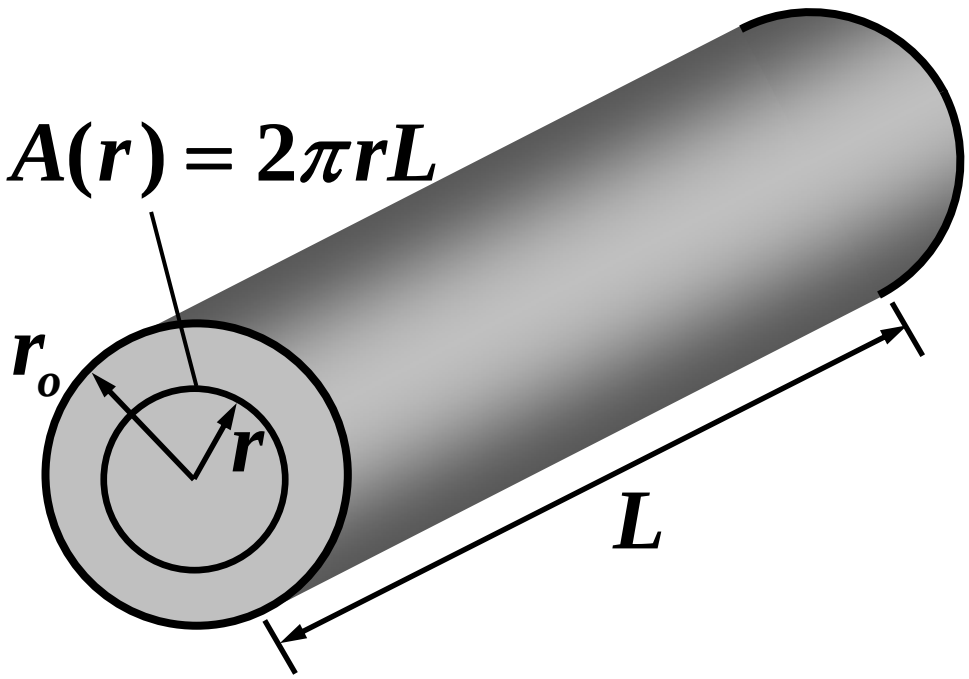


$A(x)$ = const.

$$\frac{d^2T}{dx^2}=0$$

Radial System

- **Cylindrical system**



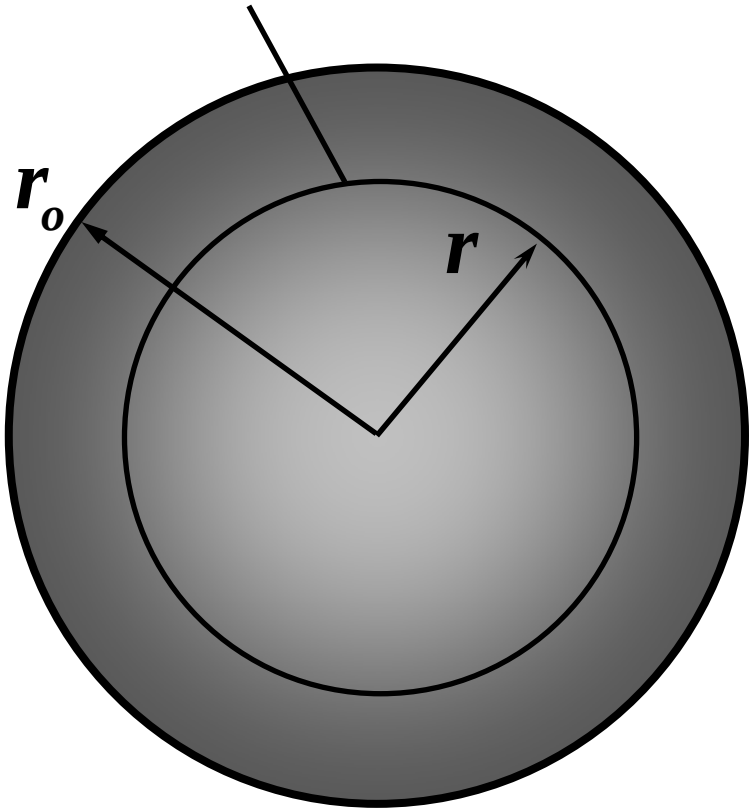
$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] = 0$$

$$\frac{d}{dr}\left[(2\pi rL)\frac{dT}{dr}\right] = 0$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

- **Spherical system**

$$A(r) = 4\pi r^2$$



$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] = 0$$

$$\frac{d}{dr}\left[\left(4\pi r^2\right)\frac{dT}{dr}\right] = 0$$

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

Summary

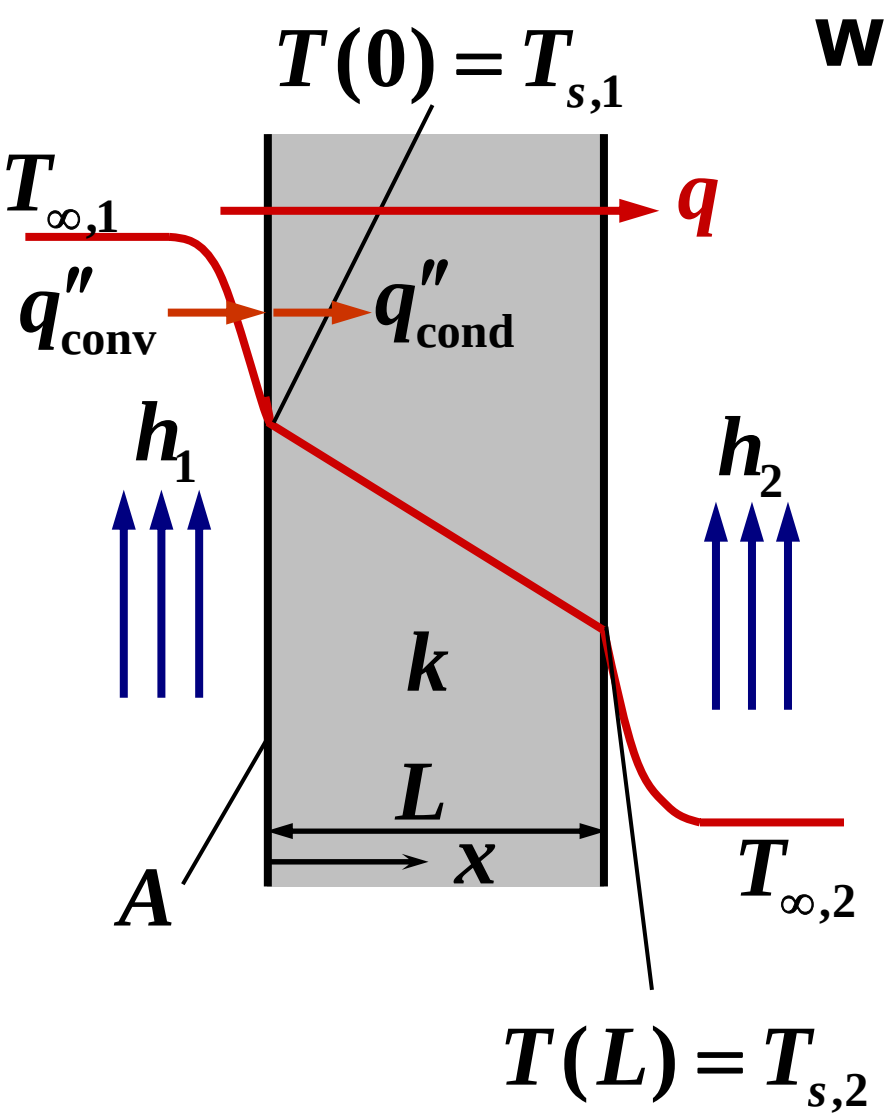
$$\frac{d}{dr}\left(r^n \frac{dT}{dr} \right) = 0$$

$n = 0$: **plane wall**

$n = 1$: **cylinder**

$n = 2$: **sphere**

Plane Wall



with no heat generation

$$\frac{d^2T}{dx^2} = 0$$

boundary conditions

$$-k\left.\frac{dT}{dx}\right|_0 = h_1\left[T_{\infty,1} - T(0)\right]$$

$$-k\left.\frac{dT}{dx}\right|_L = h_2\left[T(L) - T_{\infty,2}\right]$$

$$\frac{d^2T}{dx^2} = 0 \quad \rightarrow \frac{dT}{dx} = a \quad \rightarrow T(x) = ax + b$$

$$\text{at } x = 0, \quad -k\left.\frac{dT}{dx}\right|_0 = h_1\left[T_{\infty,1} - T(0)\right]$$

$$-ka = h_1\left[T_{\infty,1} - b\right]$$

$$\text{at } x = L, \quad -k\left.\frac{dT}{dx}\right|_L = h_2\left[T(L) - T_{\infty,2}\right]$$

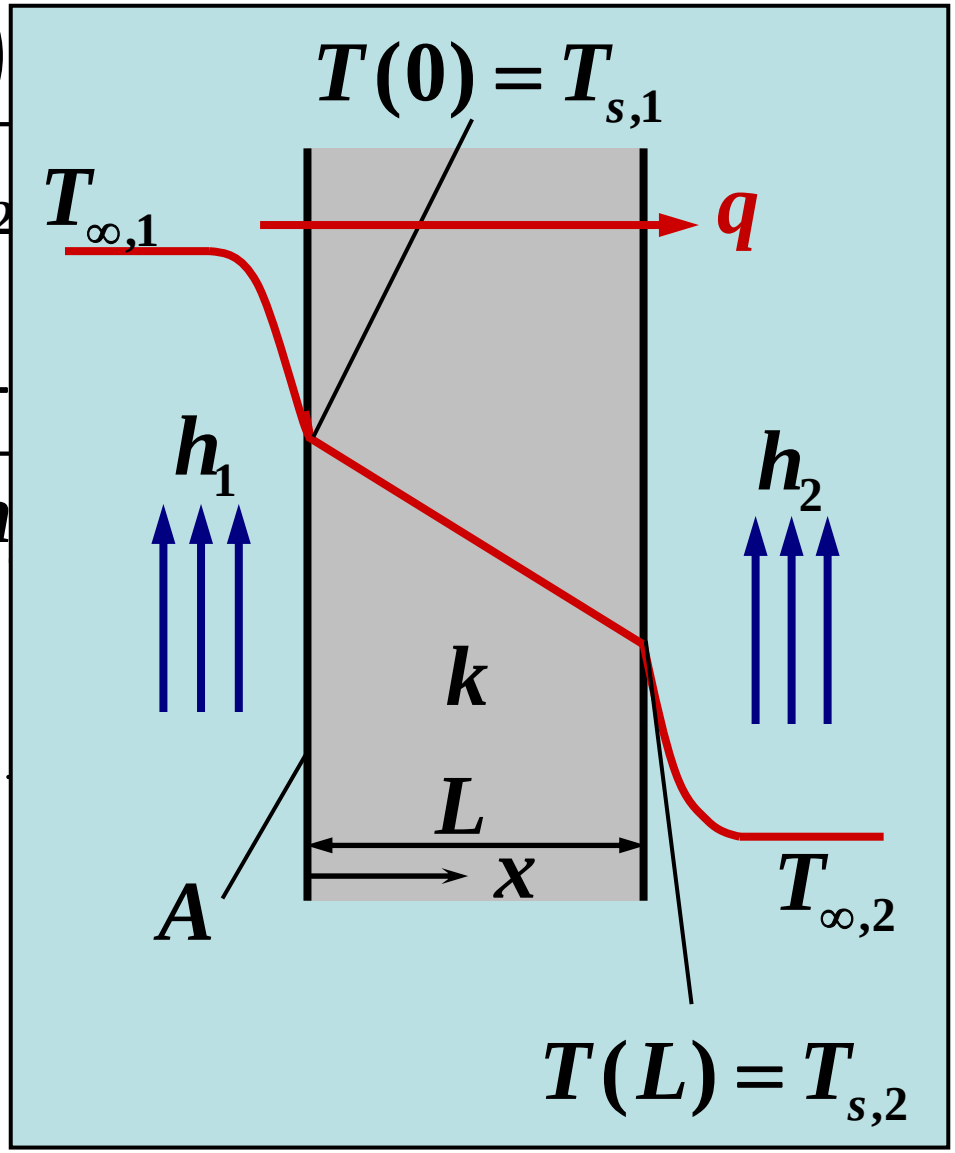
$$-ka = h_2\left[\left(aL + b\right) - T_{\infty,2}\right]$$

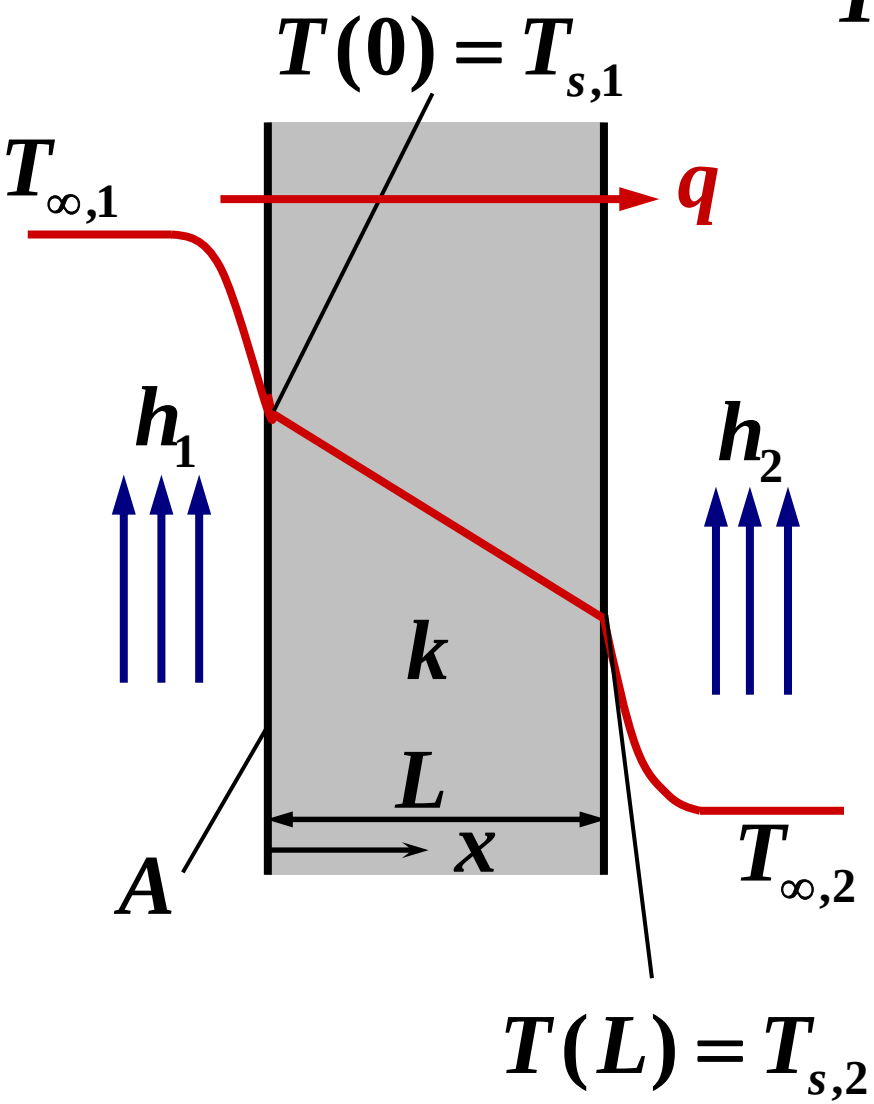
$$a = -\frac{h_1 h_2 \left(T_{\infty,1} - T_{\infty,2}\right)}{kh_1 + h_1 h_2 L + kh_2}$$

$$b = \frac{kh_2 T_{\infty,2} + h_1 \left(h_2 L +\right.}{kh_1 + h_1 h_2 L + kh}$$

$$q = -kA\frac{dT}{dx} = kAa =$$

$$= \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}}$$





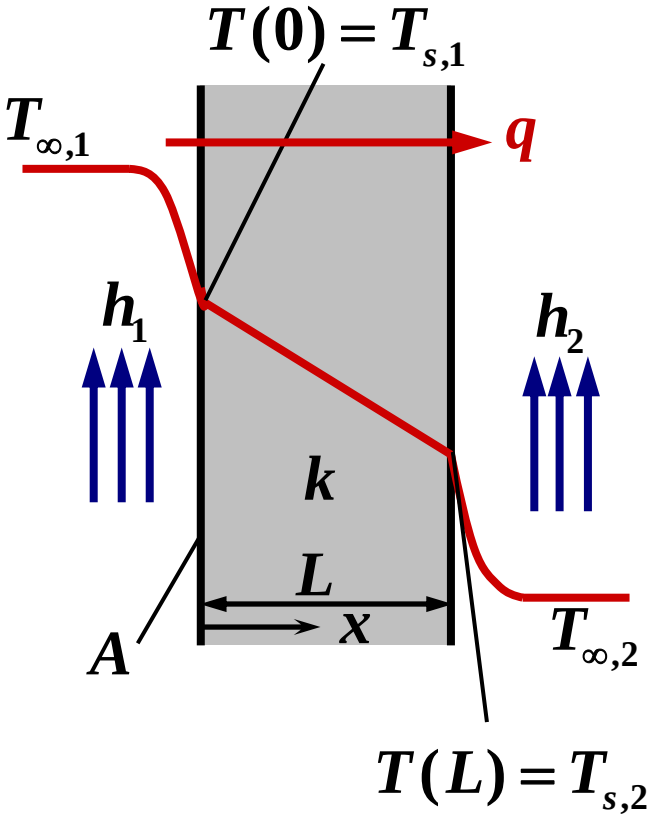
$$T_{s,1} - T_{s,2} = T(0) - T(L)$$

$$= b - (aL + b) = -aL$$

$$= \frac{L\left(T_{\infty,1} - T_{\infty,2}\right)}{\dfrac{k}{h_1} + L + \dfrac{k}{h_2}}$$

$$= \frac{T_{\infty,1} - T_{\infty,2}}{\dfrac{k}{h_1 L} + 1 + \dfrac{k}{h_2 L}}$$

Alternative Approach



$$q = -kA\frac{dT}{dx} = \text{constant}$$

$$qdx = -kAdT$$

$$\int_0^L qdx = \int_{T_{s,1}}^{T_{s,2}} \left(-kA\right)dT$$

$$qL = kA\left(T_{s,1} - T_{s,2}\right)$$

$$q = \frac{kA\left(T_{s,1} - T_{s,2}\right)}{L} = \frac{T_{s,1} - T_{s,2}}{\dfrac{L}{kA}}$$

at the boundaries

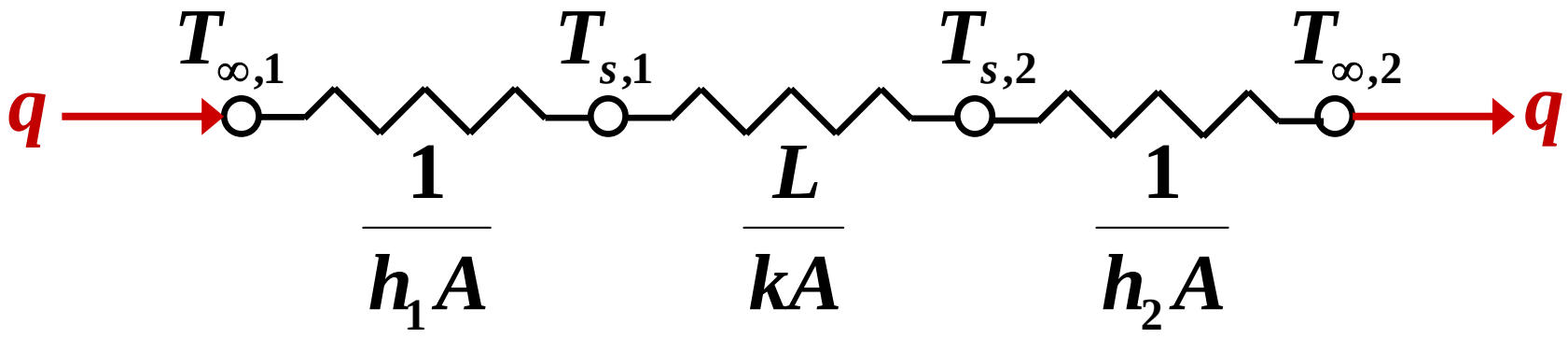
$$q = -kA\left.\frac{dT}{dx}\right|_0 = h_1 A\left(T_{\infty,1} - T_{s,1}\right) = \frac{T_{\infty,1} - T_{s,1}}{\dfrac{1}{h_1 A}}$$

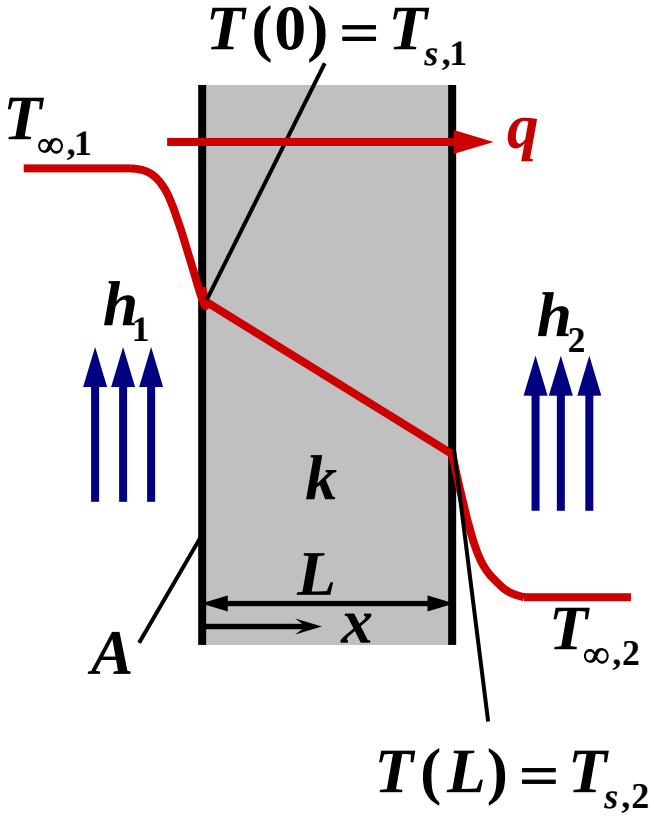
$$q = -kA\left.\frac{dT}{dx}\right|_L = h_2 A\left(T_{s,2} - T_{\infty,2}\right) = \frac{T_{s,2} - T_{\infty,2}}{\dfrac{1}{h_2 A}}$$

$$q = \frac{T_{\infty,1} - \cancel{T_{s,1}}}{1/h_1 A} = \frac{\cancel{T_{s,1}} - \cancel{T_{s,2}}}{L/kA} = \frac{\cancel{T_{s,2}} - T_{\infty,2}}{1/h_2 A}$$

$$= \frac{T_{\infty,1} - T_{\infty,2}}{\dfrac{1}{h_1 A} + \dfrac{L}{kA} + \dfrac{1}{h_2 A}}$$

Electric Network Analogy





$$q = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}}$$

thermal resistance

$$R_{t,\text{cond}} = \frac{L}{kA} \quad [\text{K/W}]$$

$$R_{t,\text{conv}} = \frac{1}{hA} \quad [\text{K/W}]$$

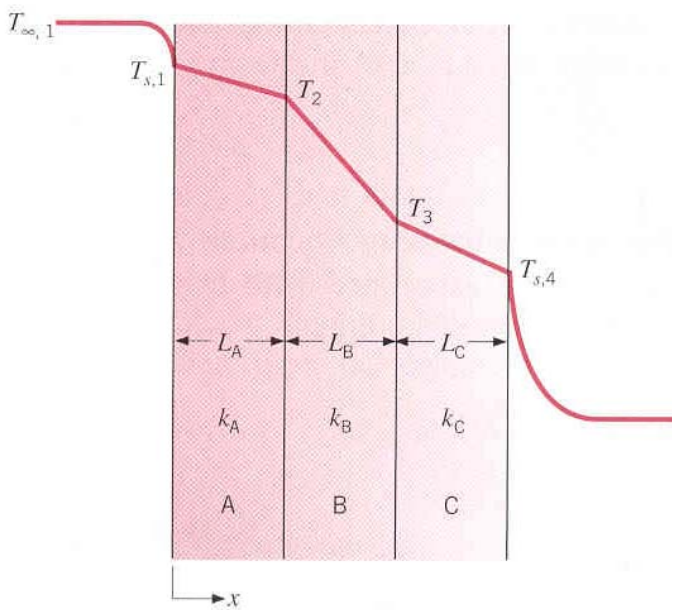
$$q = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} = \frac{\Delta T}{R_{tot}}$$

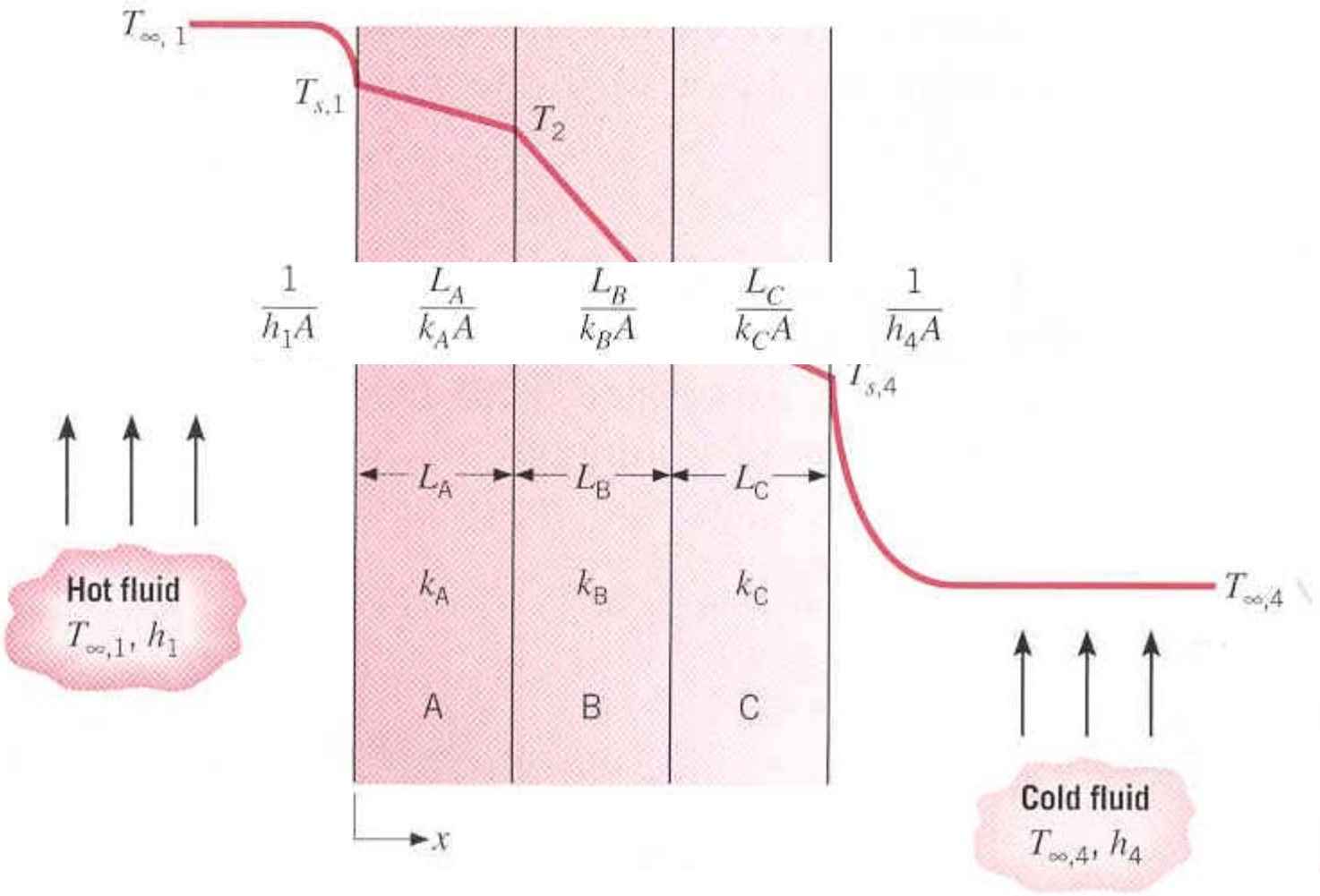
$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

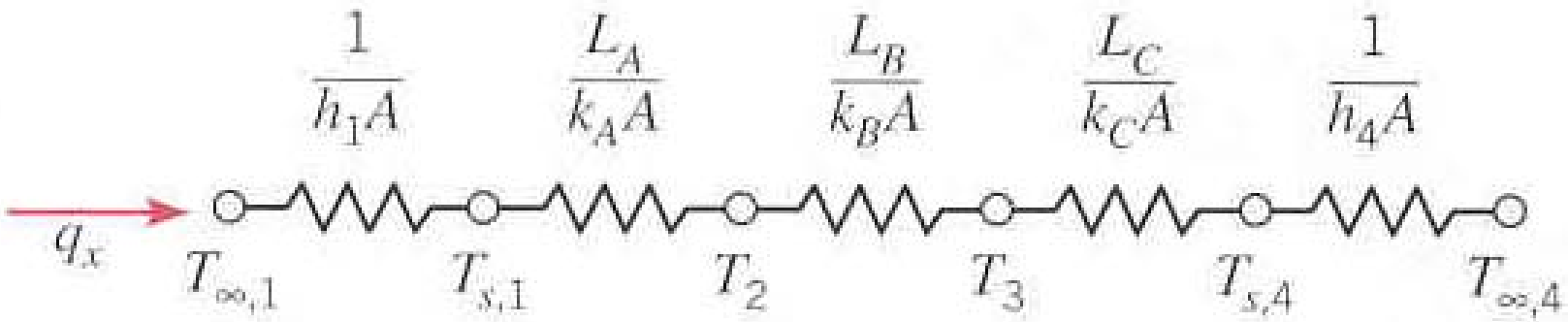
- **overall heat transfer coefficient**

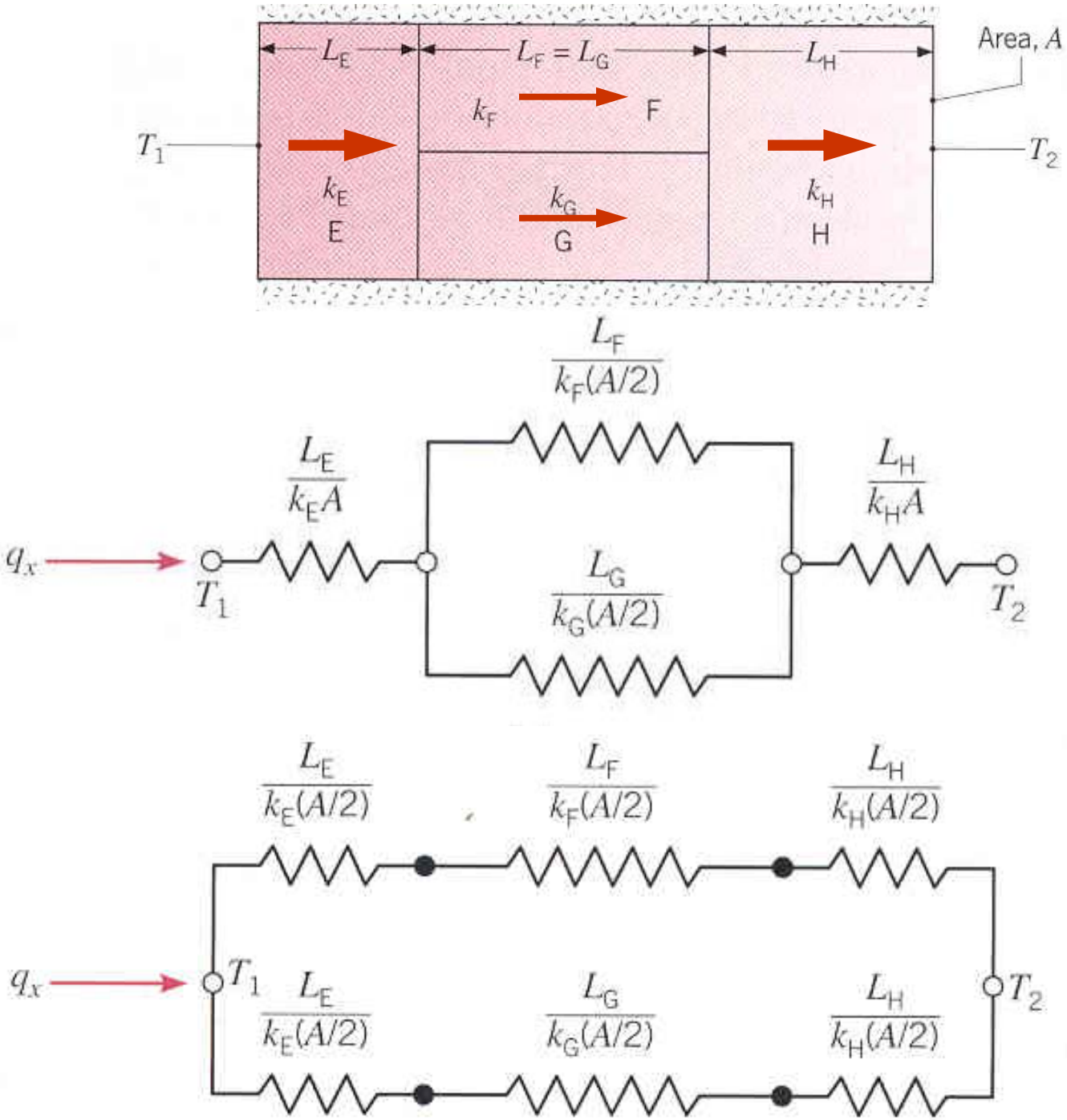
$$q \equiv UA\Delta T = \frac{\Delta T}{R_{tot}}$$

$$U = \frac{1}{AR_{tot}} = \frac{1}{\frac{1}{h_h} + \sum \frac{L_i}{k_i} + \frac{1}{h_c}}$$

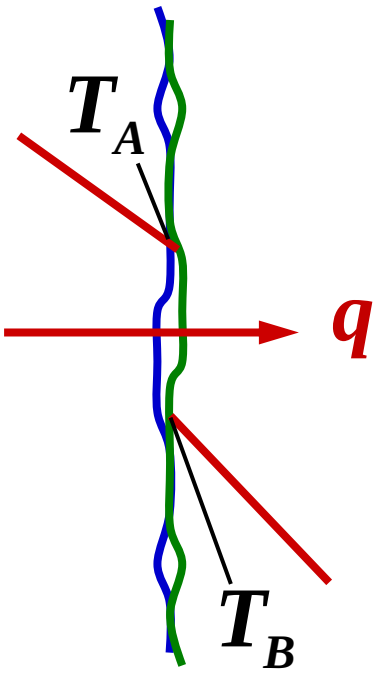



$T_{\infty,1}$
$T_{s,1}$
T_2
$\frac{1}{h_1 A}$
$\frac{L_A}{k_A A}$
$\frac{L_B}{k_B A}$
$\frac{L_C}{k_C A}$
$\frac{1}{h_4 A}$
$T_{s,4}$
L_A
L_B
L_C
k_A
k_B
k_C
A
B
C
$T_{\infty,4}$
Hot fluid
$T_{\infty,1}$, h_1
Cold fluid
$T_{\infty,4}$, h_4
x


$\frac{1}{h_1 A}$
$\frac{L_A}{k_A A}$
$\frac{L_B}{k_B A}$
$\frac{L_C}{k_C A}$
$\frac{1}{h_4 A}$
q_x
$T_{\infty,1}$
$T_{s,1}$
T_2
T_3
$T_{s,4}$
$T_{\infty,4}$


L_E
$L_F = L_G$
L_H
Area, A
k_F
F
T_1
T_2
k_E
E
k_G
G
k_H
H
$\frac{L_F}{k_F(A/2)}$
$\frac{L_E}{k_E A}$
$\frac{L_H}{k_H A}$
q_x
T_1
$\frac{L_G}{k_G(A/2)}$
T_2
$\frac{L_E}{k_E(A/2)}$
$\frac{L_F}{k_F(A/2)}$
$\frac{L_H}{k_H(A/2)}$
q_x
T_1
$\frac{L_E}{k_E(A/2)}$
$\frac{L_G}{k_G(A/2)}$
$\frac{L_H}{k_H(A/2)}$
T_2

- **thermal contact resistance**



$$q = \frac{T_A - T_B}{R_{t,c}}, \qquad q = q''A$$

$$q'' = \frac{T_A - T_B}{AR_{t,c}} \equiv \frac{T_A - T_B}{R''_{t,c}}, \quad R_{t,c} = \frac{R''_{t,c}}{A}$$

$$R''_{t,c} = \frac{T_A - T_B}{q''} \quad [\mathrm{Km^2/W}]$$

surface roughness, contact pressure

$$q = \frac{\Delta T}{R_{\mathrm{tot}}} = \frac{\Delta T}{\sum R_{t,\mathrm{conv}} + \sum R_{t,\mathrm{cond}} + \sum R_{t,c}}$$

TABLE 3.1 Thermal contact resistance for (*a*) metallic interfaces under vacuum conditions and (*b*) aluminum interface (10-μm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

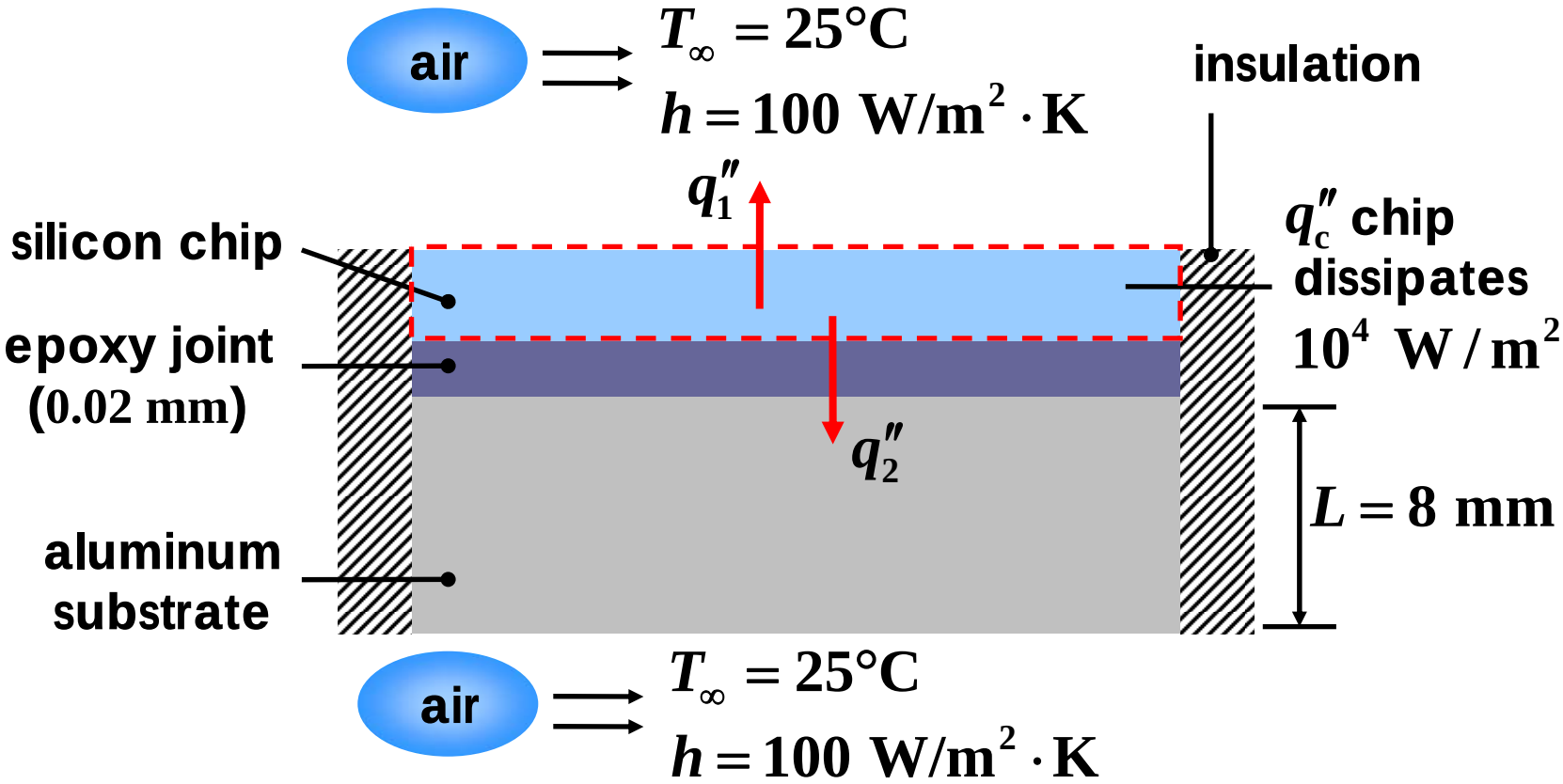
Thermal Resistance, $R''_{t,c} \times 10^4$ (m$^2 \cdot$ K/W)

(*a*) Vacuum Interface			(*b*) Interfacial Fluid	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

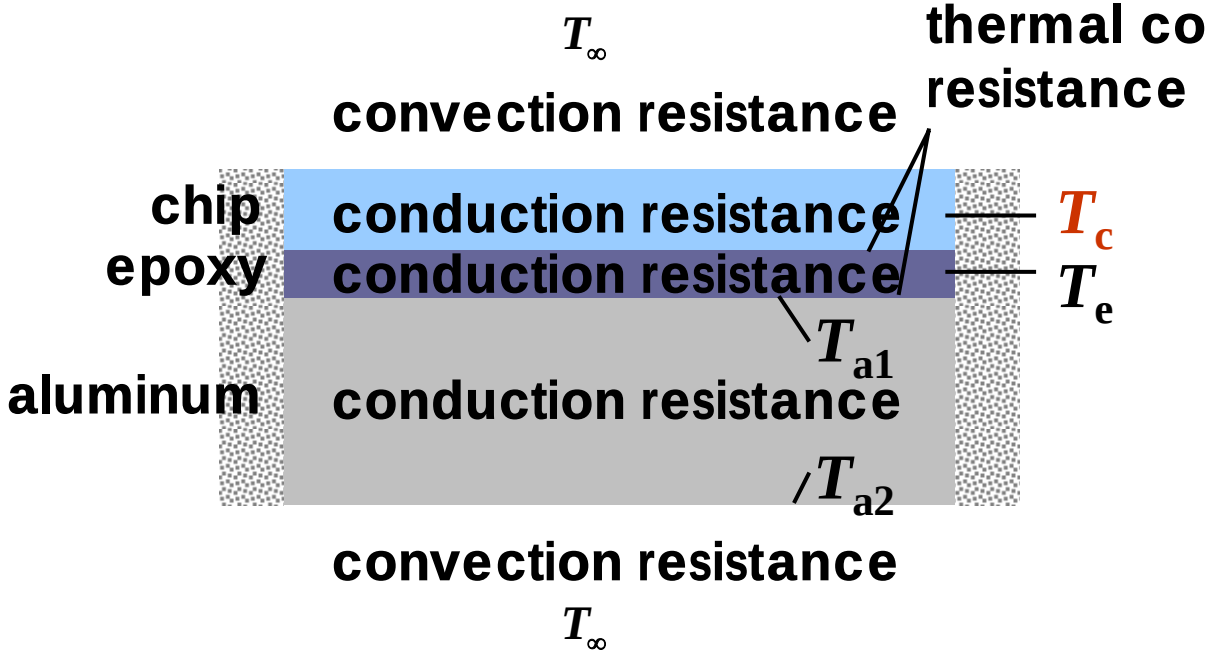
TABLE 3.2 Thermal resistance of representative solid/solid

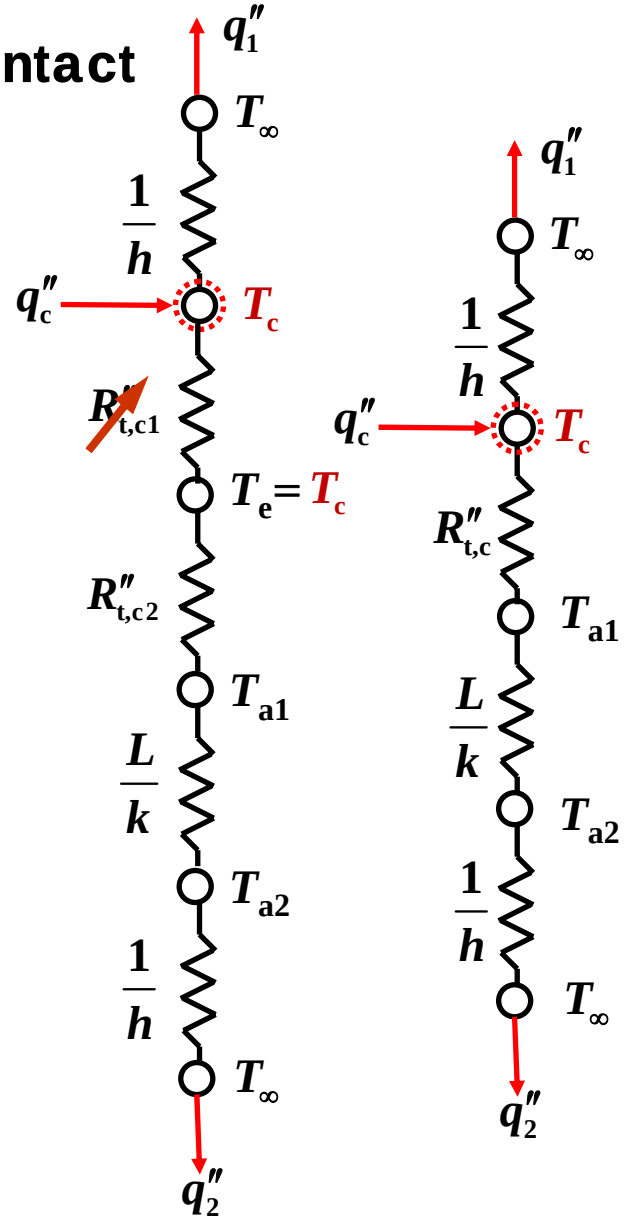
Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

Example 3.2



Find whether the chip operates below allowable temperature ($85°C$).





Assumptions :

Negligible conduction resistances of chip and epoxy.

Negligible thermal contact resistance between chip and epoxy.

TABLE 3.2 Thermal resistance of representative solid/solid

Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07
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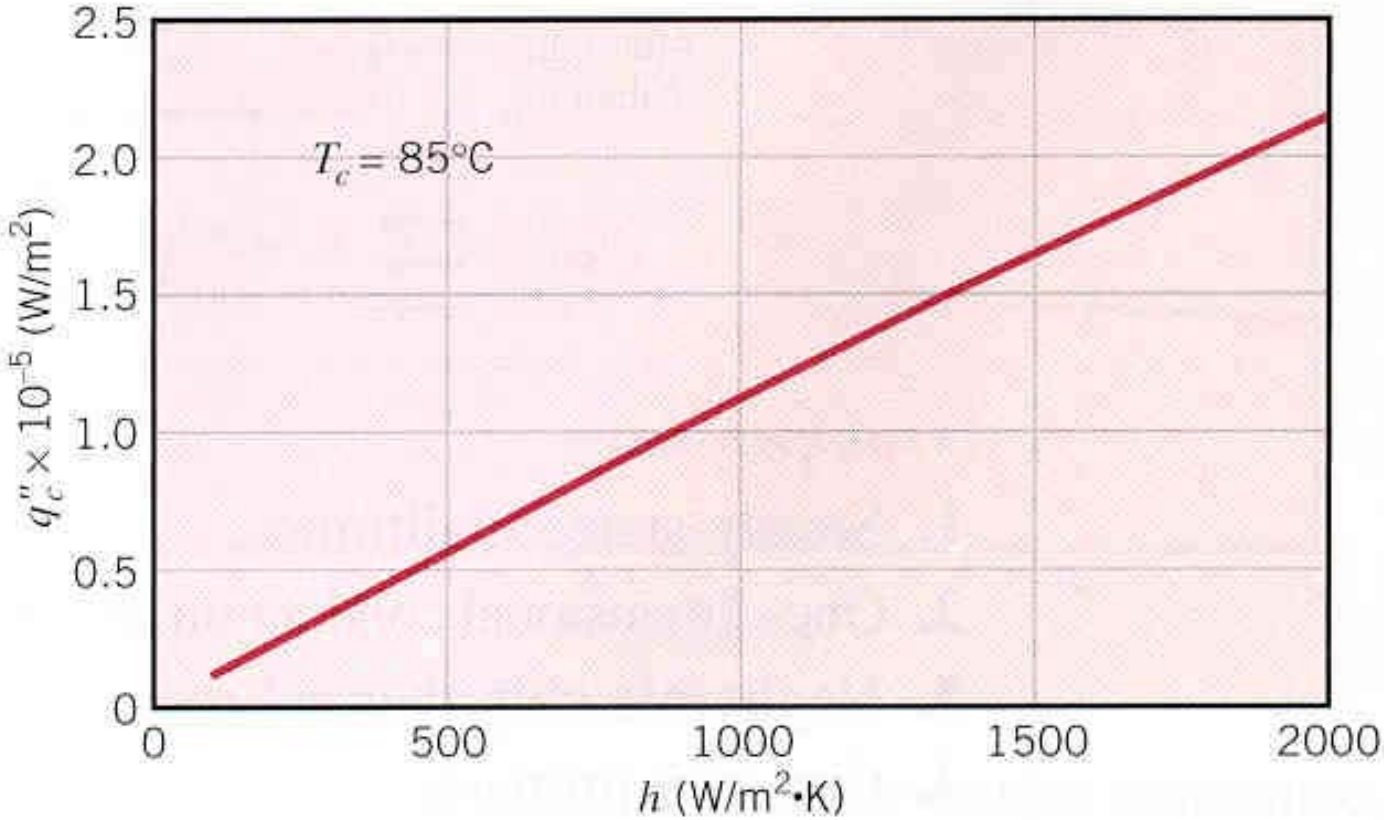
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9
Brass/brass with 15-μm tin solder	0.025–0.14

$$R''_{t,c} = 0.9\times10^{-4}\,\mathrm{m}^2\cdot\mathrm{K/W},\ 1/h = 0.01\mathrm{m}^2\cdot\mathrm{K/W},\ L/k = 3.36\times10^{-6}\,\mathrm{m}^2\cdot\mathrm{K/W}$$

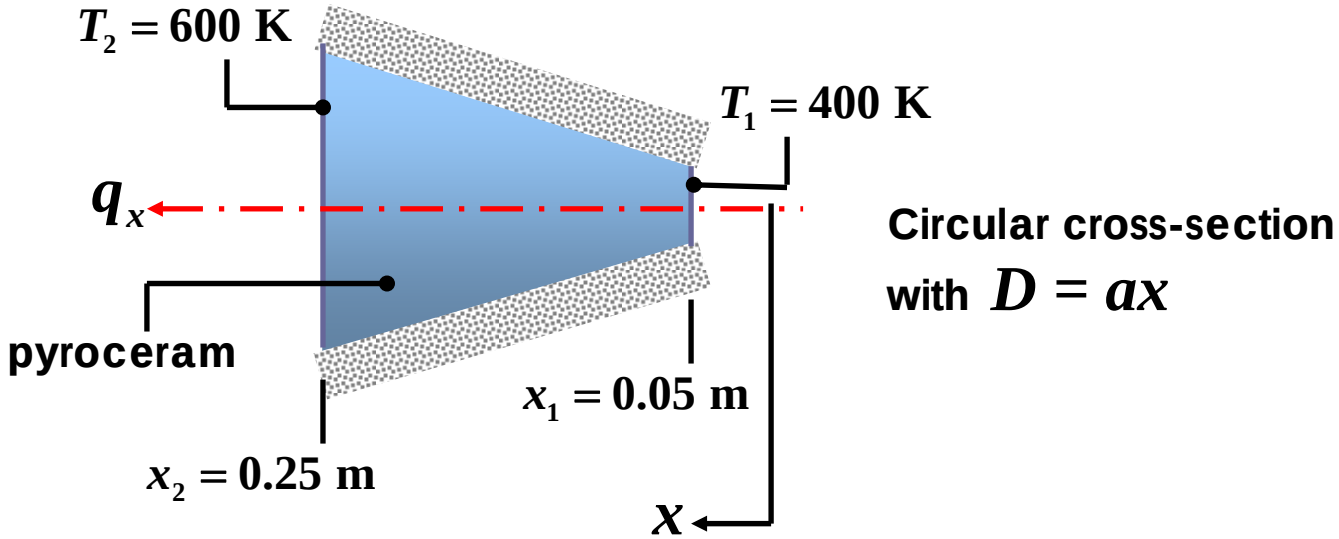


$$h \to \infty:$$

$$q'' = h\left(T_s - T_\infty\right)$$

$$T_s \to T_\infty$$

Example 3.4



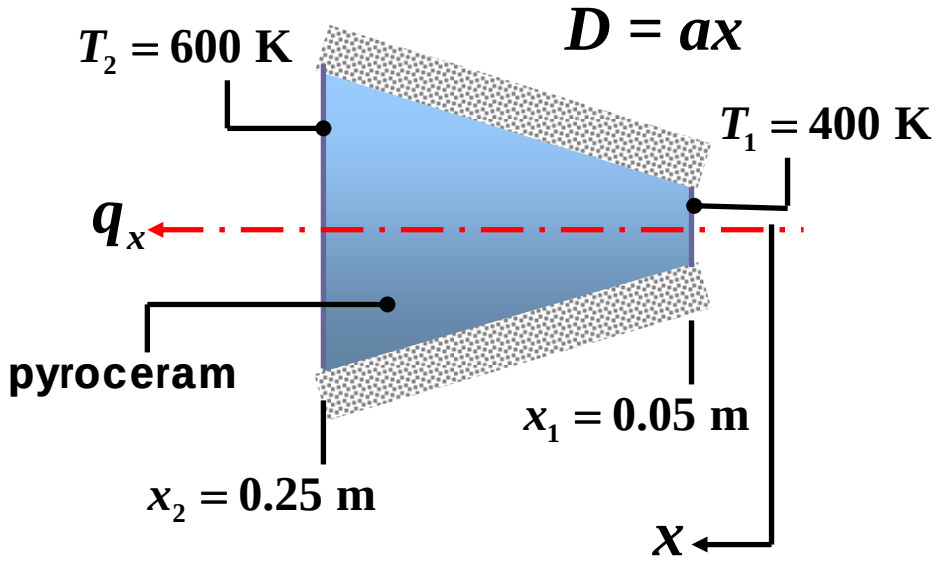
Find

1) Temperature distribution $T(x)$
2) Heat transfer rate q_x

Properties

Table A.2, pyroceram (500 K): $k = 3.46$ W/m·K

1) $q_x = -kA(x)\dfrac{dT}{dx} = \text{constant}$



With $A(x) = \dfrac{\pi D^2}{4} = \dfrac{\pi a^2 x^2}{4}$

$$q_x = -k\frac{\pi a^2 x^2}{4}\frac{dT}{dx}$$

Integrating from x_1 to any x and solving for T

$$-\int_{x_1}^{x}\frac{-4q_x}{\pi a^2 x^2 k}dx = \int_{T_1}^{T(x)} dT, \qquad T(x) = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x}\right)$$

Set $T(x_2) = T_2$ and solving for q_x

$$T_2 = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x_2}\right), \qquad q_x = \frac{\pi a^2 k\left(T_1 - T_2\right)}{4\left[\left(1/x_1\right) - \left(1/x_2\right)\right]}$$

$$T(x) = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x}\right), \qquad q_x = \frac{\pi a^2 k\left(T_1 - T_2\right)}{4\left[\left(1/x_1\right) - \left(1/x_2\right)\right]}$$

Substituting q_x into the expression for $T(x)$

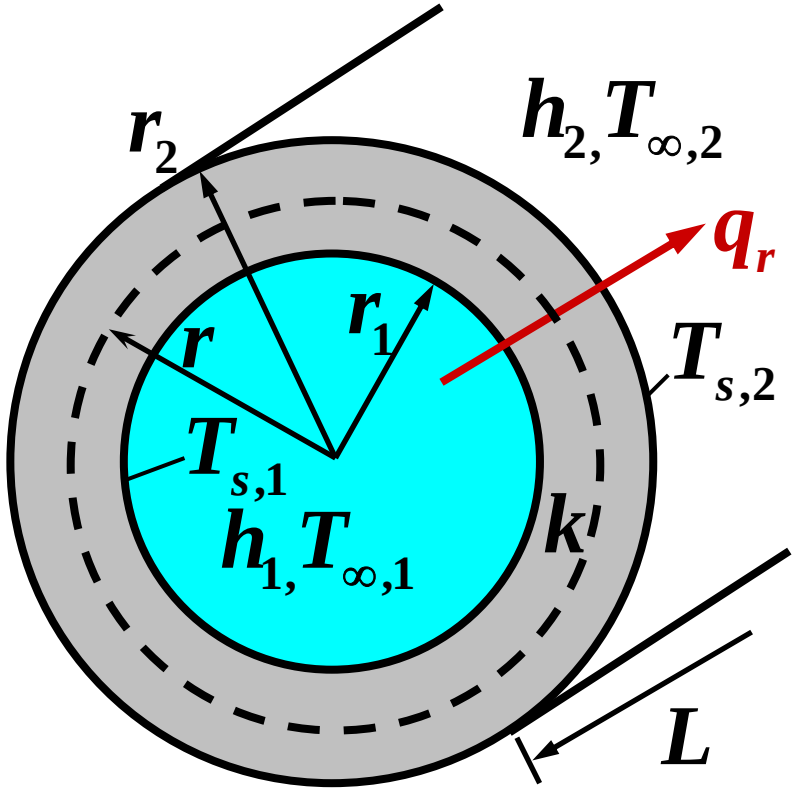
$$T(x) = T_1 - \left(T_1 - T_2\right)\left(\frac{1/x - 1/x_1}{1/x_1 - 1/x_2}\right)$$

2) Substituting numerical values into the result.

$$q_x = \frac{\pi(0.25)^2 \times 3.46\ \mathrm{W/m \cdot K}\left(400 - 600\right)\mathrm{K}}{4\left(\frac{1}{0.05\ \mathrm{m}} - \frac{1}{0.25\ \mathrm{m}}\right)} = -2.12\mathrm{W}$$

Radial System

- **Cylinder (hollow)**



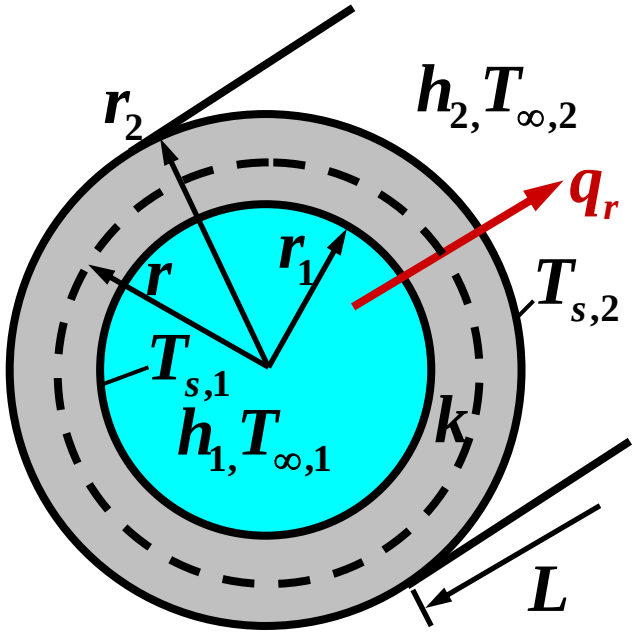
$$q_r = -k\left(2\pi r L\right)\frac{dT}{dr} = \text{const.}$$

$$\int_{r_1}^{r_2} \frac{q_r}{r}\,dr = \int_{T_{s,1}}^{T_{s,2}} \left[-k\left(2\pi L\right)\right] dT$$

$$q_r \ln\frac{r_2}{r_1} = k\left(2\pi L\right)\left(T_{s,1} - T_{s,2}\right)$$

$$q_r = \frac{T_{s,1} - T_{s,2}}{\dfrac{1}{2\pi k L}\ln\dfrac{r_2}{r_1}}$$

at the boundaries



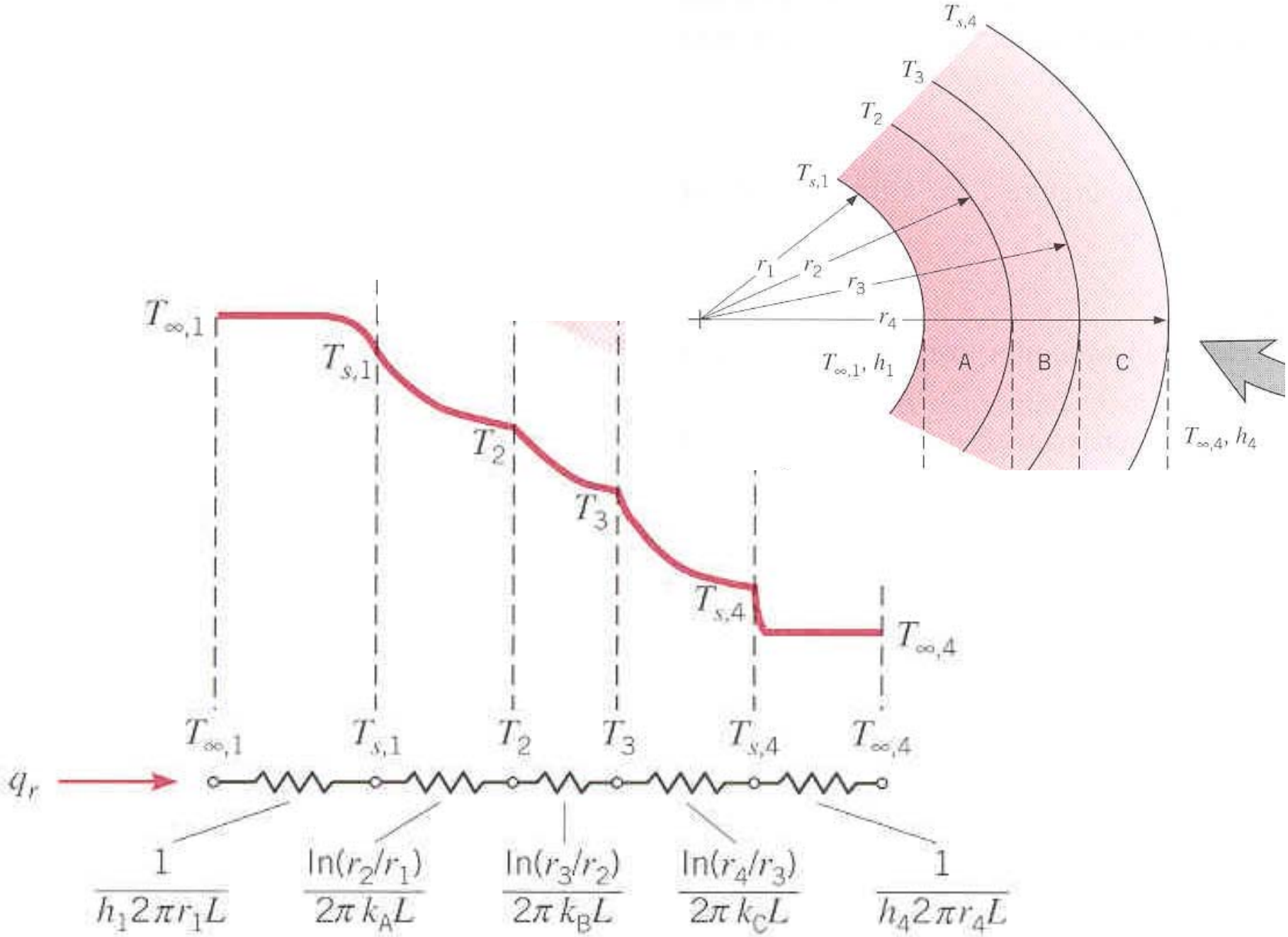
at $r = r_1$,

$$q_r = h_1\left(2\pi r_1 L\right)\left(T_{\infty,1} - T_{s,1}\right)$$

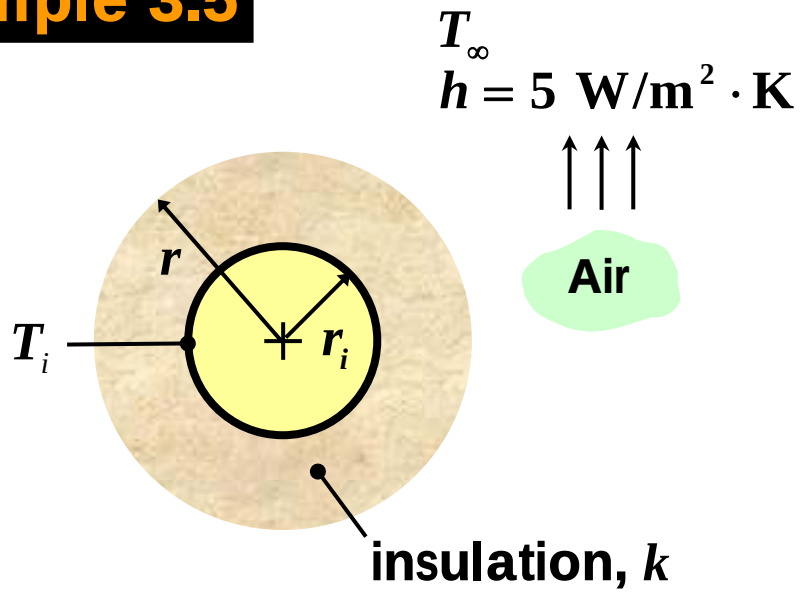
$$= \frac{T_{\infty,1} - T_{s,1}}{\dfrac{1}{2\pi r_1 h_1 L}}$$

at $r = r_2$, $\quad q_r = h_2\left(2\pi r_2 L\right)\left(T_{s,2} - T_{\infty,2}\right) = \dfrac{T_{s,2} - T_{\infty,2}}{\dfrac{1}{2\pi r_2 h_2 L}}$

$$q_r = \frac{T_{\infty,1} - T_{\infty,2}}{\dfrac{1}{2\pi r_1 h_1 L} + \dfrac{\ln\left(r_2 / r_1\right)}{2\pi k L} + \dfrac{1}{2\pi r_2 h_2 L}}$$


$T_{s,4}$
T_3
T_2
$T_{s,1}$
r_1
r_2
r_3
r_4
$T_{\infty,1}, h_1$
A
B
C
$T_{\infty,4}, h_4$
$T_{\infty,1}$
$T_{s,1}$
T_2
T_3
$T_{s,4}$
$T_{\infty,4}$
q_r
$\frac{1}{h_1 2\pi r_1 L}$
$\frac{\ln(r_2/r_1)}{2\pi k_A L}$
$\frac{\ln(r_3/r_2)}{2\pi k_B L}$
$\frac{\ln(r_4/r_3)}{2\pi k_C L}$
$\frac{1}{h_4 2\pi r_4 L}$

Example 3.5



Competing factors

Conduction resistance: increases with the addition of insulation

But heat transfer area also increases

Find:

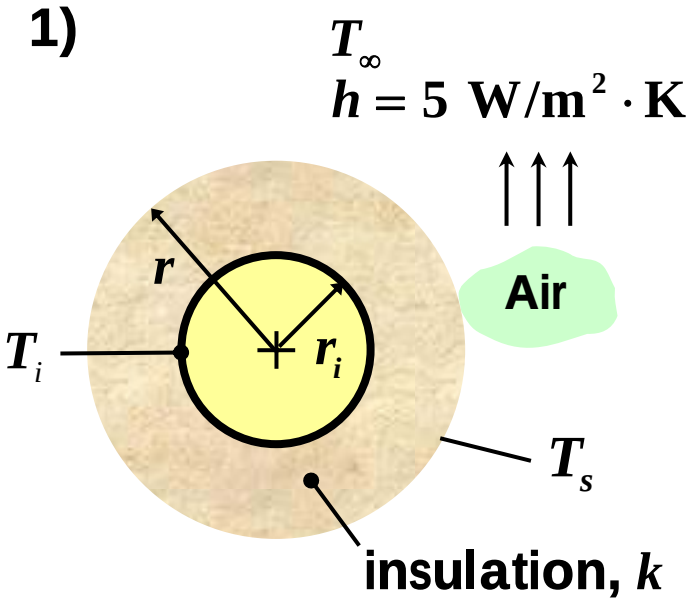
1) **Whether there exists an optimum insulation thickness that minimizes the heat transfer rate**
2) **Thermal resistance associated with using cellular glass insulation of varying thickness**

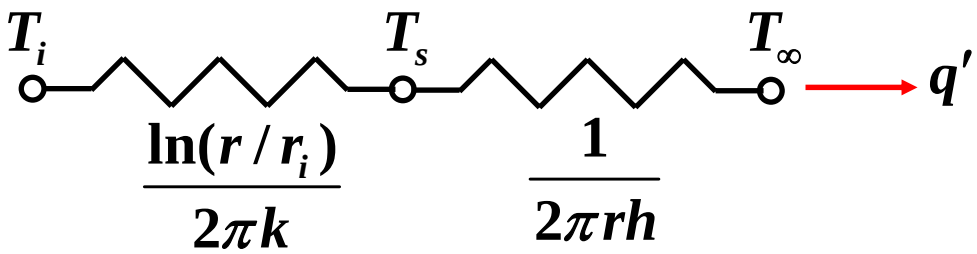
Assumptions:

Negligible tube wall thermal resistance

$$\frac{\ln\left(r_o / r_i\right)}{2\pi k L} \approx 0$$

1)





$$q' = \frac{T_i - T_\infty}{R'_{\text{tot}}}, \quad R'_{\text{tot}} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

Try to find out r which makes heat transfer rate minimum.
That is, r which makes the thermal resistance maximum.

$$\frac{dR'_{\text{tot}}}{dr} = \frac{1}{2\pi k r} - \frac{1}{2\pi h r^2} = 0 \quad \longrightarrow \quad r = \frac{k}{h}$$

Check whether the thermal resistance has its maximum.

$$\frac{d^2 R'_{\text{tot}}}{dr^2} = -\frac{1}{2\pi k r^2} + \frac{1}{\pi h r^3}$$

$$R'_{tot} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}, \qquad \frac{d^2 R'_{tot}}{dr^2} = -\frac{1}{2\pi k r^2} + \frac{1}{\pi h r^3}$$

at $r = \dfrac{k}{h}$, $\left.\dfrac{d^2 R'_{tot}}{dr^2}\right|_{r=k/h} = \dfrac{h^2}{2\pi k^3} > 0 \quad \rightarrow R'_{tot}$ has its **minimum!**

Optimum insulation thickness does not exist.

critical insulation thickness : $r_{cr} = \dfrac{k}{h}$

Ex) When $h = 5\ \mathrm{W/m^2 \cdot K}$, $k = 0.055\ \mathrm{W/m \cdot K}$ and $r_i = 5\ \mathrm{mm}$

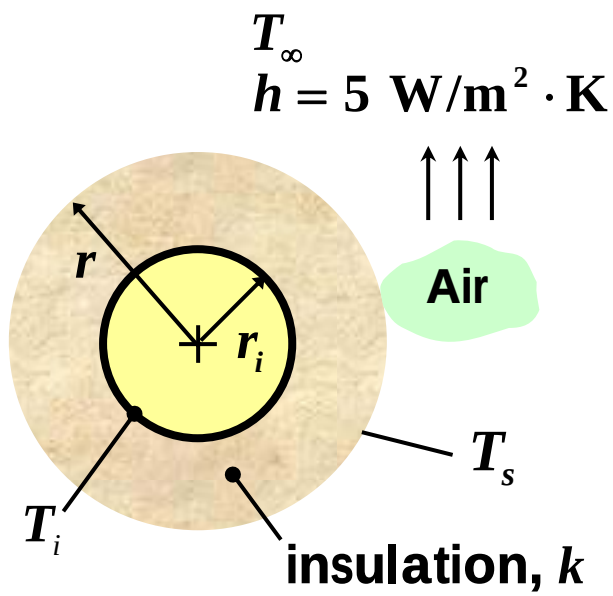
$$r_{cr} = \frac{k}{h} = \frac{0.055}{5} = 0.011\ \mathrm{m} = 11\ \mathrm{mm}$$

insulation thickness $= r_{cr} - r_i = 6\ \mathrm{mm}$

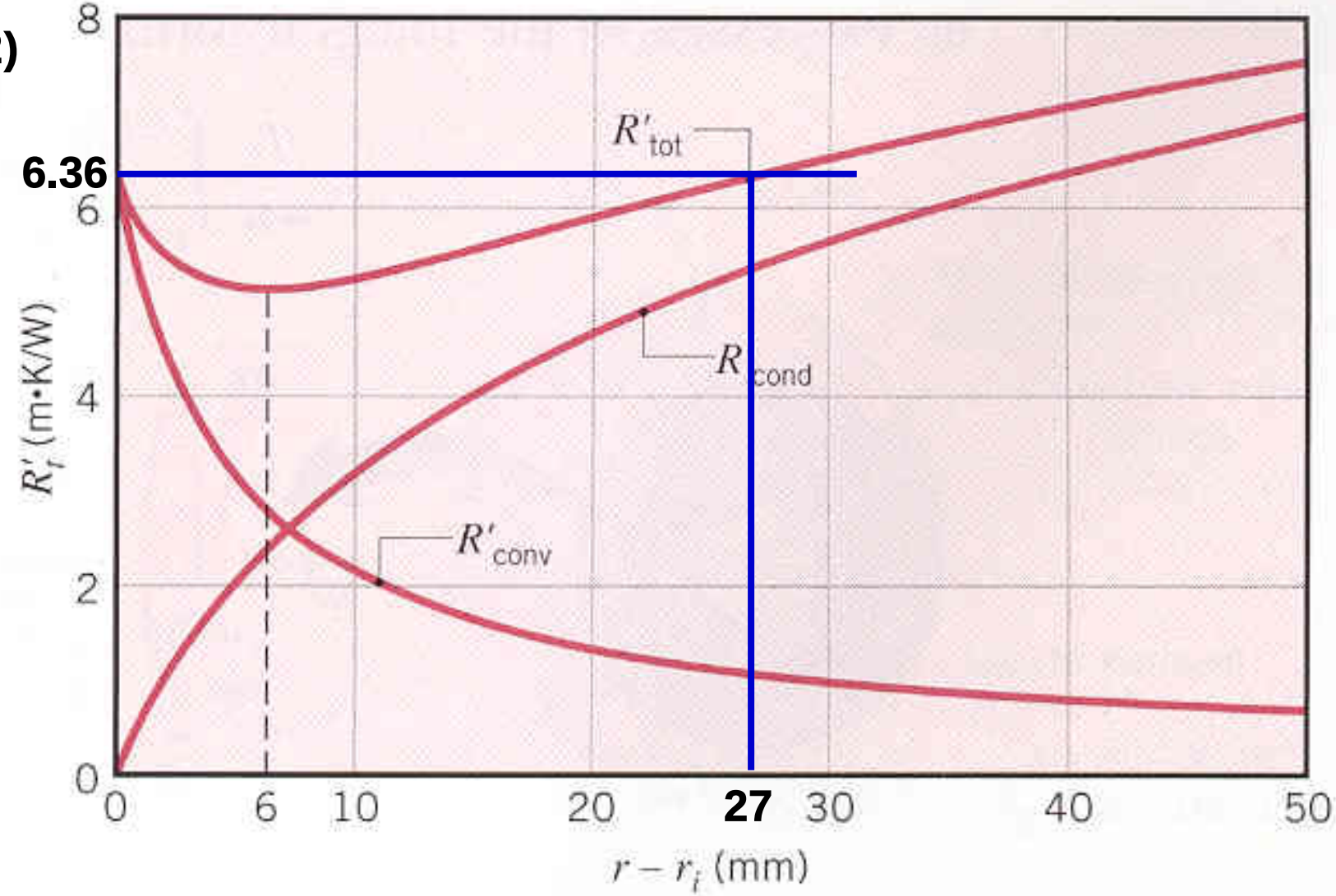
no insulation: $R'_{tot} = \dfrac{1}{2\pi r_i h} = \dfrac{1}{2\pi \times 5 \times 10^{-3} \times 5} = 6.36$

When insulation thickness is 27 mm

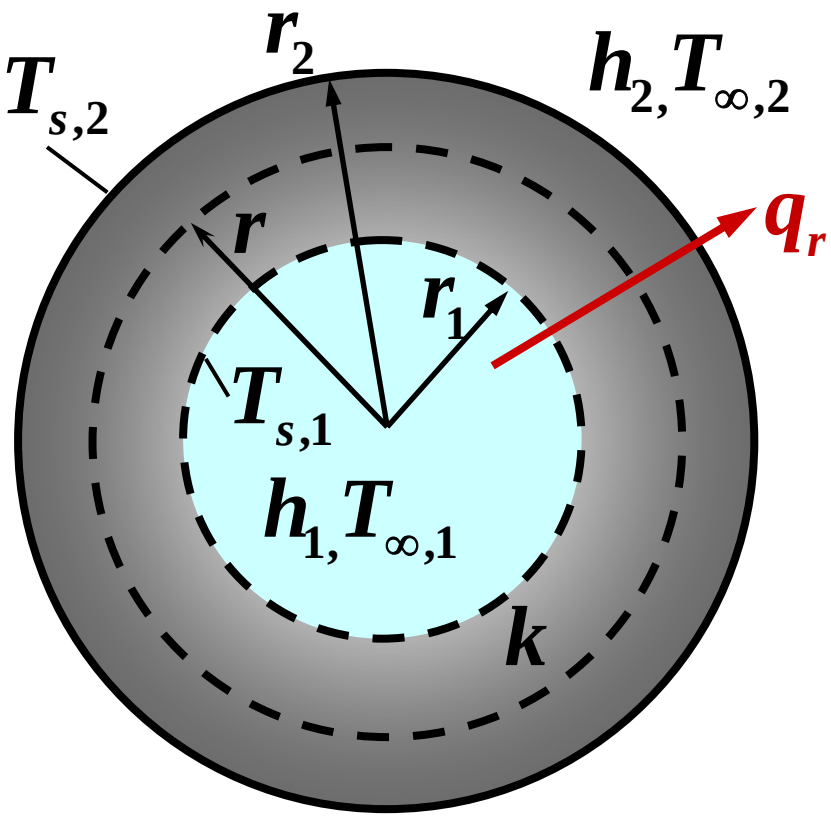
$$R'_{tot} = \frac{\ln(32/5)}{2\pi \times 0.055} + \frac{1}{2\pi \times 32 \times 10^{-3} \times 5} = 6.37$$



2)



- **Sphere (hollow)**



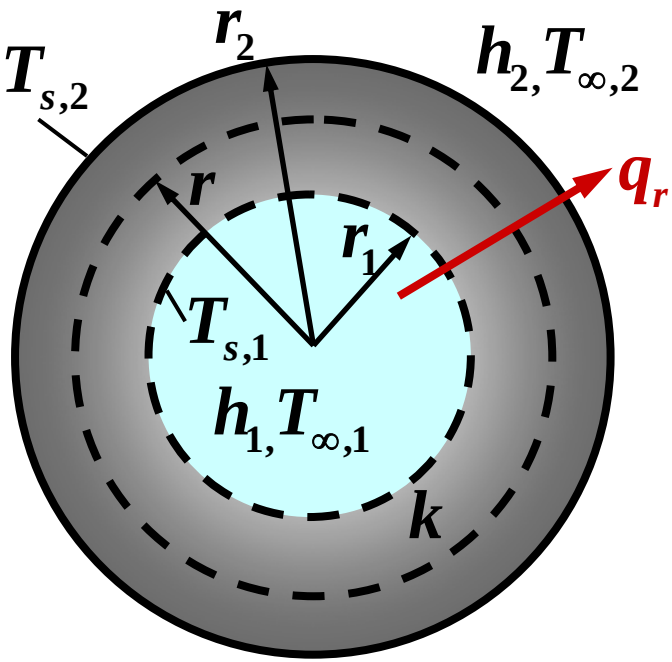
$$q_r = -k\left(4\pi r^2\right)\frac{dT}{dr} = \text{const.}$$

$$\int_{r_1}^{r_2} \frac{q_r}{4\pi k r^2}\,dr = -\int_{T_{s,1}}^{T_{s,2}} dT$$

$$\frac{q_r}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = T_{s,1} - T_{s,2}$$

$$q_r = \frac{T_{s,1} - T_{s,2}}{\dfrac{1}{4\pi k}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)}$$

at the boundaries



at $r = r_1$,

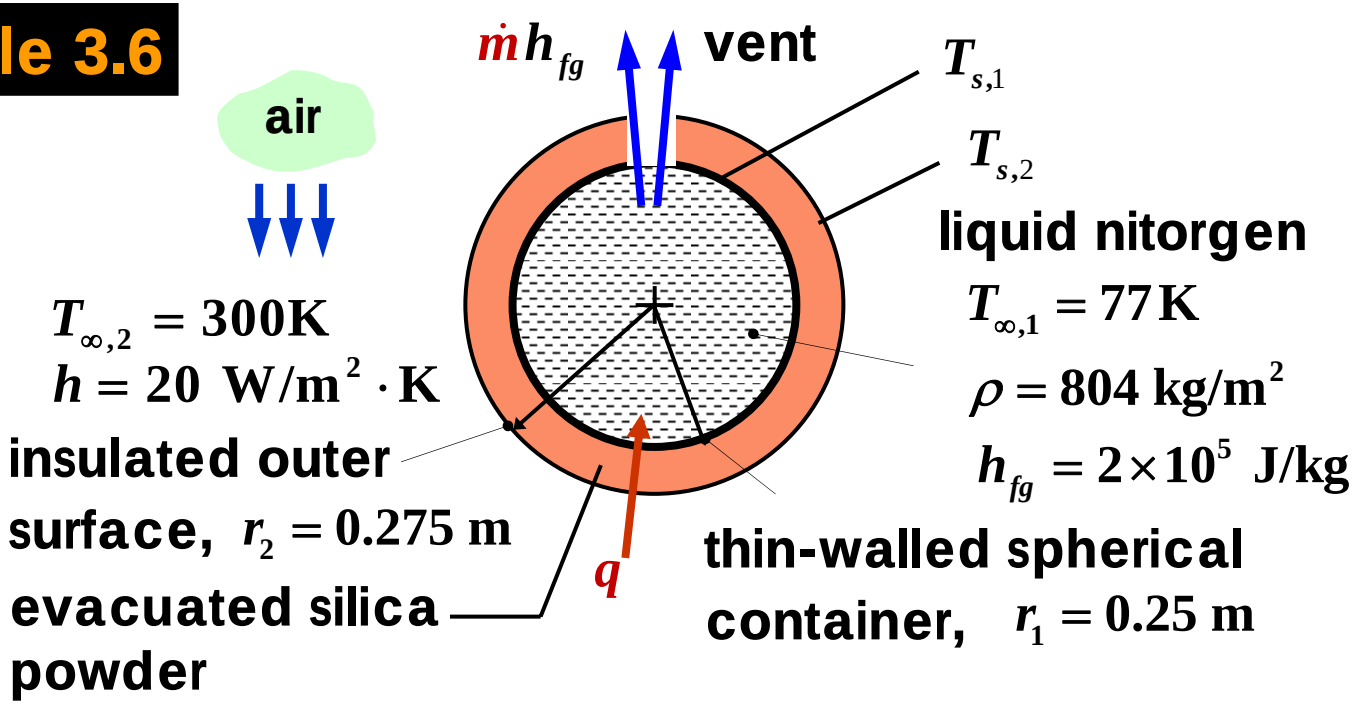
$$q_r = h_1\left(4\pi r_1^2\right)\left(T_{\infty,1} - T_{s,1}\right)$$

$$= \frac{T_{\infty,1} - T_{s,1}}{\dfrac{1}{4\pi r_1^2 h_1}}$$

at $r = r_2$, $\quad q_r = h_2\left(4\pi r_2^2\right)\left(T_{s,2} - T_{\infty,2}\right) = \dfrac{T_{s,2} - T_{\infty,2}}{\dfrac{1}{4\pi r_2^2 h_2}}$

$$q_r = \frac{T_{\infty,1} - T_{\infty,2}}{\dfrac{1}{4\pi r_1^2 h_1} + \dfrac{1}{4\pi k}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right) + \dfrac{1}{4\pi r_2^2 h_2}}$$

Example 3.6



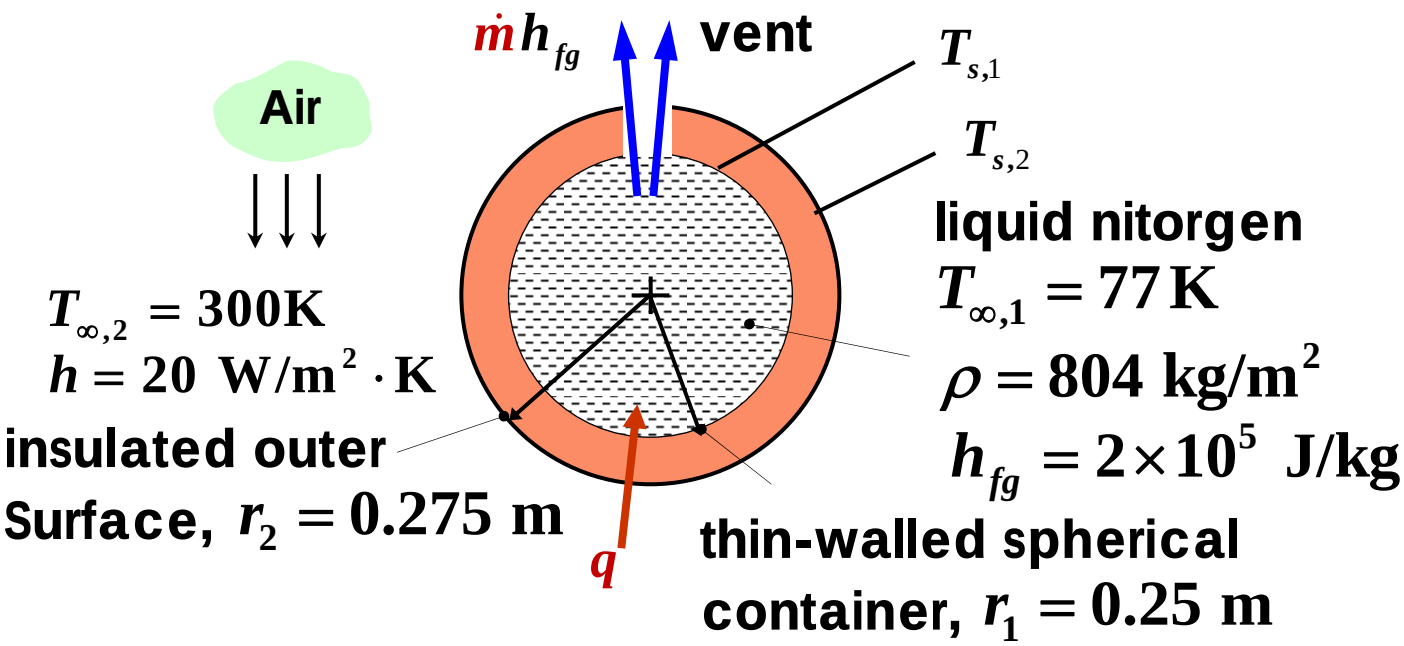
Find:

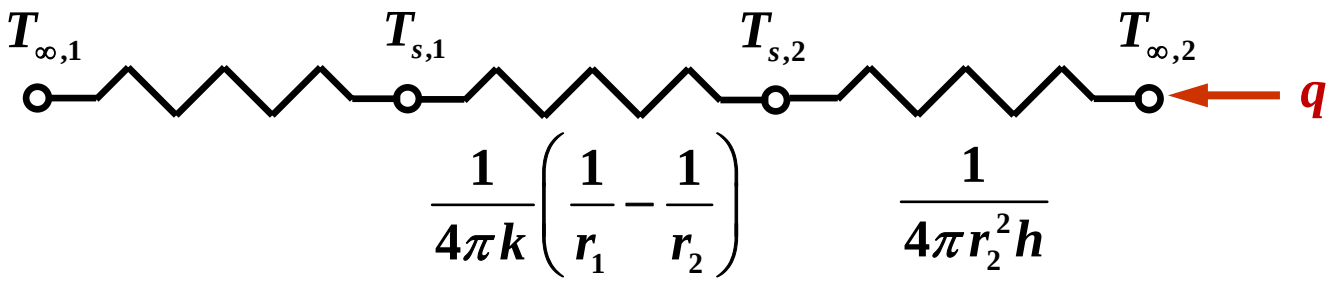
1) The rate of heat transfer to the nitrogen, q
2) The mass rate of nitrogen boil-off, $\dot{m}$

Assumption:

Negligible resistance to heat transfer through the container wall and from the container to the nitrogen, that is, $T_{s,1} = T_{\infty,1}$

1)



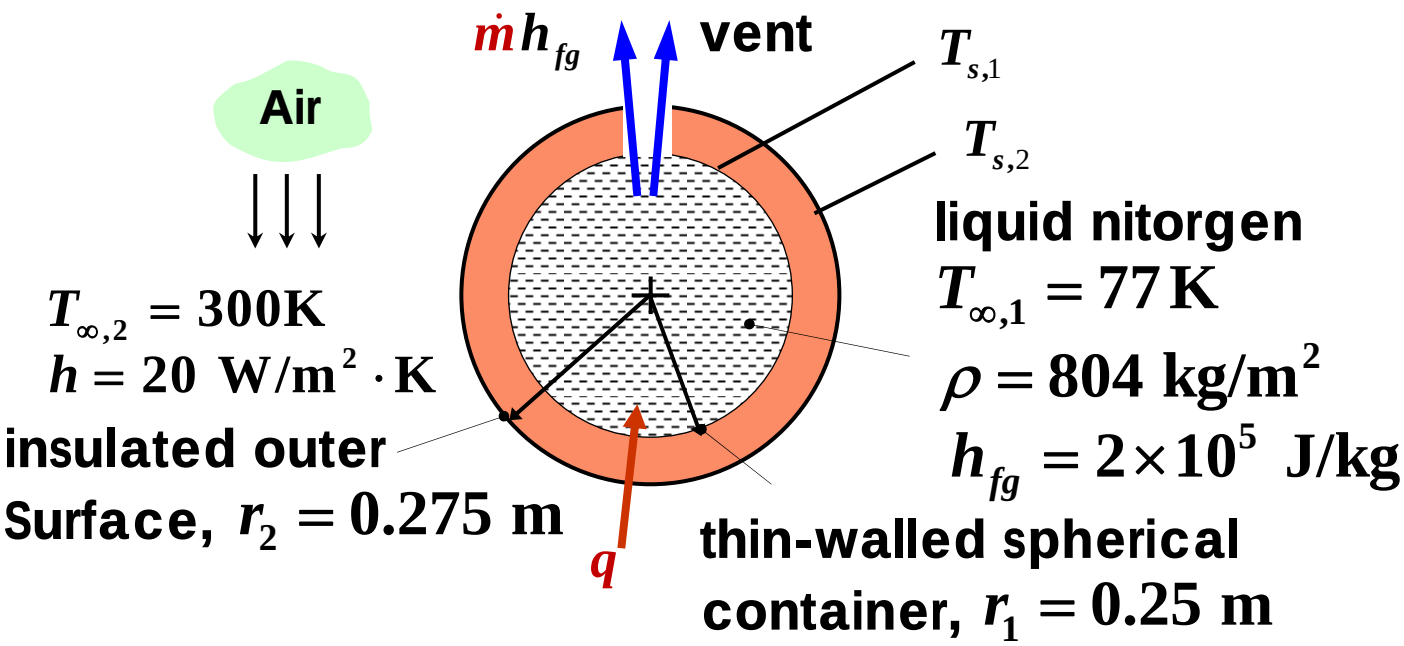


The rate of heat transfer to the liquid nitrogen :

$$q = \frac{T_{\infty,2} - T_{\infty,1}}{\frac{1}{4\pi k}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+\frac{1}{4\pi r_2^2 h}} = 13.06\,\mathrm{W}$$

Table A.3, evacuated silica powder at 300 K

$k = 0.0017\ \mathrm{W/m\cdot K}$



2) Energy balance

$$\dot{E}_{\text{in}} + \cancel{\dot{E}_{\text{g}}} - \dot{E}_{\text{out}} = \cancel{\dot{E}_{\text{st}}}$$

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \ \rightarrow q = \dot{m}h_{fg}$$

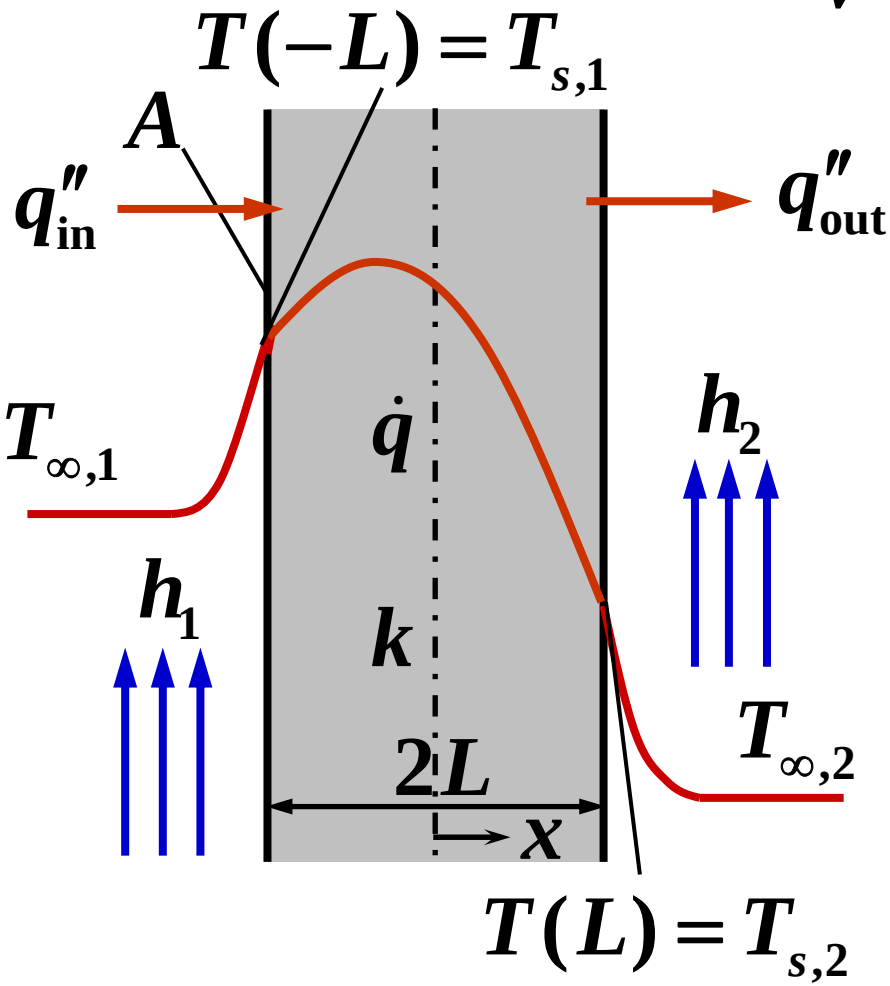
$$\dot{m} = \frac{q}{h_{fg}} = \frac{13.06\ \mathrm{W}}{2\times10^5\ \mathrm{J/kg}} = 6.53\times10^{-5}\ \mathrm{kg/s} = 5.64\ \mathrm{kg/day}$$

$$\dot{V} = \frac{\dot{m}}{\rho} = 7\ \text{liters/day}$$

Conduction with Heat Generation

Ex) electrical energy → thermal energy

$$\dot{E}_g = I^2 R_e \rightarrow \dot{q} = \frac{I^2 R_e}{V}$$

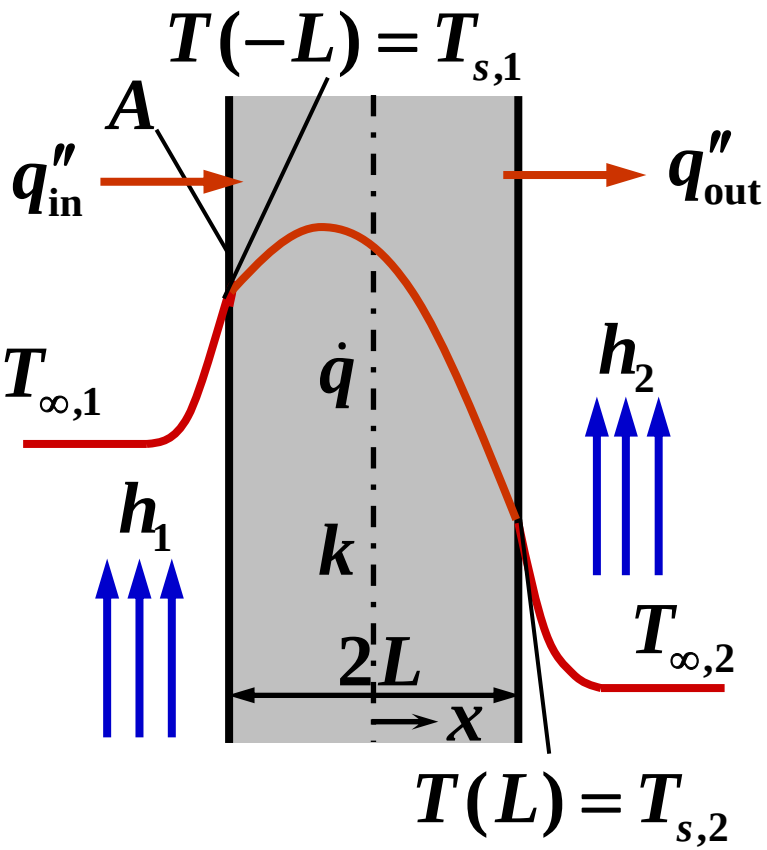


Plane Wall

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{q} = 0$$

$$\frac{dq''}{dx} = \dot{q}$$



$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + C_1$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

boundary conditions

$$-k\left.\frac{dT}{dx}\right|_{-L} = h_1\left[T_{\infty,1} - T(-L)\right]$$

$$-k\left.\frac{dT}{dx}\right|_{L} = h_2\left[T(L) - T_{\infty,2}\right]$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + C_1, \quad T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

$$-k\left.\frac{dT}{dx}\right|_{-L} = h_1\left[T_{\infty,1} - T(-L)\right]$$

$$-\dot{q}L - C_1 k = h_1\left[T_{\infty,1} + \frac{\dot{q}}{2k}L^2 + C_1 L - C_2\right]$$

$$-k\left.\frac{dT}{dx}\right|_{L} = h_2\left[T(L) - T_{\infty,2}\right]$$

$$\dot{q}L - C_1 k = h_2\left[-\frac{\dot{q}}{2k}L^2 + C_1 L + C_2 - T_{\infty,1}\right]$$

$$C_1 = -\frac{h_1 h_2 \left(T_{\infty,1} - T_{\infty,2}\right) + \dot{q}L\left(h_1 - h_2\right)}{h_1\left(k + h_2 L\right) + h_2\left(k + h_1 L\right)}$$

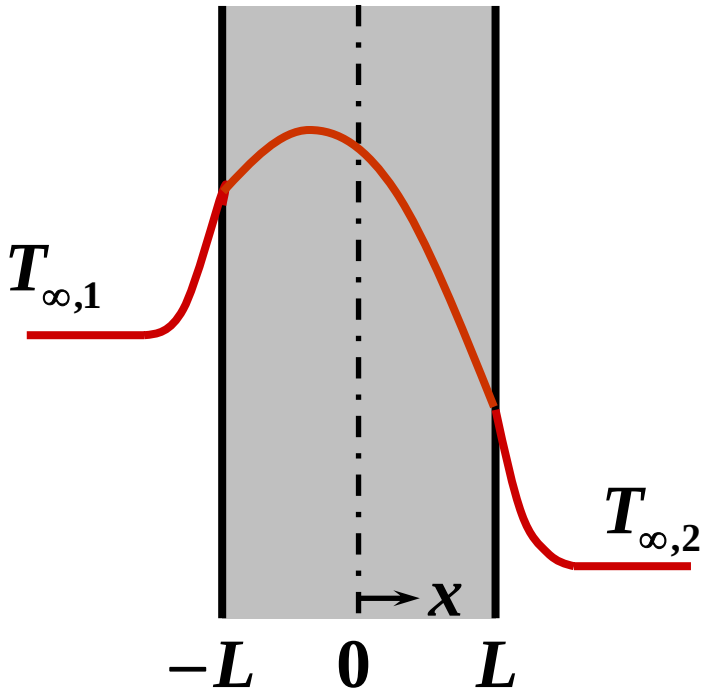
$$C_2 = \frac{\left(k + h_1 L\right)\left(\frac{h_2 \dot{q} L^2}{2k} + h_2 T_{\infty,2} + \dot{q}L\right)}{h_1\left(k + h_2 L\right) + h_2\left(k + h_1 L\right)}$$

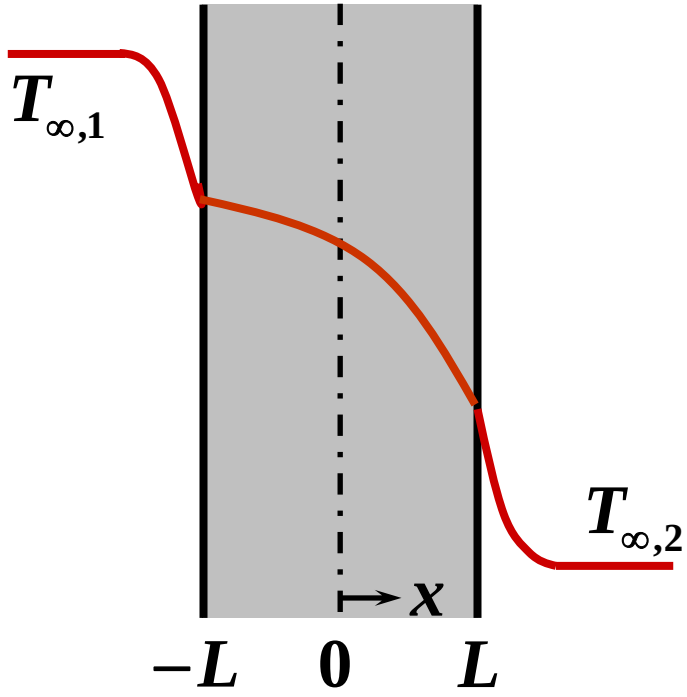
$$+ \frac{\left(k + h_2 L\right)\left(\frac{h_1 \dot{q} L^2}{2k} + h_1 T_{\infty,1} + \dot{q}L\right)}{h_1\left(k + h_2 L\right) + h_2\left(k + h_1 L\right)}$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

$$= -\frac{\dot{q}}{2k}\left(x - \frac{k}{\dot{q}}C_1\right)^2 + C_2 + \frac{k}{2\dot{q}}C_1^2$$

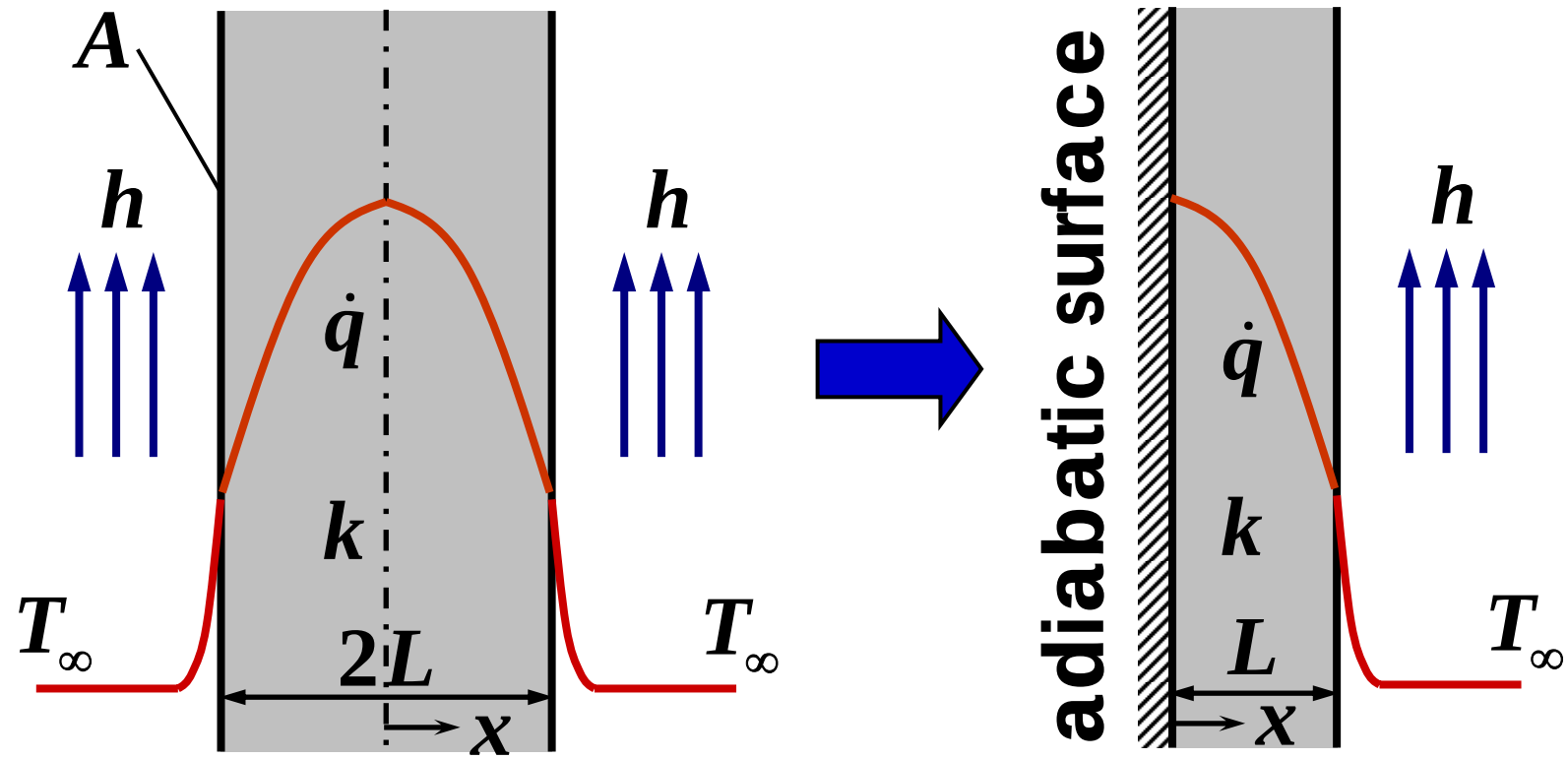
$$-L \le \frac{kC_1}{\dot{q}} \le 0 \qquad\qquad -L \ge \frac{kC_1}{\dot{q}}$$





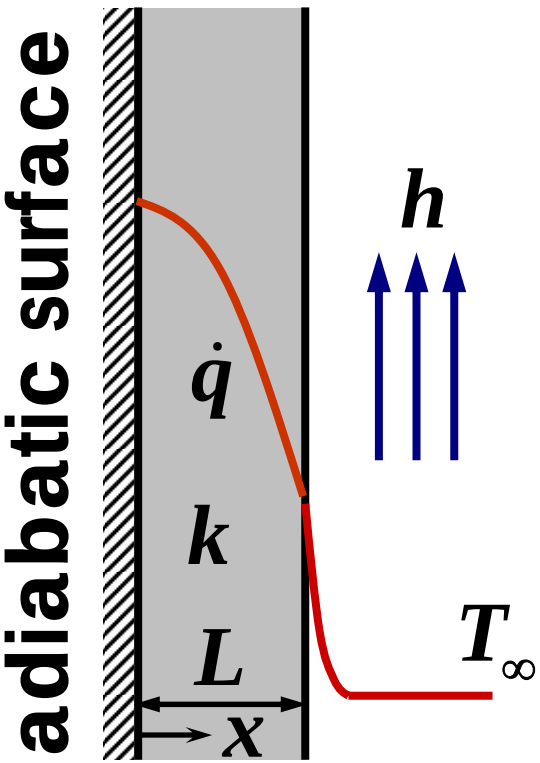
Symmetric case: $T_{\infty,1} = T_{\infty,2} = T_\infty,\ h_1 = h_2 = h$

$$\frac{dT}{dx} = 0 \quad \text{at } x = 0$$



$T(-L) = T_s \quad T(L) = T_s$

$T(L) = T_s$



$$T(L) = T_s$$

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + C_1$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

boundary conditions

$$\left.\frac{dT}{dx}\right|_0 = 0, \qquad T(L) = T_s$$

(T_s is still unknown.)

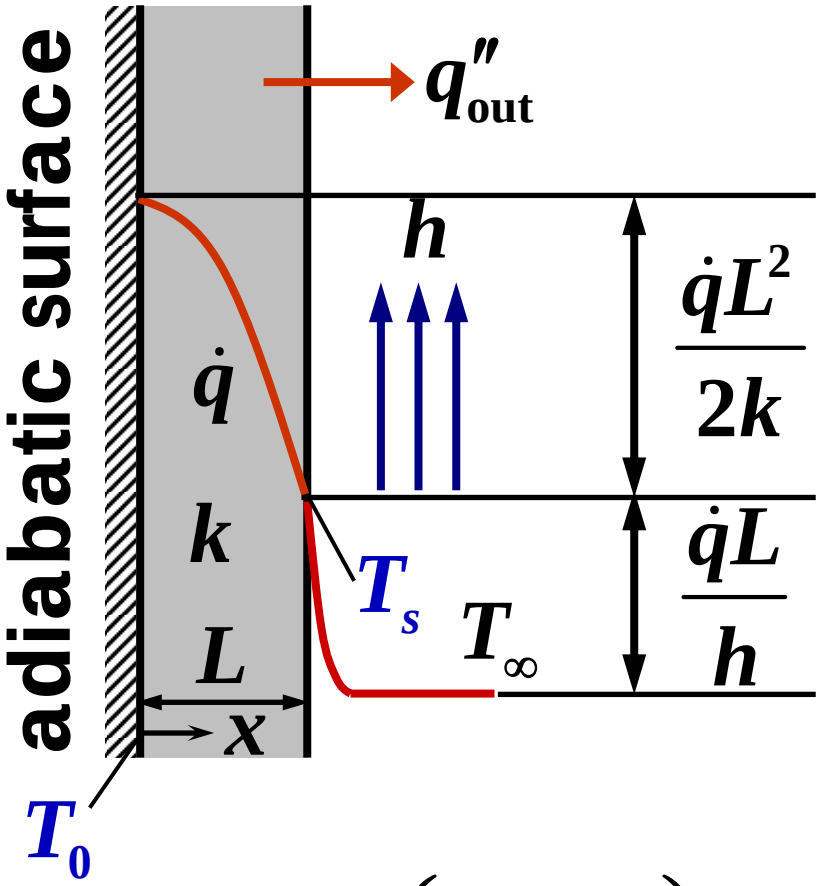
$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + C_1, \quad \left.\frac{dT}{dx}\right|_0 = 0 \quad \rightarrow C_1 = 0$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2 = -\frac{\dot{q}}{2k}x^2 + C_2$$

$$T(L) = T_s = -\frac{\dot{q}}{2k}L^2 + C_2 \rightarrow C_2 = \frac{\dot{q}}{2k}L^2 + T_s$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + \frac{\dot{q}}{2k}L^2 + T_s = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$

$$q''(x) = -k\frac{dT}{dx} = -k\frac{\dot{q}L^2}{2k}\left(-\frac{2x}{L^2}\right) = \dot{q}x$$



$$q''(x) = \dot{q}x$$

$$q''(L) = \dot{q}L$$

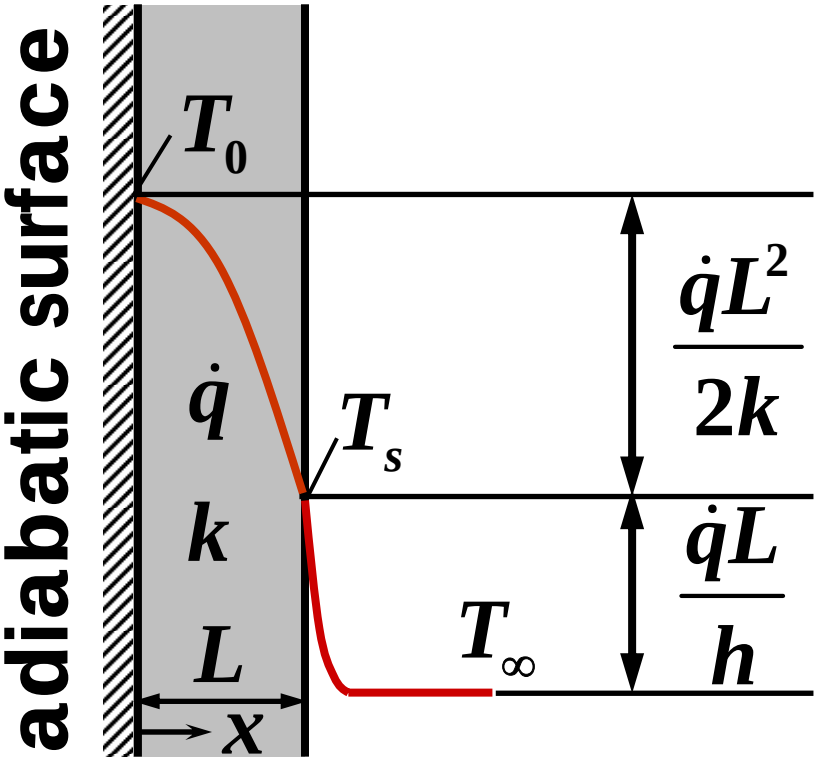
$$q''_{out} = q''(L) = \dot{q}L$$

$$\dot{q}L = h\left(T_s - T_\infty\right)$$

$$T_s = \frac{\dot{q}L}{h} + T_\infty$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{\dot{q}L}{h} + T_\infty$$

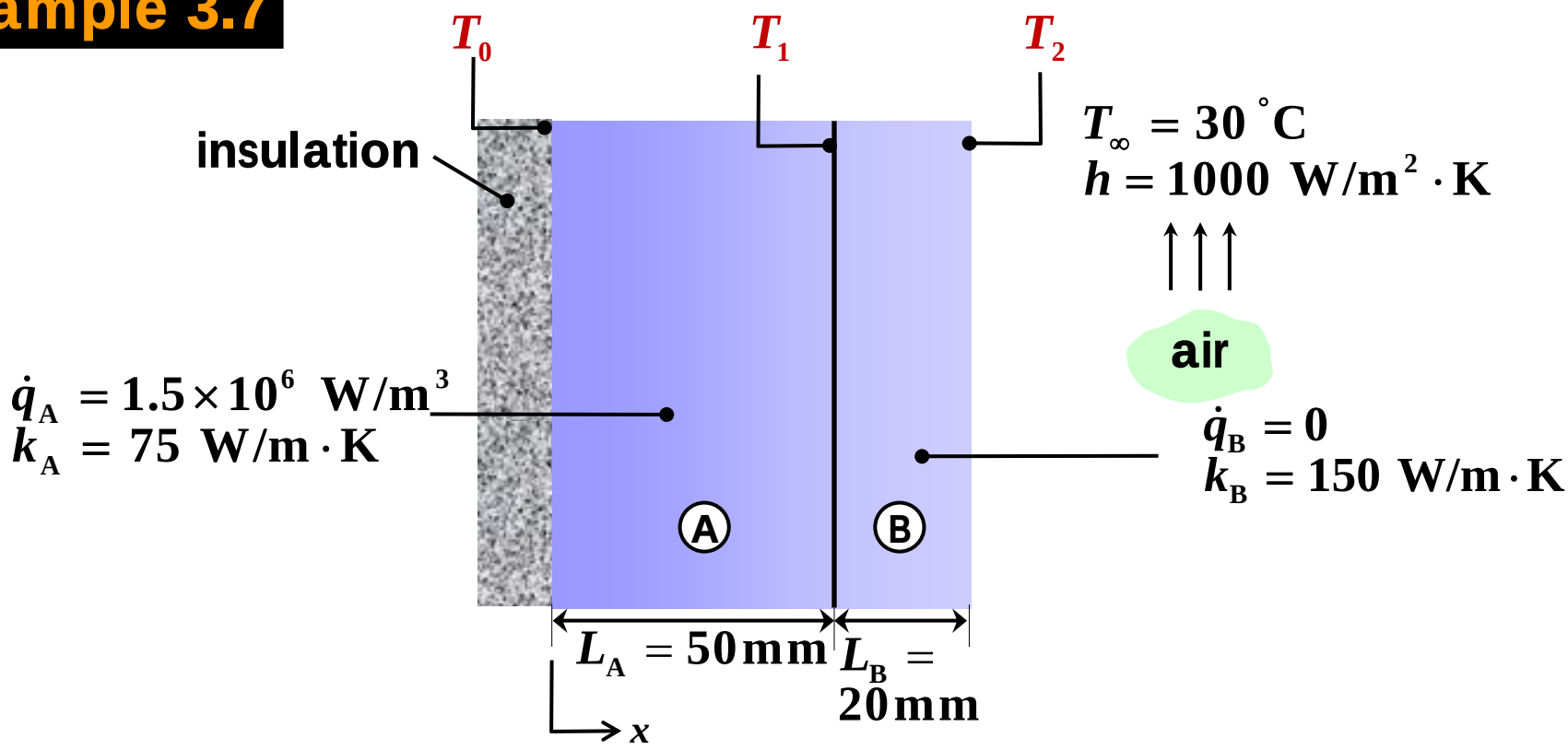
$$T_0 = T(0) = \frac{\dot{q}L^2}{2k} + \frac{\dot{q}L}{h} + T_\infty$$



$$T_0 - T_s = \frac{\dot{q}L^2}{2k} \quad \text{as } k \to \infty,\ T_s \to T_0$$

$$T_s - T_\infty = \frac{\dot{q}L}{h} \quad \text{as } h \to \infty,\ T_s \to T_\infty$$

Example 3.7



Find:

1) Sketch of steady-states temperature distribution in the composite
2) Inner and outer surface temperatures of the composite

Assumption:

Negligible contact resistances between walls

1)

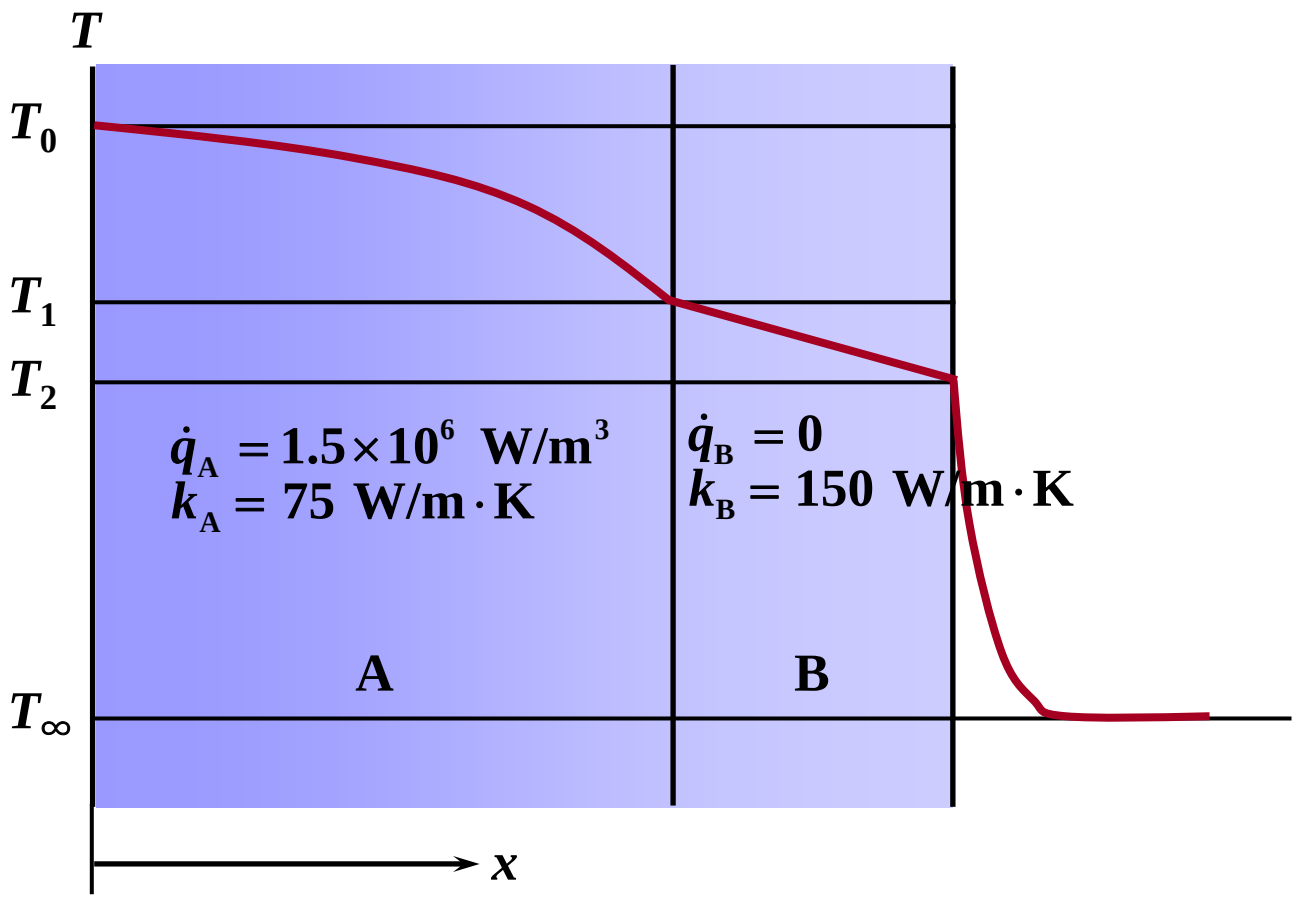
(a) Parabolic in material A

(b) Zero slope at insulated boundary at $x = 0$

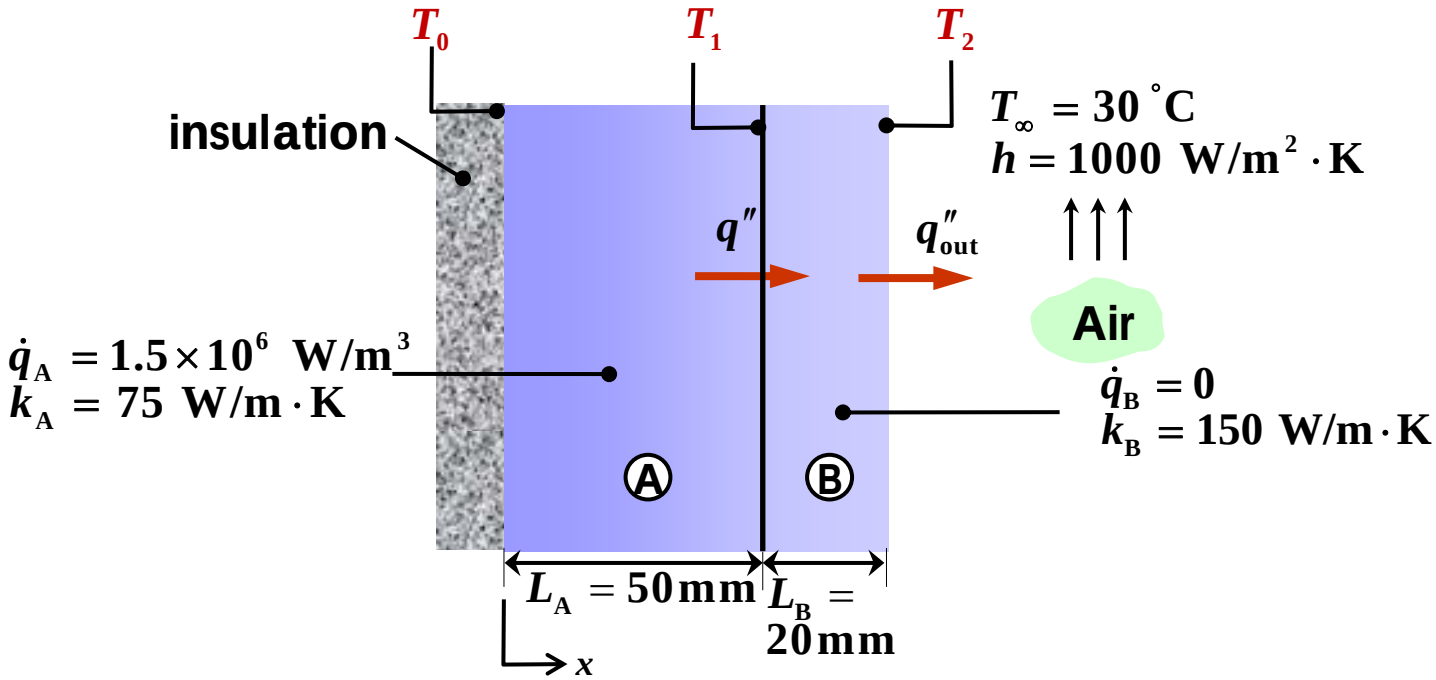
(c) Linear in material B

(d) Slope change : $k_B/k_A = 2$ at the interface

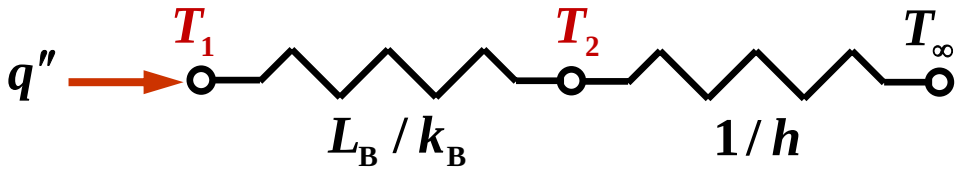
(e) Large gradient near the surface



2)



Material A, having heat generation, cannot be represented by a thermal circuit element.



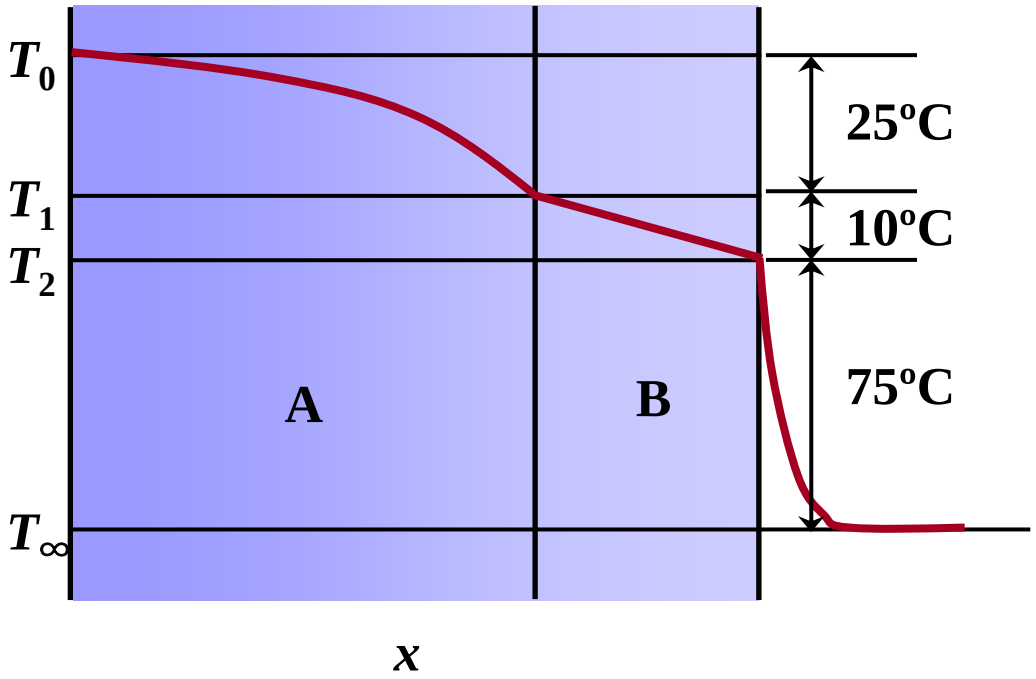
$$q'' = \frac{T_1 - T_2}{L_B / k_B} = \frac{T_2 - T_\infty}{1/h} = \dot{q}L_A, \quad T_0 = \frac{\dot{q}L_A^2}{2k_A} + T_1$$

$$q'' = \frac{T_1 - T_2}{L_B / k_B} = \frac{T_2 - T_\infty}{1/h} = \dot{q} L_A, \quad T_0 = \frac{\dot{q} L_A^2}{2k_A} + T_1$$

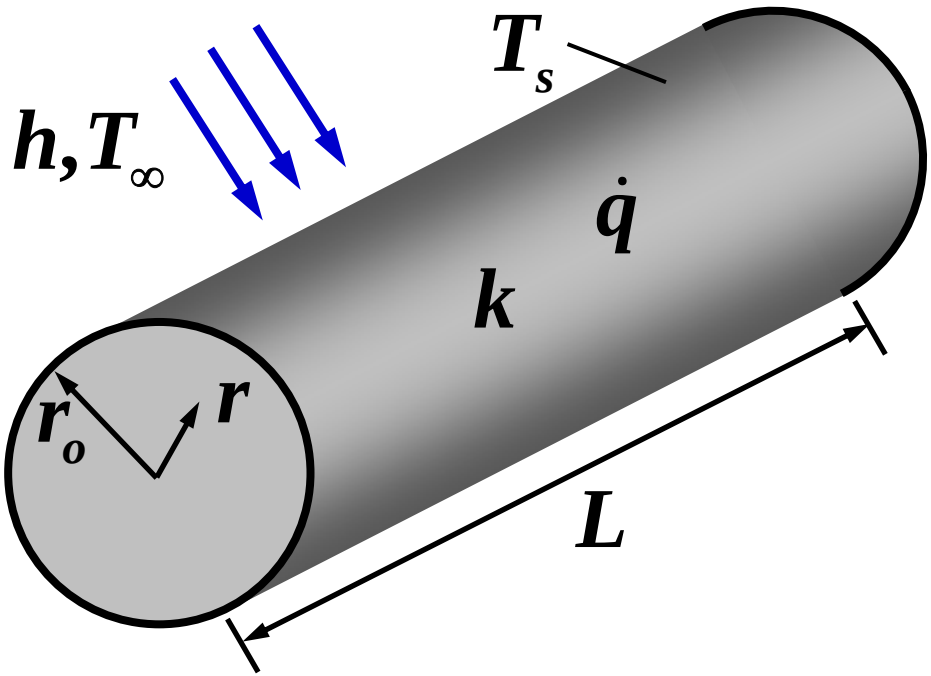
$$T_2 = T_\infty + \frac{\dot{q} L_A}{h} = 105^\circ \mathrm{C}$$

$$T_1 = T_2 + \frac{\dot{q} L_A L_B}{k_B} = 115^\circ \mathrm{C}$$

$$T_0 = \frac{\dot{q} L_A^2}{2k_A} + T_1 = 140^\circ \mathrm{C}$$



Radial System



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

boundary conditions

$$\left.\frac{dT}{dr}\right|_0=0, \quad T(r_o)=T_s$$

$$\frac{dT}{dr}=-\frac{\dot{q}}{2k}r+\frac{C_1}{r}$$

$$T(r)=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2$$

$$\frac{dT}{dr} = -\frac{\dot{q}}{2k} r + \frac{C_1}{r}, \qquad \left.\frac{dT}{dr}\right|_0 = 0 \quad \rightarrow C_1 = 0$$

$$T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln r + C_2 \; = -\frac{\dot{q}}{4k} r^2 + C_2$$

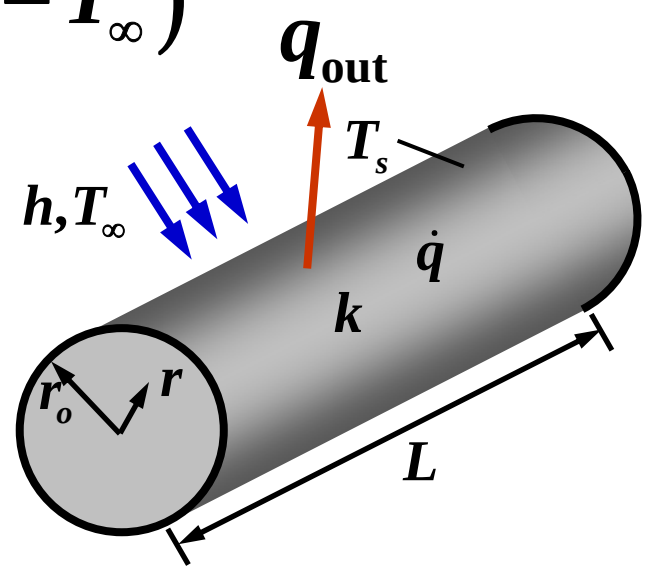
$$T(r_o) = T_s \; = -\frac{\dot{q}}{4k} r_o^2 + C_2 \rightarrow C_2 = \frac{\dot{q}}{4k} r_o^2 + T_s$$

$$T(r) = -\frac{\dot{q}}{4k} r^2 + \frac{\dot{q}}{4k} r_o^2 + T_s \; = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

$$q(r) = -k\left(2\pi r L\right)\frac{dT}{dr} \; = -k\left(2\pi r L\right)\left(-\frac{\dot{q}}{2k} r\right)$$

$$= \pi L \dot{q} r^2$$

$$q_{\text{out}} = \dot{q}\left(\pi r_o^2 L\right) = h\left(2\pi r_o L\right)\left(T_s - T_\infty\right)$$



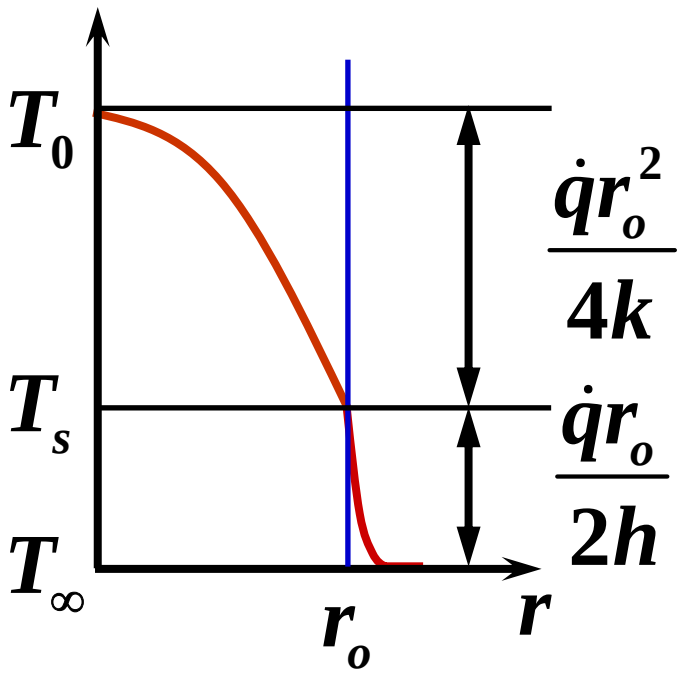
$$T_s = T_\infty + \frac{\dot{q} r_o}{2h}$$

$$T(r) = \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$$

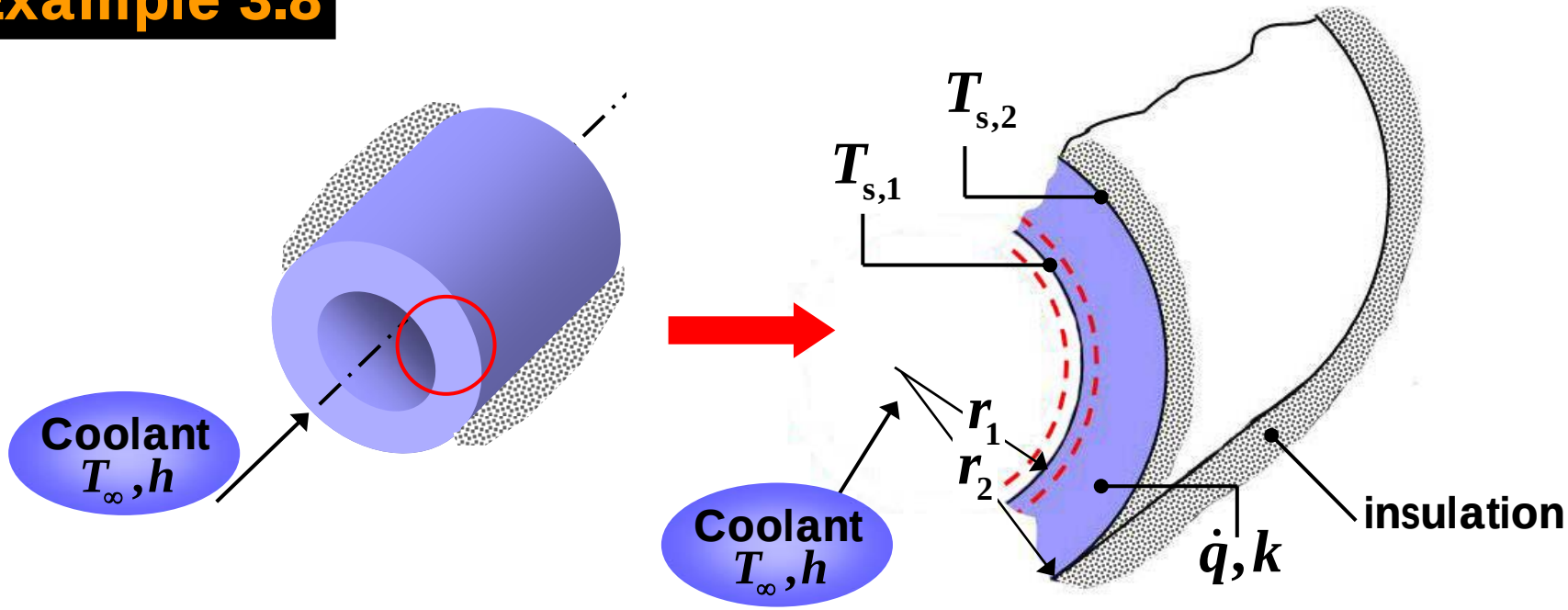
$$= \frac{\dot{q} r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + \frac{\dot{q} r_o}{2h} + T_\infty$$

$$T_0 = T(0) = \frac{\dot{q} r_o^2}{4k} + \frac{\dot{q} r_o}{2h} + T_\infty$$

$$T_0 - T_s = \frac{\dot{q} r_o^2}{4k}, \quad T_s - T_\infty = \frac{\dot{q} r_o}{2h}$$



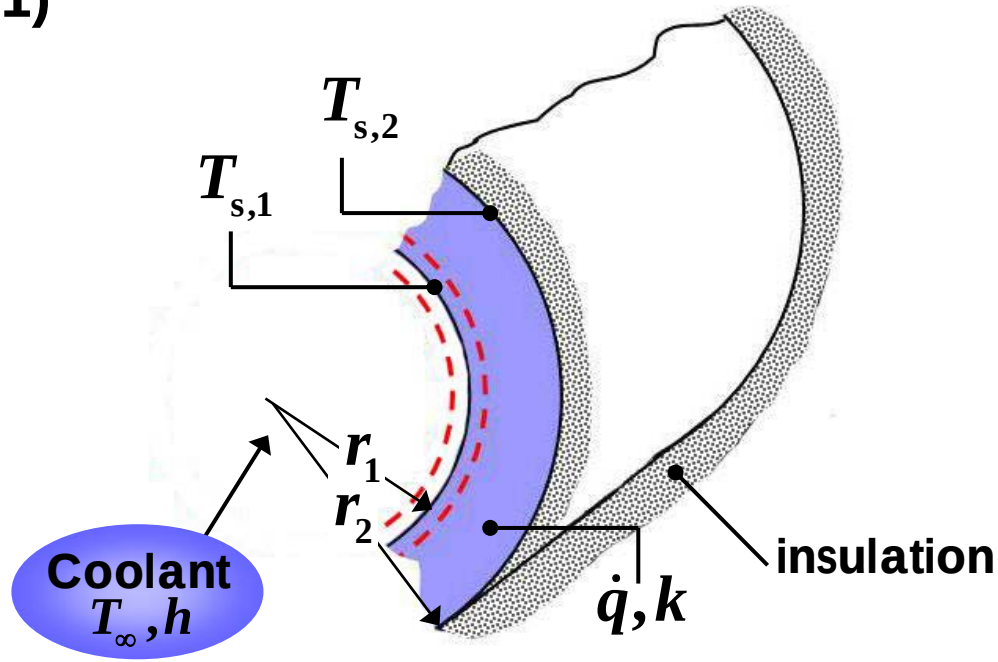
Example 3.8



Find:

1) General solution for the temperature distribution $T(r)$
2) Appropriate B.C. and the corresponding form of $T(r)$:
   at $r = r_2$, maximum permissible temperature $T_{s,2}$ is specified.
3) Heat removal rate, q' by the coolant
4) Convection heat transfer coefficient at the inner surface, h

1)



$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

$$\frac{dT}{dr}=-\frac{\dot{q}}{2k}r+\frac{C_1}{r}$$

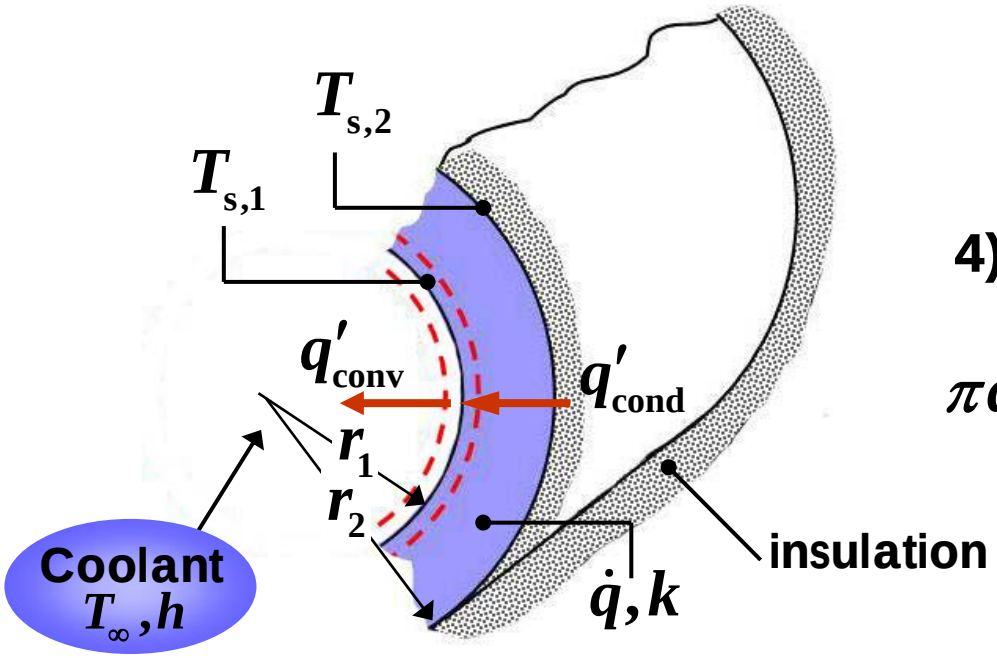
$$T(r)=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2$$

2) Two boundary conditions at the outer wall (r_2).

$$\left.\frac{dT}{dr}\right|_{r_2}=0, \qquad T(r_2)=T_{s,2}$$

$$C_1=\frac{\dot{q}}{2k}r_2^{\,2}, \qquad C_2=T_{s,2}+\frac{\dot{q}}{4k}r_2^{\,2}-\frac{\dot{q}}{2k}r_2^{\,2}\ln r_2$$

$$T(r)=T_{s,2}+\frac{\dot{q}}{4k}\left(r_2^{\,2}-r^2\right)-\frac{\dot{q}}{2k}r_2^{\,2}\ln\frac{r_2}{r}$$



3) Heat removal rate

$$q' = \pi \dot{q}\left(r_2^{\,2} - r_1^{\,2}\right)$$

4) energy balance at $r = r_1$

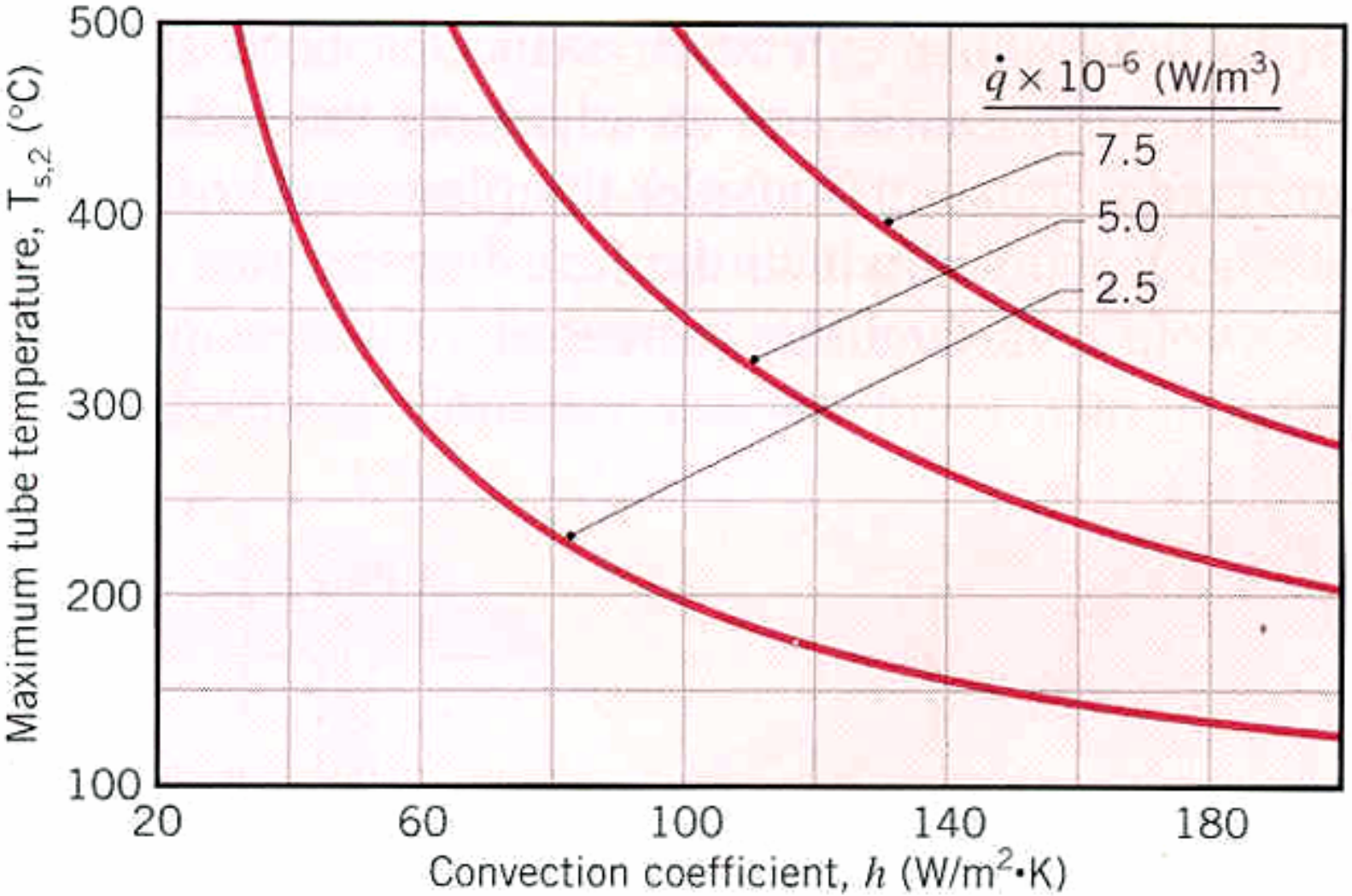
$$\pi \dot{q}\left(r_2^{\,2} - r_1^{\,2}\right) = h\left(2\pi r_1\right)\left(T_{s,1} - T_\infty\right)$$

$$h = \frac{\dot{q}\left(r_2^{\,2} - r_1^{\,2}\right)}{2r_1\left(T_{s,1} - T_\infty\right)}$$

$$T(r) = T_{s,2} + \frac{\dot{q}}{4k}\left(r_2^{\,2} - r^2\right) - \frac{\dot{q}}{2k} r_2^{\,2} \ln\frac{r_2}{r}$$

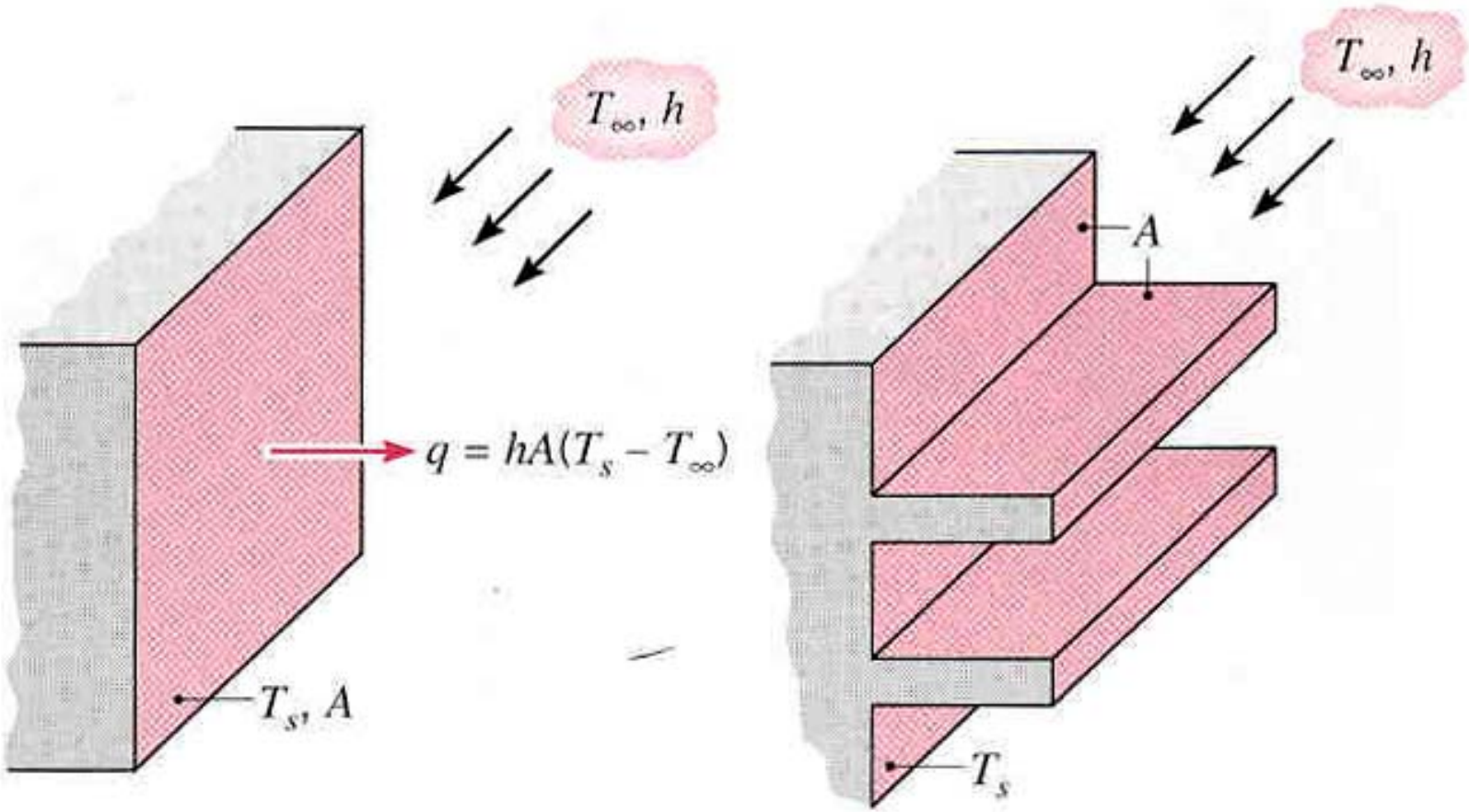
$$T_{s,1} = T(r_1) = T_{s,2} + \frac{\dot{q}}{4k}\left(r_2^{\,2} - r_1^{\,2}\right) - \frac{\dot{q}}{2k} r_2^{\,2} \ln\frac{r_2}{r_1}$$

$$k = 5\ \mathrm{W/m \cdot K}$$
$$r_1 = 20\ \mathrm{mm} \quad \text{and} \quad r_2 = 25\ \mathrm{mm}$$
$$T_\infty = 80°\mathrm{C} \quad \text{and} \quad T(r_1) = T_{s,1} = 336.5°\mathrm{C}$$

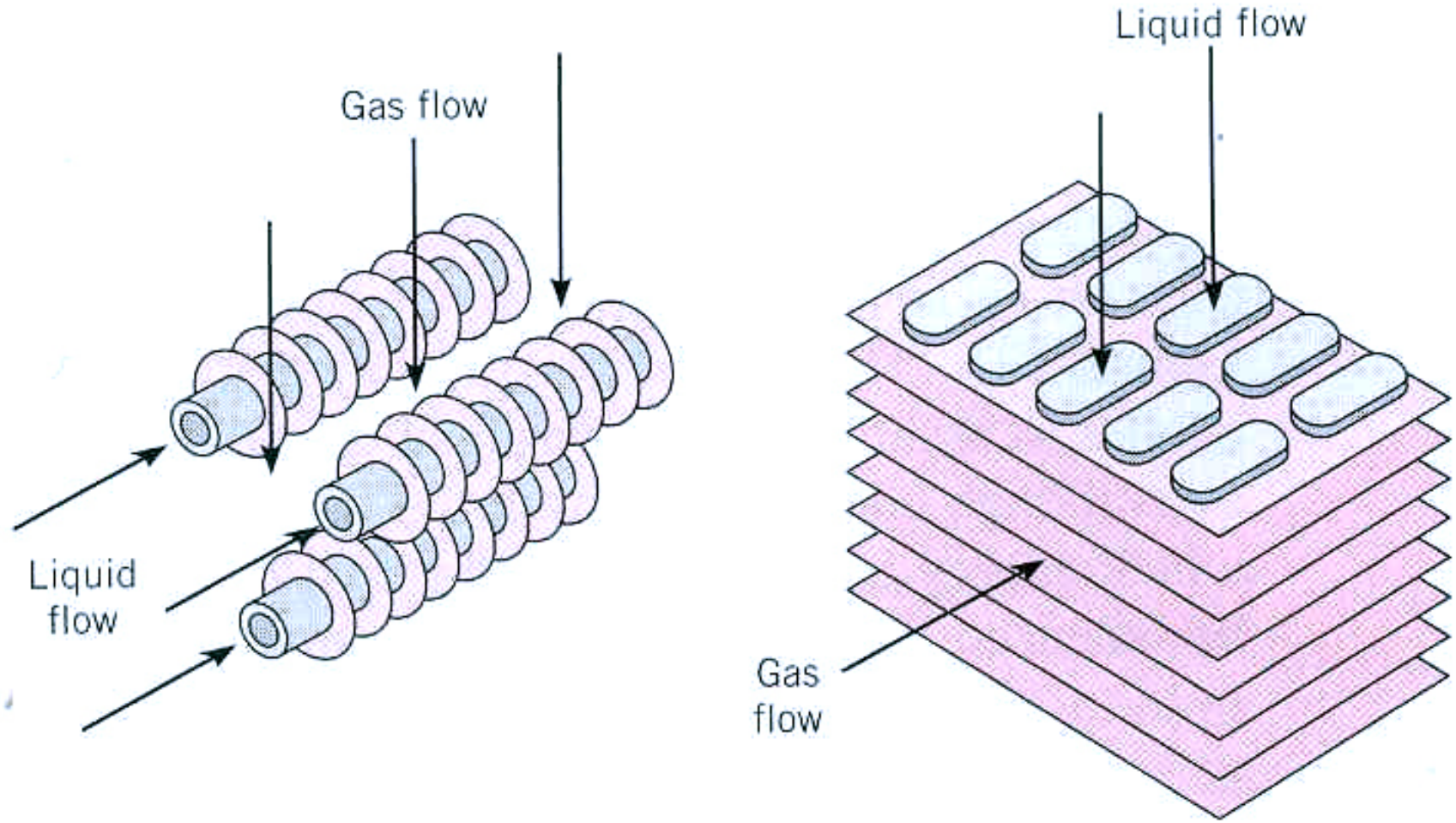


Heat Transfer from Extended Surfaces

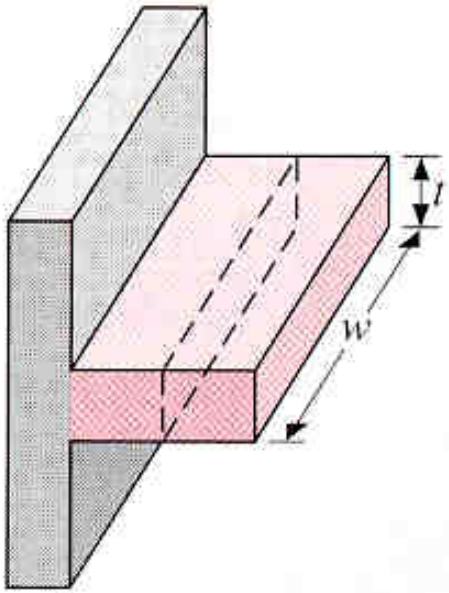
Heat Transfer Enhancement by Fins



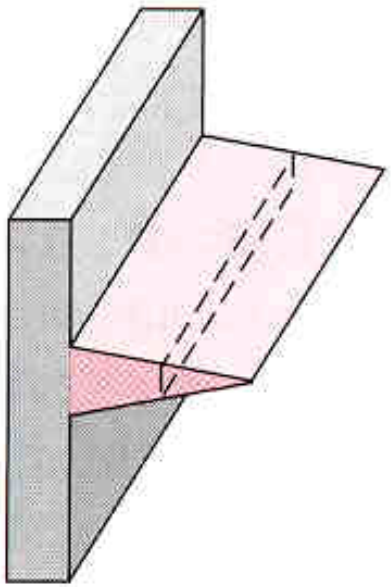
Bare surface **Finned surface**



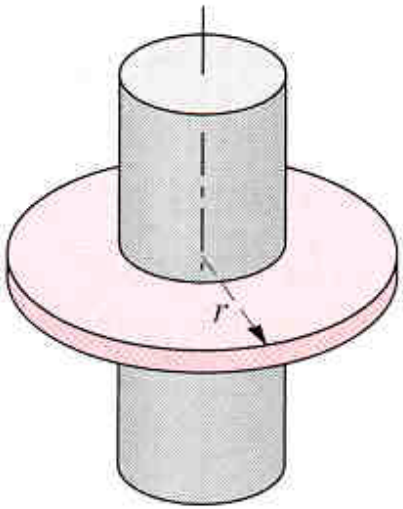
Typical finned-tube heat exchangers



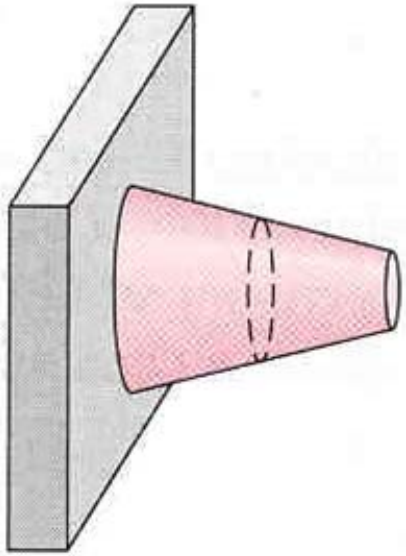
Straight fin of uniform cross section



Straight fin of nonuniform cross section

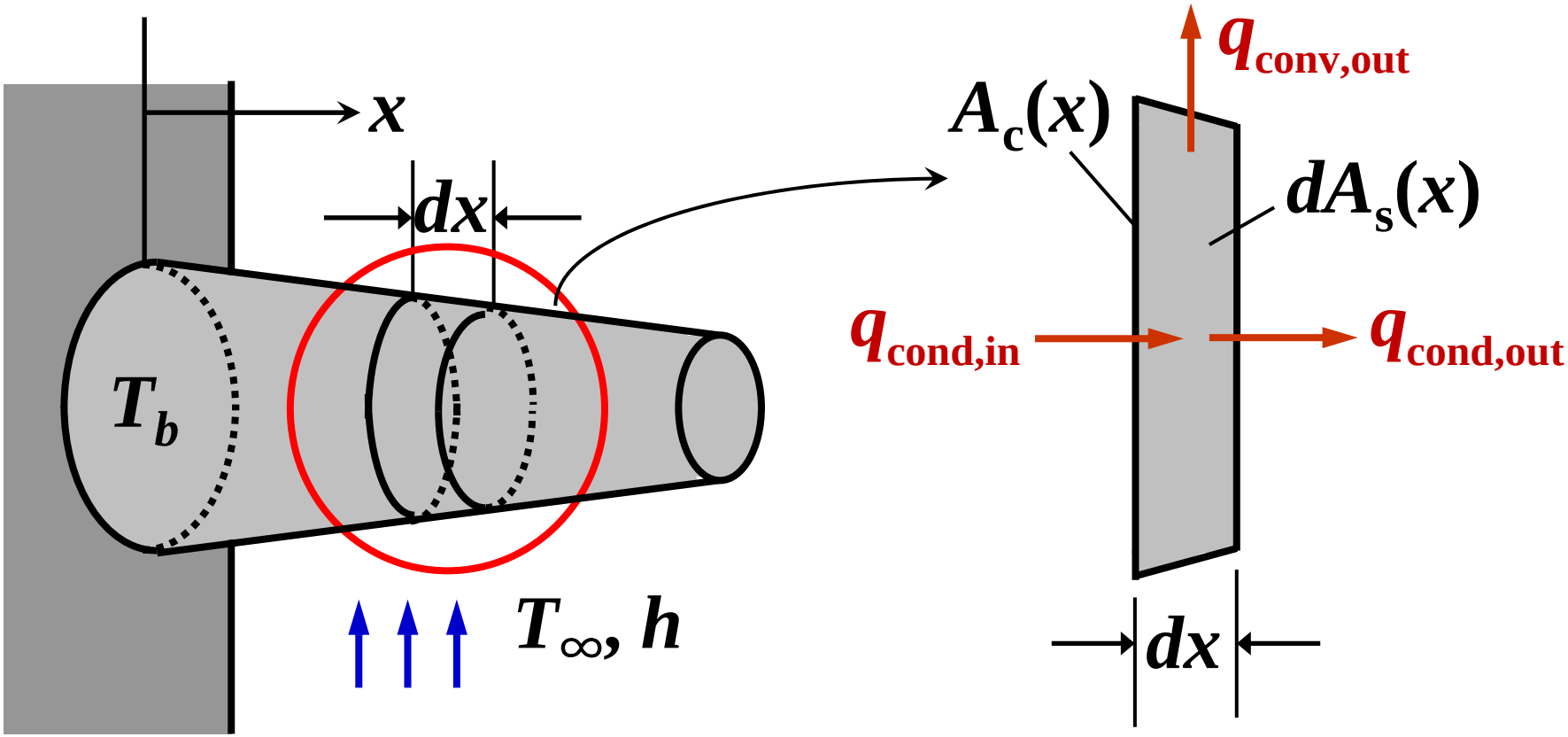


Annular fin

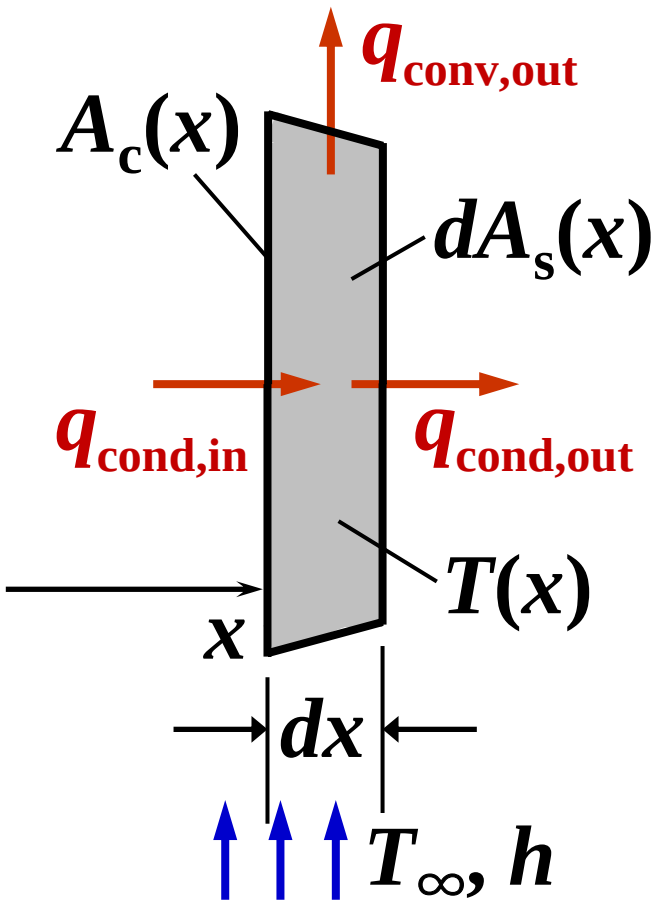


Pin fin

Equation for Extended Surfaces



$$q_{cond,in} = q_{cond,out} + q_{conv,out}$$



$$q_{\text{cond,in}} = q_{\text{cond,out}} + q_{\text{conv,out}}$$

$$q_{\text{cond,in}} = q_x = \cancel{-kA_c(x)\frac{dT}{dx}}$$

$$q_{\text{cond,out}} = q_{x+dx} = \cancel{-kA_c(x)\frac{dT}{dx}} + \frac{d}{dx}\left[-kA_c(x)\frac{dT}{dx}\right]dx$$

$$q_{\text{conv,out}} = dq_{\text{conv}} = hdA_s(x)\left[T(x) - T_\infty\right]$$

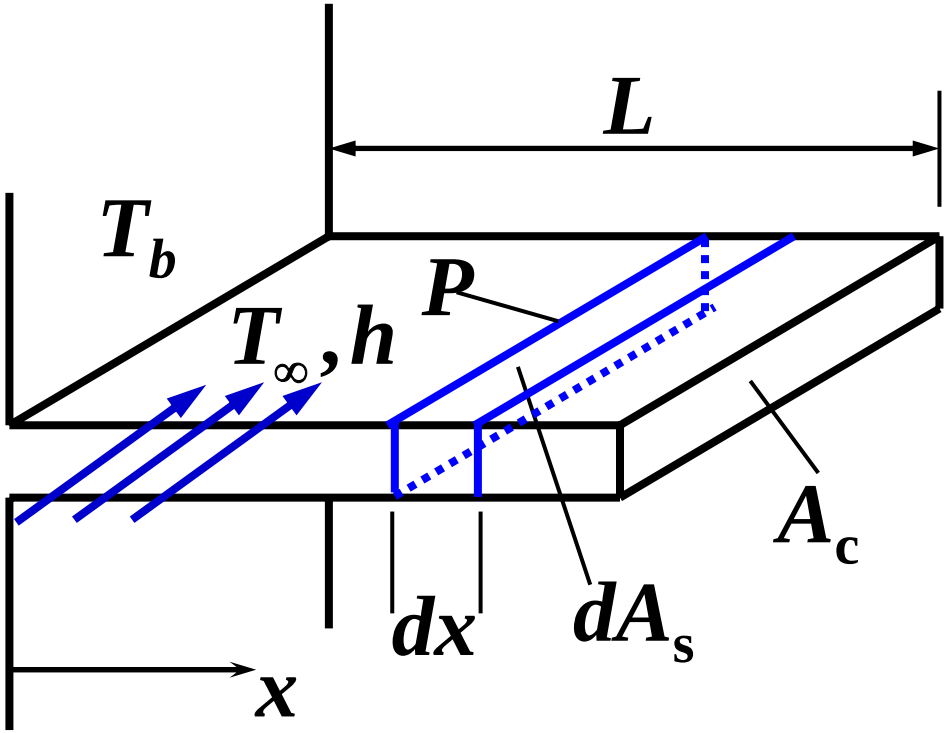
$$\frac{d}{dx}\left[-kA_c(x)\frac{dT}{dx}\right]dx + hdA_s(x)\left[T(x)-T_\infty\right]=0$$

$$\frac{d}{dx}\left[kA_c(x)\frac{dT}{dx}\right]-h\frac{dA_s}{dx}\left[T(x)-T_\infty\right]=0$$

When k = constant,

$$\frac{d}{dx}\left[A_c(x)\frac{dT}{dx}\right]-\frac{h}{k}\frac{dA_s}{dx}\left[T(x)-T_\infty\right]=0$$

Fins of Uniform Cross-Sectional Area



P: fin perimeter

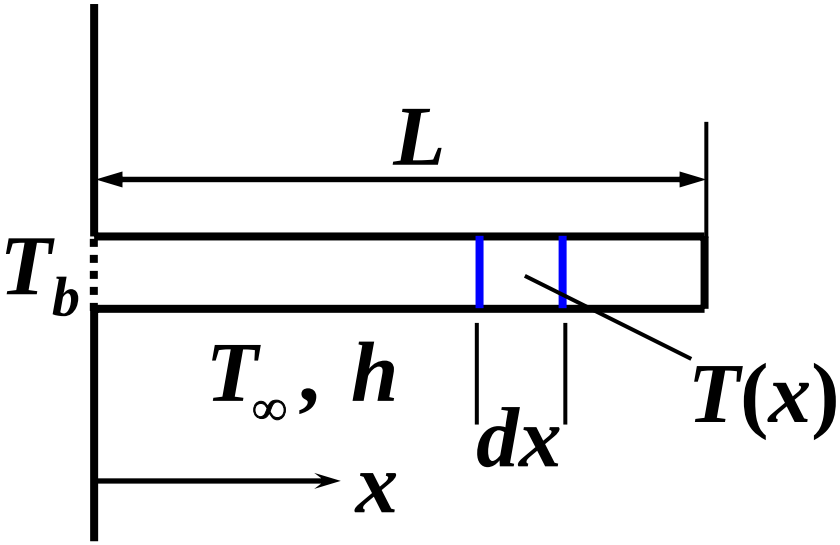
$A_c(x)$ = constant,

and $dA_s = Pdx$

$$\frac{d}{dx}\left[A_c(x)\frac{dT}{dx}\right] - \frac{h}{k}\frac{dA_s}{dx}\left[T(x) - T_\infty\right] = 0$$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}\left[T(x) - T_\infty\right] = 0$$

excess temperature : $\theta(x) = T(x) - T_\infty$



$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}\left[T(x) - T_\infty\right] = 0$$

$$\rightarrow \frac{d^2\theta}{dx^2} - m^2\theta = 0$$

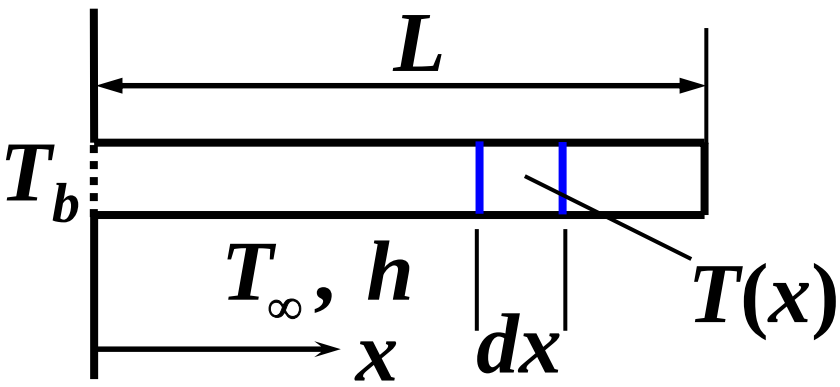
where $m^2 \equiv \dfrac{hP}{kA_c}$

$$\theta(x) = C_1 e^{-mx} + C_2 e^{mx} = D_1 \sinh(mx) + D_2 \cosh(mx)$$

boundary conditions

at $x = 0$**:**

$$T(0) = T_b \rightarrow \theta(0) = T(0) - T_\infty = T_b - T_\infty \equiv \theta_b$$



$$\theta(x) = T(x) - T_\infty$$

at $x = L$: 3 cases

1) very long fin ($L \to \infty$):

$$T(L) \approx T_\infty \to \theta(\infty) = T_\infty - T_\infty = 0$$

2) convection tip:

$$-k\left.\frac{dT}{dx}\right|_L = h\left[T(L) - T_\infty\right] \to \left.\frac{d\theta}{dx}\right|_L + \frac{h}{k}\theta(L) = 0$$

3) negligible heat loss: adiabatic tip

$$-k\left.\frac{dT}{dx}\right|_L = 0 \to \left.\frac{d\theta}{dx}\right|_L = 0$$

Temperature distribution

1) long fin:

$$\frac{\theta(x)}{\theta_b} = e^{-mx}$$

2) convection tip:

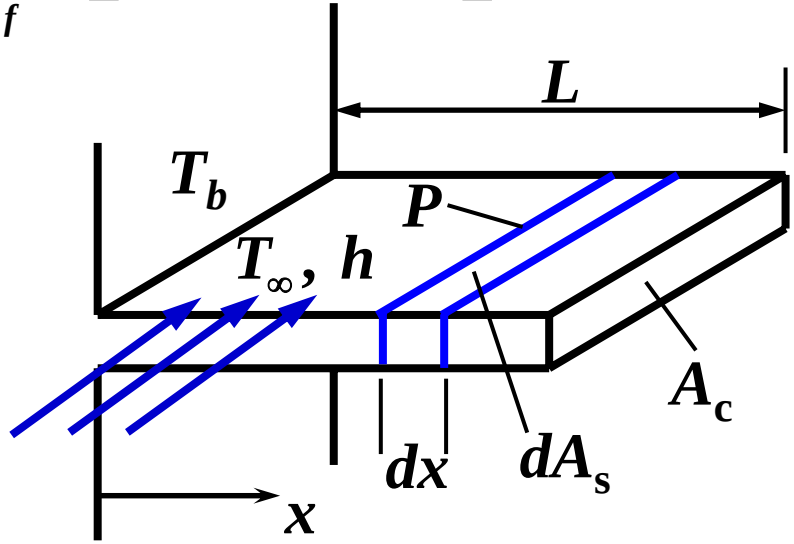
$$\frac{\theta(x)}{\theta_b} = \frac{\cosh m(L-x) + \left(h/mk\right)\sinh m(L-x)}{\cosh mL + \left(h/mk\right)\sinh mL}$$

3) adiabatic tip:

$$\frac{\theta(x)}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

Total heat loss by the fin

$$q_f = -kA_{c,b}\left.\frac{dT}{dx}\right|_{x=0} \quad \text{or } q = \int_{A_f} h\left[T(x) - T_\infty\right]dA$$



1) long fin:

$$q_f = \sqrt{hPkA_c}\,\theta_b \equiv M$$

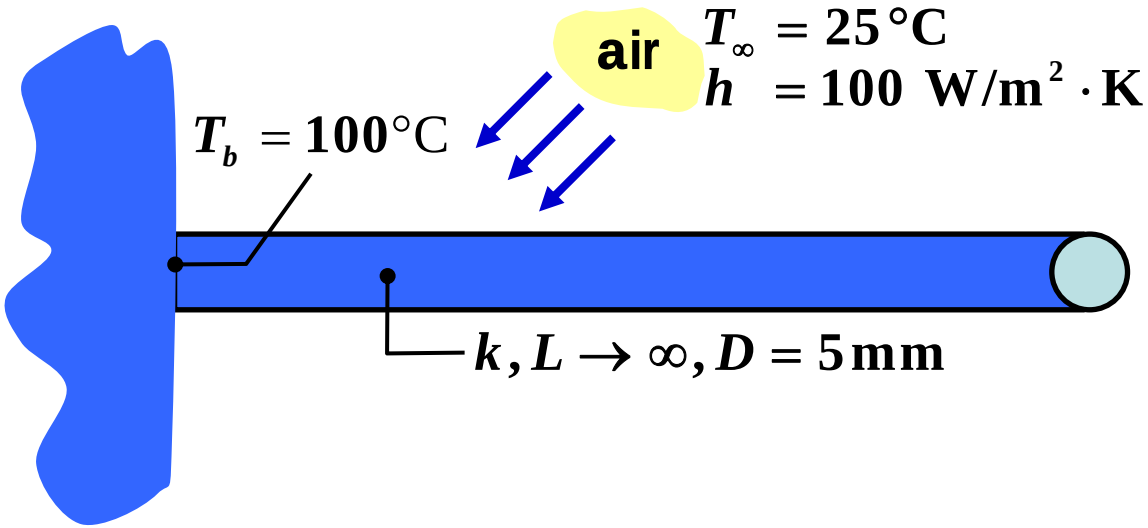
2) convection tip:

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

3) adiabatic tip:

$$q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL = M \tanh mL$$

Example 3.9



Find:

1) Temperature distribution $T(x)$ and heat loss q_f when the fin is constructed from: a) pure copper, b) 2024 aluminum alloy, and c) type AISI 316 stainless steel.
2) Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the heat loss.

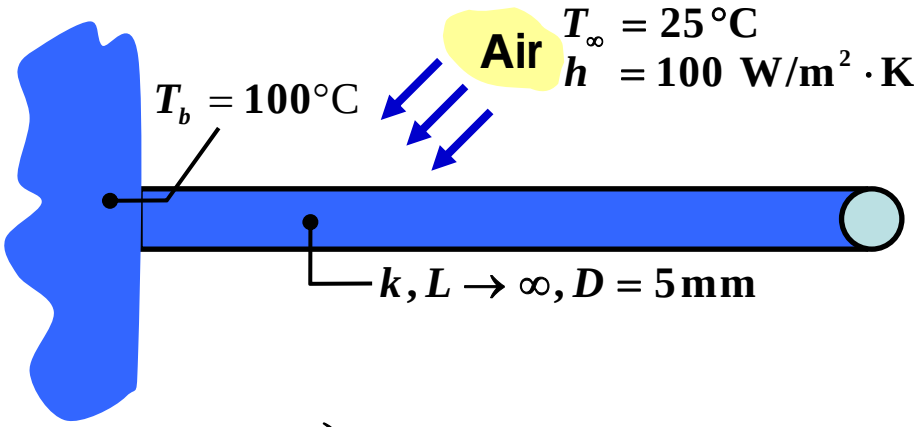
Assumption:

very long fin

1) For a very long fin

$$\frac{\theta(x)}{\theta_b} = e^{-mx} = \frac{T(x) - T_\infty}{T_b - T_\infty}$$

$$\rightarrow T(x) = T_\infty + \left(T_b - T_\infty\right) e^{-mx}$$



$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{4h}{kD}} \quad \left(P = \pi D, A_c = \frac{\pi D^2}{4}\right)$$

heat loss: $q_f = \sqrt{hPkA_c}\,\theta_b$

conductivity at $T = \dfrac{T_b + T_\infty}{2}$

Copper: k = 398 W/m·K

Aluminum alloy: k = 180 W/m·K

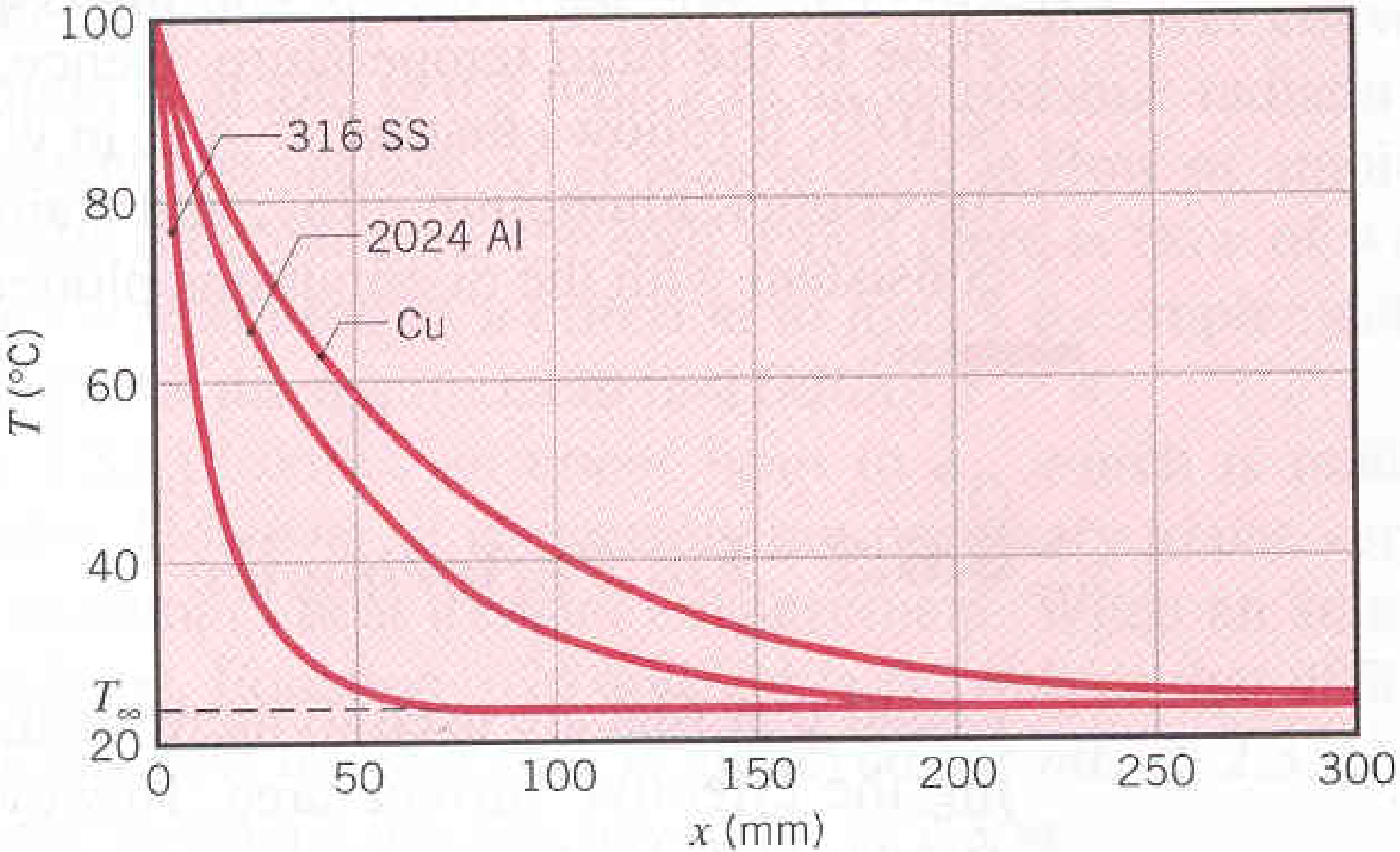
Stainless steel: k = 14 W/m·K

q_f

Copper: 8.3 W

Aluminum alloy: 5.6 W

Stainless steel: 1.6 W


100
80
60
40
T_∞
20
T (°C)
316 SS
2024 Al
Cu
0
50
100
150
200
250
300
x (mm)

2) For the adiabatic condition at the tip

$$q_f = M \tanh mL \quad \text{(long fin: } q_f = M\text{)}$$

To get an accuracy over 99%

$$\tanh mL \geq 0.99 \quad \text{or} \quad mL \geq 2.65$$

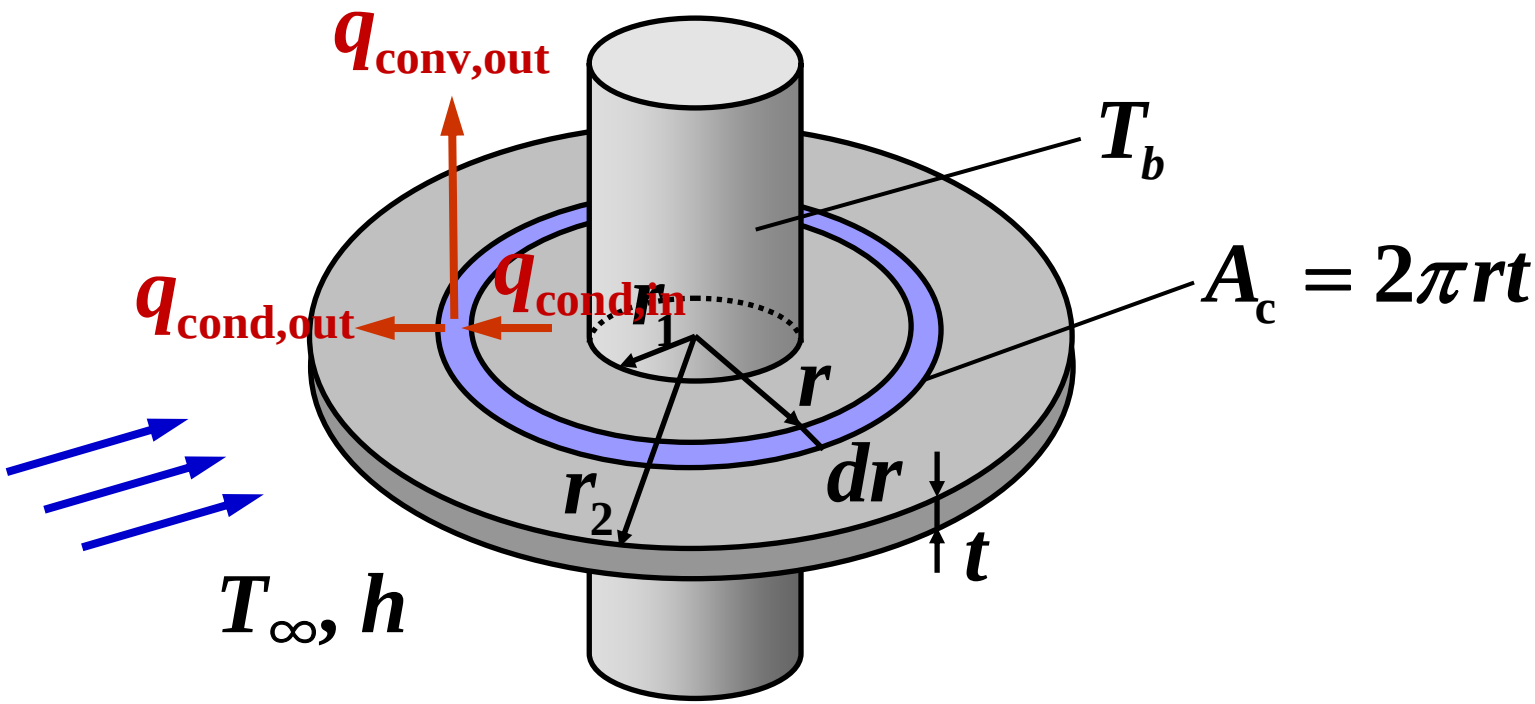
$$L \geq \frac{2.65}{m} = 2.65\left(\frac{kA_c}{hP}\right)^{1/2}$$

Copper: 0.19 m

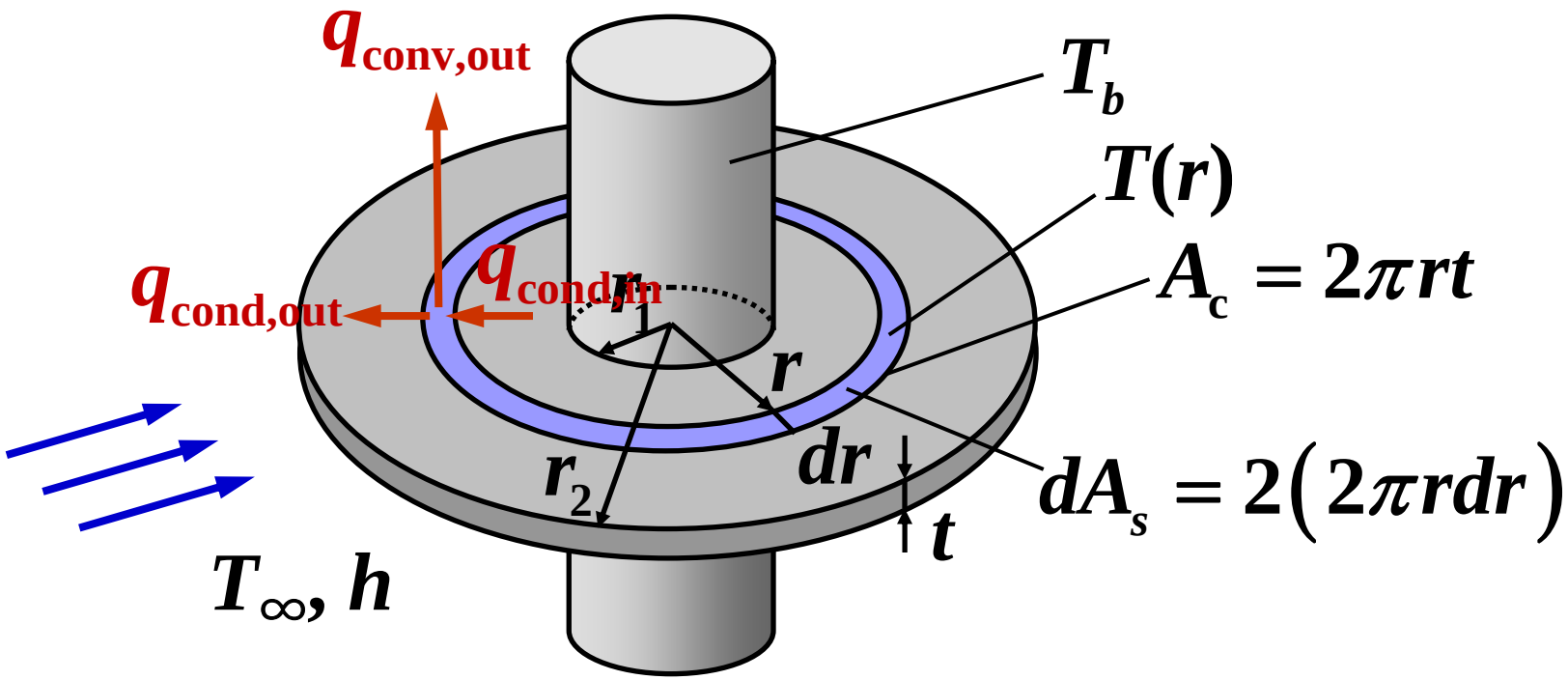
Aluminum alloy: 0.13 m

Stainless steel: 0.04 m

Fins of Nonuniform Cross-Sectional Area : Annular Fin



$$q_{cond,in} = q_{cond,out} + q_{conv,out}$$



$$q_{\text{cond,in}} = q_r = -k(2\pi rt)\frac{dT}{dr}$$

$$q_{\text{cond,out}} = q_{r+dr} = -k(2\pi rt)\frac{dT}{dr} + \frac{d}{dr}\left[-k(2\pi rt)\frac{dT}{dr}\right]dr$$

$$q_{\text{conv,out}} = dq_{\text{conv}} = h2(2\pi r dr)\left[T(r) - T_\infty\right]$$

$$\frac{d}{dr}\left[k\left(2\pi rt\right)\frac{dT}{dr}\right]-h\left(4\pi r\right)\left[T(r)-T_\infty\right]=0$$

When k **= const,** $\frac{d}{dr}\left(r\frac{dT}{dr}\right)-\frac{2h}{kt}r\left[T(r)-T_\infty\right]=0$

in terms of excess temperature $\theta(r)=T(r)-T_\infty$

$$\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)-m^2r\theta=0 \quad \textbf{where} \quad m^2\equiv\frac{2h}{kt}$$

boundary conditions: when an adiabatic tip is presumed

$$\theta(r_1)=\theta_b, \qquad \left.\frac{d\theta}{dr}\right|_{r=r_2}=0$$

Solutions to generalized Bessel equations

$$\frac{d}{dr}\left(r^{\alpha}\frac{d\theta}{dr}\right)+\gamma^{2}r^{\beta}\theta=0,\quad \beta-\alpha+2\neq 0$$

General Solution: $\theta(r)=r^{\nu/\mu}Z_{\nu}\left(|\gamma|\mu r^{1/\mu}\right),$

where $\nu=(1-\alpha)/(\beta-\alpha+2),\ \mu=2/(\beta-\alpha+2),\ \nu/\mu=(1-\alpha)/2$

Particular solutions

γ: real	ν: real	J_{ν} and $J_{-\nu}$ (or Y_{ν})
	ν: zero or integer	J_{n} and Y_{n}
γ: imaginary	ν: fractional	I_{ν} and $I_{-\nu}$ (or K_{ν})
	ν: zero or integer	I_{n} and K_{n}

present case: $\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)-m^2 r\theta=0$

$$\frac{d}{dr}\left(r^{\alpha}\frac{d\theta}{dr}\right)+\gamma^2 r^{\beta}\theta=0,\quad \beta-\alpha+2\neq 0$$

$$\alpha=1,\ \beta=1,\ \gamma^2=-m^2 \qquad \theta(r)=r^{\nu/\mu}Z_{\nu}\left(|\gamma|\mu r^{1/\mu}\right)$$

Thus, $\nu=0,\ \mu=1,\ \nu/\mu=0,\ \gamma=\pm mi$

$$\theta(r)=Z_0(mr)\rightarrow\ \theta(r)=C_1 I_0(mr)+C_2 K_0(mr)$$

boundary conditions:

$$\theta(r_1)=C_1 I_0(mr_1)+C_2 K_0(mr_1)=\theta_b$$

$$\left.\frac{d\theta}{dr}\right|_{r=r_2}=C_1\left.\frac{dI_0(mr)}{dr}\right|_{r=r_2}+C_2\left.\frac{dK_0(mr)}{dr}\right|_{r=r_2}=0$$

$$\frac{d}{dx}\left\{x^{-n}I_n(x)\right\} = x^{-n}I_{n+1}(x),$$

$$\frac{d}{dx}\left\{x^{-n}K_n(x)\right\} = -x^{-n}K_{n+1}(x)$$

$$\frac{dI_0(mr)}{dx} = mI_1(mr), \quad \frac{dK_0(mr)}{dx} = -mK_1(mr)$$

$$C_1\left.\frac{dI_0(mr)}{dr}\right|_{r=r_2} + C_2\left.\frac{dK_0(mr)}{dr}\right|_{r=r_2} = 0$$

$$C_1 I_1(mr_2) - C_2 K_1(mr_2) = 0$$

$$C_1 = \frac{K_1(mr_2)\theta_b}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)},$$

$$C_2 = \frac{I_1(mr_2)\theta_b}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}$$

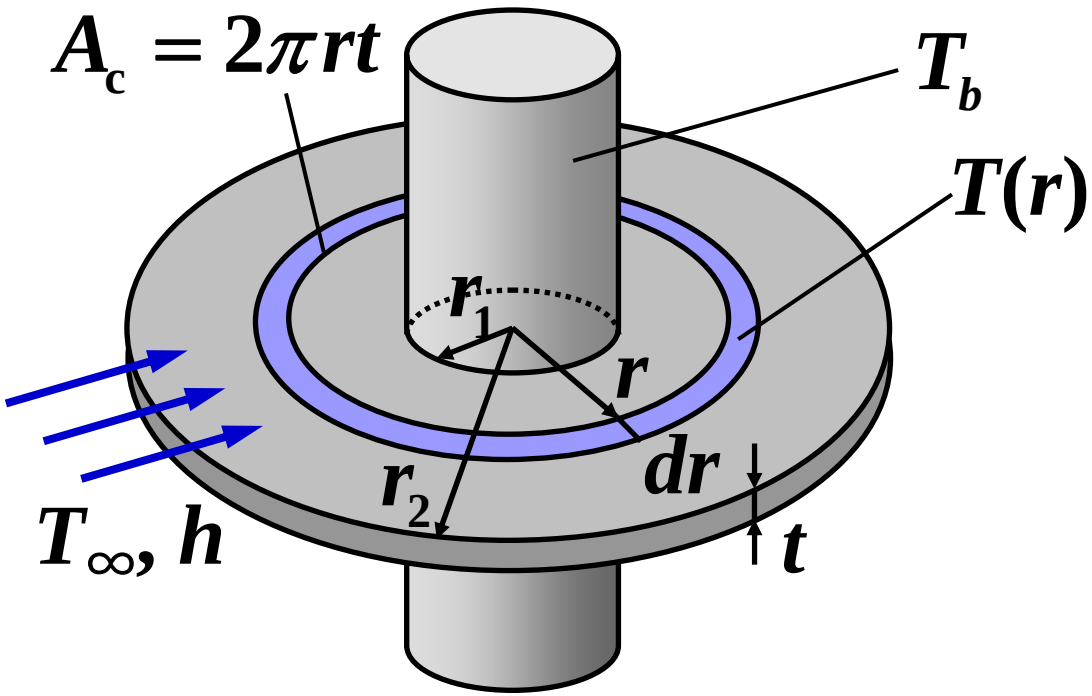
$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)$$

$$\frac{\theta(r)}{\theta_b} = \frac{K_1(mr_2)I_0(mr) + I_1(mr_2)K_0(mr)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}$$

Heat loss from the fin

$$q_f = -kA_{c,b}\left.\frac{dT}{dr}\right|_{r=r_1}$$

$$= -k\left(2\pi r_1 t\right)\left.\frac{d\theta}{dr}\right|_{r=r_1}$$



$$\frac{\theta(r)}{\theta_b} = \frac{K_1(mr_2)I_0(mr) + I_1(mr_2)K_0(mr)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}$$

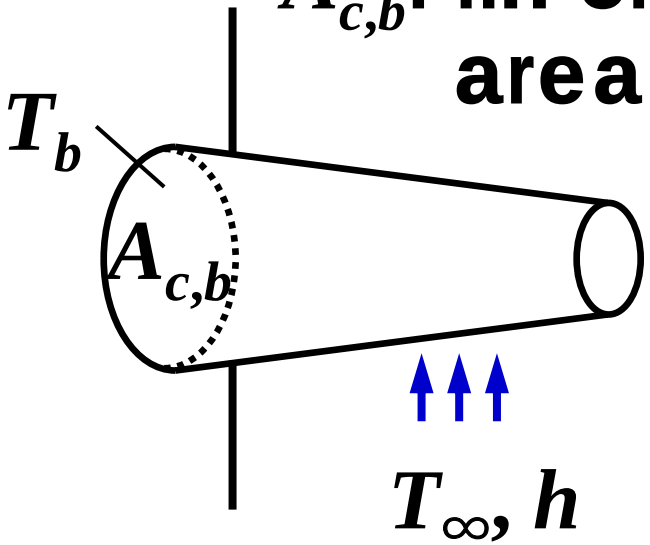
$$q_f = 2\pi k r_1 t \theta_b \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}$$

Fin Performance

- fin effectiveness
- fin resistance
- fin efficiency

1. Fin effectiveness

$A_{c,b}$: fin cross-sectional area at the base



heat loss without fin

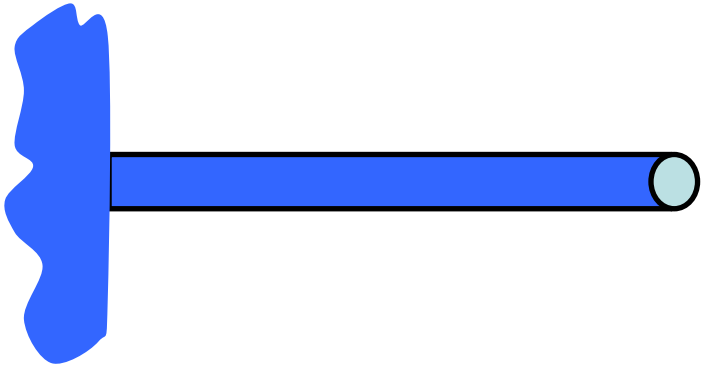
$$q_b = hA_{c,b}\left(T_b - T_\infty\right) = hA_{c,b}\theta_b$$

fin effectiveness: $\varepsilon_f \equiv \dfrac{q_f}{hA_{c,b}\theta_b}$

design criteria: $\varepsilon_f \geq 2$ (rule of a thumb)

Assume *h*s are the same for with or without fin.

Ex) long straight fin with uniform cross-sectional area



$$q_f = \sqrt{hPkA_c}\,\theta_b$$

$$\varepsilon_f = \frac{\sqrt{hPkA_c}\,\theta_b}{hA_{c,b}\theta_b} = \sqrt{\frac{kP}{hA_c}}$$

In order to get high fin performance

- **large k material**
- **installation of fins at the lower h side**
- **thin shape**

$$\varepsilon_f = \sqrt{\frac{kP}{hA_c}}$$

provides upper limit of ε_f, which is reached as L approaches infinity.

long fin $q_f = \sqrt{hPkA_c}\,\theta_b$

adia. tip $q_f = \sqrt{hPkA_c}\,\theta_b \tanh mL, \quad m = \sqrt{\frac{hP}{kA_c}}$

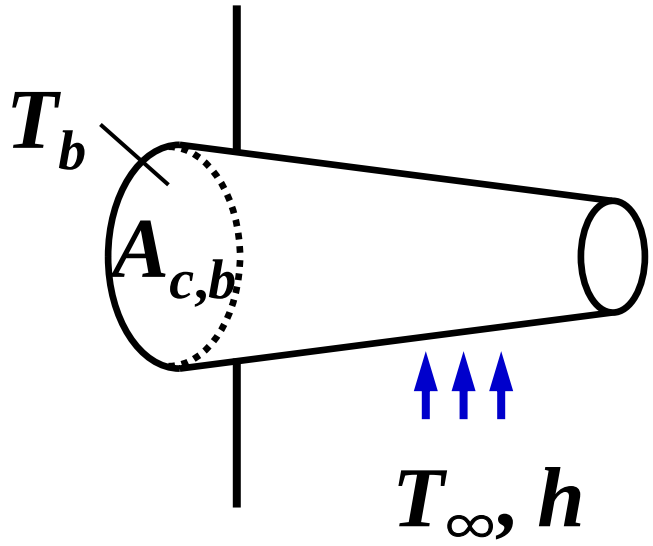
Practically q_f for the adiabatic tip reaches 98% of heat transfer when mL = 2.3.

Thus, the fin length longer than $L = 2.3/m$ is not effective.

2. Fin resistance

$$q_f = \frac{\theta_b}{R_{t,f}}$$

$$\rightarrow R_{t,f} \equiv \frac{\theta_b}{q_f}$$



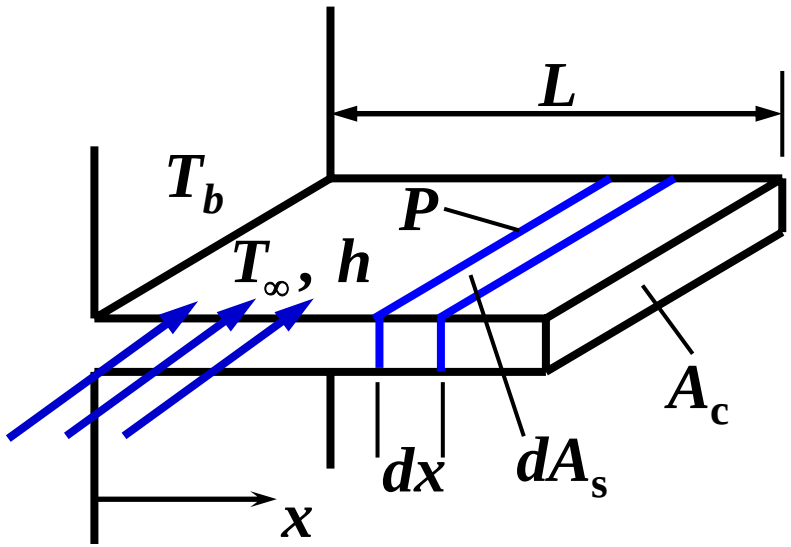
θ_b: **driving potential**

Thermal resistance due to convection at the exposed base: $R_{t,b}$

$$R_{t,b} = \frac{\theta_b}{q_b} = \frac{\theta_b}{hA_{c,b}\theta_b} = \frac{1}{hA_{c,b}}$$

3. Fin efficiency

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b}$$



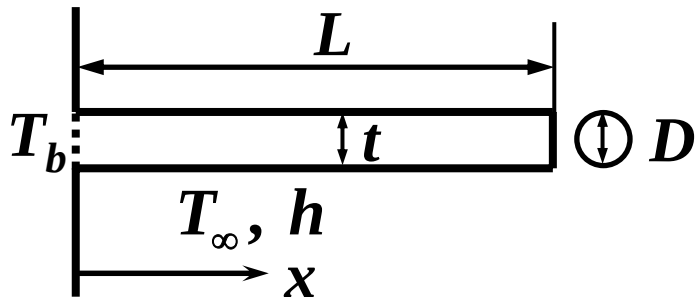
$q_{\max}$: heat loss when the whole fin is assumed at T_b

Ex) straight fin of uniform cross-sectional area with an adiabatic tip

$$\eta_f = \frac{q_f}{hA_f\theta_b} = \frac{\sqrt{hPkA_c}\,\theta_b \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

For an active tip, the above relation can be used with fin length correction.

rectangular fin: $L_c = L + \dfrac{t}{2}$



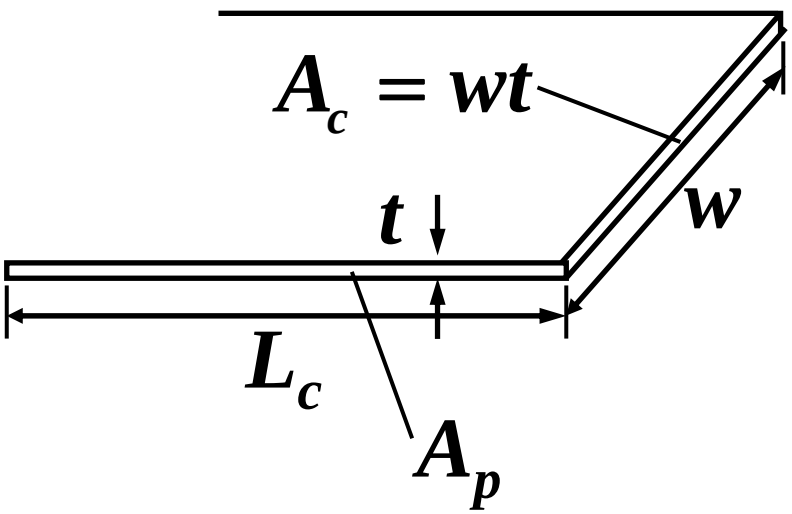
pin fin: $L_c = L + \dfrac{D}{4}$

$$q_f = M \tanh mL_c, \quad \eta_f = \frac{\tanh mL_c}{mL_c} = f\left(mL_c\right)$$

Errors can be negligible if

$$\left(ht/k\right) \text{ or } \left(hD/2k\right) \le 0.0625$$

When $w >> t$, $P \sim 2w$,



$$mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c$$

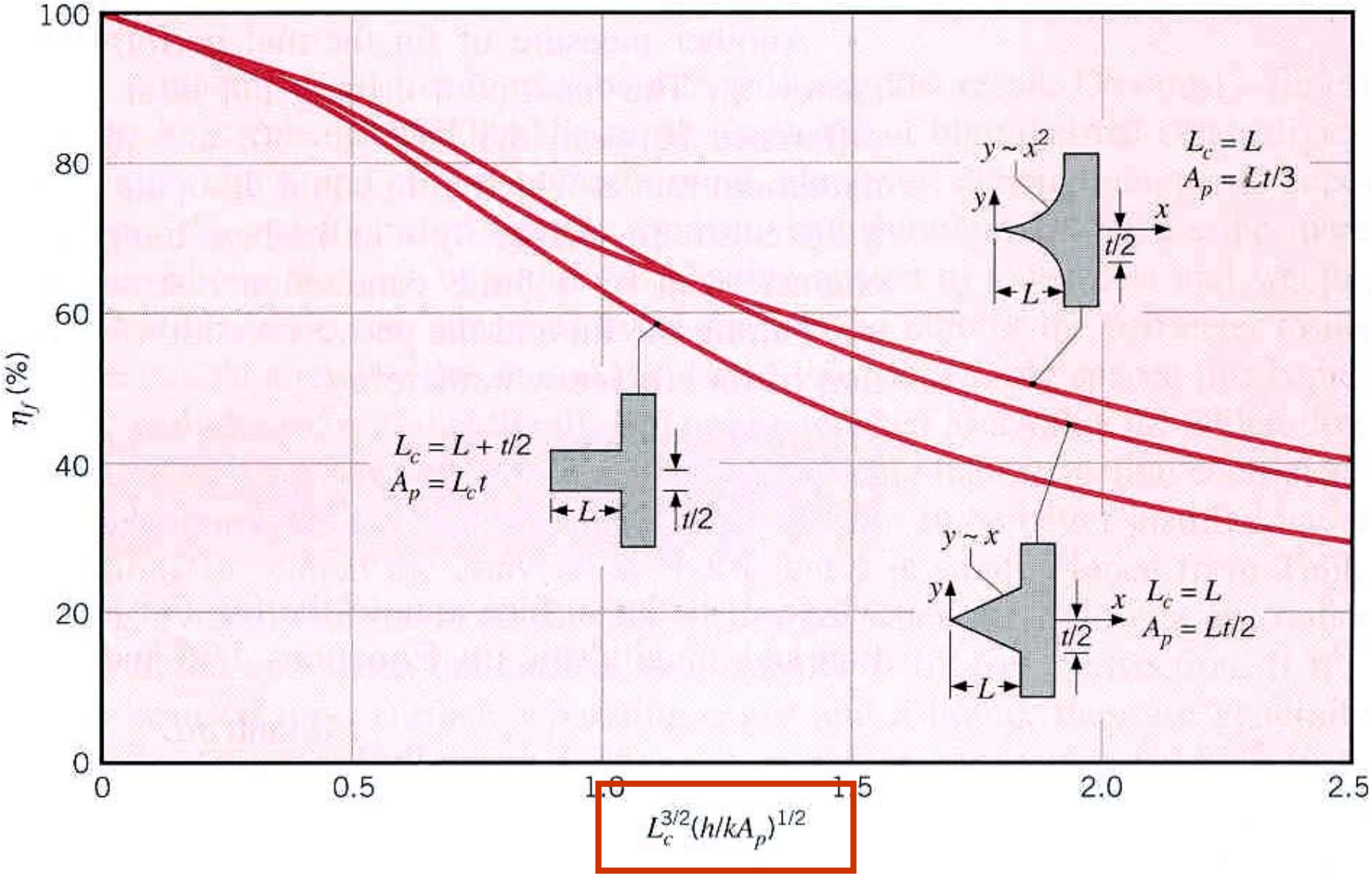
$$= \left(\frac{h2w}{kwt}\right)^{1/2} L_c = \left(\frac{2h}{kt}\right)^{1/2} L_c$$

Corrected fin profile area $A_p = L_c t \rightarrow t = \dfrac{A_p}{L_c}$

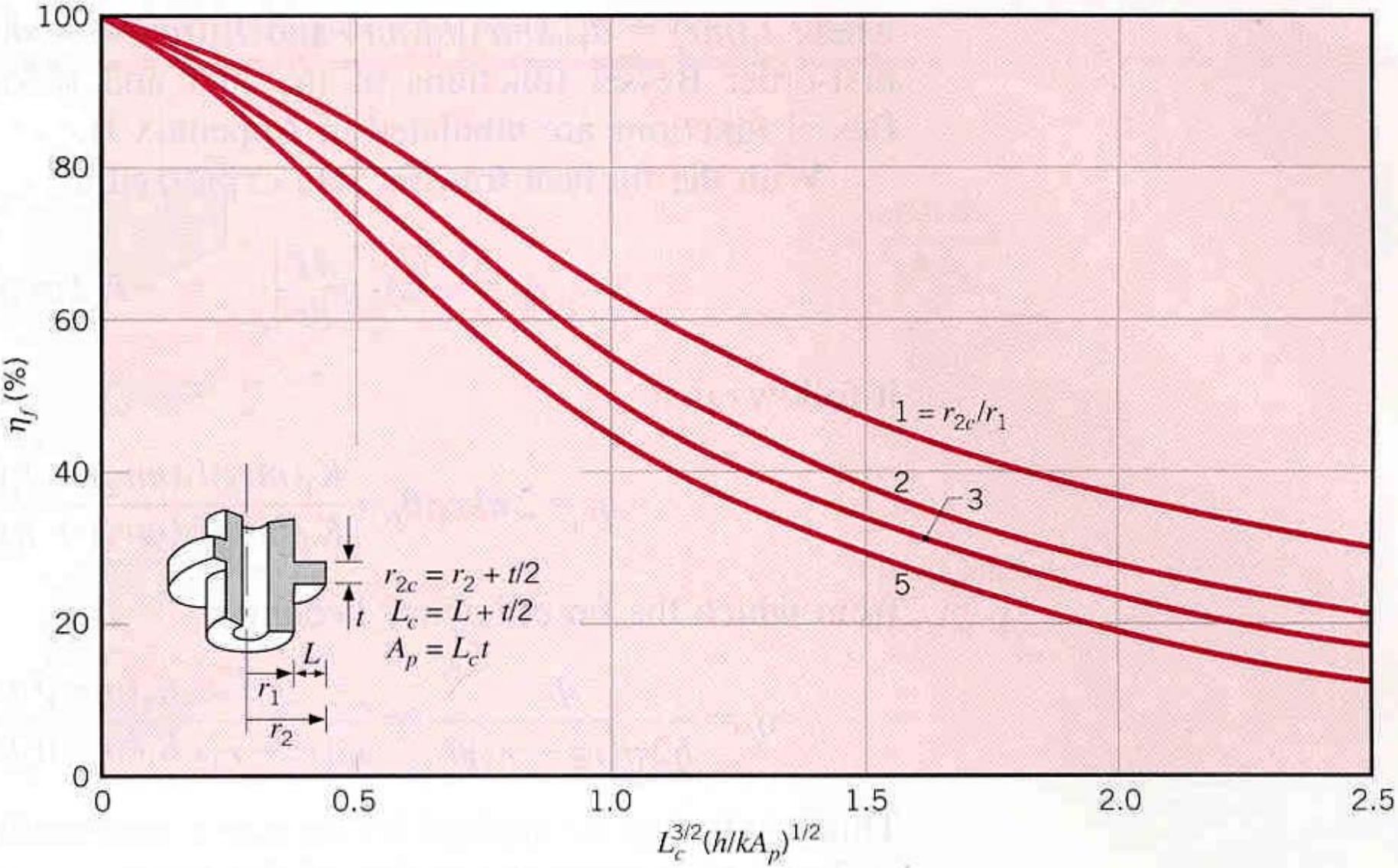
$$mL_c = \left(\frac{2h}{kt}\right)^{1/2} L_c = \left(\frac{2hL_c}{kA_p}\right)^{1/2} L_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2}$$

$$\eta_f = f\left(L_c^{3/2}\left(2h/kA_p\right)^{1/2}\right)$$

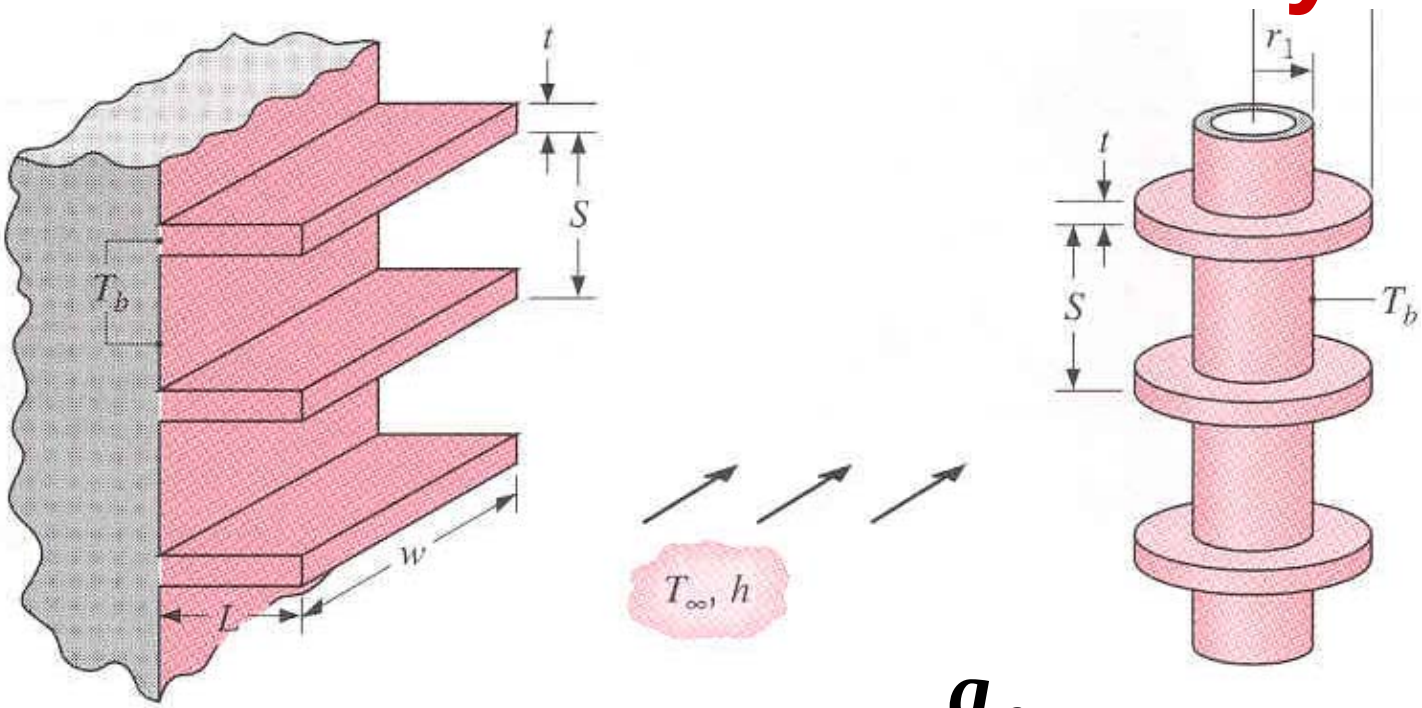
Efficiency of straight fins
(rectangular, triangular, and parabolic profile)



Efficiency of annular fins of rectangular profile



Overall Surface Efficiency



single fin efficiency: $\eta_f = \dfrac{q_f}{hA_f\theta_b}$

overall efficiency of array of fins: $\eta_o = \dfrac{q_t}{q_{\max}} = \dfrac{q_t}{hA_t\theta_b}$

A_t **: area of fins + exposed portion of the base** $A_t = NA_f + A_b$

$$q_t = Nq_f + hA_b\theta_b$$

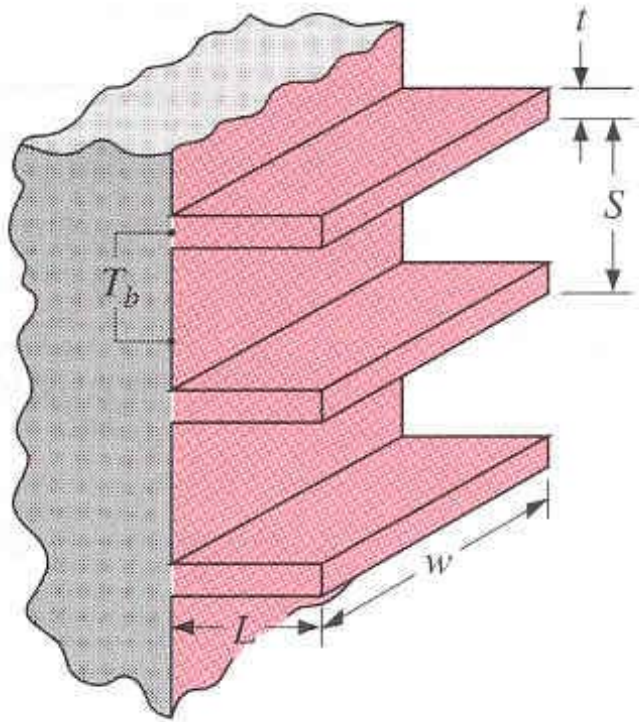
$$\left(\eta_f = \frac{q_f}{hA_f\theta_b}\right)$$

$$= N\eta_f hA_f\theta_b + hA_b\theta_b$$

$$= h\theta_b\left[N\eta_f A_f + \left(A_t - NA_f\right)\right]$$

$$= hA_t\theta_b\left[\frac{NA_f}{A_t}\eta_f + 1 - \frac{NA_f}{A_t}\right]$$

$$= hA_t\theta_b\left[1 - \frac{NA_f}{A_t}\left(1 - \eta_f\right)\right]$$

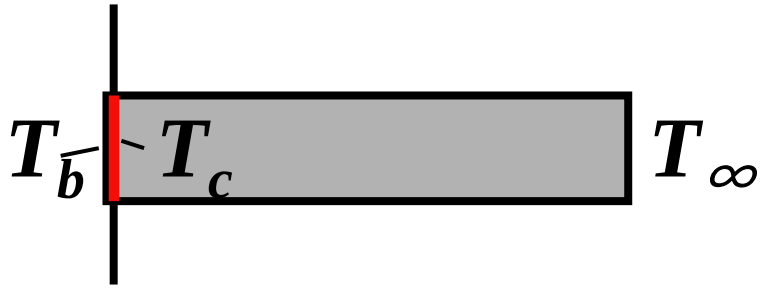


$$\eta_o = \frac{q_t}{hA_t\theta_b} = 1 - \frac{NA_f}{A_t}\left(1 - \eta_f\right)$$

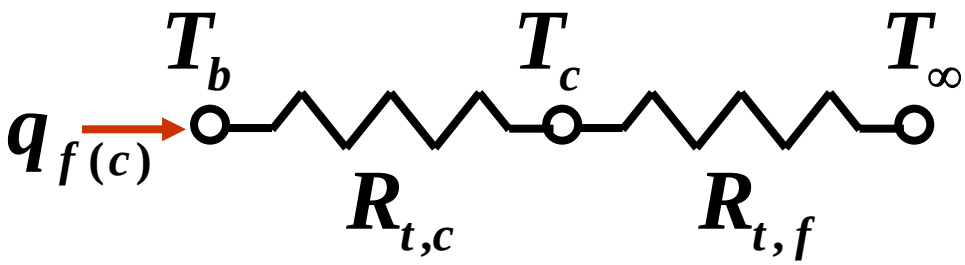
$$A_t = NA_f + A_b$$

In the case of press fit:
thermal contact resistance

$$\eta_{o(c)} = \frac{q_{t(c)}}{q_{\max}} = \frac{q_{t(c)}}{hA_t\theta_b}$$



$$q_{t(c)} = Nq_{f(c)} + hA_b\theta_b$$



$$q_{f(c)} = \frac{\theta_b}{R_{t,c} + R_{t,f}}$$

$$q''_{f(c)} = \frac{T_b - T_c}{R''_{t,c}} \rightarrow q_{f(c)} = q''_{f(c)}A_{c,b} = \frac{T_b - T_c}{R''_{t,c} / A_{c,b}}$$

$$\rightarrow R_{t,c} = R''_{t,c} / A_{c,b}$$

$$R_{t,f} = \frac{\theta_b}{q_f}, \quad \eta_f = \frac{q_f}{hA_f\theta_b} \rightarrow R_{t,f} = \frac{1}{\eta_f hA_f}$$

$$q_{f(c)} = \frac{\theta_b}{R''_{t,c}/A_{c,b} + 1/\eta_f hA_f}$$

$$q_{t(c)} = Nq_{f(c)} + hA_b\theta_b$$

$$= \frac{N\theta_b}{R''_{t,c}/A_{c,b} + 1/\eta_f hA_f} + hA_b\theta_b$$

$$= \frac{N\eta_f hA_f\theta_b}{1 + \eta_f hA_f R''_{t,c}/A_{c,b}} + hA_b\theta_b$$

$$= h\theta_b\left[\frac{N\eta_f A_f}{1 + \eta_f hA_f R''_{t,c}/A_{c,b}} + A_b\right]$$

let $1+\eta_f hA_f R''_{t,c} / A_{c,b} \equiv C_1$

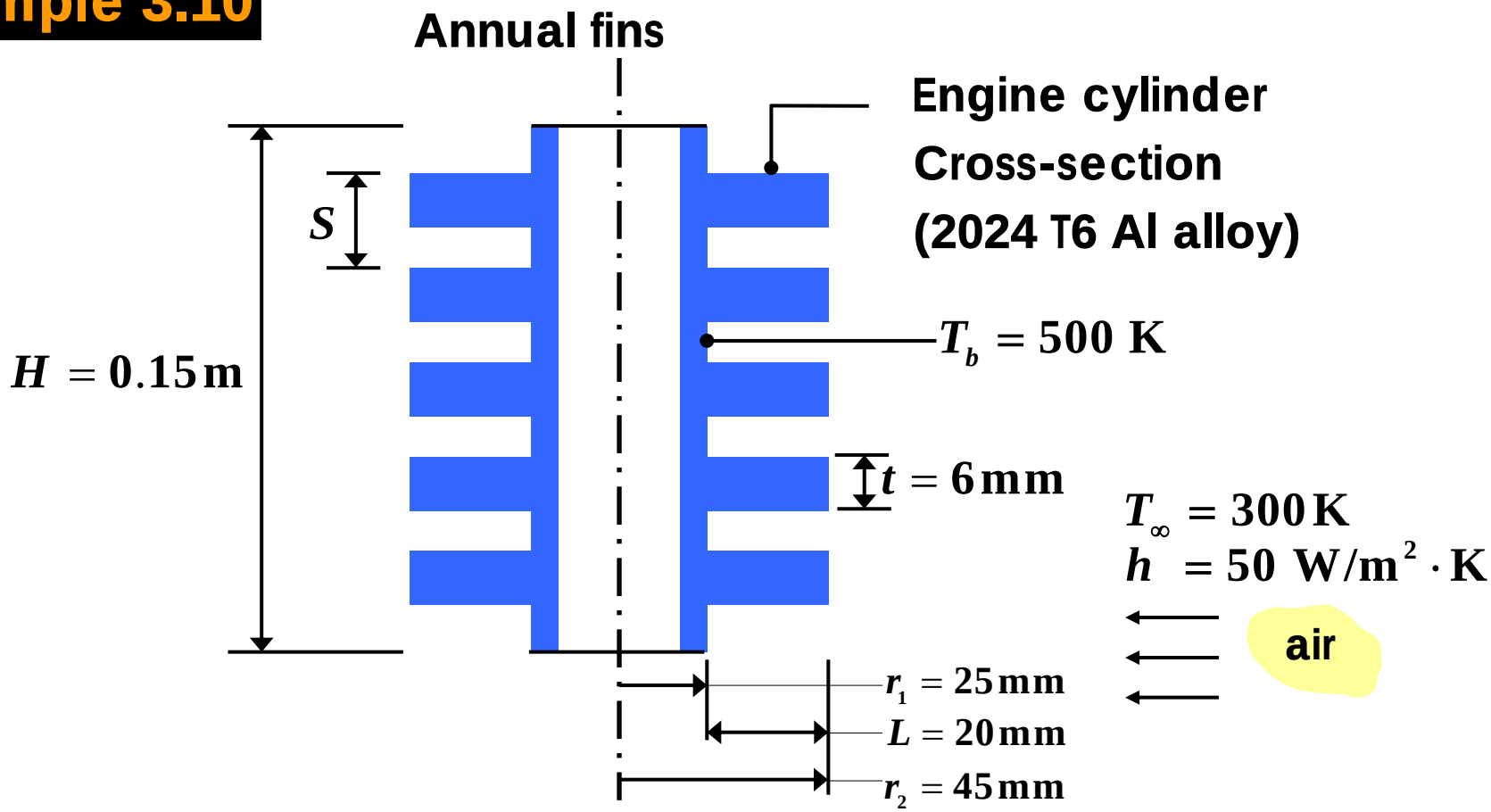
$$q_{t(c)} = h\theta_b \left[\frac{N\eta_f A_f}{C_1} + A_b \right]$$

$$= hA_t\theta_b \left[\frac{N\eta_f A_f}{C_1 A_t} + \frac{A_t - NA_f}{A_t} \right]$$

$$= hA_t\theta_b \left[1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1} \right) \right]$$

$$\eta_{o(c)} = \frac{q_{t(c)}}{q_{\max}} = \frac{q_{t(c)}}{hA_t\theta_b} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{\eta_f}{C_1} \right)$$

Example 3.10

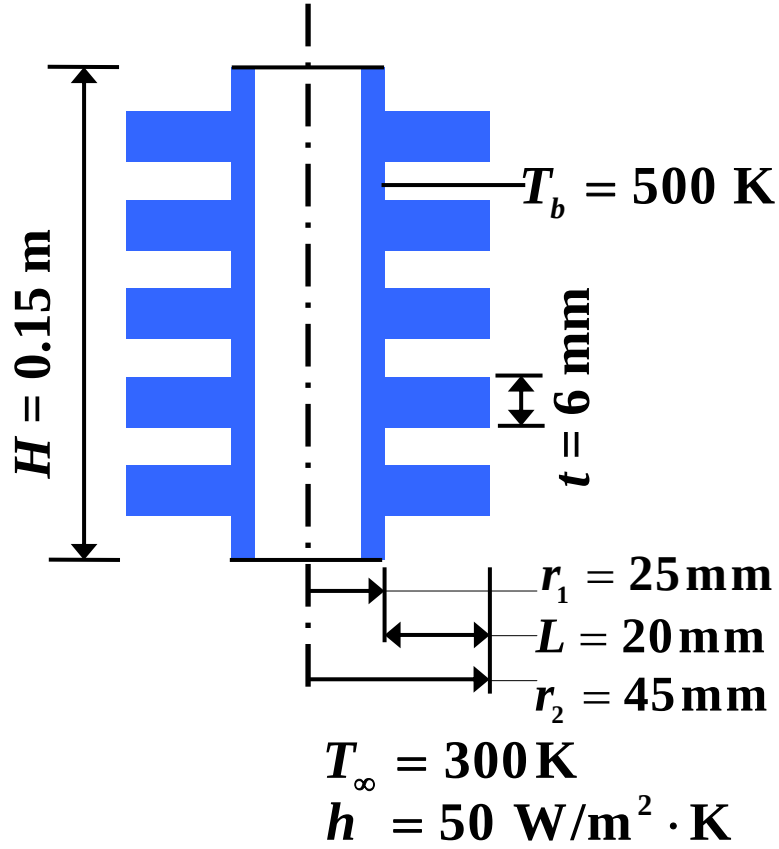


Find:

Increase in heat transfer, $\Delta q = q_t - q_{wo}$, associated with using fins

1) Heat transfer rate

η_f: known



$$q_t = hA_t\left[1-\frac{NA_f}{A_t}\left(1-\eta_f\right)\right]\theta_b$$

$$A_t = NA_f + A_b$$

$$= NA_f + 2\pi r_1\left(H - Nt\right)$$

$$A_f = 2\pi\left(r_{2c}^2 - r_1^2\right)$$

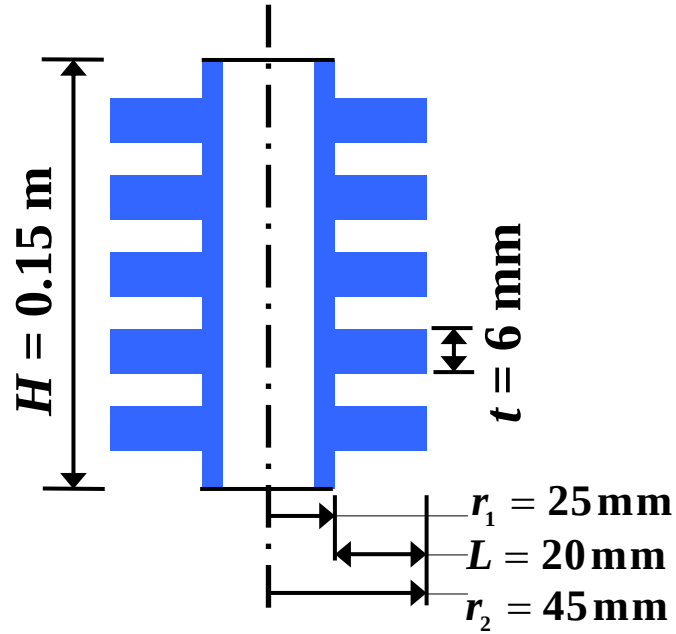
$$r_{2c} = r_2 + \frac{t}{2} = 0.045 + \frac{0.006}{2} = 0.048 \text{ m}$$

$$A_f = 2\pi\left(0.048^2 - 0.025^2\right) = 0.0105 \text{ m}^2$$

$$A_t = 5\times 0.0105 + 2\pi\times 0.025\left(0.15 - 5\times 0.006\right) = 0.0716 \text{ m}^2$$

To get η_f, use Fig. 3.19.

Parameters: $\dfrac{r_{2c}}{r_1}$, $L_c^{3/2}\left(h/kA_p\right)^{1/2}$



2024 T6 Al

$k = 186$ at 400 K

$$\frac{r_{2c}}{r_1} = \frac{0.048}{0.025} = 1.92$$

$$L_c = L + \frac{t}{2} = 0.02 + \frac{0.006}{2} = 0.023 \text{ m}$$

$$A_p = L_c t = 0.023 \times 0.006 = 1.38 \times 10^{-4} \text{ m}^2$$

$$L_c^{3/2}\left(h/kA_p\right)^{1/2} = \left(0.023\right)^{3/2}\left(\frac{50}{186 \times 1.38 \times 10^{-4}}\right)^{1/2} = 0.15$$

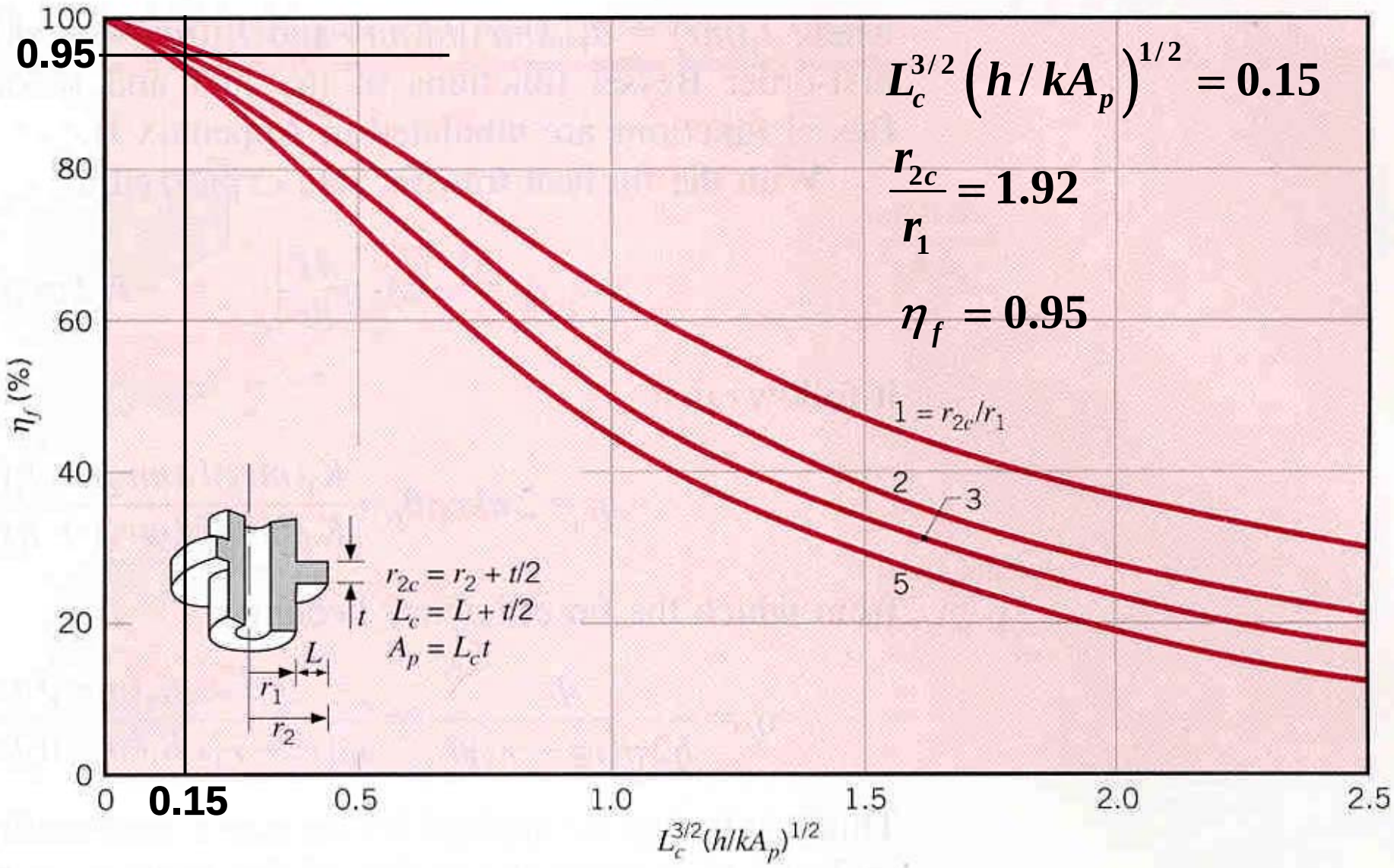


FIGURE 3.19 Efficiency of annular fins of rectangular profile.

$$q_t = hA_t\left[1-\frac{NA_f}{A_t}\left(1-\eta_f\right)\right]\theta_b$$

$$= 50\times 0.0716\left[1-\frac{5\times 0.0105}{0.0716}\left(1-0.95\right)\right]\left(500-300\right) = 690\ \text{W}$$

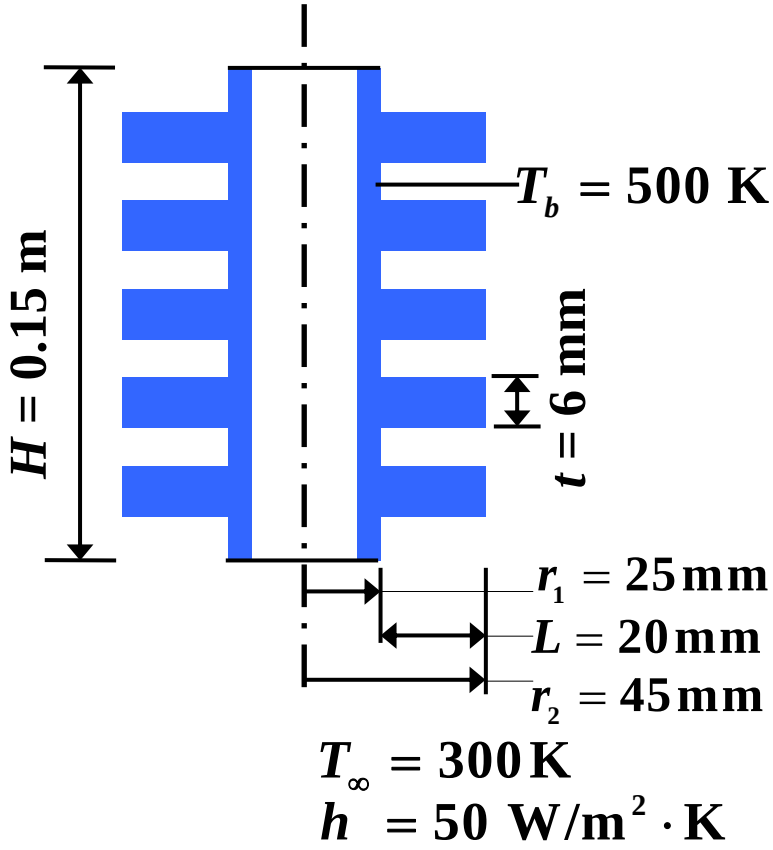
Without fins

$$q_{\text{wo}} = h\left(2\pi r_1 H\right)\theta_b = 236\ \text{W}$$

The amount of increase in heat transfer

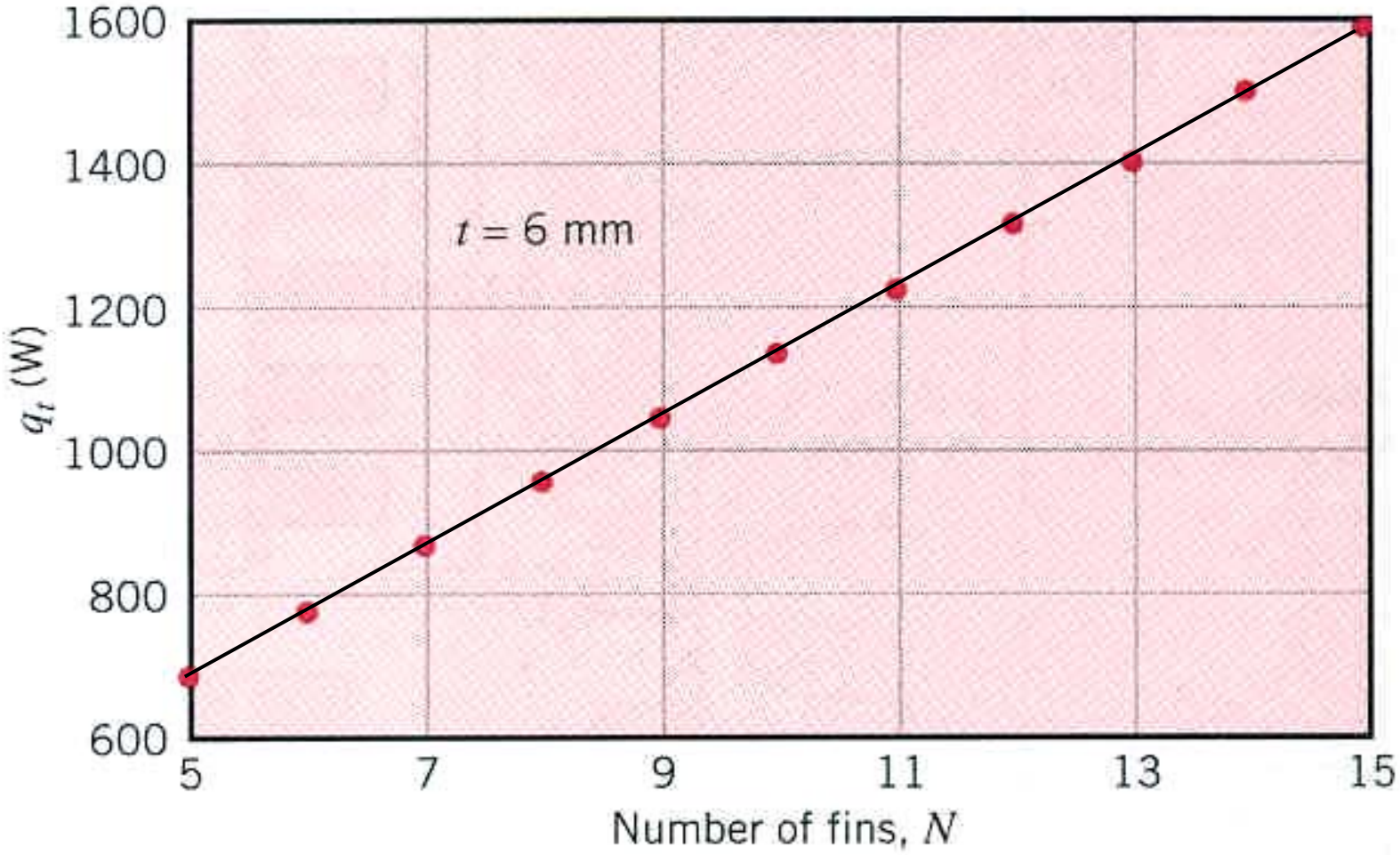
$$\Delta q = q_t - q_{\text{wo}} = 454\,\text{W}$$

$$\frac{q_t}{q_{\text{wo}}} = \frac{690}{236} = 2.67$$

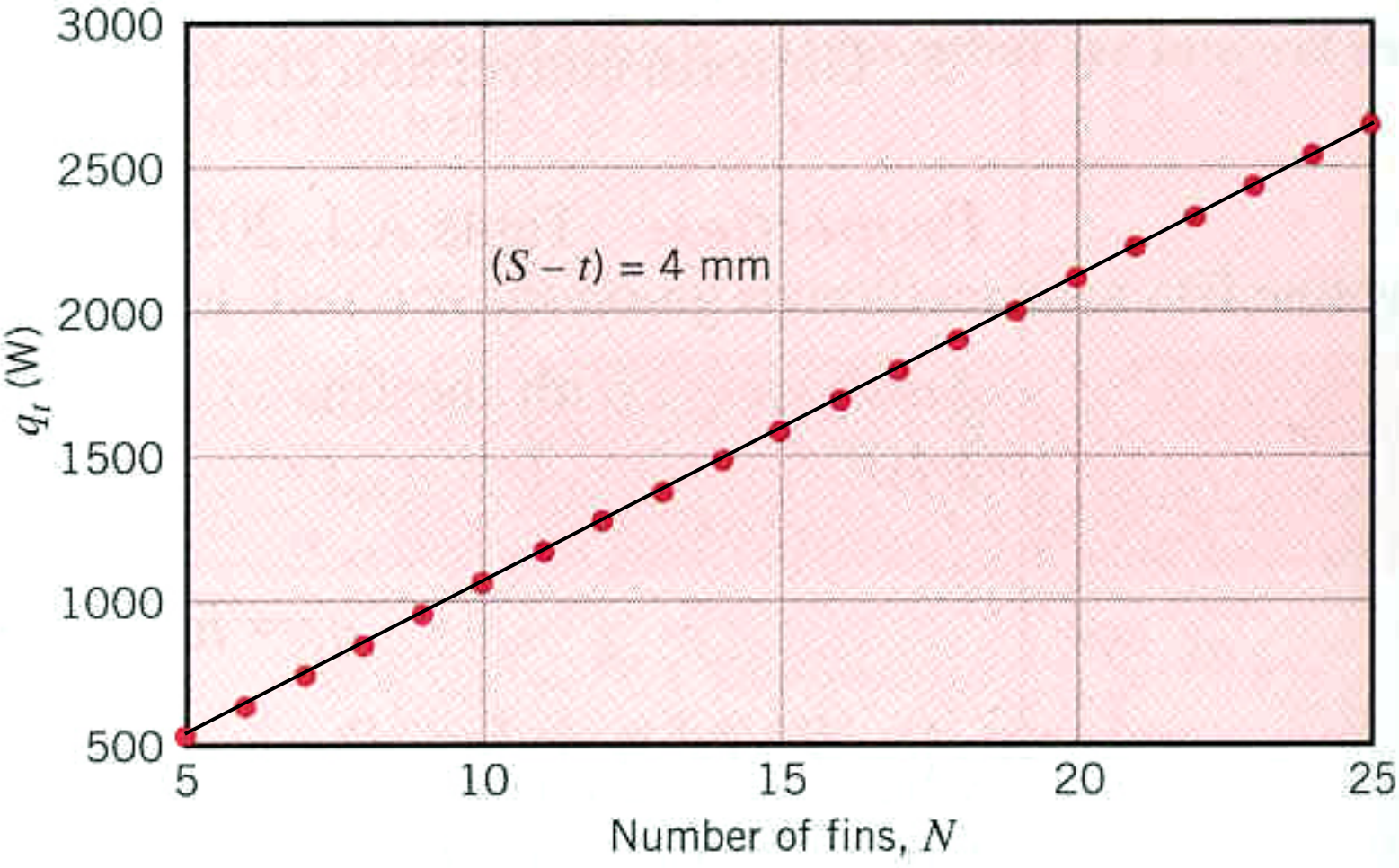


Comments:

Fixed fin thickness : 6 mm, Minimum fin gap : 4 mm ➔ $N_{max} = H/S = 15$

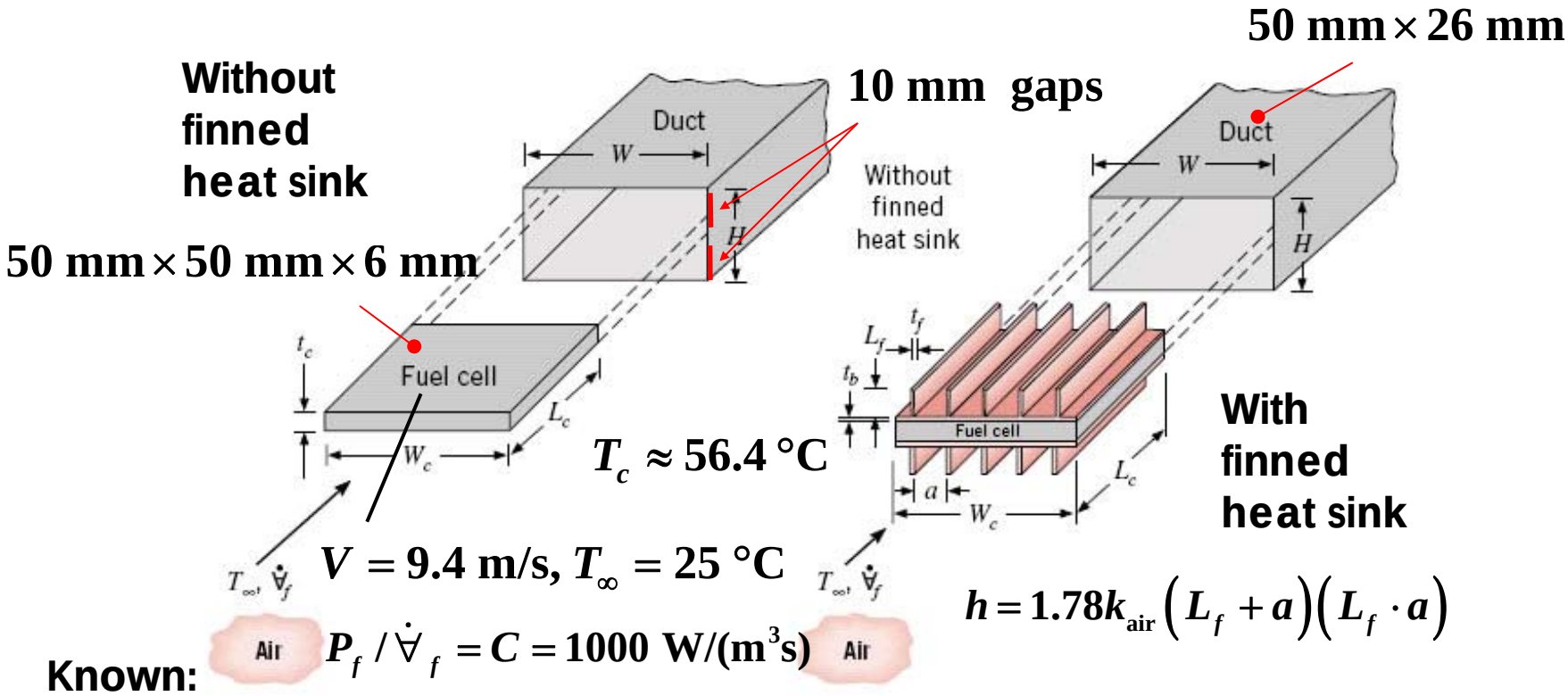


Fixed fin gap : 4 mm, Minimum fin thickness : 2 mm ➔ $N_{max} = H/S = 25$



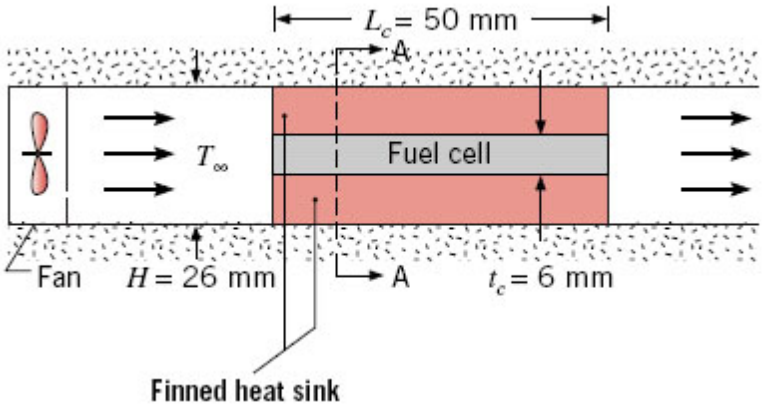
Example 3.11

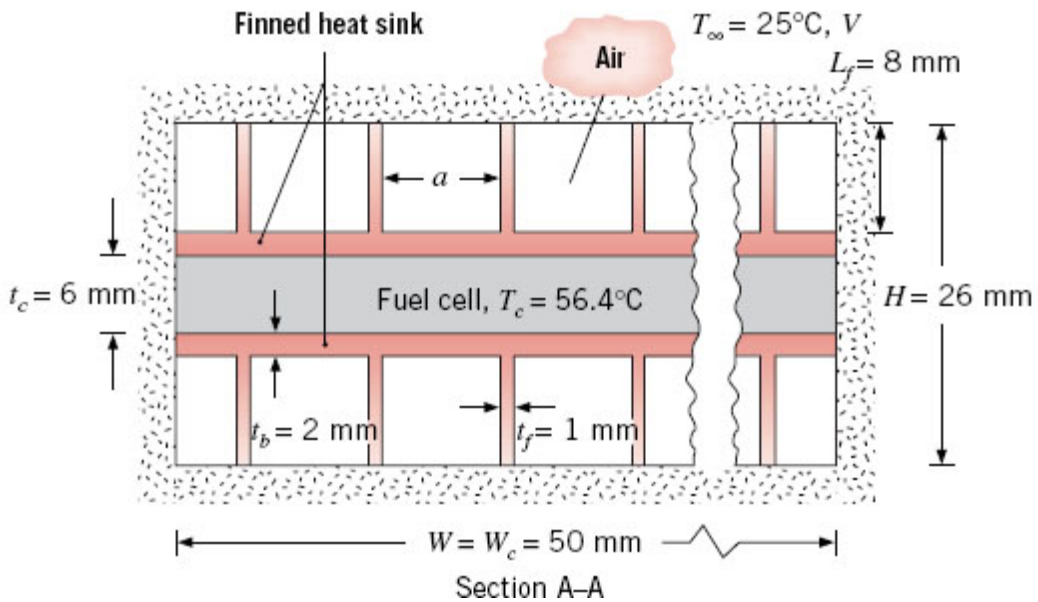
Hydrogen-air **Proton Exchange Membrane (PEM) fuel cell**



Known:

1) **Dimensions of a fuel cell and finned heat sink**
2) **Fuel cell operating temperature**
3) **Rate of thermal energy generation:** 11.25 W
4) **Power production:** $P = 9$ W
5) **Relationship between the convection coefficient and the air channel dimensions**



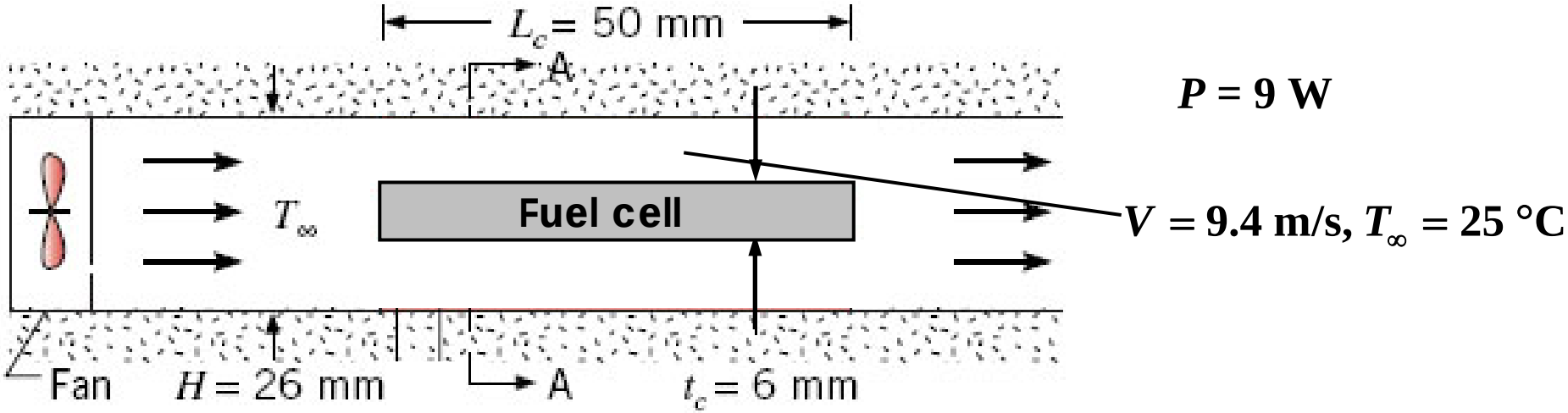


Find:

1) **Net power of the fuel cell-fan system for no heat sink, $P_{net} = P - P_f$**
2) **# of fins N needed to reduce the fan power consumption by 50%**

Assumptions:

1) **Steady-state conditions**
2) **Negligible heat transfer from the edges of the fuel cell, as well as from the front and back faces of the finned heat sink**
3) **1D heat transfer through the heat sink**
4) **Adiabatic fin tips**
5) **Negligible radiation when the heat sink is in place.**



1. $P_{net} = P - P_f = P - C\dot{\forall}_f$

Fan power consumption: $P_f / \dot{\forall}_f = C = 1000\ \text{W/(m}^3\text{s)}$

Volumetric flow rate of cooling air: $\dot{\forall}_f = VA_c, \quad A_c = W\left(H - t_c\right)$

$$\dot{\forall}_f = V\left[W\left(H - t_c\right)\right] = 9.4\,\text{m/s} \times \left[0.05\,\text{m} \times (0.026\,\text{m} - 0.006\,\text{m})\right]$$

$$= 9.4 \times 10^{-3}\,\text{m}^3/\text{s}$$

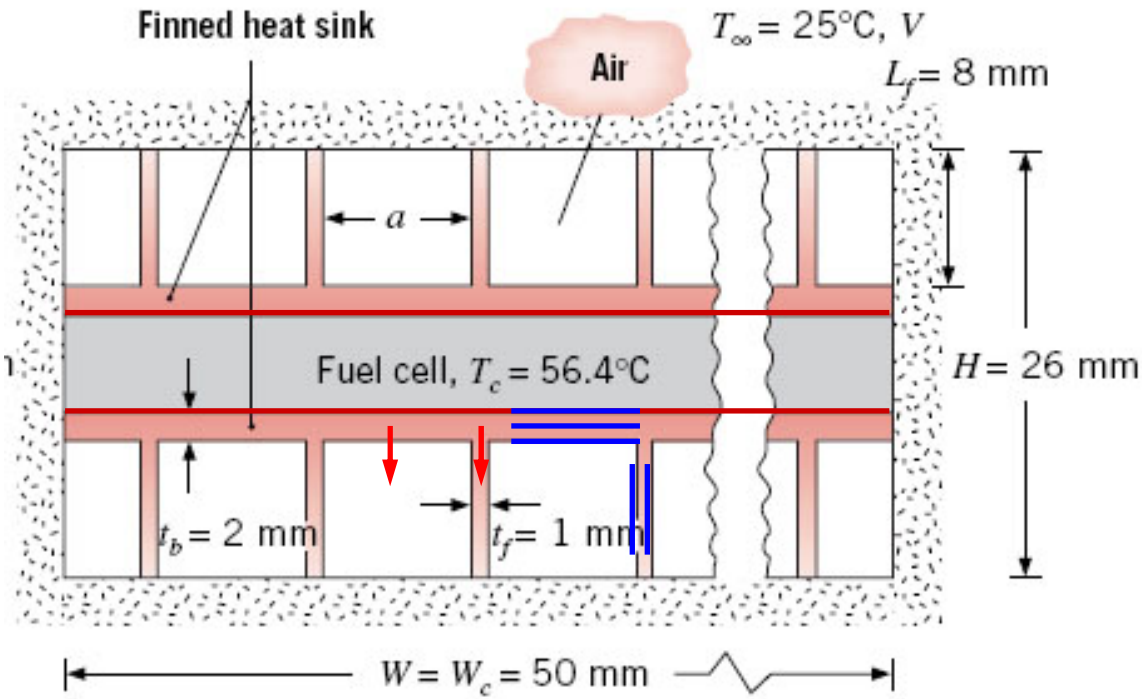
$$P_{net} = P - C\dot{\forall}_f = 9.0\,\text{W} - 1000\,\text{W/(m}^3\text{/s)} \times 9.4 \times 10^{-3}\,\text{m}^3/\text{s}$$

$$= 9.0\,\text{W} - 9.4\,\text{W} = -0.4\,\text{W}$$

The fan consumes more power than is generated by the fuel cell, and the system cannot produce net power.

2. To reduce the fan power consumption by 50%, $P_f = 9.4\ \mathrm{W}/2 = 4.7\ \mathrm{W}$

$$P_f / \dot{\forall}_f = C = 1000\ \mathrm{W/(m^3 s)} \rightarrow \dot{\forall}_f = 4.7\times10^{-3}\ \mathrm{m^3/s}$$



Aluminum fined sink:

k = 200 W/m·K

$$R''_{t,c} = 10^{-3}\ \mathrm{m^2 \cdot K/W}$$

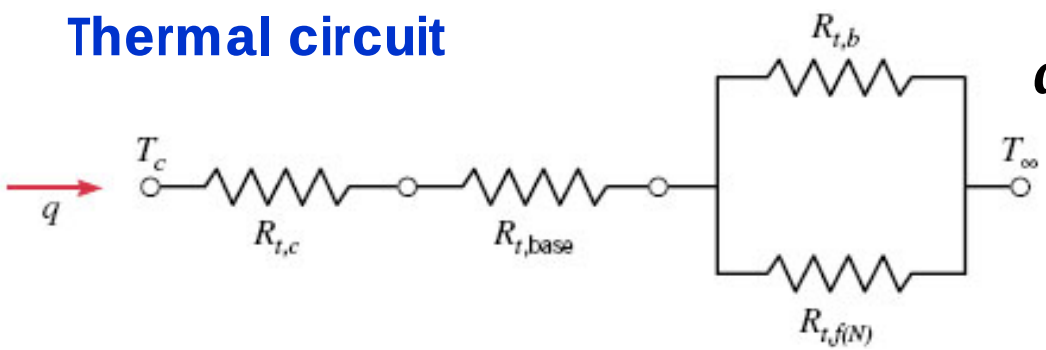
$$L_c = 50\ \mathrm{mm}$$

$$h = 1.78 k_{\mathrm{air}} \left(L_f + a\right)\left(L_f \cdot a\right)$$

contact joint + finned sink base + exposed base of finned side + fins

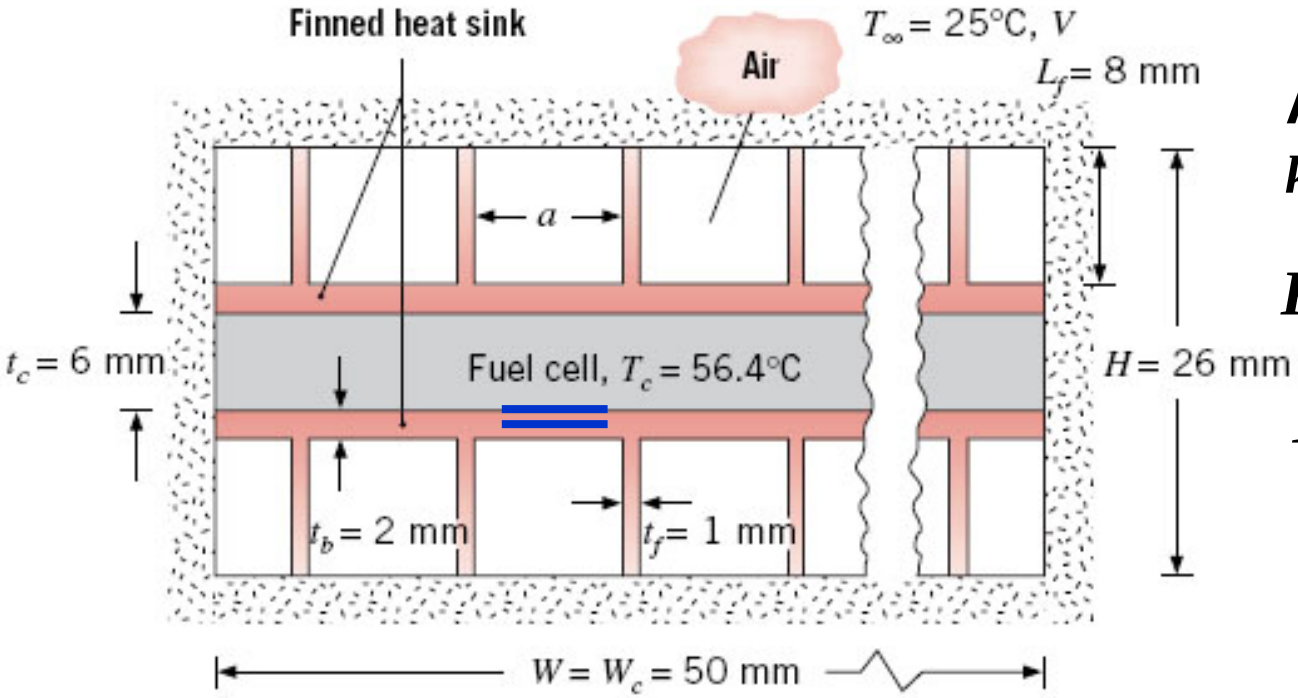
(contact) (conduction) (convection) (conv+cond)

Thermal circuit



$$q = \frac{T_c - T_\infty}{R_{t,c} + R_{t,\mathrm{base}} + R_{\mathrm{equiv}}} = 11.25\ \mathrm{W}$$

$$R_{\mathrm{equiv}} = \left[R_{t,b}^{-1} + R_{t,f(N)}^{-1}\right]^{-1}$$



Aluminum fined sink:

$k = 200$ W/m·K

$$R''_{t,c} = 10^{-3}\ \mathrm{m}^2 \cdot \mathrm{K/W}$$

$$L_c = 50\ \mathrm{mm}$$

$R_{t,c}$: $$q'' = \frac{T_c - T_b}{R''_{t,c}},\quad q = Aq'' = \frac{T_c - T_b}{R''_{t,c}/A} = \frac{T_c - T_b}{R_{t,c}}$$

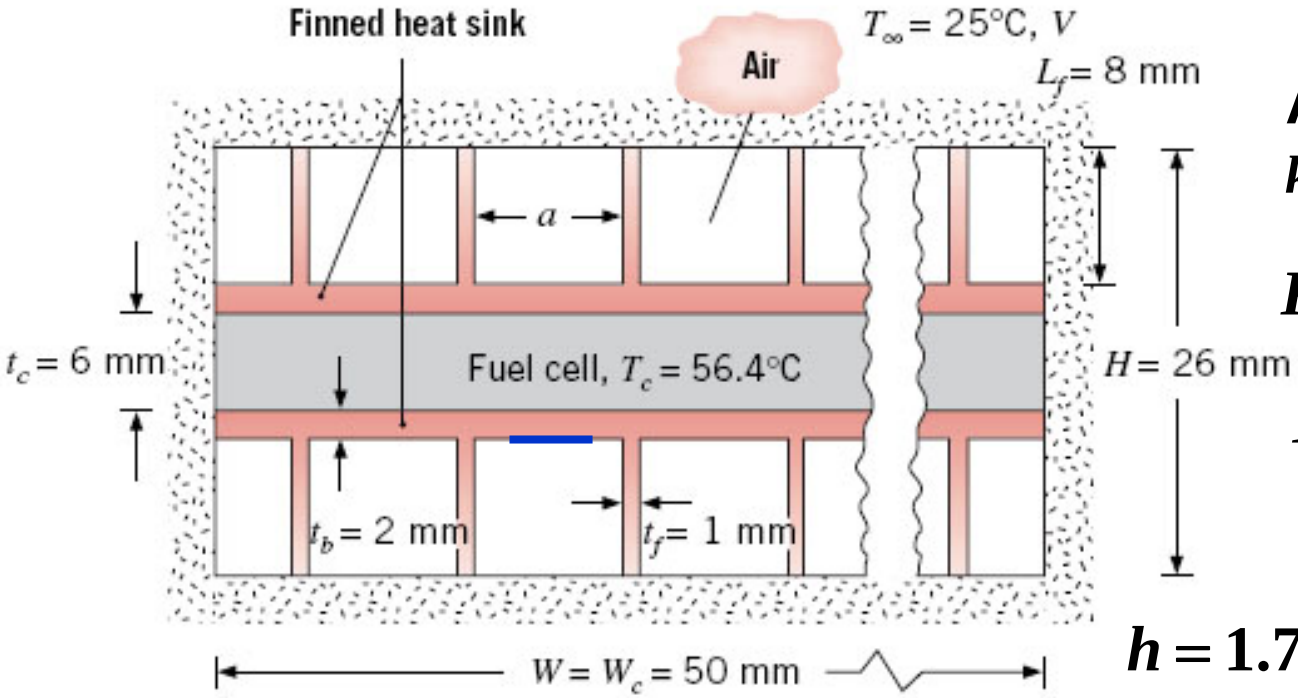
$$R_{t,c} = R''_{t,c}/A = R''_{t,c}/2L_cW_c$$

: 2 sides of the heat sink assembly

$$= (10^{-3}\mathrm{m}^2 \cdot \mathrm{K/W})/(2 \times 0.05\,\mathrm{m} \times 0.05\,\mathrm{m}) = 0.2\,\mathrm{K/W}$$

$R_{t,\mathrm{base}}$: $$R_{t,\mathrm{base}} = L/kA = t_b/(2kL_cW_c)$$

$$= (0.02\,\mathrm{m})/(2 \times 200\,\mathrm{W/m \cdot K} \times 0.05\,\mathrm{m} \times 0.05\,\mathrm{m}) = 0.002\,\mathrm{K/W}$$



Aluminum fined sink:

$k = 200$ W/m·K

$$R''_{t,c} = 10^{-3}\ \mathrm{m}^2 \cdot \mathrm{K/W}$$

$$L_c = 50\ \mathrm{mm}$$

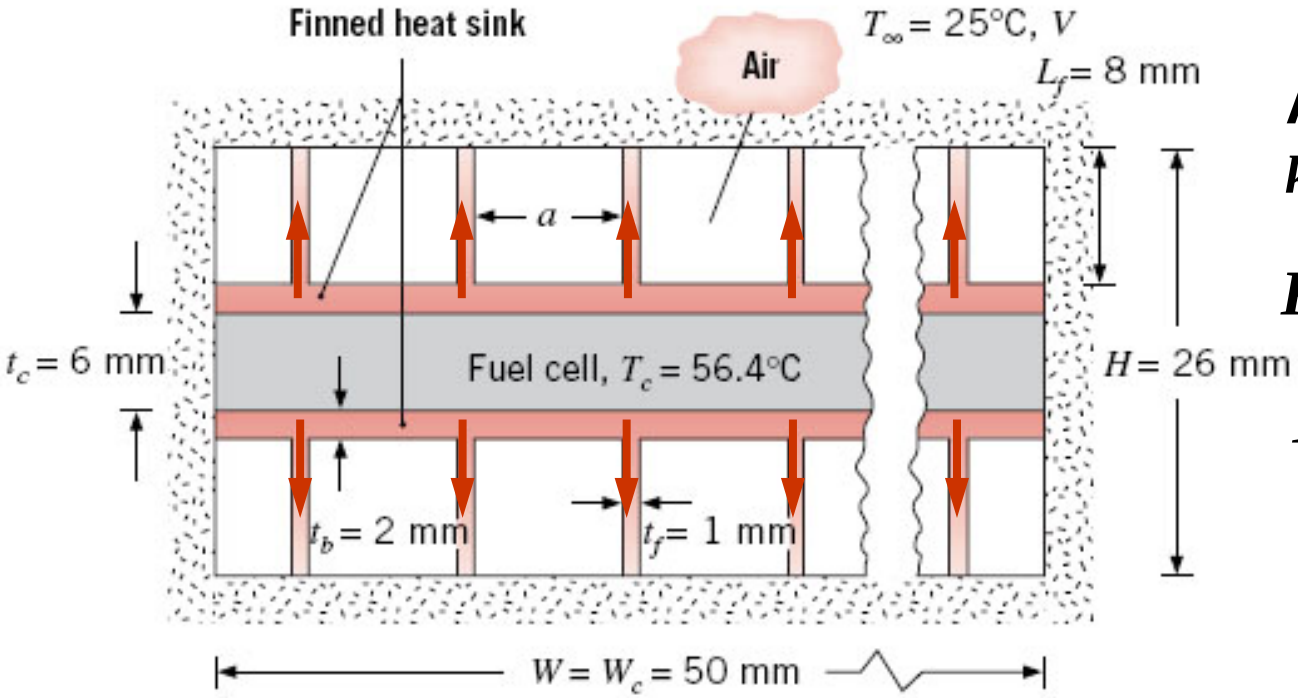
$$h = 1.78 k_{\mathrm{air}} \left(L_f + a\right)\left(L_f \cdot a\right)$$

$R_{t,b}$: $$R_{t,b} = \frac{1}{hA} = \frac{1}{h\left(2W_c - Nt_f\right)L_c}$$

$$= \frac{1}{h \times (2 \times 0.05\,\mathrm{m} - N \times 0.001\,\mathrm{m}) \times 0.05\ \mathrm{m}}$$

$$= \frac{1}{h\left(0.005 - 0.0005\,N\right)}$$

→ N, h are needed to evaluate $R_{t,b}$



Aluminum fined sink:

$k = 200$ W/m·K

$$R''_{t,c} = 10^{-3}\ \mathrm{m}^2 \cdot \mathrm{K/W}$$

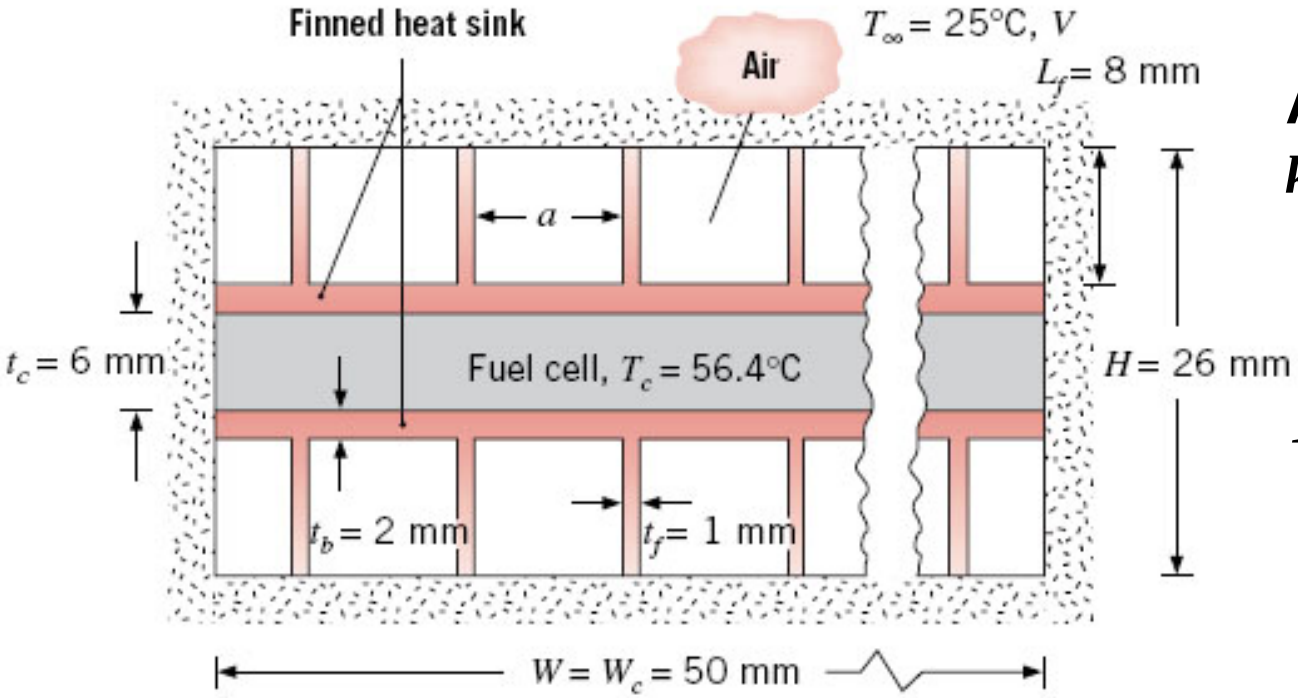
$$L_c = 50\ \mathrm{mm}$$

$$R_{t,f(N)}: \quad q_{f(N)} = \frac{T_b - T_\infty}{R_{t,f(N)}} = N q_f = \frac{N\left(T_b - T_\infty\right)}{R_{t,f}} \rightarrow R_{t,f(N)} = \frac{R_{t,f}}{N}$$

For a single fin, $R_{t,f} = \theta_b / q_f$

For a fin with an insulated fin tip, $q_f = \sqrt{hPkA_c}\,\theta_b \times \tanh mL_f$

$$R_{t,f} = \frac{\theta_b}{q_f} = \frac{1}{\sqrt{hPkA_c} \times \tanh mL_f}, \qquad m = \sqrt{\frac{hP}{kA_c}}$$



Aluminum fined sink:

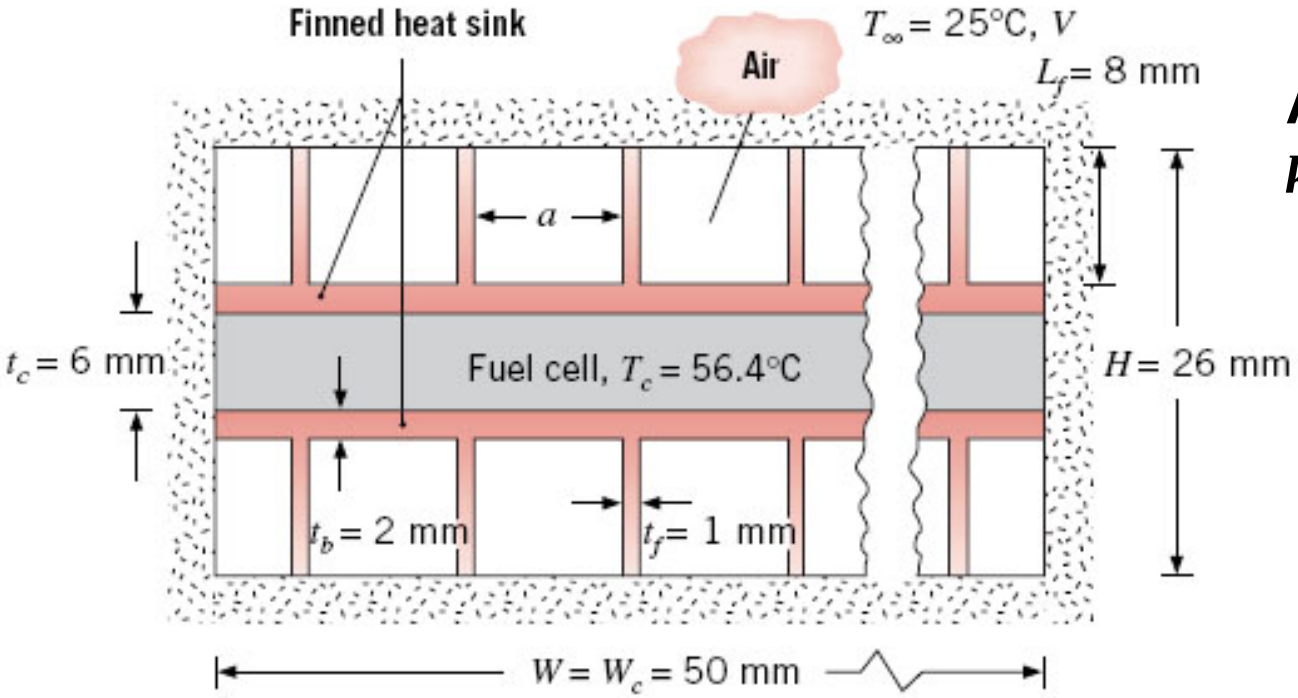
$k = 200$ W/m·K

$L_c = 50$ mm

$$P = 2\left(L_c + t_f\right) = 2\times(0.05\,\text{m} + 0.001\,\text{m}) = 0.102\,\text{m}$$

$$A_c = L_c t_f = 0.05\,\text{m}\times 0.001\,\text{m} = 0.00005\,\text{m}^2$$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h\times 0.102\,\text{m}}{200\,\text{W/m}\cdot\text{K}\times 0.00005\,\text{m}^2}} = \sqrt{10.2h}$$



Aluminum fined sink:

$k = 200$ W/m·K

$$R_{t,f} = \frac{1}{\sqrt{hPkA_c} \times \tanh mL_f}$$

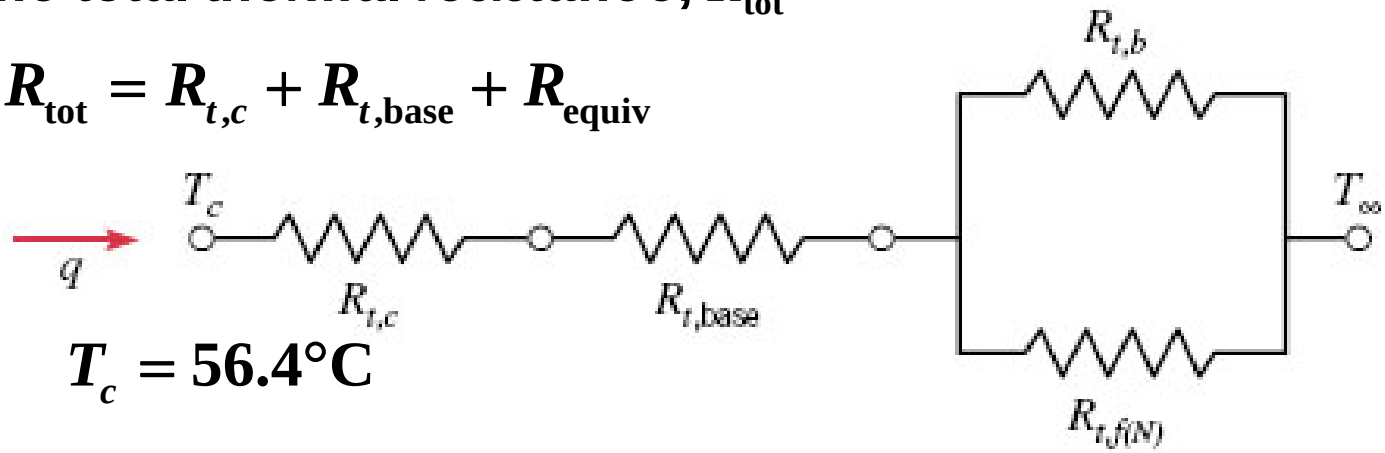
$$= \frac{1}{\sqrt{h \times 0.102\,\text{m} \times 200\,\text{W/m}\cdot\text{K} \times 0.00005\,\text{m}^2} \times \tanh(\sqrt{10.2h} \times 0.008\,\text{m})}$$

For N fins, $$R_{t,f(N)} = \frac{R_{t,f}}{N} = \frac{1}{N\sqrt{0.00102h}\,\tanh(0.008\sqrt{10.2h})}$$

→ N, h are also needed to evaluate $R_{t,f}$

The total thermal resistance, R_{tot}

$$R_{tot} = R_{t,c} + R_{t,base} + R_{equiv}$$



$$T_\infty = 25°C$$

$$T_c = 56.4°C$$

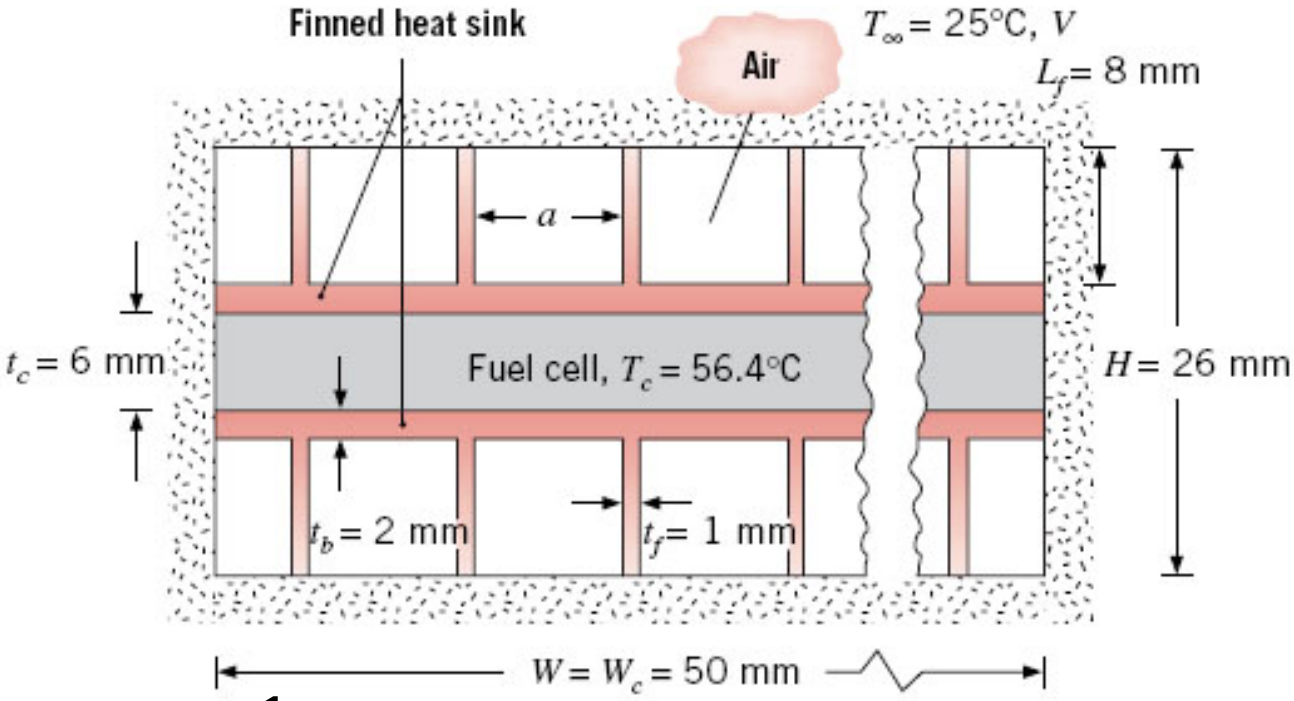
The equivalent fin resistance, R_{eqiuv}

$$R_{equiv} = \left[R_{t,b}^{-1} + R_{t,f(N)}^{-1} \right]^{-1}$$

$$q = \frac{T_c - T_\infty}{R_{tot}} = \frac{T_c - T_\infty}{R_{t,c} + R_{t,base} + R_{equiv}} = 11.25\ \text{W}$$

$$R_{equiv} = \frac{T_c - T_\infty}{q} - \left(R_{t,c} + R_{t,base} \right)$$

$$= \frac{56.4°C - 25°C}{11.25\,\text{W}} - (0.2 + 0.02)\,\text{K/W} = 2.59\,\text{K/W}$$

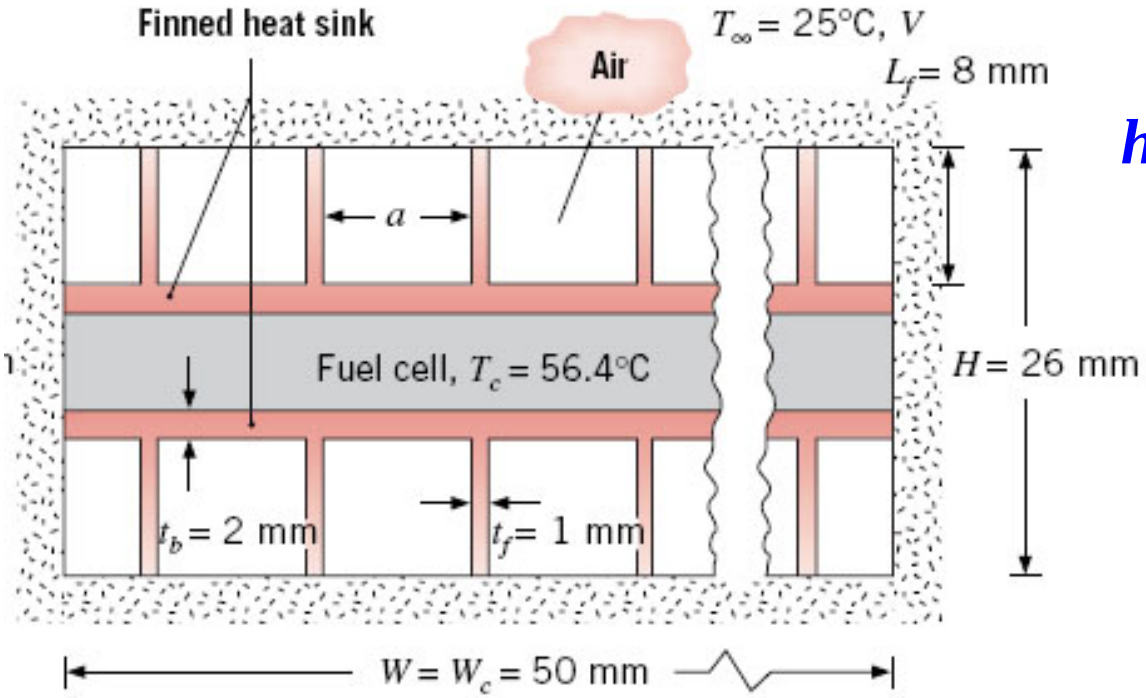


$$\frac{1}{R_{t,b}^{-1} + R_{t,f(N)}^{-1}} = 2.59\,\text{K/W}$$

$$R_{t,b} = \frac{1}{h\left(0.005 - 0.0005\,N\right)}, \quad R_{t,f(N)} = \frac{1}{N\sqrt{0.00102h}\,\tanh(0.008\sqrt{10.2h})}$$

$$h = 1.78k_{\text{air}}\left(L_f + a\right)\left(L_f \cdot a\right)$$

$$a = \frac{2W_c - Nt_f}{N} = \frac{2\times 0.05\,\text{m} - N\times 0.001\,\text{m}}{N} = \frac{0.1 - 0.001N}{N}$$



$$h = 1.78 k_{air}\left(L_f + a\right)\left(L_f \cdot a\right)$$

Properties: Table A.4. air

$$(\bar{T} = 300\,\mathrm{K})$$

$$k_{air} = 0.0263\,\mathrm{W/m \cdot K}$$

$$h = 1.78 \times 0.0263\left(0.08 + \frac{0.1 - 0.001N}{N}\right)\left(0.08 \times \frac{0.1 - 0.001N}{N}\right)$$

$$\frac{1}{R_{t,b}^{-1} + R_{t,f(N)}^{-1}} = 2.59\,\mathrm{K/W}$$

$$R_{t,b} = \frac{1}{h\left(0.005 - 0.0005N\right)}$$

$$R_{t,f(N)} = \frac{1}{N\sqrt{0.00102h}\,\tanh(0.008\sqrt{10.2h})}$$

For $N = 22$, 11 on top and 11 on the bottom,

$$a = 0.0035\,\text{m},\ h = 19.1\,\text{W/m}^2\cdot\text{K},\ m = 13.9\,\text{m}^{-1}$$

$$R_{t,f(N)} = 2.94\,\text{K/W},\ R_{t,b} = 13.5\,\text{K/W}$$

$$R_{\text{equiv}} = 2.41\,\text{K/W},\ R_{\text{tot}} = 2.61\,\text{K/W}$$

$$T_c = 54.4°\text{C} \quad (T_c \approx 56.4\,°\text{C})$$

$$P_{\text{net}} = P - P_f = 9.0\,\text{W} - 4.7\,\text{W} = 4.3\,\text{W}$$

For $N = 20$, $T_c = 58.9°\text{C}$

For $N = 24$, $T_c = 50.7\,°\text{C}$