⊙ zero-equation models

- Mixing-length models : Prandtl (1925)

$$\nu_t \propto \tilde{V} \, L$$

$\tilde{V}$: mean fluct vel.

L: mixing length l_m

shear layers with only one significant turb. stress $\overline{uv}$ and mean vel. grad. $\partial U/\partial y$ $\rightarrow$ $\tilde{V} = l_m \left|\frac{\partial U}{\partial y}\right|$

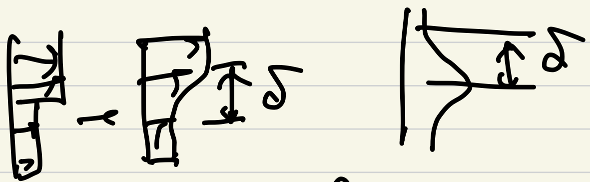
then, $\boxed{\nu_t = l_m^2 \left|\frac{\partial U}{\partial y}\right|}$: Prandtl mixing-length hypothesis

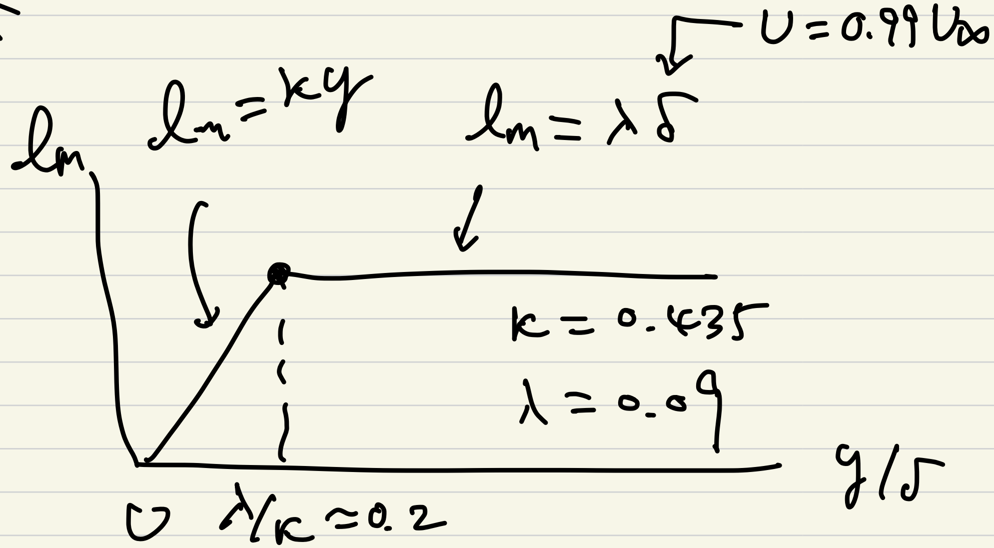
l_m : unknown

the mixing-length hypothesis has been applied with great success, at least for relatively simple flows, because l_m can be specified by simple empirical formulae in many situations.

In free shear layers, $l_m \sim$ const across the shear layer and $l_m \propto \delta$.

Flow	plane mixing layer	plane jet	round jet	radial jet	plane wake
l_m/δ	0.07	0.09	0.075	0.125	0.16





In wall boundary layers,

Nikuradse : $\frac{l_m}{R} = 0.14 - 0.08\left(1-\frac{y}{R}\right)^2 - 0.06\left(1-\frac{y}{R}\right)^4$ R: pipe radius

very close to the wall (viscosity effect)

$$l_m = \kappa y \left[1 - \exp\left(y^+/A^+\right)\right], \quad A^+ = 26$$

van Driest's damping ft.

General flow $\nu_t = l_m^2 \left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} \right]^{\frac{1}{2}}$ ← difficult to define l_m.

Heat & mass transfer

$$\Gamma = \frac{\nu_t}{\sigma_t}$$

$\sigma_t \simeq 0.9$ near-wall flows

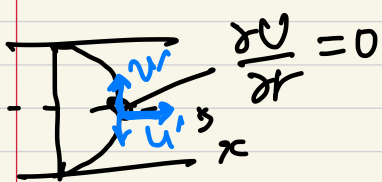
0.5 plane jet & mixing layers

0.7 round jet

Major shortcoming of the mixing-length model

- when $\partial U_i / \partial x_j = 0$, $\nu_t = 0$ and $\Gamma = 0$

↑ not true in reality

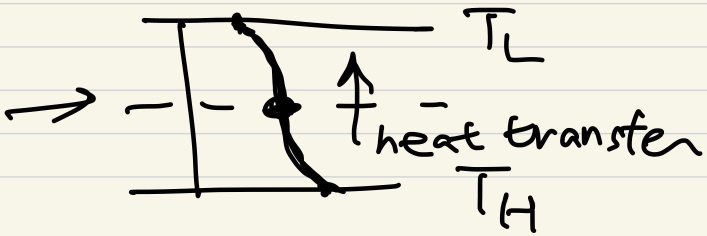


e.g. $\nu_t = 0.8\, \nu_{t\,max}$ @ pipe centerline

for velocity, it is ok because $\overline{uv} = 0$ there.

but for temperature, $\Gamma = 0$ $\overline{u\theta} \neq 0$

↓ unrealistic



- Mixing-length model implies that turb. is in a state of local equil. $\longrightarrow P \simeq \varepsilon$

  there can be no influence from other points via transport of turb. quantities.

  this inability to account for transport of turb. quantities is the main reason for the shortcomings of the mixing length model. ↳ no diffusive transport

  the model also neglects the conv. transport and thus predicts zero ν_t and Γ in grid turbulence and unrealistically low values in recirculating flows.

→ the mixing-length model is not suitable when processes of convective and diffusive transport of turb. are important.

⊙ One-equation model

- Give up the direct link bet. the fluct. vel. scale and mean vel. grad. $\quad \tilde{v} \sim l_m \left| \frac{\partial U}{\partial y} \right|$
- Determine this scale from a transport eq.

Models using the eddy viscosity concept.

velocity scale? $\quad k \longrightarrow \sqrt{k}$

turb. kinetic energy.

$$\Rightarrow \quad \nu_t = S_\mu' \sqrt{k}\, L \quad : \text{Kolmogorov-Prandtl expression}$$

↑ empirical const. $\qquad$ ('42) $\quad$ ('45)

Determine k by solving a transport eq.

$$u_i \left(\frac{\partial u_i}{\partial t} + - \cdots \right)$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left(\underbrace{\tfrac{1}{2} u_i u_i}_{= k}\right) = - - - .$$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = -\frac{\partial}{\partial x_i}\left[\underbrace{\overline{u_i\left(\frac{1}{2}u_j u_j + \frac{p}{\rho}\right)}}_{\text{diff. transport}}\right] \underbrace{- \overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{\overset{P}{\text{production}}} \underbrace{- \beta g_i \overline{u_i \phi}}_{\overset{G}{\text{buoyant prod/destruction}}} \underbrace{- \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\varepsilon}$$

conv. transport

modeling

$$-\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i} \qquad \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij}$$

$$-\overline{u_i\phi} = \Gamma\frac{\partial \Phi}{\partial x_i} = \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i} \qquad C_D\frac{k^{3/2}}{L}$$

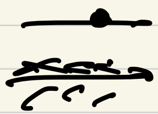
from dimensional argument

$$\Rightarrow \frac{\partial k}{\partial t} + U_j\frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i} - C_D\frac{k^{3/2}}{L}$$

good for high Re # flow

$C_\mu' C_D \simeq 0.08$, $\sigma_k \simeq 1$

The model is not applicable to viscous sublayer.



when $\partial k/\partial t = 0$ and conv. & diff. transport are negligible,

$$P \simeq \varepsilon$$

$$\left.\begin{array}{l} \rightarrow \quad \nu_t \left(\frac{\partial U}{\partial y}\right)^2 = C_D \frac{k^{\frac{3}{2}}}{L} \\ \nu_t = C_\mu' \sqrt{k}\, L \end{array}\right] \rightarrow \nu_t = \underbrace{\left(\frac{C_\mu'^3}{C_D}\right)^{\frac{1}{2}}}_{= l_m} L^2 \left|\frac{\partial U}{\partial y}\right|$$

L needs to be specified to complete turb. model.

→ but L is not easier to prescribe than l_m!

⇒ move to 2-eq. models which determine the length scales from a transport eq.

Two-equation models

equation for L is needed.

a) length scale eqs.

Dependent variables used for L in the literature:

$$\left\{\begin{array}{l} \varepsilon \sim k^{3/2}/L \\ kL,\ k^{\frac{1}{2}}/L,\ k/L^2 \end{array}\right.$$

⇒ eqs. for these variables are very similar among themselves.

$$\rightarrow \frac{\partial Z}{\partial t} + U_j \frac{\partial Z}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\sqrt{k}L}{\sigma_Z}\frac{\partial Z}{\partial x_j}\right) + C_{Z_1}\frac{Z}{k}P - C_{Z_2} Z \frac{\sqrt{k}}{L} + S$$

conv. diff. prod destruction

different depending on the choice of Z
important near the wall.

Near the wall, the gradient assumption for diffusion works better for $Z = \varepsilon$.

Also, ε-eq. does not require S.

$\Rightarrow$ ε-eq. became popular.

b) k–ε model

- ε-eq: $\varepsilon = \nu\overline{(\partial u_i/\partial x_j)^2}$

use grad. assumption

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}, \quad \Gamma = \frac{\nu_t}{\sigma_t}$$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + \overbrace{\nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}^{P} + \overbrace{\beta g_j \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_j}}^{G} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + C_{1\varepsilon}\frac{\varepsilon}{k}(P+G)(1+C_{3\varepsilon}R_f) - C_{2\varepsilon}\frac{\varepsilon^2}{k}$$

($C_\mu = 0.09$, $C_{1\varepsilon} = 1.44$, $C_{2\varepsilon} = 1.92$ ⟸ determined from extensive examination of free shear flows.
$\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

"standard k-ε model"

Constants $C_{1\varepsilon}$ and $C_{2\varepsilon}$ affect sols.

There have been many modifications in the constants.

standard k-ε model → eddy viscosity is the same for all the Reynolds stresses
(isotropic eddy viscosity)

this is not valid for square duct flow.

secondary flow ← not produced by standard k-ε model.

→ non-isotropic ν_t → algebraic stress model

- $k-\varepsilon$ model: most widely tested and successfully appied.

<u>$k-\omega$ SST model</u>

0-eq. → 1-eq. → 2-eq → more eq. models

1-eq.: k, L

2-eq: k, ε