STEADY-STATE, MULTI-DIMENSIONAL CONDUCTION

- **Analytical Method:**
  - **Separation of Variables**
- **Conduction Factor and Dimensionless Conduction Heat Rate**
- **Numerical Method:**
  - **Finite Difference Method**
  - **Finite Volume Method**

Analytical Method

Separation of Variables

$$\nabla^2 T(x,y) + \frac{\dot{q}}{k} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{k} = 0$$

+ Boundary conditions

1) Equation : linear and homogeneous

2) Boundary condition : homogeneous at least in one direction

⇓

Eigenvalue problem in the direction of homogeneous boundary condition

Sturm-Liouville Theory

Sturm-Liouville System : Hermitian operator
(self-adjoint operator)

$$L \equiv \frac{1}{w(x)}\left\{\frac{d}{dx}\left[P(x)\frac{d}{dx}\right]+R(x)\right\}, \quad w(x)>0 \quad \textbf{over} \quad (a<x<b)$$

$w(x)$ **: weighting function**

Corresponding eigenvalue problem

$$Lu(x)+\lambda u(x)=\frac{1}{w(x)}\left\{\left(P(x)u'(x)\right)'+R(x)u(x)\right\}+\lambda u(x)=0$$

or $$\left(P(x)u'(x)\right)'+R(x)u(x)+\lambda w(x)u(x)=0$$

boundary conditions : $\alpha u(a)+\beta u'(a)=0,\ \gamma u(b)+\delta u'(b)=0$

Definition of inner product : $\left(f(x), g(x)\right) = \int_a^b w(x) f(x) g(x) dx$

$\left(Lu, v\right) =$ **boundary terms** $+\left(L^* v, u\right)$

$$Lu(x) = \frac{1}{w(x)}\left\{\left(P(x)u'(x)\right)' + R(x)u(x)\right\}$$

$$\left(Lu, v\right) = \int_a^b \left[\left(Pu'\right)' + Ru\right] v dx = \left[Pu'v\right]_a^b - \int_a^b Pu'v' dx + \int_a^b Ruvdx$$

$$= \left[Pu'v\right]_a^b - \left\{\left[Pv'u\right]_a^b - \int_a^b \left(Pv'\right)' udx\right\} + \int_a^b Ruvdx$$

$$= \left[Pu'v - Puv'\right]_a^b + \int_a^b u\left[\left(Pv'\right)' + Rv\right] dx$$

= **boundary terms** $+\left(L^* v, u\right)$ **thus** $L^* = L$

boundary terms

$$P(b)u'(b)v(b) - P(b)u(b)v'(b) - P(a)u'(a)v(a) + P(a)u(a)v'(a)$$

$$\alpha u(a) + \beta u'(a) = 0,\ \gamma u(b) + \delta u'(b) = 0$$

$$= P(a)u(a)\frac{1}{\beta}\left[\alpha v(a) + \beta v'(a)\right] - P(b)u(b)\frac{1}{\delta}\left[\gamma v(b) + \delta v'(b)\right]$$

i) $\alpha v(a) + \beta v'(a) = 0,\ \gamma v(b) + \delta v'(b) = 0$

homogeneous boundary conditions $\left(Lu, v\right) = \left(L^{*}v, u\right)$

ii) **when** $P(a) = 0$ **or** $P(b) = 0,\ P(a) = P(b) = 0,$

u **and** u' **can be finite.**

iii) **when** $P(a) = P(b),\ u = u'$ **either at** a **or** b

Ex) $L = \frac{d}{dx}$

$$(Lu, v) = \int_a^b u'v\,dx = [uv]_a^b - \int_a^b uv'\,dx = [uv]_a^b + \int_a^b u(-v')\,dx$$

$$= \textbf{boundary terms} + \int_a^b u\left(-\frac{dv}{dx}\right)dx$$

$$L^* = -\frac{d}{dx} \neq L$$

$$L = \frac{d^2}{dx^2}, \qquad u(a) = 0, u(b) = 0$$

$$(Lu, v) = \int_a^b u''v\,dx = [u'v]_a^b - \int_a^b u'v'\,dx = [u'v]_a^b - \left\{[uv']_a^b - \int_a^b uv''\,dx\right\}$$

$$= u'(b)v(b) - u'(a)v(a) - u(b)v'(b) + u(a)v'(a) + \int_a^b uv''\,dx$$

$$L^* = \frac{d^2}{dx^2} = L$$

When $v(a) = 0, v(b) = 0,\ (Lu, v) = (L^*v, u)$

$L = \frac{d^2}{dx^2}$ **: self-adjoint operator**

Properties of Hermitian operator

1. **eigenvalues : all real**
2. **eigenfunctions : orthogonal**
3. **eigenfunctions : a complete set**

$\phi(x)$ **: eigenfunction**

orthogonality: $(\phi_m, \phi_n) = \int_a^b w\phi_m\phi_n dx = 0$ **when** $m \neq n$

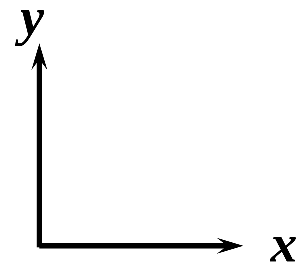
$f(x)$ **: at least piecewise continuous function**

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x), \quad c_n = \frac{\int_a^b w(x) f(x)\phi_n(x)dx}{\int_a^b w(x)\phi_n^2(x)dx}$$

$\lim_{n\to\infty}\int_a^b \left[f(x) - \sum_{n=0}^{\infty} c_n\phi_n(x)\right]^2 w(x)dx = 0$ **: least square convergence**

Example : Laplace equation $\nabla^2 u = 0$

1) in (x,y) coordinate system (Cartesian)



$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with homogeneous boundary conditions in the x direction

assume $u(x,y) = X(x)Y(y)$, **then** $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \quad X'' + \lambda X = 0$

This is a special case of S-L system with

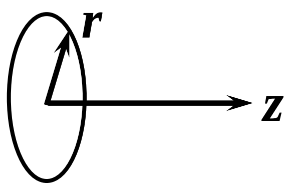
$$w(x) = 1,\ \ P(x) = 1,\ \ R(x) = 0$$

$$Lu(x) + \lambda u(x) = \frac{1}{w(x)}\left\{\left(P(x)u'(x)\right)' + R(x)u(x)\right\} + \lambda u(x) = 0$$

eigenfunctions : trigonometric functions, $\sin x$ **or** $\cos x$

Ex) $\int_a^b \sin nx \sin mx\, dx = 0, \quad n \neq m$

2) in (r,z) coordinate system (cylindrical)



$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0$$

with homogeneous boundary conditions in the r direction

assume $u(r,z) = F(r)G(z)$, then $\frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = -\frac{G''}{G} = -\lambda$

$$F'' + \frac{1}{r}F' + \lambda F = 0 \quad \text{or} \quad r^2F'' + rF' + \lambda r^2 F = 0 \quad \text{or} \quad \frac{1}{r}\left(rF'\right)' + \lambda F = 0$$

This is a special case of S-L system with

$$w(x) = r,\ \ P(x) = r,\ \ R(x) = 0$$

$$Lu(x) + \lambda u(x) = \frac{1}{w(x)}\left\{\left(P(x)u'(x)\right)' + R(x)u(x)\right\} + \lambda u(x) = 0$$

Remark : transform with $\lambda r = x$

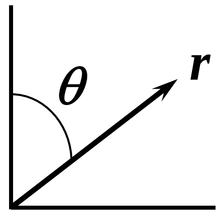
$$x^2F'' + xF' + x^2F = 0$$

Bessel equation : $x^2y'' + xy' + (x^2 - v^2)y = 0$

eigenfunctions : Bessel functions   Ex) $\int_a^b xJ(nx)J(mx)dx = 0,\ \ n \neq m$

3) in (r,θ) coordinate system (spherical)

$$\nabla^2 u(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0$$



with homogeneous boundary conditions in the θ direction

assume $u(r,\theta) = H(r)\Theta(\theta)$

equation for $\Theta : \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \lambda\Theta = 0$

or with $x = \sin\theta$, $\frac{d}{dx}\left[\left(1-x^2\right)\frac{d\Theta}{dx}\right] + \lambda\Theta = 0, \ |x| \le 1$

or $\left(1-x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0$

This is a special case of S-L system with

$w(x) = 1, \ P(x) = 1 - x^2, \ R(x) = 0$

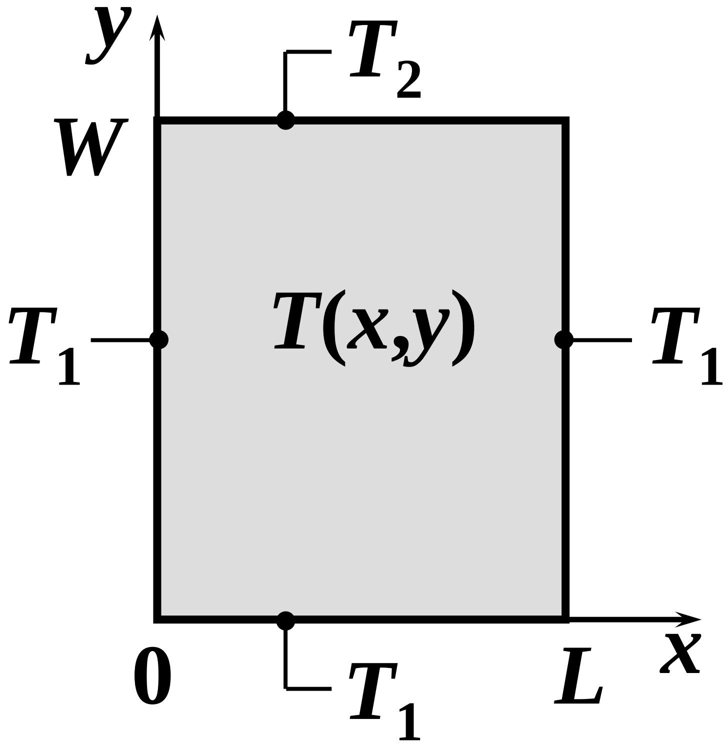
Remark Legendre equation : $\left(1-x^2\right)y'' - 2xy' + n\left(n+1\right)y = 0$

eigenfunctions : Legendre functions

Two dimensional conduction

in a thin rectangular plate or a long rectangular rod

with no heat generation



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

b.c. $T(0, y) = T_1, \quad T(L, y) = T_1$

$T(x, 0) = T_1, \quad T(x, W) = T_2$

Let $\theta(x, y) = \dfrac{T(x, y) - T_1}{T_2 - T_1}$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

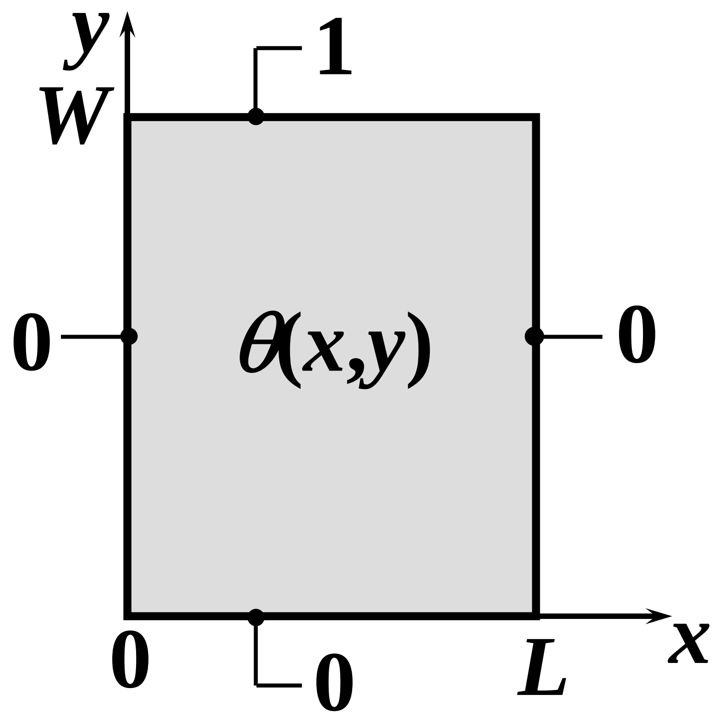
b.c. $\theta(0, y) = 0, \quad \theta(L, y) = 0$

$\theta(x, 0) = 0, \quad \theta(x, W) = 1$

Let $\theta(x,y) = X(x)Y(y)$

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}$$

$$= X''(x)Y(y) + X(x)Y''(y) = 0$$

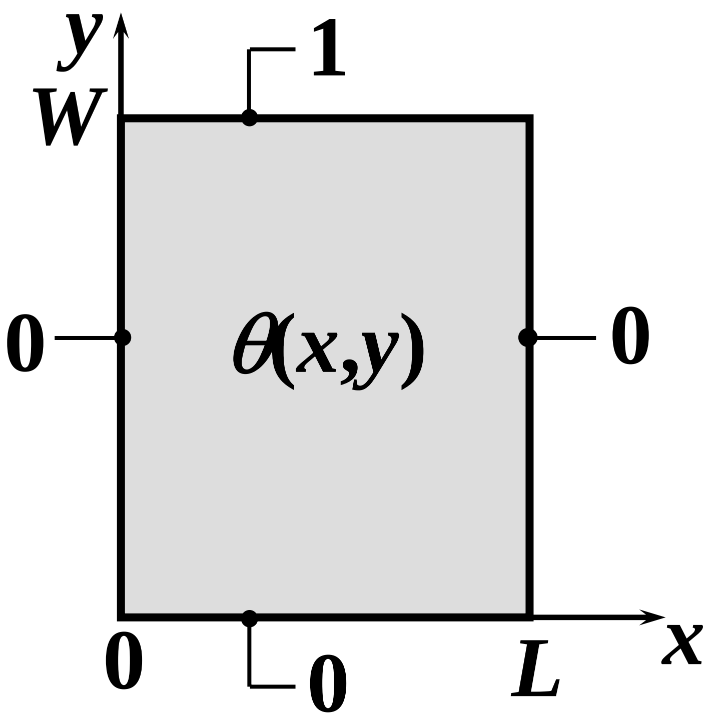


Dividing both sides by XY,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

or $\dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)}$ = constant $\equiv -\lambda^2$

$$X''(x) + \lambda^2 X(x) = 0, \quad Y''(y) - \lambda^2 Y(y) = 0$$



boundary conditions

$$\theta(0,y)=0,\quad \theta(L,y)=0$$

$$\theta(x,0)=0,\quad \theta(x,W)=1$$

$$\theta(0,y)=X(0)Y(y)=0 \rightarrow X(0)=0$$

$$\theta(L,y)=X(L)Y(y)=0 \rightarrow X(L)=0$$

$$\theta(x,0)=X(x)Y(0)=0 \rightarrow Y(0)=0$$

$$\theta(x,W)=X(x)Y(W)=1$$

For $X(x)$: $X''(x) + \lambda^2 X(x) = 0$

b.c. $X(0) = 0, \quad X(L) = 0$

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x$$

$$X(0) = 0 = C_2, \; X(L) = 0 = C_1 \sin \lambda L$$

To be non-trivial

$$\lambda L = n\pi \rightarrow \lambda_n = \frac{n\pi}{L} \quad (n = 1, 2, 3, \cdots)$$: eigenvalue

$$X_n(x) = a_n \sin \frac{n\pi}{L} x$$

$$\phi_n(x) = \sin \frac{n\pi}{L} x$$: eigenfunction

For $Y(y)$: $Y''(y) - \lambda^2 Y(y) = 0$

b.c. $Y(0) = 0$

$$Y(y) = C_3 \sinh \lambda y + C_4 \cosh \lambda y$$

$$Y(0) = 0 = C_4$$

$$Y_n(y) = b_n \sinh \frac{n\pi}{L} y$$

particular solution:

$$\theta_n(x, y) = X_n(x) Y_n(y)$$

$$= c_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

$$\theta(x,y)=\sum_{n=1}^{\infty}\theta_n(x,y)=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

$$\theta(x,W)=1=\sum_{n=1}^{\infty}c_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L}$$

Multiply both sides by $\sin\left(m\pi x/L\right)$ **and integrate from** $x=0$ **to** L

$$\int_0^L\sin\frac{m\pi x}{L}dx=\int_0^L\sum_{n=1}^{\infty}c_n\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L}dx$$

$$=\sum_{n=1}^{\infty}c_n\sinh\frac{n\pi W}{L}\int_0^L\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$$

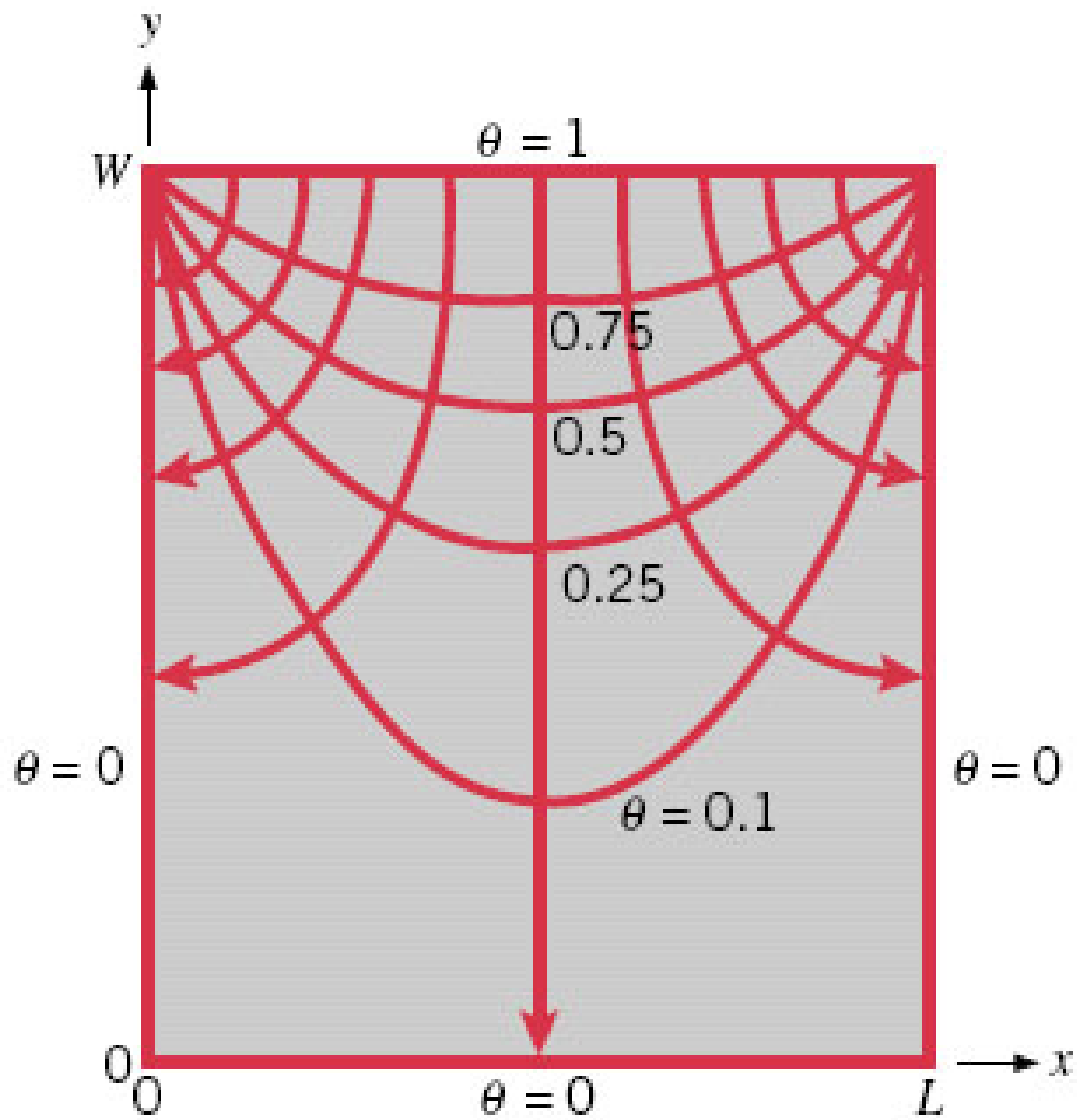
$$=c_m\sinh\frac{m\pi W}{L}\int_0^L\sin^2\frac{m\pi x}{L}dx$$

$$c_m = \frac{\left(\int_0^L \sin\frac{m\pi x}{L}dx\right) / \left(\int_0^L \sin^2\frac{m\pi x}{L}dx\right)}{\sinh\frac{m\pi W}{L}}$$

$$= \frac{2\left[(-1)^{m+1}+1\right]}{m\pi\sinh\frac{m\pi W}{L}}$$

Solution: $\theta(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$

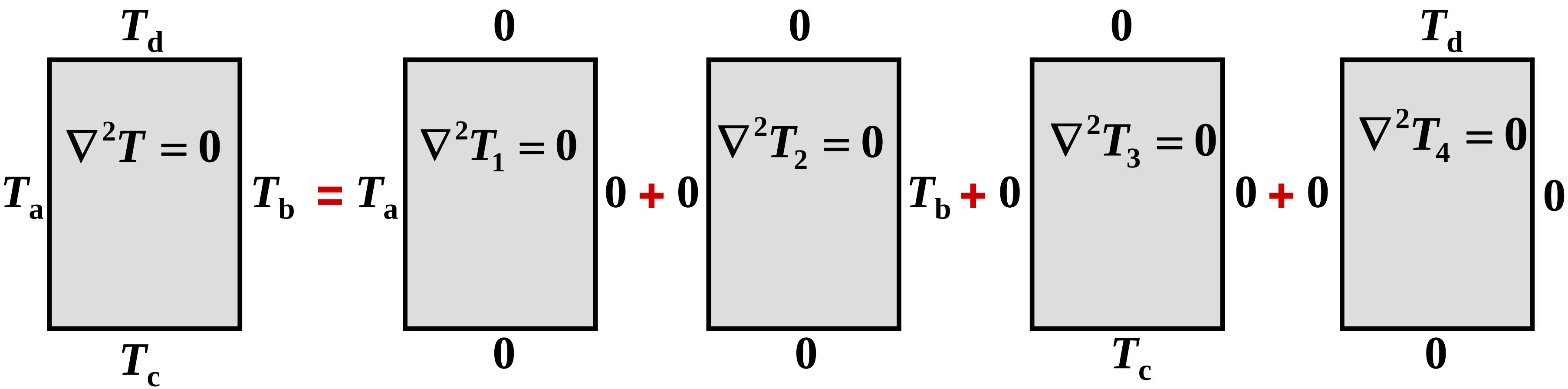
where $c_n = \frac{2\left[(-1)^{n+1}+1\right]}{n\pi\sinh\frac{n\pi W}{L}}$



y
W
θ = 1
0.75
0.5
0.25
θ = 0
θ = 0
θ = 0.1
0
0
θ = 0
L
x

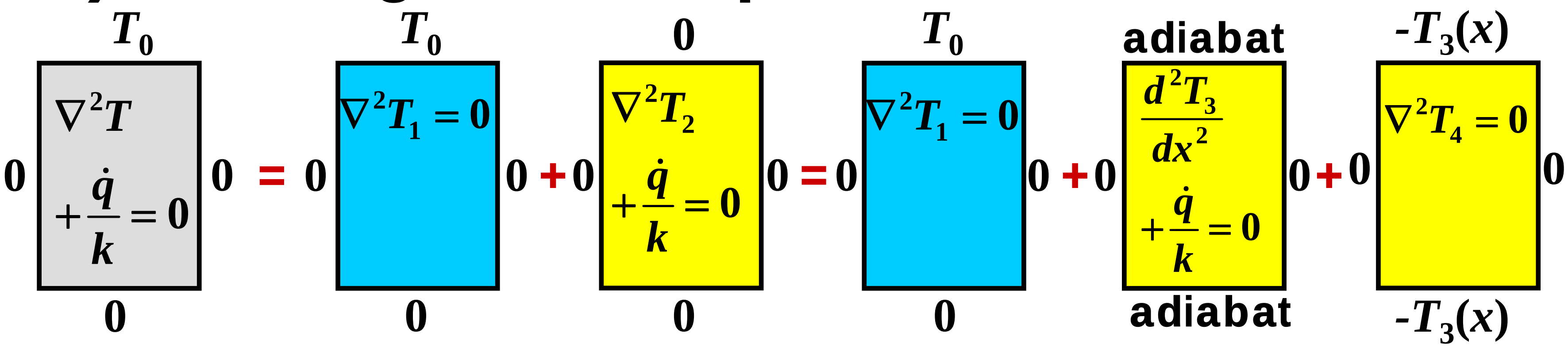
Method of Superposition

1) inhomogeneous boundary condition



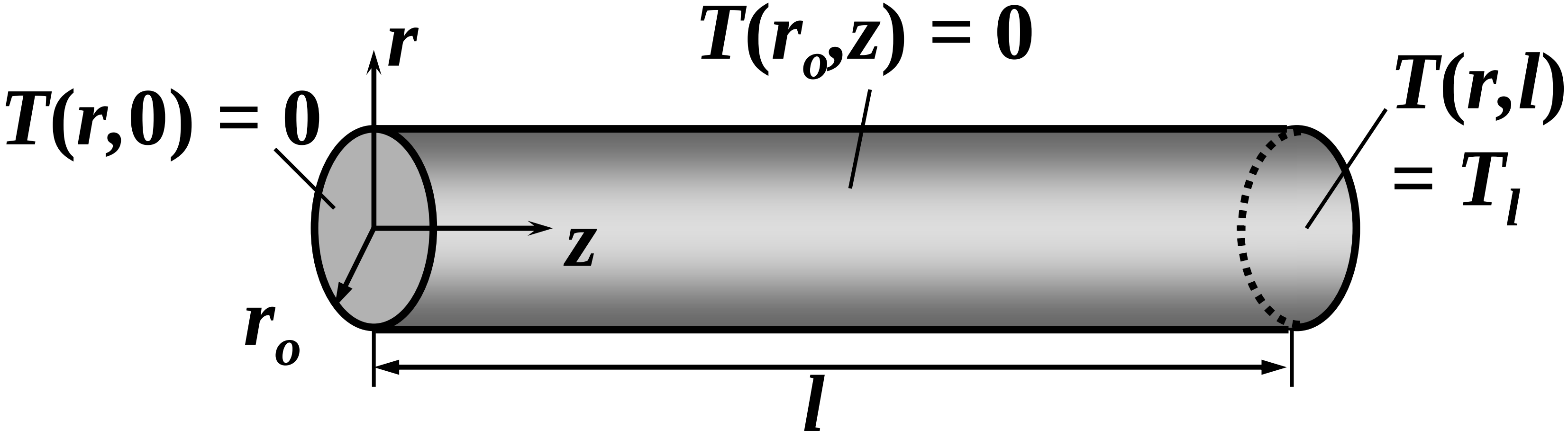
$$T = T_1 + T_2 + T_3 + T_4$$

2) inhomogeneous equation



$$T = T_1 + T_2 = T_1 + T_3 + T_4$$

Cylindrical Rod



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=0$$

boundary conditions

r : $T(0,z)=$ finite or $\dfrac{\partial T}{\partial r}(0,z)=0,\ \ T(r_o,z)=0$

z : $T(r,0)=0,\ \ T(r,l)=T_l$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=0$$

assume $T(r,z)=R(r)Z(z)$

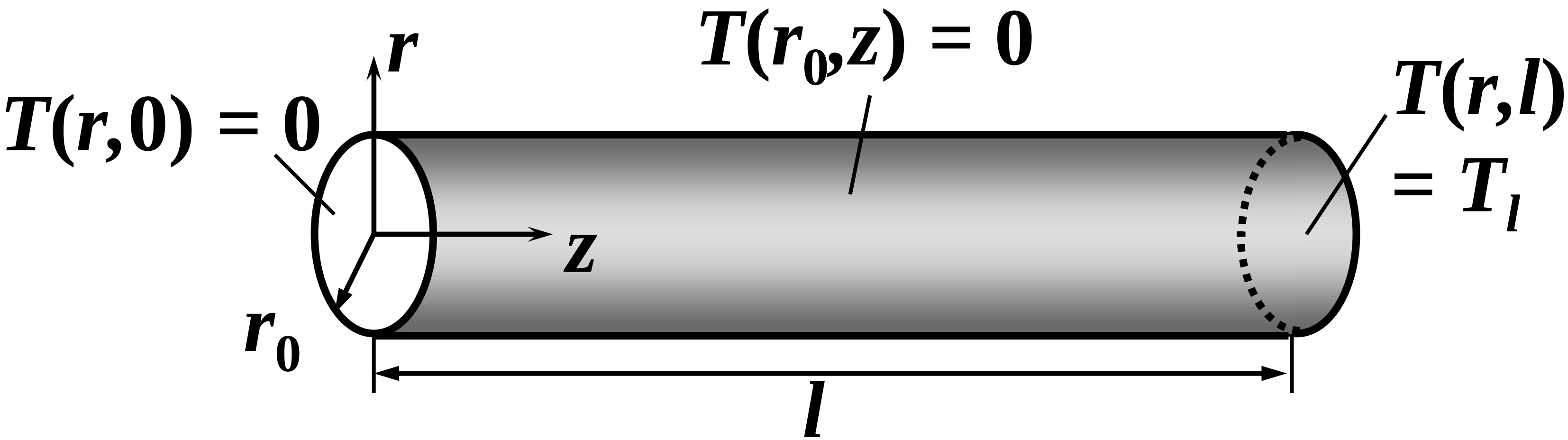
$$\frac{1}{r}(rR')'\,Z+RZ''=0,\quad \frac{1}{r}(R'+rR'')Z+RZ''=0$$

$$R''Z+\frac{1}{r}R'Z+RZ''=0,\quad \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{Z''}{Z}=0$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}=-\frac{Z''}{Z}=-\lambda^2$$

$$R''+\frac{1}{r}R'+\lambda^2 R=0,\qquad Z''-\lambda^2 Z=0$$

Boundary conditions



$$T(0,z) = R(0)Z(z) = \text{finite} \rightarrow R(0) = \text{finite}$$

$$T(r_0,z) = R(r_0)Z(z) = 0 \rightarrow R(r_0) = 0$$

$$T(r,0) = R(r)Z(0) = 0 \rightarrow Z(0) = 0$$

$$T(r,l) = R(r)Z(l) = T_l$$

For $R(r)$: $R'' + \frac{1}{r} R' + \lambda^2 R = 0$

b.c. $R(0)$ = finite, $R(r_0) = 0$

Let $\lambda r = x$, then $x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + x^2 R = 0$

$$\left[x^2 y'' + xy' + m^2 \left(x^2 - v^2 \right) y = 0 \rightarrow y = AJ_v(mx) + BY_v(mx) \right]$$

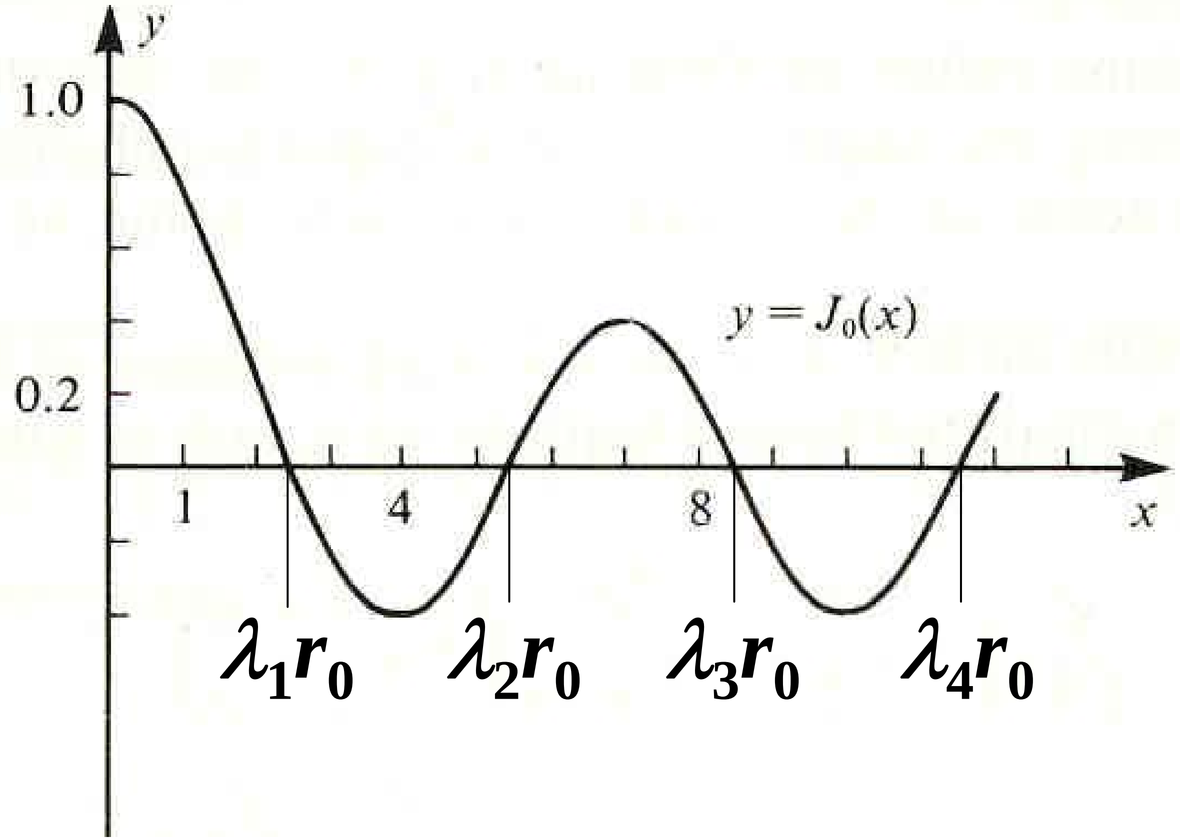
$$R(x) = C_1 J_0(x) + C_2 Y_0(x)$$

or $R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$

$R(0)$ = finite: $Y_0(x) \rightarrow -\infty$ as $x \rightarrow 0$ thus $C_2 = 0$

$$R(r_0) = C_1 J_0(\lambda r_0) = 0$$

$\rightarrow \lambda_n$ such that $J_0(\lambda_n r_0) = 0$



$$R_n(r) = a_n J_0(\lambda_n r)$$

eigenfunction: $\phi_n(r) = J_0(\lambda_n r)$

For $Z(z)$: $Z'' - \lambda^2 Z = 0$

b.c. $Z(0) = 0$

$$Z(z) = C_3 \sinh \lambda z + C_4 \cosh \lambda z$$

$$Z(0) = C_4 = 0$$

$$Z_n(z) = b_n \sinh \lambda_n z$$

$$T(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \sinh \lambda_n z$$

$$T(r,l) = T_l = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \sinh \lambda_n l$$

$$c_n = \frac{T_l \int_0^{r_0} r J_0(\lambda_n r)\,dr}{\sinh \lambda_n l \int_0^{r_0} r J_0^2(\lambda_n r)\,dr}$$

$$\int_0^{r_0} rJ_0(\lambda_n r)dr = \frac{r_0}{\lambda_n} J_1(\lambda_n r_0)$$

$$\int_0^{r_0} rJ_0^2(\lambda_n r)dr = \frac{r_0^2}{2}\lambda_n^2 J_1^2(\lambda_n r_0) + \frac{r_0^2}{2} J_0^2(\lambda_n r_0) - \frac{n^2}{2\lambda_n^2} J_0^2(\lambda_n r_0) + \frac{n^2}{2\lambda_n^2}$$

Solution: $T(r,z) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)\sinh\lambda_n z$

where $c_n = \dfrac{T_l \int_0^{r_0} rJ_0(\lambda_n r)dr}{\sinh\lambda_n l \int_0^{r_0} rJ_0^2(\lambda_n r)dr}$

Conduction Shape Factor and Dimensionless Heat Transfer Rate

Conduction Shape Factor

$q \equiv S k \Delta T_{1-2}$ S: shape factor

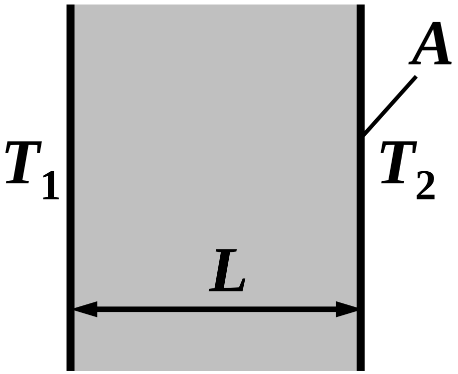
S : determined analytically

two-dimensional conduction resistance

$$q = S k \Delta T_{1-2} = \frac{\Delta T_{1-2}}{1/Sk}$$

$$R_{t,\mathrm{cond(2D)}} = \frac{1}{Sk}$$

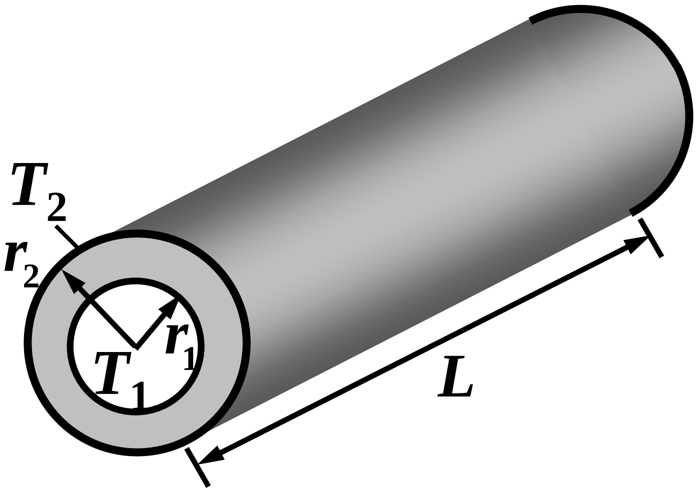
Ex) Plane wall



$$q = kA\frac{\Delta T_{1-2}}{L} = Sk\Delta T_{1-2}$$

$$S = \frac{A}{L}$$

Cylindrical wall

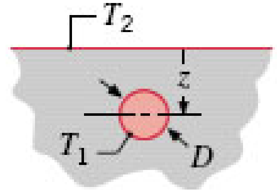
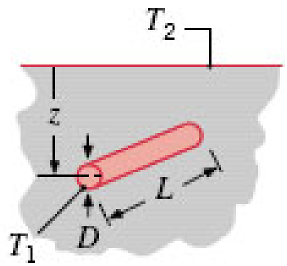
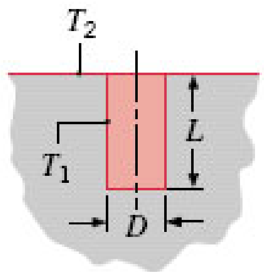


$$q = \frac{\Delta T_{1-2}}{\frac{1}{2\pi kL}\ln\frac{r_2}{r_1}} = Sk\Delta T_{1-2}$$

$$S = \frac{2\pi L}{\ln\left(r_2/r_1\right)}$$

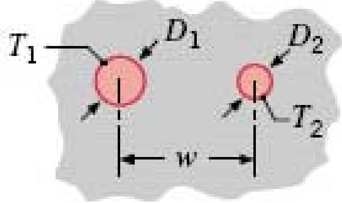
Conduction shape factors and dimensionless conduction heat rates for selected systems

(a) Shape factors [$q = Sk(T_1 - T_2)$]

System	Schematic	Restrictions	Shape Factor
Case 1 Isothermal sphere buried in a semi-infinite medium		$z > D/2$	$\dfrac{2\pi D}{1 - D/4z}$
Case 2 Horizontal isothermal cylinder of length L buried in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\cosh^{-1}(2z/D)}$
		$L \gg D$ $z > 3D/2$	$\dfrac{2\pi L}{\ln(4z/D)}$
Case 3 Vertical cylinder in a semi-infinite medium		$L \gg D$	$\dfrac{2\pi L}{\ln(4L/D)}$

Case 4

Conduction between two cylinders of length L in infinite medium

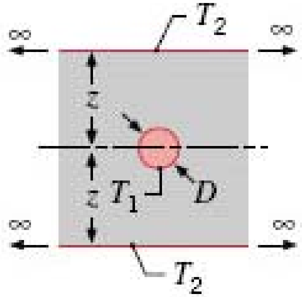


$L \gg D_1, D_2$
$L \gg w$

$$\frac{2\pi L}{\cosh^{-1}\left(\dfrac{4w^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$

Case 5

Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width

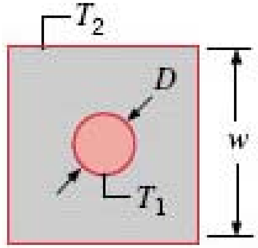


$z \gg D/2$
$L \gg z$

$$\frac{2\pi L}{\ln(8z/\pi D)}$$

Case 6

Circular cylinder of length L centered in a square solid of equal length

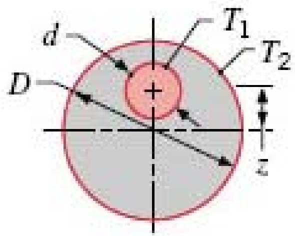


$w > D$
$L \gg w$

$$\frac{2\pi L}{\ln(1.08\, w/D)}$$

Case 7

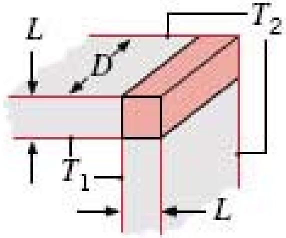
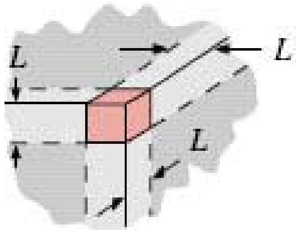
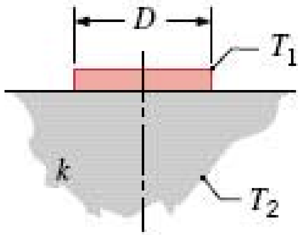
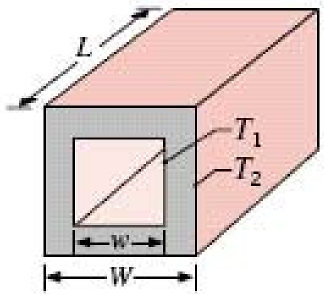
Eccentric circular cylinder of length L in a cylinder of equal length



$D > d$
$L \gg D$

$$\frac{2\pi L}{\cosh^{-1}\left(\dfrac{D^2 + d^2 - 4z^2}{2Dd}\right)}$$

TABLE 4.1 *Continued*

System	Schematic	Restrictions	Shape Factor
Case 8 Conduction through the edge of adjoining walls		$D > 5L$	$0.54D$
Case 9 Conduction through corner of three walls with a temperature difference ΔT_{1-2} across the walls		$L \ll$ length and width of wall	$0.15L$
Case 10 Disk of diameter D and temperature T_1 on a semi-infinite medium of thermal conductivity k and temperature T_2		None	$2D$
Case 11 Square channel of length L		$\frac{W}{w} < 1.4$ $\frac{W}{w} > 1.4$ $L \gg W$	$\frac{2\pi L}{0.785 \ln (W/w)}$ $\frac{2\pi L}{0.930 \ln (W/w) - 0.050}$

Dimensionless Heat Transfer Rate

Objects at isothermal temperature (T_1) embedded with an infinite medium of uniform temperature (T_2)

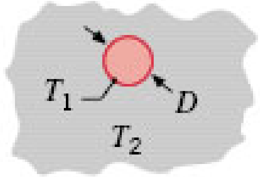
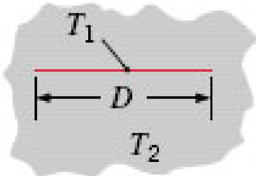
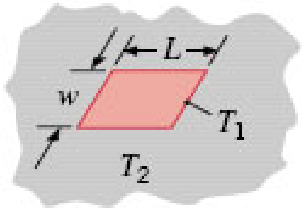
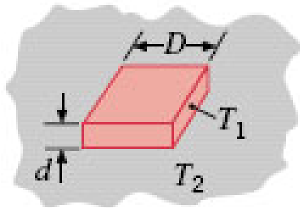
Characteristic length $L_c \equiv \left(A_s / 4\pi\right)^{1/2}$

Dimensionless conduction heat rate

$$q_{ss}^{*} \equiv \frac{qL_c}{kA_s\left(T_1 - T_2\right)}$$

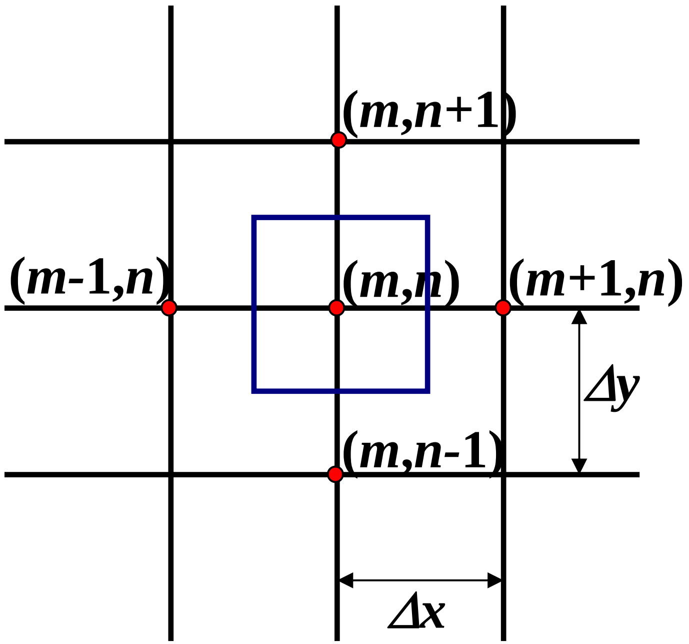
or $$q = \frac{q_{ss}^{*} kA_s\left(T_1 - T_2\right)}{L_c}$$

(b) Dimensionless conduction heat rates [$q = q^*_{ss} kA_s(T_1 - T_2)/L_c$; $L_c \equiv (A_s/4\pi)^{1/2}$]

System	Schematic	Active Area, A_s	q^*_{ss}
Case 12 Isothermal sphere of diameter D and temperature T_1 in an infinite medium of temperature T_2		πD^2	1
Case 13 Infinitely thin, isothermal disk of diameter D and temperature T_1 in an infinite medium of temperature T_2		$\frac{\pi D^2}{2}$	$\frac{2\sqrt{2}}{\pi} = 0.900$
Case 14 Infinitely thin rectangle of length L, width w, and temperature T_1 in an infinite medium of temperature T_2		$2wL$	0.932
Case 15 Cuboid shape of height d with a square footprint of width D and temperature T_1 in an infinite medium of temperature T_2		$2D^2 + 4Dd$	d/D — q^*_{ss} 0.1 — 0.943 1.0 — 0.956 2.0 — 0.961 10 — 1.111

Numerical Method

Finite Difference Method



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

discretization

$$T(x, y) = T_{m,n}$$

$$T(x - \Delta x, y) = T_{m-1,n} \qquad T(x + \Delta x, y) = T_{m+1,n}$$

$$T(x, y - \Delta y) = T_{m,n-1} \qquad T(x, y + \Delta y) = T_{m,n+1}$$

$$T(x - \Delta x, y) = T(x, y) - \left.\frac{\partial T}{\partial x}\right)_{x,y}(\Delta x) + \frac{1}{2}\left.\frac{\partial^2 T}{\partial x^2}\right)_{x,y}(\Delta x)^2 - \frac{1}{6}\left.\frac{\partial^3 T}{\partial x^3}\right)_{x,y}(\Delta x)^3 + O\left[(\Delta x)^4\right]$$

$$T(x + \Delta x, y) = T(x, y) + \left.\frac{\partial T}{\partial x}\right)_{x,y}(\Delta x) + \frac{1}{2}\left.\frac{\partial^2 T}{\partial x^2}\right)_{x,y}(\Delta x)^2 + \frac{1}{6}\left.\frac{\partial^3 T}{\partial x^3}\right)_{x,y}(\Delta x)^3 + O\left[(\Delta x)^4\right]$$

$$T(x+\Delta x, y) + T(x-\Delta x, y) = 2T(x, y) + \left.\frac{\partial^2 T}{\partial x^2}\right)_{x,y} (\Delta x)^2 + O\left[(\Delta x)^4\right]$$

or

$$\left.\frac{\partial^2 T}{\partial x^2}\right)_{x,y} = \frac{T(x+\Delta x, y) + T(x-\Delta x, y) - 2T(x, y)}{(\Delta x)^2} + O\left[(\Delta x)^2\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + O\left[(\Delta x)^2\right]$$

Similarly,

$$\left.\frac{\partial^2 T}{\partial y^2}\right)_{x,y} = \frac{T(x, y+\Delta y) + T(x, y-\Delta y) - 2T(x, y)}{(\Delta y)^2} + O\left[(\Delta y)^2\right]$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} + O\left[(\Delta y)^2\right]$$

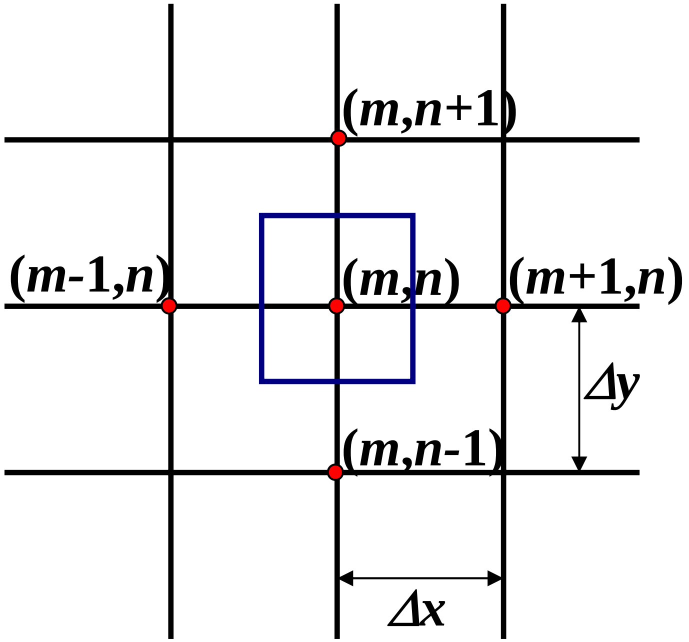
$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}$$

$$+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}+O\left[(\Delta x)^2+(\Delta y)^2\right]=0$$

If we neglect $O\left[(\Delta x)^2+(\Delta y)^2\right]$ **(truncation error) and when** $\Delta x=\Delta y$

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Finite Volume Method
(Energy Balance Method)



$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\dot{q}}{k}=0$$

$$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\dot{q}}{k}\right]dxdy=0$$

$$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\partial^2 T}{\partial x^2}dxdy=\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\left[\frac{\partial T}{\partial x}\right]_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}dy=\Delta y\left[\left.\frac{\partial T}{\partial x}\right)_{x+\frac{\Delta x}{2}}-\left.\frac{\partial T}{\partial x}\right)_{x-\frac{\Delta x}{2}}\right]$$

Assumption of linear temperature profile

$$\left.\frac{\partial T}{\partial x}\right)_{x+\frac{\Delta x}{2}}=\frac{T_{m+1,n}-T_{m,n}}{\Delta x},\quad \left.\frac{\partial T}{\partial x}\right)_{x-\frac{\Delta x}{2}}=\frac{T_{m,n}-T_{m-1,n}}{\Delta x}$$

Then, $$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\partial^2 T}{\partial x^2}dxdy = \Delta y\left[\frac{T_{m+1,n}-T_{m,n}}{\Delta x}-\frac{T_{m,n}-T_{m-1,n}}{\Delta x}\right]$$

$$=\frac{\Delta y}{\Delta x}\left[T_{m+1,n}+T_{m-1,n}-2T_{m,n}\right]$$

Similarly, $$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\partial^2 T}{\partial y^2}dxdy = \Delta x\left[\frac{T_{m,n+1}-T_{m,n}}{\Delta y}-\frac{T_{m,n}-T_{m,n-1}}{\Delta y}\right]$$

$$=\frac{\Delta x}{\Delta y}\left[T_{m,n+1}+T_{m,n-1}-2T_{m,n}\right]$$

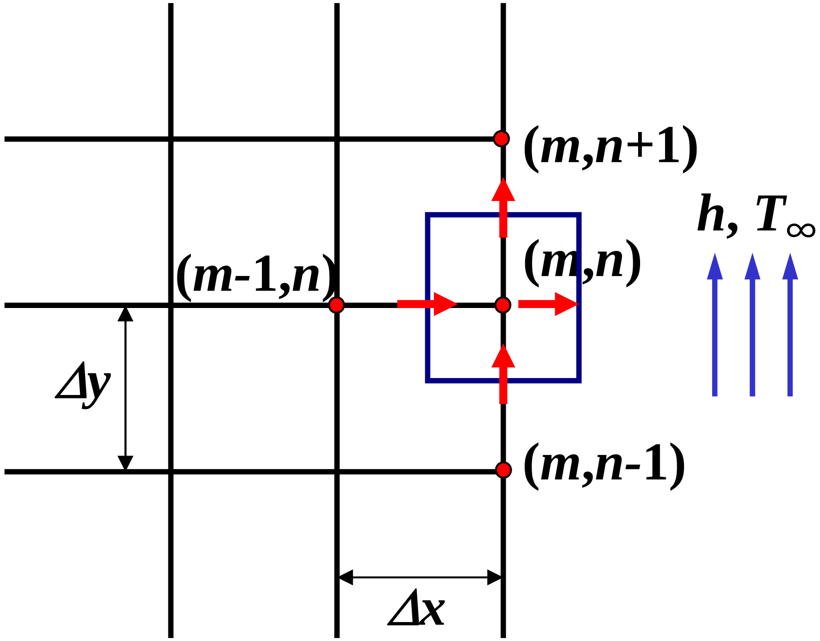
$$\int_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}\frac{\dot{q}}{k}dxdy = \frac{\dot{q}}{k}(\Delta x)(\Delta y)$$

$$\frac{\Delta y}{\Delta x}\left[T_{m,n+1}+T_{m,n-1}-2T_{m,n}\right]+\frac{\Delta x}{\Delta y}\left[T_{m,n+1}+T_{m,n-1}-2T_{m,n}\right]+\frac{\dot{q}}{k}(\Delta x)(\Delta y)=0$$

If $\Delta x = \Delta y$, $T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}+\dfrac{\dot{q}(\Delta x)(\Delta y)}{k}-4T_{m,n}=0$

Convection Boundary Conditions

1) Side



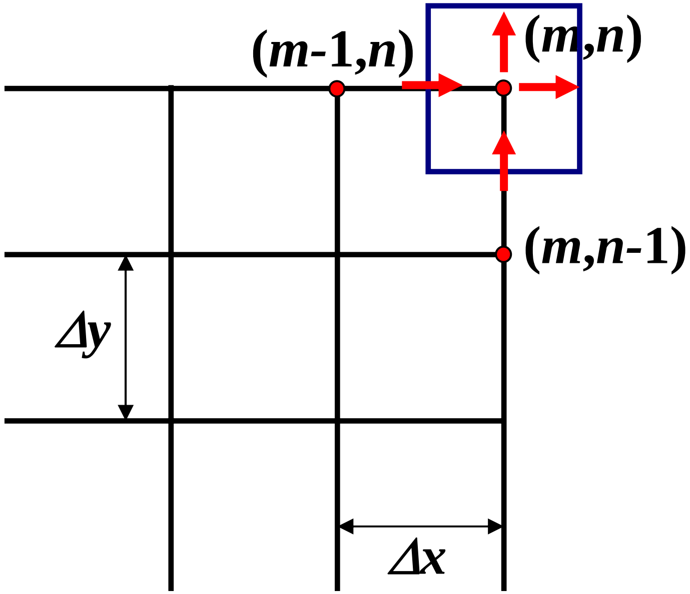
$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\frac{\Delta x}{2}\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$= k\frac{\Delta x}{2}\frac{T_{m,n} - T_{m,n+1}}{\Delta y} + h\Delta y\left(T_{m,n} - T_{\infty}\right)$$

If $\Delta x = \Delta y, T_{m,n}\left(\frac{h\Delta x}{2} + 2\right) - \frac{h\Delta x}{k}T_{\infty} - \frac{1}{2}\left(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right) = 0$

or $0 \times T_{m+1,n} + T_{m-1,n} + \frac{1}{2}T_{m,n+1} + \frac{1}{2}T_{m,n-1} - \left(\frac{h\Delta x}{k} + 2\right)T_{m,n} = -\frac{h\Delta x}{k}T_{\infty}$

2) Corner



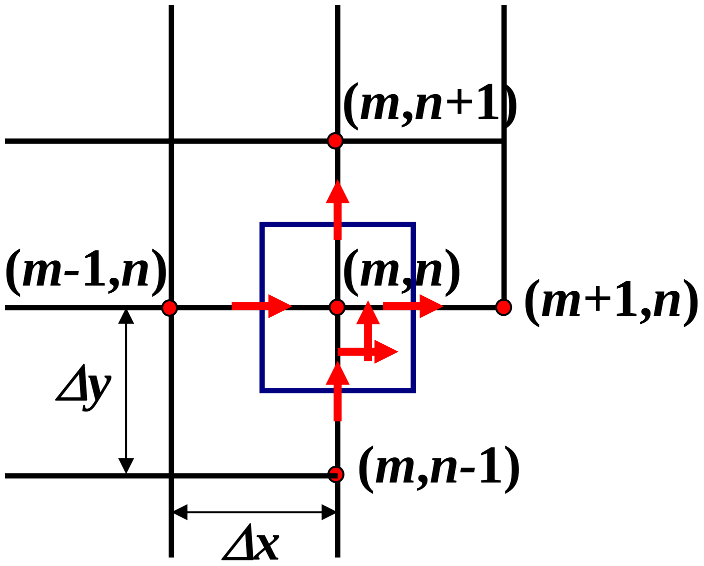
$$k\frac{\Delta y}{2}\frac{T_{m-1,n}-T_{m,n}}{\Delta x}+k\frac{\Delta x}{2}\frac{T_{m,n-1}-T_{m,n}}{\Delta y}$$

$$=h\frac{\Delta x}{2}\left(T_{m,n}-T_{\infty}\right)+h\frac{\Delta y}{2}\left(T_{m,n}-T_{\infty}\right)$$

If $\Delta x=\Delta y,$

$$0\times T_{m+1,n}+T_{m-1,n}+0\times T_{m,n+1}+T_{m,n-1}-2\left(1+\frac{h\Delta x}{k}\right)T_{m,n}=-\frac{2h\Delta x}{k}T_{\infty}$$

3) Internal corner



$$k(\Delta y)\frac{T_{m-1,n}-T_{m,n}}{\Delta x}-k\left(\frac{\Delta y}{2}\right)\frac{T_{m,n}-T_{m+1,n}}{\Delta x}$$

$$+k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}-k(\Delta x)\frac{T_{m,n}-T_{m,n+1}}{\Delta y}$$

$$+h\left(\frac{\Delta x}{2}\right)\left(T_\infty-T_{m,n}\right)-h\left(\frac{\Delta y}{2}\right)\left(T_{m,n}-T_\infty\right)=0$$

If $\Delta x=\Delta y,$

$$\left(T_{m-1,n}-T_{m,n}\right)-\frac{1}{2}\left(T_{m,n}-T_{m+1,n}\right)+\frac{1}{2}\left(T_{m,n-1}-T_{m,n}\right)$$

$$-\left(T_{m,n}-T_{m,n+1}\right)+\frac{h}{k}\frac{\Delta x}{2}\left(T_\infty-T_{m,n}\right)-\frac{h}{k}\frac{\Delta x}{2}\left(T_{m,n}-T_\infty\right)=0$$

or $T_{m-1,n}+T_{m,n+1}+\frac{1}{2}\left(T_{m+1,n}+T_{m,n-1}\right)+\frac{h\Delta x}{k}T_\infty-\left(3+\frac{h\Delta x}{k}\right)T_{m,n}=0$

Solution Techniques

$$
\begin{aligned}
a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1n}T_n &= c_1 \\
a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2n}T_n &= c_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad & \vdots \quad \vdots \\
a_{n1}T_1 + a_{n2}T_2 + a_{n3}T_3 + \cdots + a_{nn}T_n &= c_n
\end{aligned}
$$

or

$$
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}
=
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
$$

$$\underset{\sim}{A}\,\underset{\sim}{T} = \underset{\sim}{C}$$

1) Analytical method (matrix inversion) → Cramer's rule

$$T_1 = \frac{D_1}{\det\left[\underset{\sim}{A}\right]},\ T_2 = \frac{D_2}{\det\left[\underset{\sim}{A}\right]}, \cdots$$

If $n = 10$, 3×10^6 operations. If $n = 25$, IBM360 10^{17} years

2) Direct (elimination) method ($n < 40$)

$$\begin{aligned}
a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1n}T_n &= c_1 \\
a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2n}T_n &= c_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{n1}T_1 + a_{n2}T_2 + a_{n3}T_3 + \cdots + a_{nn}T_n &= c_n
\end{aligned}$$

Gauss-Jordan elimination method

Augmented matrix of $\underset{\sim}{A}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & c_n \end{bmatrix} \xrightarrow{\text{operations}} \begin{bmatrix} a'_{11} & 0 & \cdots\cdots & 0 & c'_1 \\ 0 & a'_{22} & \cdots\cdots & 0 & c'_2 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots\cdots & a_{nn} & c_n \end{bmatrix}$$

$$T_1 = \frac{c'_1}{a'_{11}},\ T_2 = \frac{c'_2}{a'_{22}}, \cdots, T_n = \frac{c'_n}{a'_{nn}}$$

$$
\begin{aligned}
a_{11}T_1 + a_{12}T_2 + a_{13}T_3 &= c_1 \\
a_{21}T_1 + a_{22}T_2 + a_{23}T_3 &= c_2 \\
a_{31}T_1 + a_{32}T_2 + a_{33}T_3 &= c_3
\end{aligned}
$$

$$
\begin{aligned}
a_{11}a_{21}T_1 + a_{12}a_{21}T_2 + a_{13}a_{21}T_3 &= c_1 a_{21} \\
a_{11}a_{21}T_1 + a_{11}a_{22}T_2 + a_{11}a_{23}T_3 &= c_2 a_{11} \\
a_{31}T_1 + a_{32}T_2 + a_{33}T_3 &= c_3
\end{aligned}
$$

$$
\begin{aligned}
a_{11}a_{31}T_1 + a_{12}a_{31}T_2 + a_{13}a_{31}T_3 &= c_1 a_{31} \\
0T_1 + a'_{22}T_2 + a'_{23}T_3 &= c'_2 \\
a_{11}a_{31}T_1 + a_{11}a_{32}T_2 + a_{11}a_{33}T_3 &= c_3 a_{11}
\end{aligned}
$$

○ : pivot element

$$
\begin{aligned}
a'_{11}T_1 + a'_{12}T_2 + a'_{13}T_3 &= c'_1 \\
0T_1 + a'_{22}T_2 + a'_{23}T_3 &= c'_2 \\
0T_1 + a'_{32}T_2 + a'_{33}T_3 &= c'_3
\end{aligned}
$$

$$
\begin{aligned}
a'_{22}a'_{11}T_1 + a'_{22}a'_{12}T_2 + a'_{22}a'_{13}T_3 &= c'_1 a'_{22} \\
0T_1 + a'_{12}a'_{22}T_2 + a'_{12}a'_{23}T_3 &= c'_2 a'_{12} \\
0T_1 + a'_{32}T_2 + a'_{33}T_3 &= c'_3
\end{aligned}
$$

$$
\begin{aligned}
a''_{11}T_1 + 0T_2 + a''_{13}T_3 &= c''_1 \\
0T_1 + a'_{32}a'_{22}T_2 + a'_{32}a'_{23}T_3 &= c'_2 a'_{32} \\
0T_1 + a'_{22}a'_{32}T_2 + a'_{22}a'_{33}T_3 &= c'_3 a'_{22}
\end{aligned}
$$

$$
\begin{aligned}
a''_{11}T_1 + 0T_2 + a''_{13}T_3 &= c''_1 \\
0T_1 + a''_{22}T_2 + a''_{23}T_3 &= c''_2 \\
0T_1 + 0T_2 + a''_{33}T_3 &= c''_3
\end{aligned}
$$

$$
\begin{aligned}
a'''_{11}T_1 + 0T_2 + 0T_3 &= c'''_1 \\
0T_1 + a'''_{22}T_2 + 0T_3 &= c'''_2 \\
0T_1 + 0T_2 + a'''_{33}T_3 &= c'''_3
\end{aligned}
$$

3) Gauss-Seidel iteration method $(n > 100)$

$$a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1n}T_n = c_1$$
$$a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2n}T_n = c_2$$
$$\cdots\cdots\cdots\cdots$$
$$a_{i1}T_1 + a_{i2}T_2 + a_{i3}T_3 + \cdots + a_{in}T_n = c_i$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n1}T_1 + a_{n2}T_2 + a_{n3}T_3 + \cdots + a_{nn}T_n = c_n$$

$$a_{i1}T_1 + a_{i2}T_2 + a_{i3}T_3 + \cdots + a_{i(i-1)}T_{i(i-1)} + a_{ii}T_i + a_{i(i+1)}T_{i(i+1)} + \cdots + a_{in}T_n = c_i$$

$$T_i = \frac{c_i}{a_{ii}} - \left(\frac{a_{i1}}{a_{ii}}T_1 + \frac{a_{i2}}{a_{ii}}T_2 + \frac{a_{i3}}{a_{ii}}T_3 \cdots + \frac{a_{i(i-1)}}{a_{ii}}T_{i-1} + \frac{a_{i(i+1)}}{a_{ii}}T_{i+1} + \cdots + \frac{a_{in}}{a_{ii}}T_n \right)$$

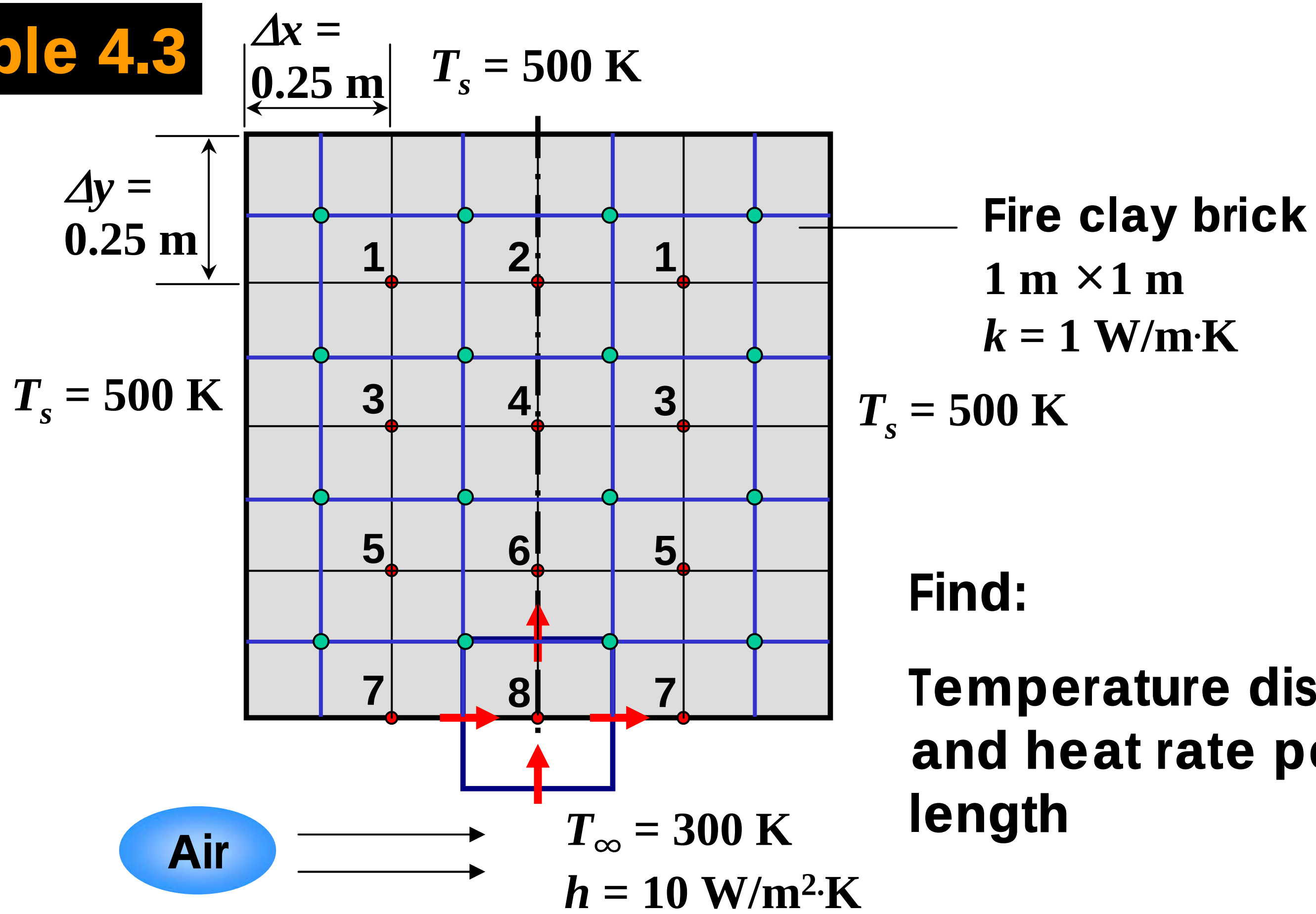
$$T_i^{(k)} = \frac{c_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

diagonal dominance → sufficient condition for convergence

$$\sum_{\substack{j=1 \\ j \neq i}} \left| a_{ij} \right| < \left| a_{ii} \right|$$

convergence criterion $\left| \dfrac{T_i^{(k)} - T_i^{(k-1)}}{T_i^{(k)}} \right| \leq \varepsilon$

Example 4.3



Find:

Temperature distribution and heat rate per unit length

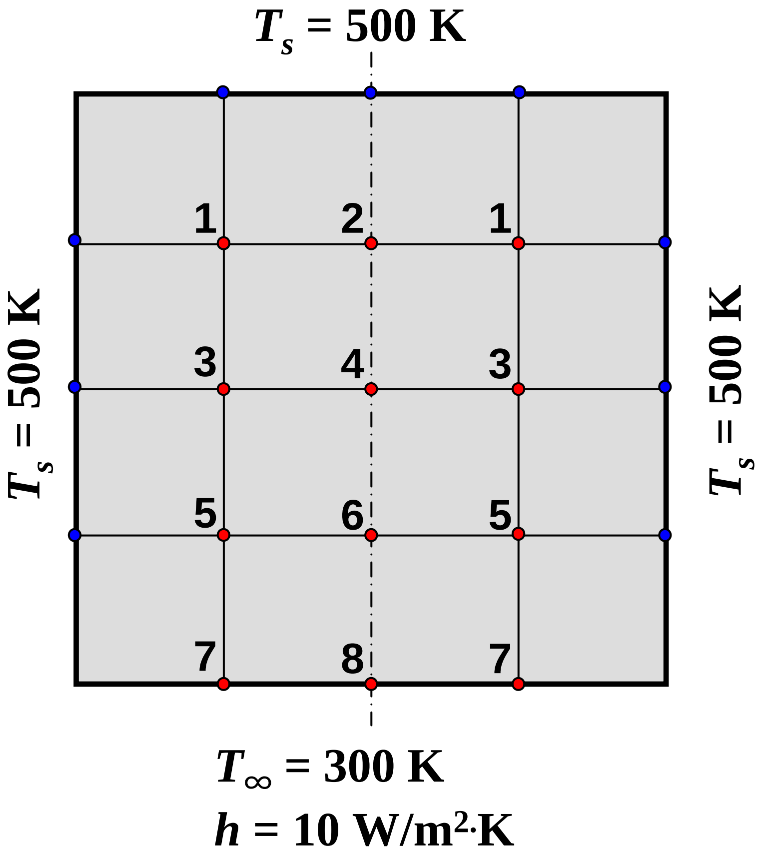
Nodes at the plane surface with convection:

$$k\frac{\Delta y}{2}\frac{T_{m-1,n}-T_{m,n}}{\Delta x}-k\frac{\Delta y}{2}\frac{T_{m,n}-T_{m+1,n}}{\Delta x}+h\Delta x\left(T_\infty-T_{m,n}\right)-k\Delta x\frac{T_{m,n}-T_{m,n+1}}{\Delta y}=0$$

When $\Delta x = \Delta y$, $T_{m-1,n}+T_{m+1,n}+2T_{m,n+1}+\frac{2h\Delta x}{k}T_\infty-\left(\frac{2h}{k}\Delta x+4\right)T_{m,n}=0$

inner nodes: $T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$

convection nodes: $T_{m-1,n} + T_{m+1,n} + 2T_{m,n+1} + 1500 - 9T_{m,n} = 0$



Node 1: $T_2 + T_3 + 1000 - 4T_1 = 0$

Node 3: $T_1 + T_4 + T_5 + 500 - 4T_3 = 0$

Node 5: $T_3 + T_6 + T_7 + 500 - 4T_5 = 0$

Node 2: $2T_1 + T_4 + 500 - 4T_2 = 0$

Node 4: $T_2 + 2T_3 + T_6 - 4T_4 = 0$

Node 6: $T_4 + 2T_5 + T_8 - 4T_6 = 0$

Node 7: $2T_5 + T_8 + 2000 - 9T_7 = 0$

Node 8: $2T_6 + 2T_7 + 1500 - 9T_8 = 0$

$$
\begin{aligned}
-4T_1 + T_2 + T_3 + 0 + 0 + 0 + 0 + 0 &= -1000 \\
2T_1 - 4T_2 + 0 + T_4 + 0 + 0 + 0 + 0 &= -500 \\
T_1 + 0 - 4T_3 + T_4 + T_5 + 0 + 0 + 0 &= -500 \\
0 + T_2 + 2T_3 - 4T_4 + 0 + T_6 + 0 + 0 &= 0 \\
0 + 0 + T_3 + 0 - 4T_5 + T_6 + T_7 + 0 &= -500 \\
0 + 0 + 0 + T_4 + 2T_5 - 4T_6 + 0 + T_8 &= 0 \\
0 + 0 + 0 + 0 + 2T_5 + 0 - 9T_7 + T_8 &= -2000 \\
0 + 0 + 0 + 0 + 0 + 2T_6 + 2T_7 - 9T_8 &= -1500
\end{aligned}
$$

$$[A]=\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & -9 \end{bmatrix} \qquad [C]=\begin{bmatrix} -1000 \\ -500 \\ -500 \\ 0 \\ -500 \\ 0 \\ -2000 \\ -1500 \end{bmatrix}$$

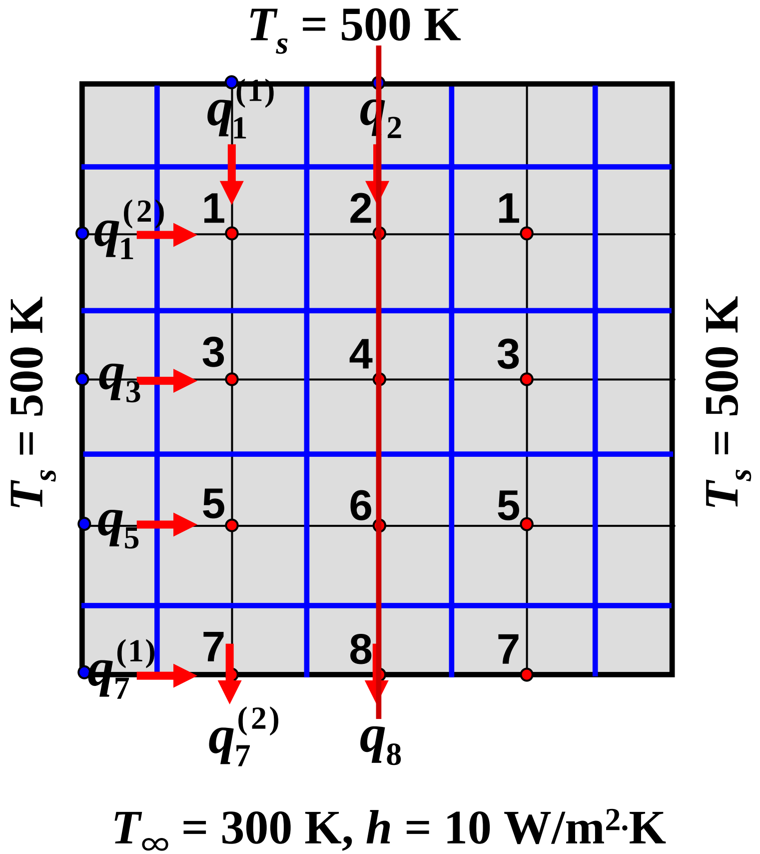
Using a standard matrix inversion routine, it is a simple matter to find the inverse of $[A]$, $[A]^{-1}$, giving

$$[T]=[A]^{-1}[C]$$

where

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} 489.30 \\ 485.15 \\ 472.07 \\ 462.01 \\ 436.95 \\ 418.74 \\ 356.99 \\ 339.05 \end{bmatrix} {}^\circ\mathrm{C}$$

Heat transfer rate



conduction in = convection out

conduction in = $q_1^{(1)} + q_1^{(2)} + q_2 + q_3 + q_5 + q_7^{(1)}$

convection out = $q_7^{(2)} + q_8$

$$\frac{q_{\mathrm{cond}}}{L} = k\left[\Delta x\frac{(T_s - T_1)}{\Delta y} + \Delta y\frac{(T_s - T_1)}{\Delta x} + \frac{\Delta x}{2}\frac{(T_s - T_2)}{\Delta y} + \Delta y\frac{(T_s - T_3)}{\Delta x} + \Delta y\frac{(T_s - T_5)}{\Delta x} + \frac{\Delta y}{2}\frac{(T_s - T_7)}{\Delta x}\right] = 191.31 \ \ \mathrm{W/m}$$

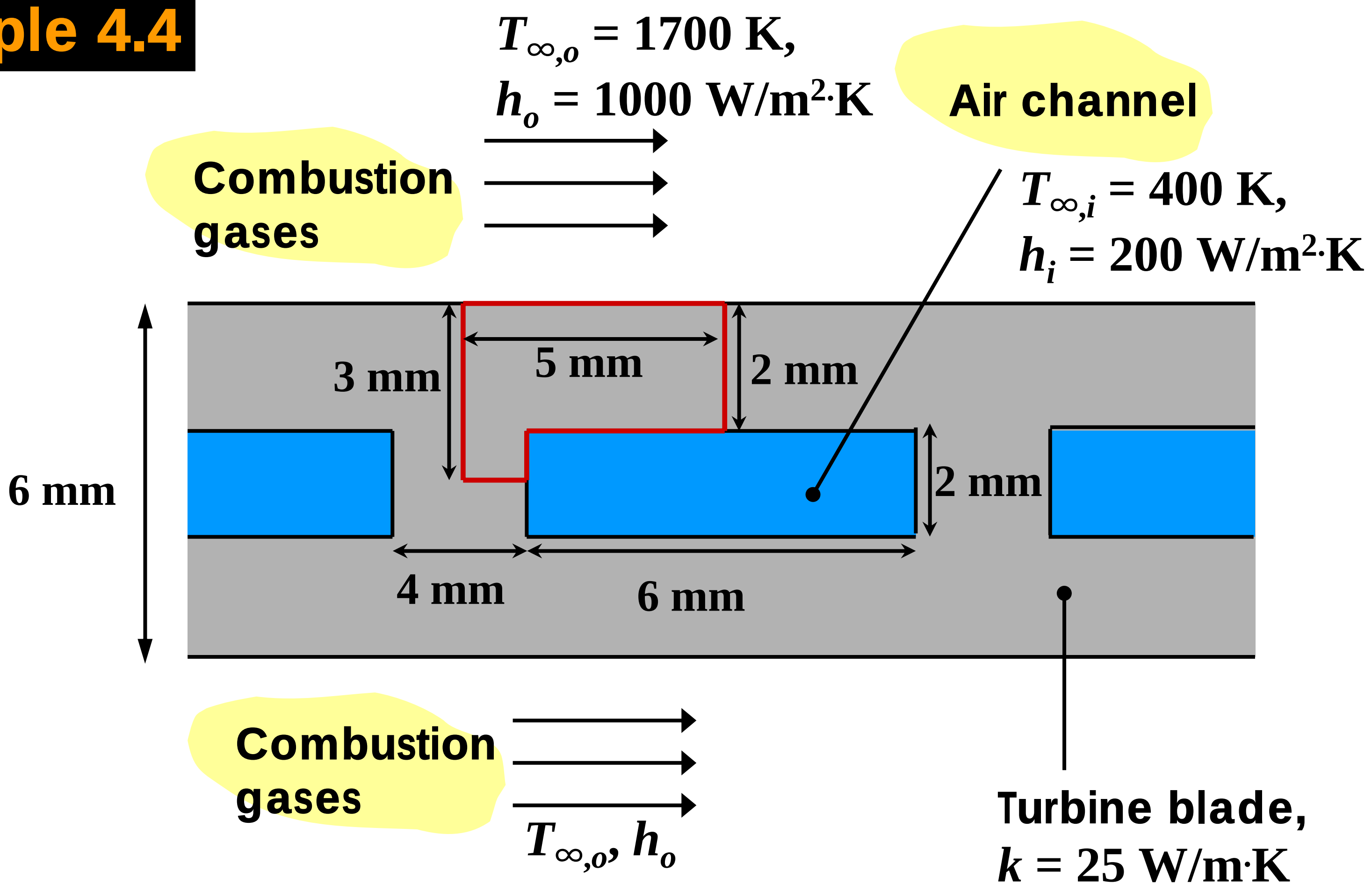
$$\frac{q_{\mathrm{conv}}}{L} = h\left[\Delta x(T_7 - T_\infty) + \frac{\Delta x}{2}(T_8 - T_\infty)\right] = 191.29 \ \ \mathrm{W/m}$$

Gauss-Seidel iteration method

$$
\begin{aligned}
T_1^{(k)} &= 0.25T_2^{(k-1)} + 0.25T_3^{(k-1)} + 250 \\
T_2^{(k)} &= 0.50T_1^{(k)} + 0.25T_4^{(k-1)} + 125 \\
T_3^{(k)} &= 0.25T_1^{(k)} + 0.25T_4^{(k-1)} + 0.25T_5^{(k-1)} + 125 \\
T_4^{(k)} &= 0.25T_2^{(k)} + 0.50T_3^{(k)} + 0.25T_6^{(k-1)} \\
T_5^{(k)} &= 0.25T_3^{(k)} + 0.25T_6^{(k-1)} + 0.25T_7^{(k-1)} + 125 \\
T_6^{(k)} &= 0.25T_4^{(k)} + 0.50T_5^{(k)} + 0.25T_8^{(k-1)} \\
T_7^{(k)} &= 0.2222T_5^{(k)} + 0.1111T_8^{(k-1)} + 222.22 \\
T_8^{(k)} &= 0.2222T_6^{(k)} + 0.2222T_7^{(k)} + 166.67
\end{aligned}
$$

k	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
0	480	470	440	430	400	390	370	350
1	477.5	471.3	451.9	441.3	428.0	411.8	356.2	337.3
2	480.8	475.7	462.5	453.1	436.6	413.9	355.8	337.7
3	484.6	480.6	467.6	457.4	434.3	415.9	356.2	338.2
4	487.1	482.9	469.7	459.6	435.5	417.2	356.6	338.6
5	488.1	484.0	470.8	460.7	436.1	417.9	356.7	338.8
6	488.7	484.5	471.4	461.3	436.5	418.3	356.9	338.9
7	489.0	484.8	471.7	461.6	436.7	418.5	356.9	339.0
8	489.1	485.0	471.9	461.8	436.8	418.6	356.9	339.0
	489.3	**485.2**	**472.1**	**462.0**	**437.0**	**418.7**	**357.0**	**339.1**

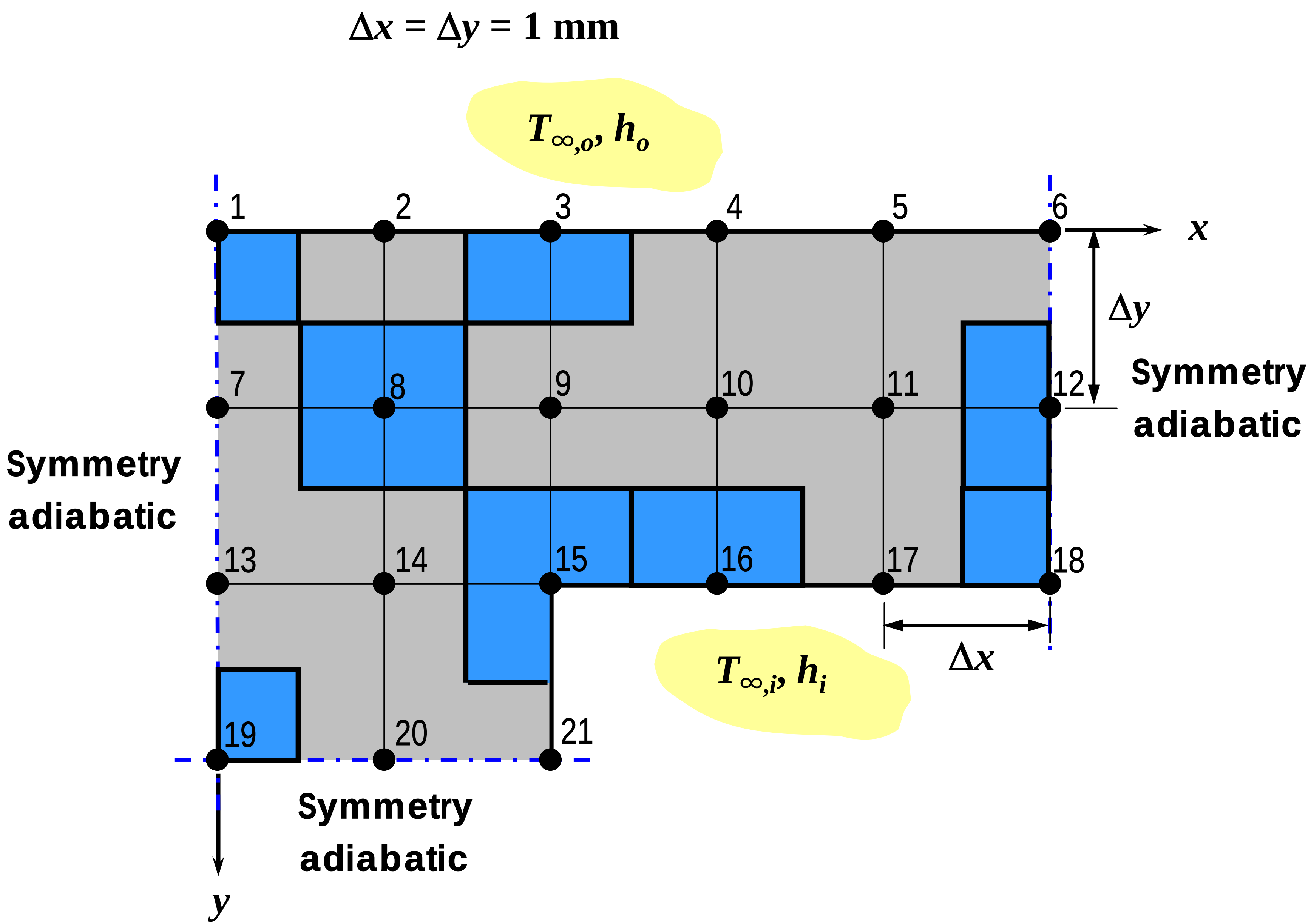
Example 4.4



Find:

1. Temperature field in the blade,
2. Location of maximum temperature
3. Heat transfer rate per unit length to the channel

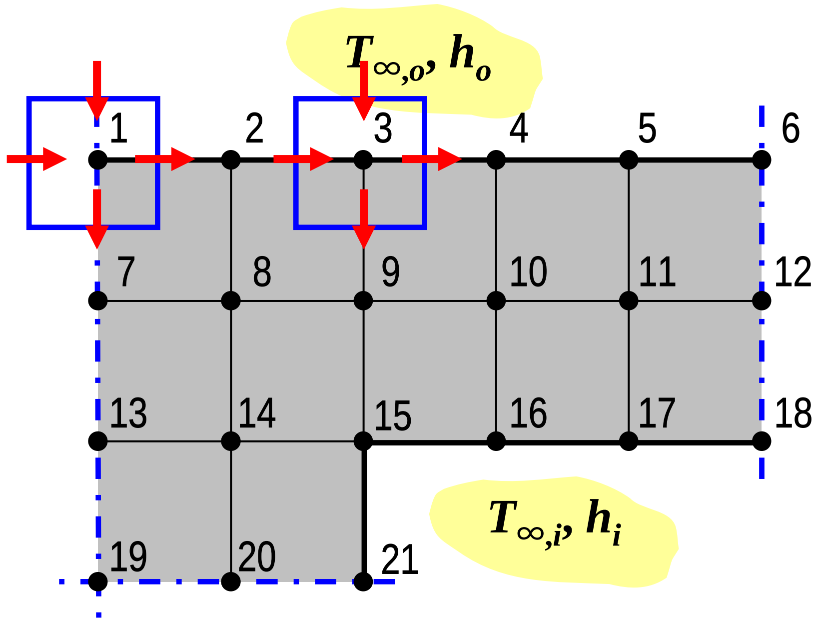
Guess locations of maximum and minimum temperatures.



Δx = Δy = 1 mm
T∞,o, ho
1
2
3
4
5
6
x
Δy
7
8
9
10
11
12
Symmetry adiabatic
Symmetry adiabatic
13
14
15
16
17
18
T∞,i, hi
Δx
19
20
21
Symmetry adiabatic
y

Node 1 :

$$T_2 + T_7 - \left(2 + \frac{h_o \Delta x}{k}\right) T_1 = -\frac{h_o \Delta x}{k} T_{\infty,o}$$



- same as node 6

Node 3 :

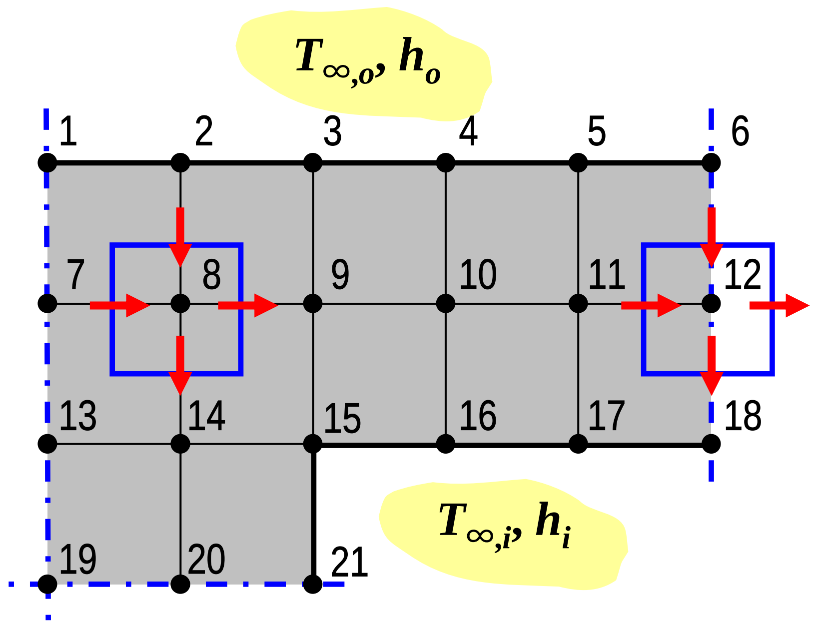
$$T_2 + T_4 + 2T_9 - 2\left(\frac{h_o \Delta x}{k} + 2\right) T_3 = -\frac{2h_o \Delta x}{k} T_{\infty,o}$$

- same as nodes 2, 4, and 5

Node 12 :

$$T_6 + 2T_{11} + T_{18} - 4T_{12} = 0$$

- same as nodes 7, 13, and 20



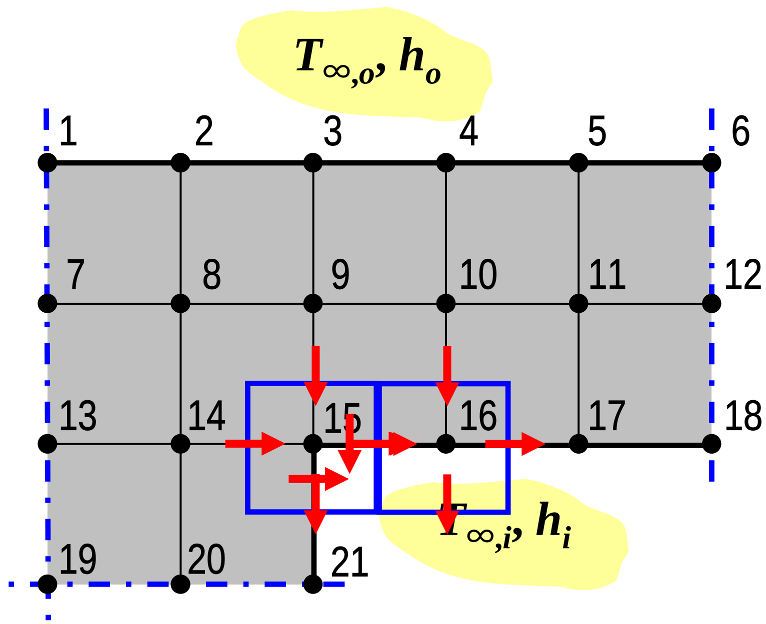
Node 8 :

$$T_2 + T_7 + T_9 + T_{14} - 4T_8 = 0$$

- same as nodes 9, 10, 11, and 14

Node 15 :

$$2T_9 + 2T_{14} + T_{16} + T_{21} - 2\left(3 + \frac{h_i \Delta x}{k}\right)T_{15} = -2\frac{h_i \Delta x}{k}T_{\infty,i}$$



Node 16 :

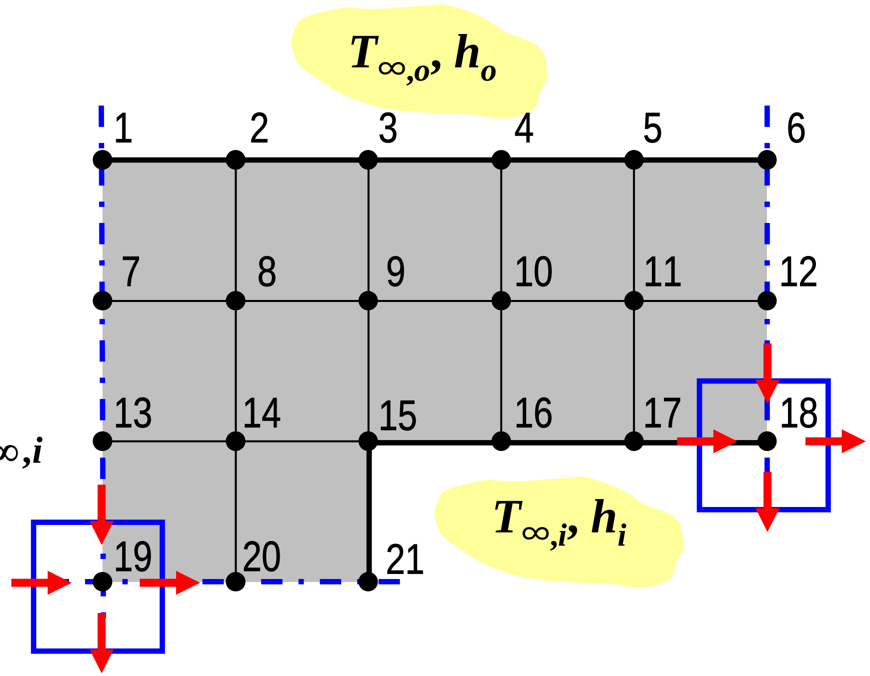
$$2T_{10} + T_{15} + T_{17} - 2\left(\frac{h_i \Delta x}{k} + 2\right)T_{16} = -2\frac{h_i \Delta x}{k}T_{\infty,i}$$

- same as node 17

Node 18 :

$$T_{12} + T_{17} - \left(2 + \frac{h_i \Delta x}{k}\right) T_{18} = -\frac{h_i \Delta x}{k} T_{\infty,i}$$

- same as nodes 21



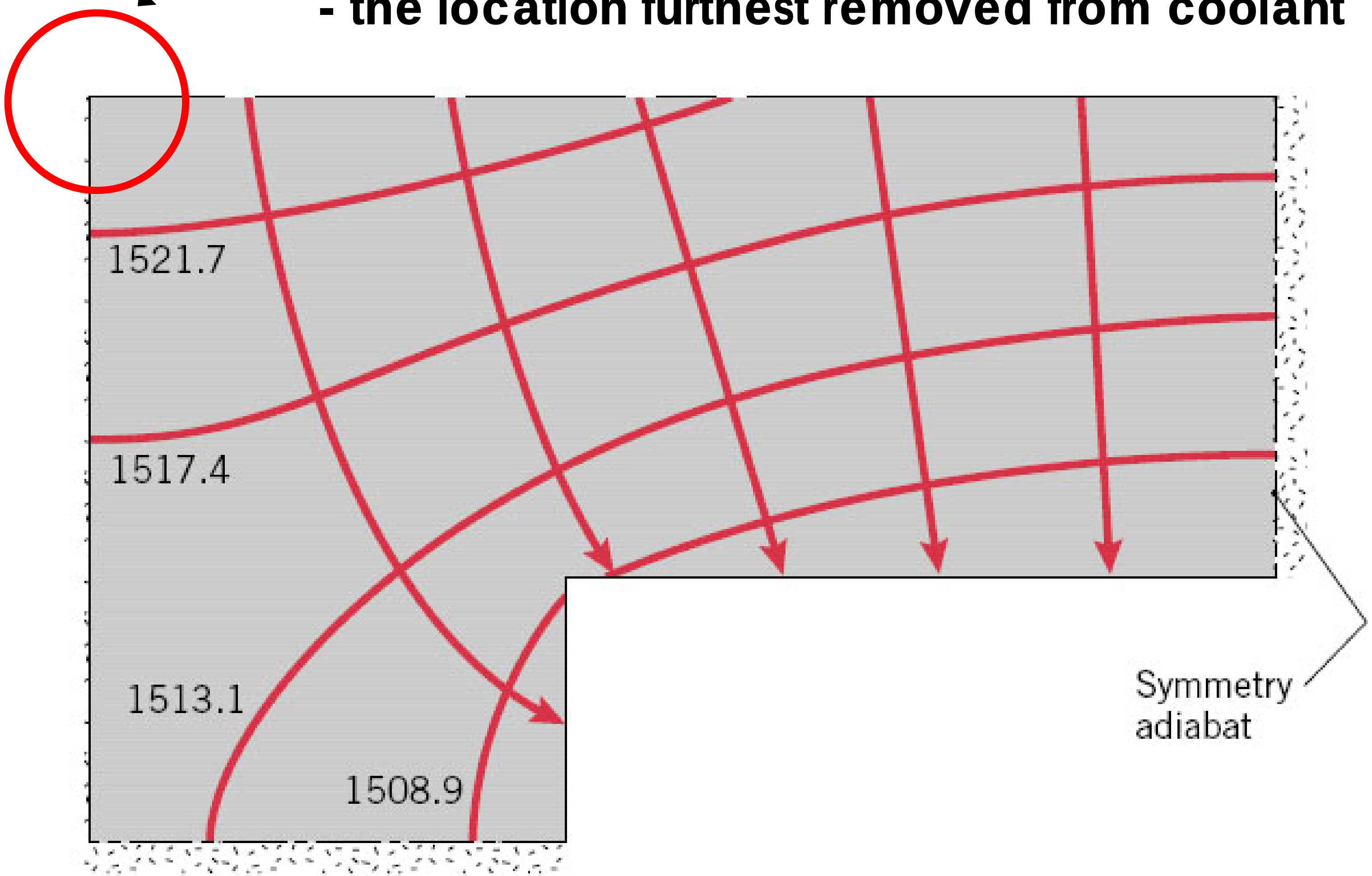
Node 19 :

$$T_{13} + T_{20} - 2T_{19} = 0$$

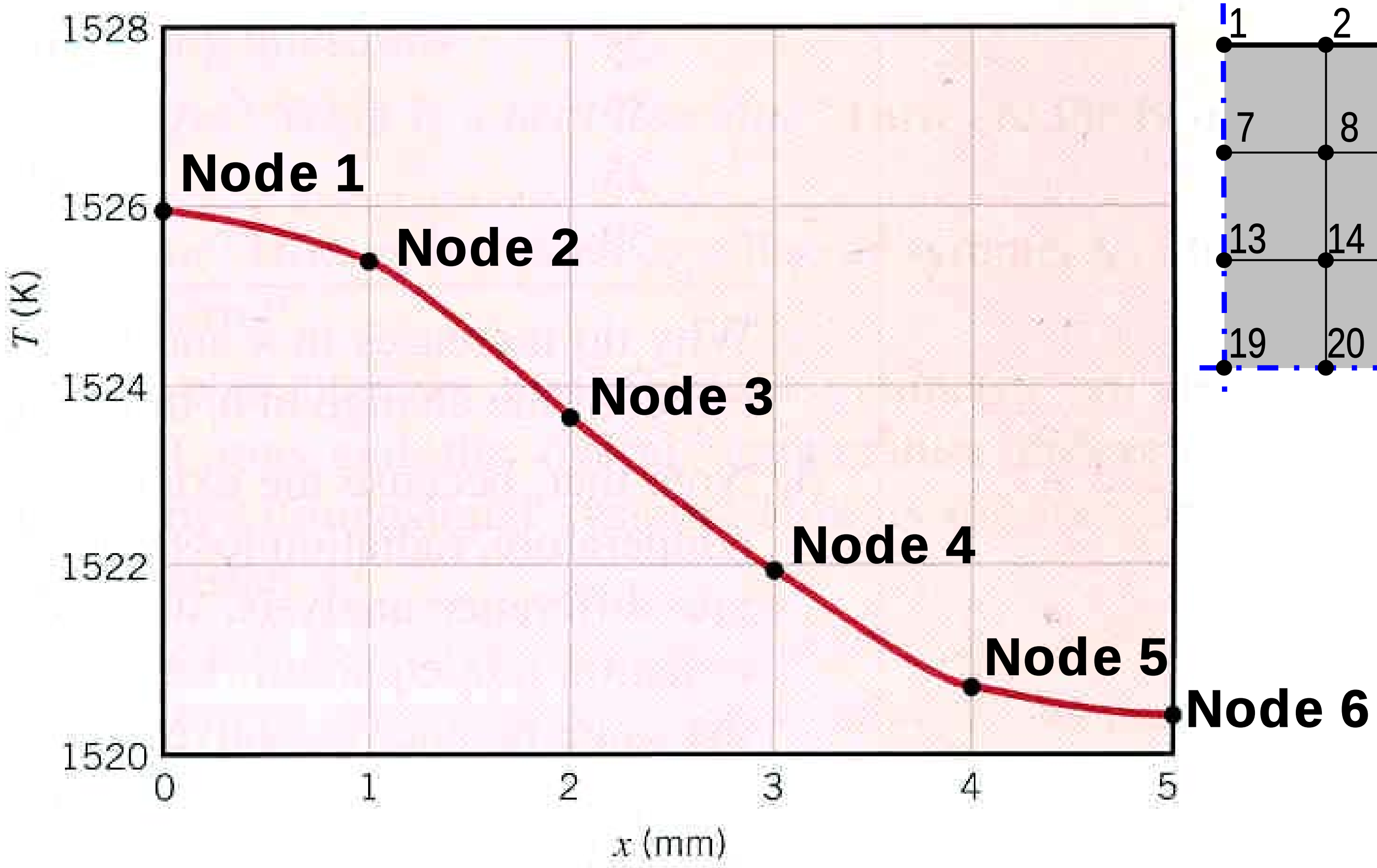
The 21 finite difference equations may be solved for the unknown temperatures and for the prescribed conditions.

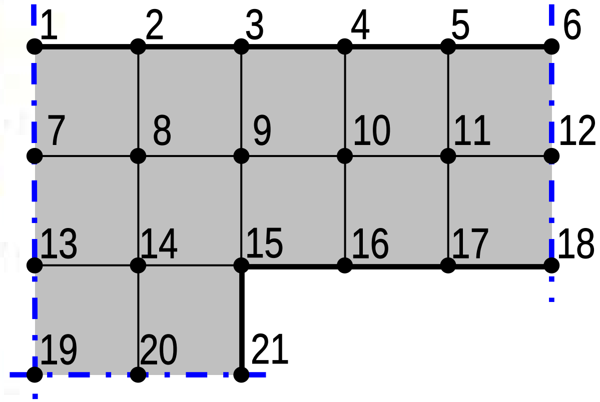
T_1 1526.0 K	T_2 1525.3 K	T_3 1523.6 K	T_4 1521.9 K	T_5 1520.8 K	T_6 1520.5 K
T_7 1519.7 K	T_8 1518.8 K	T_9 1516.5 K	T_{10} 1514.5 K	T_{11} 1513.3 K	T_{12} 1512.9 K
T_{13} 1515.1 K	T_{14} 1513.7 K	T_{15} 1509.2 K	T_{16} 1506.4 K	T_{17} 1505.0 K	T_{18} 1504.5 K
T_{19} 1513.4 K	T_{20} 1511.7 K	T_{21} 1506.0 K			

Maximum temperature (Node 1)
- the location furthest removed from coolant



Isothermal contours

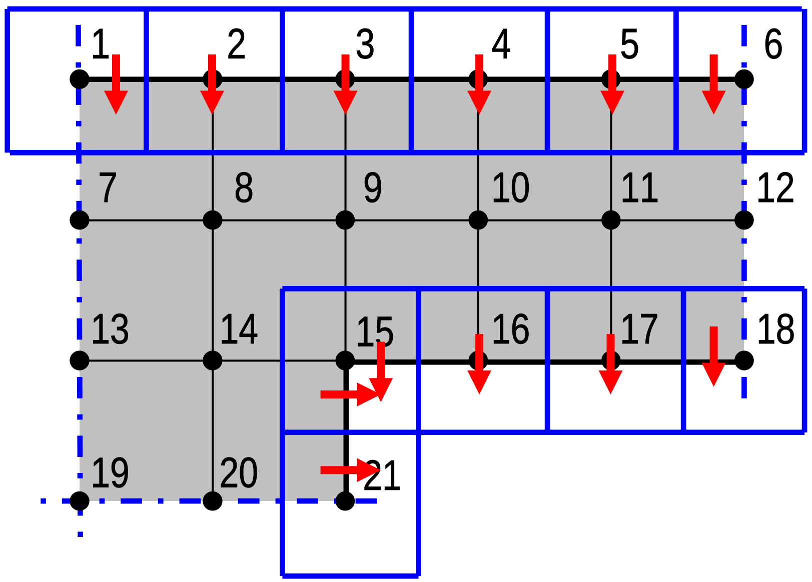




Temperatures along the surface of the turbine blade exposed to the combustion gases

heat transfer per unit length of channel :

$$q' = 4h_i[(\Delta y/2)(T_{21} - T_{\infty,i}) + (\Delta y/2 + \Delta x/2)(T_{15} - T_{\infty,i}) + (\Delta x)(T_{16} - T_{\infty,i}) + (\Delta x)(T_{17} - T_{\infty,i}) + (\Delta x/2)(T_{18} - T_{\infty,i})]$$



or, alternatively, as

$$q' = 4h_o[(\Delta x/2)(T_{\infty,o} - T_1) + (\Delta x)(T_{\infty,o} - T_2) + (\Delta x)(T_{\infty,o} - T_3) + (\Delta x)(T_{\infty,o} - T_4) + (\Delta x)(T_{\infty,o} - T_5) + (\Delta x/2)(T_{\infty,o} - T_6)]$$

In both cases,

$$\therefore q' = 3540.6 \text{ W/m}$$

Grid dependency

- **The accuracy of the solution may be improved by refining the grid.**

Ex) $\Delta x = \Delta y = 0.5\ \text{mm}$

T_1 = 1525.9 K (1526.0 K)

T_6 = 1520.5 K (1520.5 K)

T_{15} = 1509.2 K (1509.2 K)

T_{18} = 1504.5 K (1504.5 K)

T_{19} = 1513.5 K (1513.4 K)

T_{21} = 1505.7 K (1506.0 K)

$q' = 3539.9\ \text{W/m}$ (3540.6 W/m)

In the gas turbine industry, there is great interest in adopting measures that reduce blade temperature.

Alloy of larger thermal conductivity

➡ **and/or**

Increasing coolant flow through the channel (h_i)

k (W/mK)	h_i (W/m^{2}K)	T_1 (K)	q' (W/m)
25	200	1526.0	3540.6
50	200	1523.4	3563.3
25	1000	1154.5	11095.5
50	1000	1138.9	11320.7

$$(\Delta x = \Delta y = 1 \text{ mm})$$