Equilibrium

- Tokamak equilibrium
- Flux functions
- Grad-Shafranov equation
- Safety factor
- Beta
- Large aspect-ratio
- Shafranov shift
- Vacuum magnetic field
- Electric fields

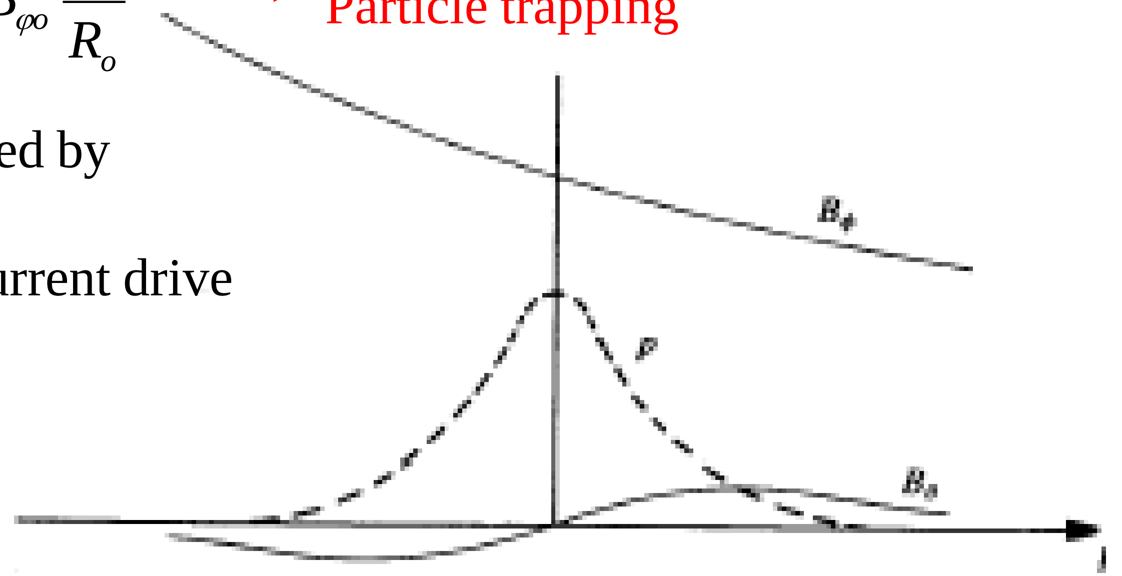
Tokamak Equilibrium

- **Internal force balance between the plasma pressure and the forces due to the magnetic field.**
- **Shape and position of the plasma determined and controlled by currents in external coils.**

Toroidal magnetic field $2\pi R B_\varphi = \mu_o I_T$

$$\Delta B_\varphi = B_{\varphi o}\left(\frac{R_o}{R_o - a} - \frac{R_o}{R_o + a}\right) \approx B_{\varphi o}\frac{2a}{R_o}$$

→ Particle trapping



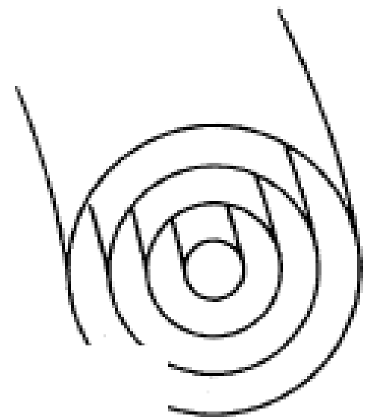
Poloidal magnetic field : provided by
plasma currents from

- inductive and non-inductive current drive
- bootstrap current

Flux Functions

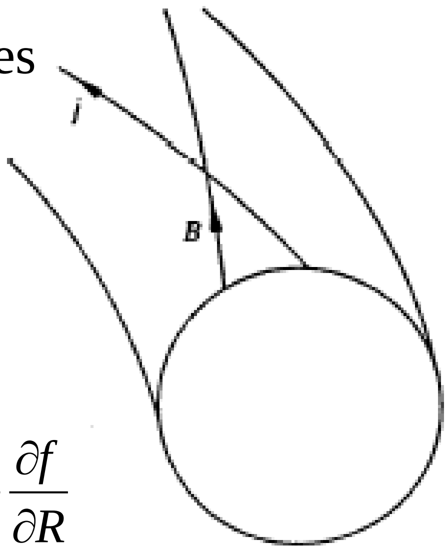
- **Magnetic flux surfaces forming a set of nested toroids : magnetic field lines lie in these surfaces**

Magnetic force balance $\vec{j} \times \vec{B} = \nabla p$

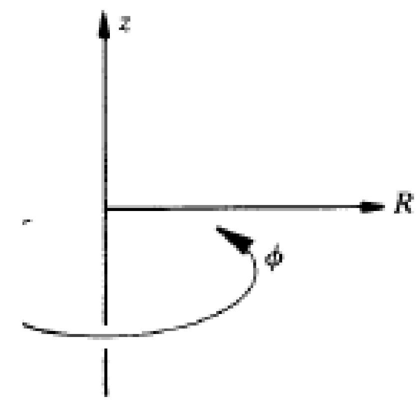


No pressure gradient along the magnetic field lines and plasma moves at sound speed of typically 10^5 - 10^6 m/sec $\vec{B} \cdot \nabla p = 0$ $\quad \vec{j} \cdot \nabla p = 0$

Current also lies in the magnetic surfaces



- **Introduce poloidal magnetic flux function, then** $\vec{B} \cdot \nabla \psi = 0$



$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z} \qquad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R} \qquad \nabla \cdot \vec{B} = 0$$

from symmetry $\quad j_R = -\frac{1}{R}\frac{\partial f}{\partial Z} \qquad j_Z = \frac{1}{R}\frac{\partial f}{\partial R}$

from Ampere's law

$$j_R = -\frac{1}{\mu_o}\frac{\partial B_\phi}{\partial Z} \qquad j_Z = \frac{1}{\mu_o R}\frac{\partial R B_\phi}{\partial R} \longrightarrow \qquad f = \frac{R B_\phi}{\mu_o}$$

Grad-Shafranov Equation

$$\vec{j}\cdot\nabla p = 0 \longrightarrow \frac{\partial f}{\partial R}\frac{\partial p}{\partial Z} - \frac{\partial f}{\partial Z}\frac{\partial p}{\partial R} = 0 \longrightarrow \nabla f \times \nabla p = 0 \longrightarrow f = f(p)$$

then $p = p(\psi) \qquad f = f(\psi)$

Axisymmetric equilibrium equation $\vec{j}\times\vec{B} = \nabla p \qquad \vec{j}_p \times \hat{\phi} B_\phi + j_\phi \hat{\phi}\times\vec{B}_p = \nabla p$

$$\vec{B}_p = \frac{1}{R}\nabla\psi\times\hat{\phi} \qquad \vec{j}_p = \frac{1}{R}\nabla f\times\hat{\phi} \qquad -\frac{B_\phi}{R}\nabla f + \frac{j_\phi}{R}\nabla\psi = \nabla p$$

then

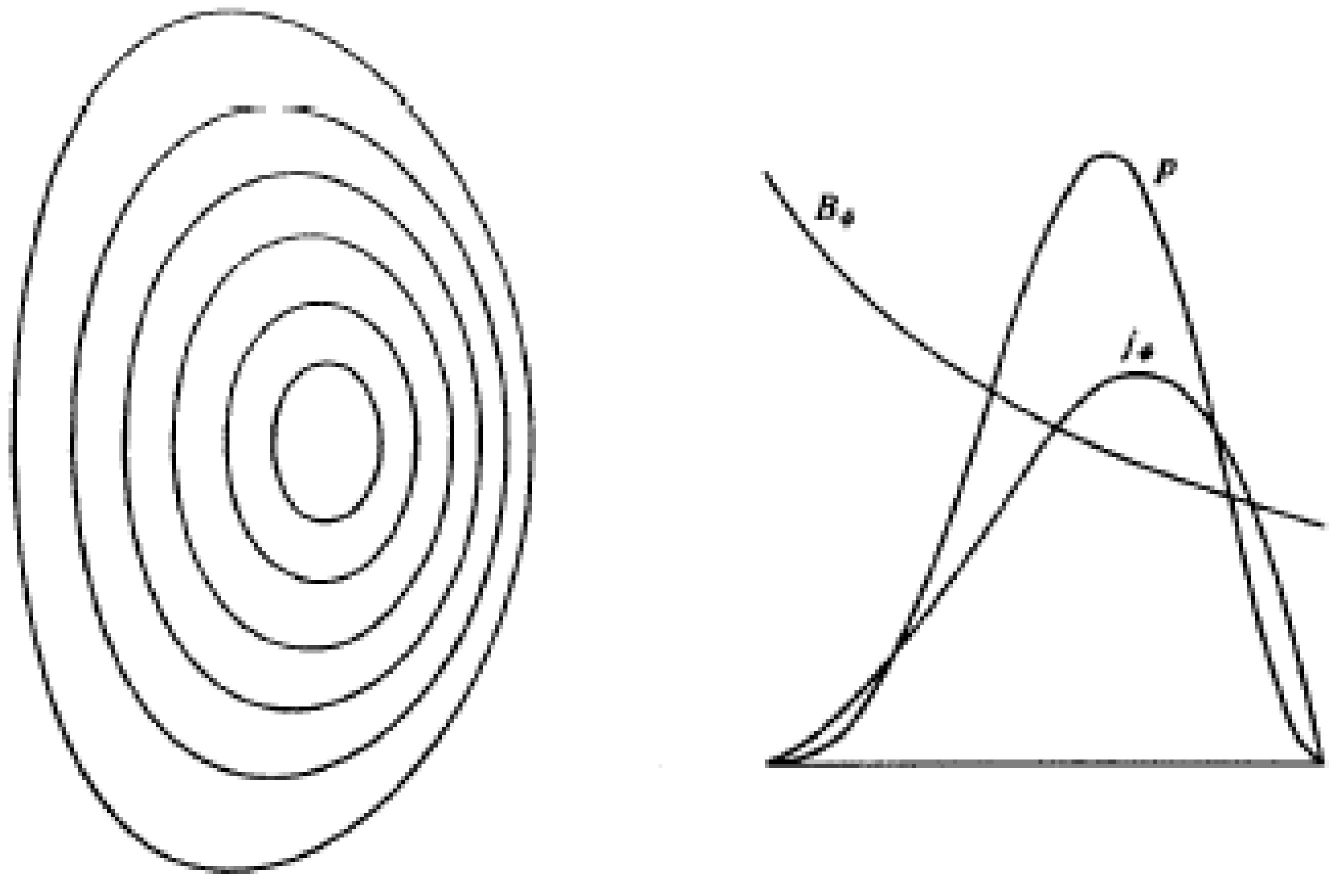
$$j_\phi = R\frac{dp}{d\psi} + B_\phi\frac{df}{d\psi} = R\left(\frac{dp}{d\psi} + \frac{\mu_o f}{R}\frac{df}{d\psi}\right)$$

with

$\mu_o\vec{j} = \nabla\times\vec{B}$ Grad-Shafranov equation becomes

$$-\mu_o R j_\phi = R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial Z^2} = -\mu_o R^2\left(\frac{dp}{d\psi} + \frac{\mu_o f}{R}\frac{df}{d\psi}\right)$$

Grad-Shafranov Equation



Equilibrium flux surfaces and plots of toroidal current density, plasma pressure, and toroidal magnetic field across the midplane

Safety Factor

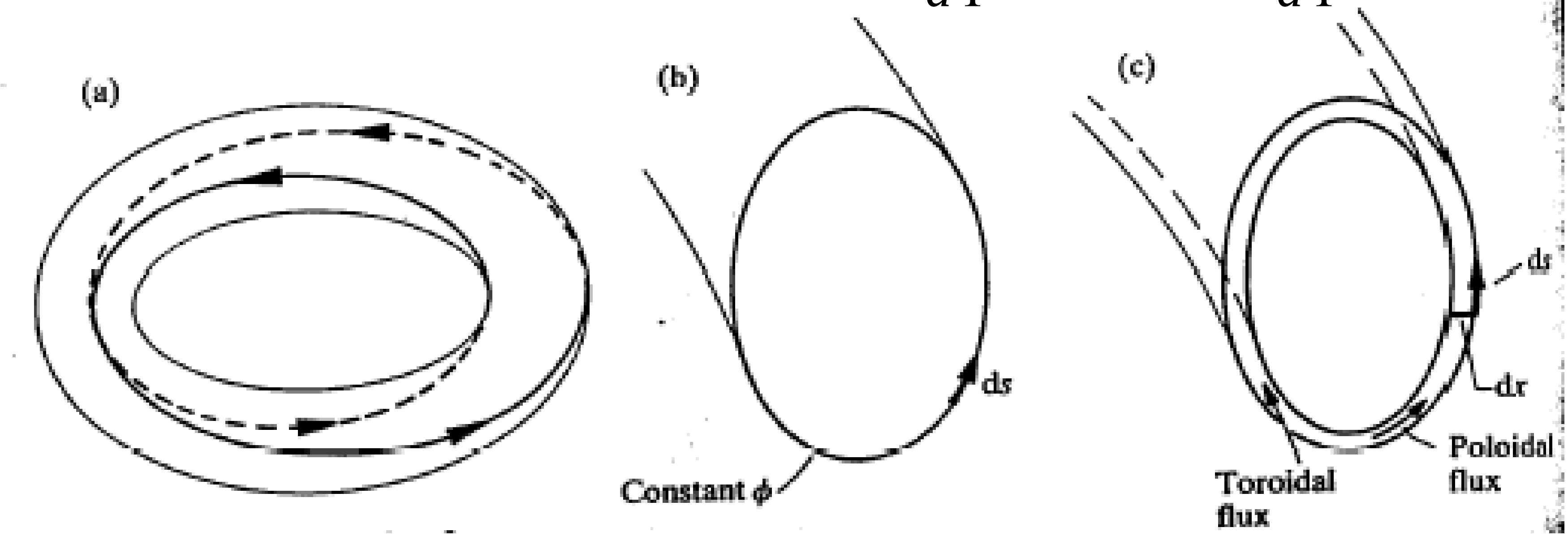
Safety factor, q, is called because of the role it plays in determining stability, i.e. higher values with greater stability.

$$q \equiv \frac{\Delta\phi}{2\pi} \qquad \iota = \frac{2\pi}{q} \qquad q = \frac{m}{n}$$

Rational surfaces : the field line joins up on itself after m toroidal and n poloidal rotations

$$\frac{Rd\phi}{ds} = \frac{B_\phi}{B_p} \longrightarrow q = \frac{1}{2\pi}\oint \frac{B_\phi}{RB_p} ds = q(\psi) \qquad q = \frac{rB_\phi}{R_o B_\theta}$$

$$d\Psi = 2\pi RB_p dx \qquad d\Phi = \oint (B_\phi dx) ds \qquad q = \frac{1}{d\Psi}\oint (B_\phi dx) ds = \frac{d\Phi}{d\Psi}$$



q Profiles

- **The direction of the magnetic field changes from surface to surface. This shear of the magnetic field has important implications for the stability of the plasma. The shear is determined by the radial change rate of q**

For large aspect-ratio and circular plasmas, $2\pi RB_\theta = \mu_o I_p(r) = \mu_o \int_0^r j(r')2\pi r'dr'$

for r=a, for r=0, $I_p(r) = \pi r^2 j(0) = \pi r^2 j_o$

$$q(r) = \frac{2\pi r^2 B_\phi}{\mu_o I_p(r) R}$$

$$q_a = \frac{2\pi a^2 B_\phi}{\mu_o I_p R} \qquad q_0 = \frac{2B_\phi}{\mu_o j_o R} \qquad \frac{q_a}{q_0} = \frac{\pi a^2 j_o}{I_p} = \frac{j_o}{< j >_a}$$

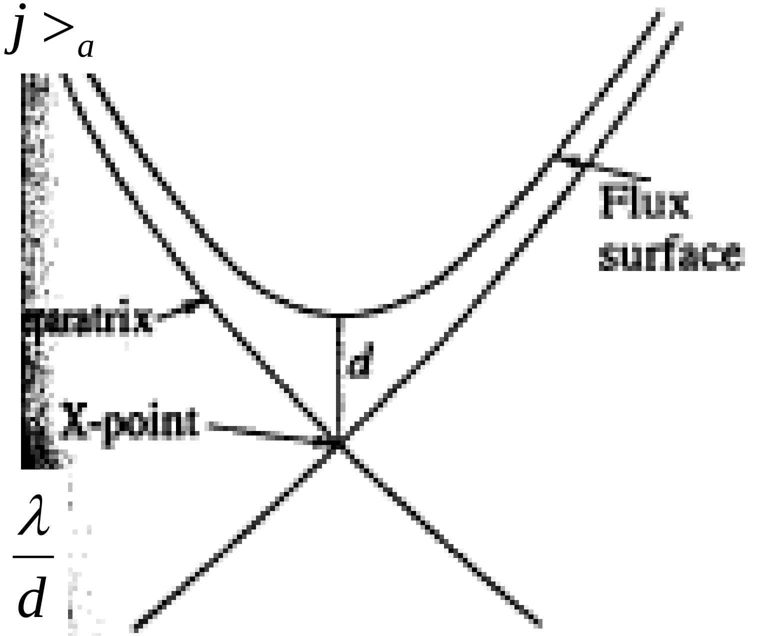
for $j = j_o(1 - r^2/a^2)^\nu$ $\quad \frac{q_a}{q_0} = \frac{j_o}{< j >_a} = \nu + 1$

- **q cylindrical**

$$q_{cyl} = \frac{2\pi rabB_\phi}{\mu_o I_p R}$$

- **q_{95}**
- **near separatrix** $(d \to 0)$

$$q \to \frac{B_\phi}{\pi R|\nabla B_p|} \ln \frac{\lambda}{d}$$



Plasma Beta

• **The efficiency of confinement of plasma pressure by the magnetic field is represented as following;**

Various definitions of beta :

$$\beta \equiv \frac{p}{B^2/2\mu_o} \qquad \beta^* = \frac{(\int p^2 d\tau / \int d\tau)^{1/2}}{B^2/2\mu_o}$$

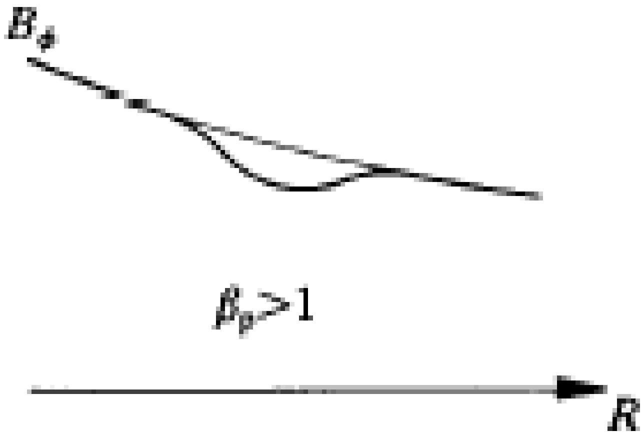
$$<\beta> = \frac{\int p d\tau / \int d\tau}{B^2/2\mu_o} \qquad \beta_p = \frac{\int p dS / \int dS}{B_a^2/2\mu_o} \qquad B_a = \frac{\mu_o I_p}{l} \qquad \beta_p = \frac{\int p dS}{\mu_o I_p^2/8\pi}$$

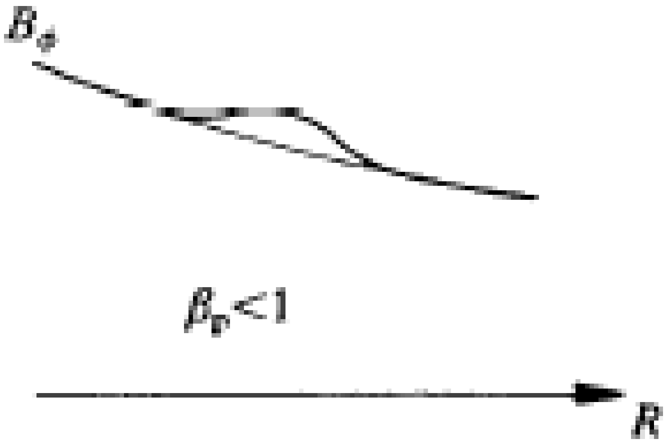
$$\frac{dp}{dr} + \frac{d}{dr}\left(\frac{B_\phi^2}{2\mu_o}\right) + \frac{B_\theta}{\mu_o r}\frac{d}{dr}(rB_\theta) = 0$$

pressure balance

poloidal beta for circular plasmas

$$\beta_p = -\frac{8\pi^2}{\mu_o I_p^2}\int_0^a \frac{dp}{dr} r^2 dr = 1 + \frac{1}{(aB_{\theta a})^2}\int_0^a \frac{dB_\phi^2}{dr} r^2 dr$$





Large Aspect-Ratio

- **tokamak equilibria for low-beta, large aspect-ratio plasmas of circular c.x.**

Ordering in terms of the inverse aspect-ratio, ε=*a/R*

$$B_\varphi = B_{\varphi o}(R_o/R)(1+0(\varepsilon^2)) \qquad B_\theta \sim \varepsilon B_{\varphi o} \qquad j_\phi \sim \varepsilon B_{\varphi o}/\mu_o a$$

$$j_\theta \sim \varepsilon^2 B_{\varphi o}/\mu_o a \qquad \beta \sim \varepsilon^2 \qquad p \sim \varepsilon^2 B_{\phi o}^3/\mu_o \qquad \beta_p \sim 1$$

Basic pressure balance equation of a cylinder $\quad \dfrac{dp}{dr} = j_\phi B_\theta - j_\theta B_{\phi o}$

Ampere's equation $\quad \mu_o j_\phi = -\dfrac{1}{r}\dfrac{d}{dr}(rB_\theta)$

$$\longrightarrow \qquad j_\theta = (j_\phi B_\theta - \frac{dp}{dr})/B_{\phi o} = -\left(\frac{B_\theta}{\mu_o r}\frac{d}{dr}(rB_\theta) + \frac{dp}{dr}\right)/B_{\phi o}$$

$$= -\left(\frac{B_\theta^2}{\mu_o r} + \frac{d}{dr}(p + \frac{B_\theta^2}{2\mu_o})\right)/B_{\phi o}$$

Large Aspect-Ratio

- **non-concentric flux surfaces with toroidal effects**

Grad-Shafranov equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi+\frac{1}{R_o+r\cos\theta}\left(\cos\theta\frac{\partial}{\partial r}-\sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\psi$$

$$=-\mu_o(R_o+r\cos\theta)^2p'(\psi)-\mu_o^2f(\psi)f'(\psi)$$

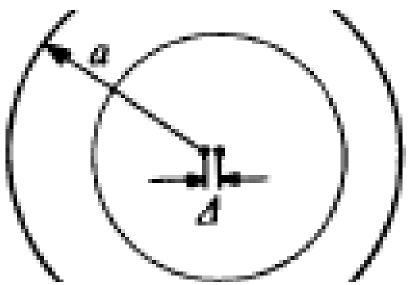
$$\psi=\psi_o+\psi_1$$

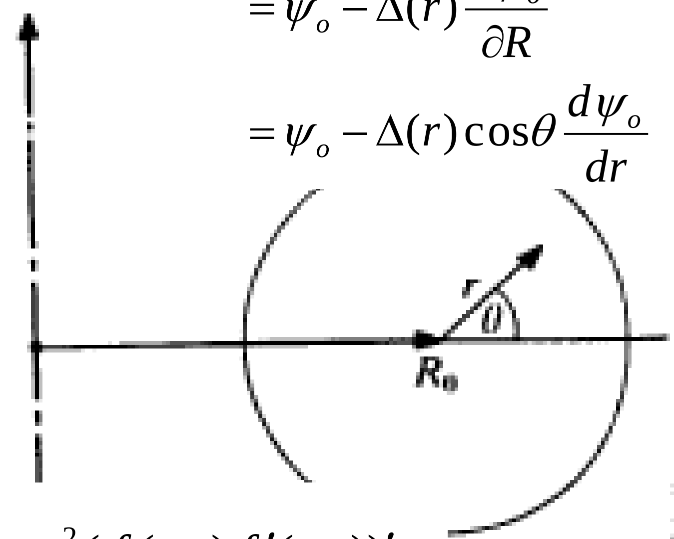
$$=\psi_o-\Delta(r)\frac{\partial\psi_o}{\partial R}$$

$$=\psi_o-\Delta(r)\cos\theta\frac{d\psi_o}{dr}$$

Expanding ψ in ε=*a/R*, $\psi=\psi_o(r)+\psi_1(r,\theta)$

$$\frac{1}{r}\frac{d}{dr}(r\frac{d\psi_o}{dr})=-\mu_oR_o^2p'(\psi_o)-\mu_o^2f(\psi_o)f'(\psi_o)$$





$$(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\psi_1-\frac{\cos\theta}{R_o}\frac{d\psi_o}{dr}=-\mu_oR_o^2p''(\psi_o)\psi_1-\mu_o^2(f(\psi_o)f'(\psi_o))'\psi_1$$

$$=-\frac{d}{dr}(\mu_oR_o^2p'(\psi_o)\psi_1-\mu_o^2(f(\psi_o)f'(\psi_o))\frac{dr}{d\psi_o}\psi_1-2\mu_oR_or\cos\theta p'(\psi_o)$$

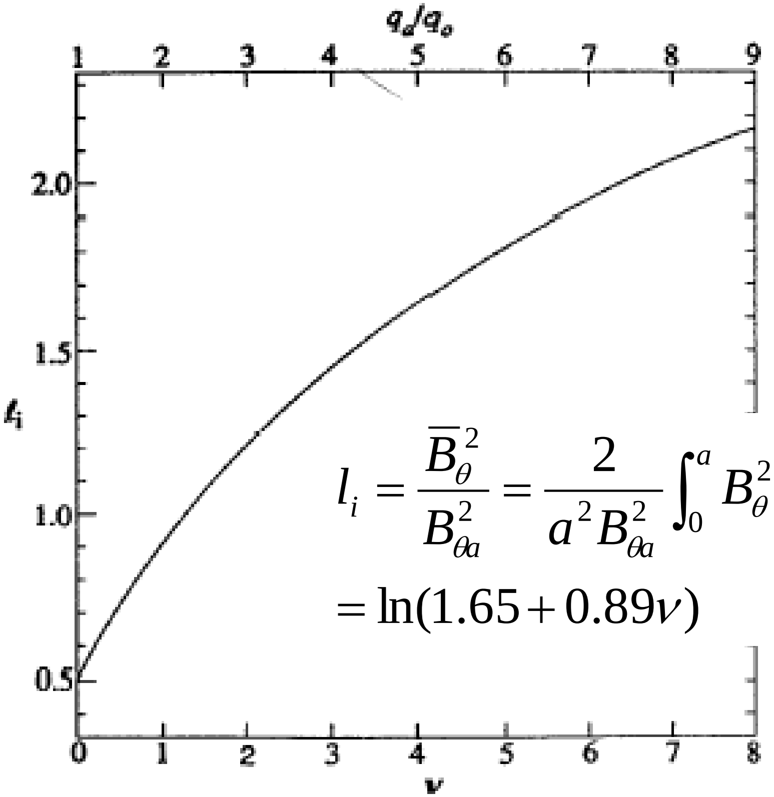
Shafranov Shift

$$\frac{d}{dr}\left(rB_{\theta o}^2\frac{d\Delta}{dr}\right)=\frac{r}{R_o}\left(2\mu_o r\frac{dp_o}{dr}-B_{\theta o}^2\right)$$

$$\frac{d\Delta}{dr}=\frac{1}{rB_{\theta o}^2}\frac{d}{dr}\left(\frac{r^3}{a^2}\beta_p B_{\theta a}^2+\frac{1}{r}\int_0^r B_\theta^2 rdr\right)$$

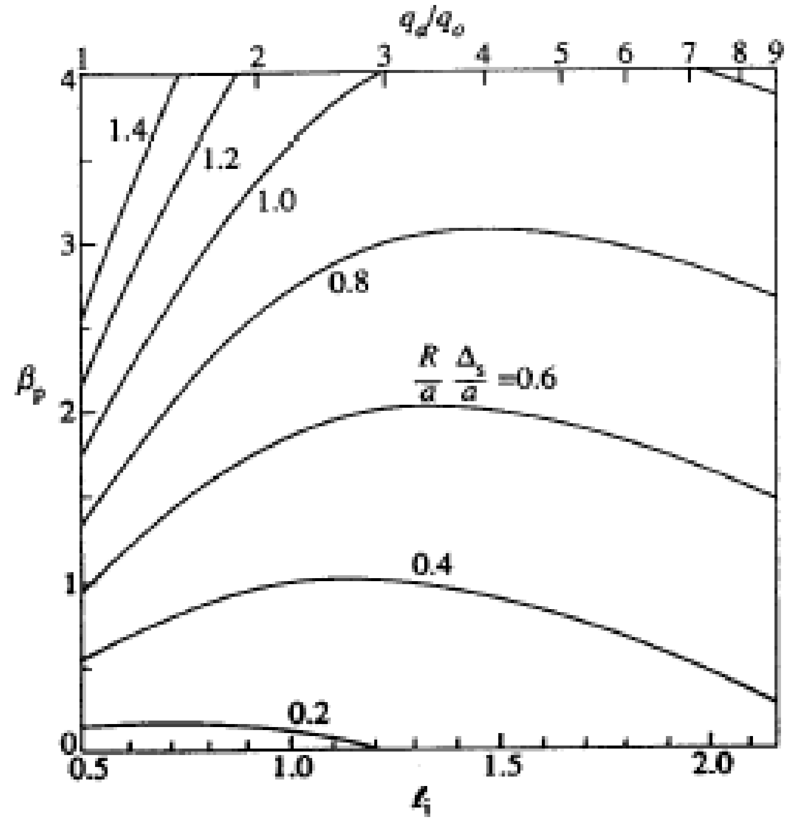
$$p=p_o\left(1-\frac{r^2}{a^2}\right)\quad j=j_o\left(1-\frac{r^2}{a^2}\right)^\nu\quad B_{\theta o}=\frac{1}{R_o}\frac{d\psi_o}{dr}$$

$$\beta_p=\frac{\overline{p}}{B_{\theta a}^2/2\mu_o}=\frac{4\mu_o}{a^2B_{\theta a}^2}\int_0^a prdr$$



$$l_i=\frac{\overline{B}_\theta^2}{B_{\theta a}^2}=\frac{2}{a^2B_{\theta a}^2}\int_0^a B_\theta^2 rdr$$

$$=\ln(1.65+0.89\nu)$$



Vacuum Magnetic Field

Using large aspect-ratio expansion, $$B_\theta = \frac{1}{R}\frac{d\psi}{dr} = \frac{1}{R_o + r\cos\theta}\frac{d\psi}{dr}$$

$$\psi = \psi_o - \Delta(r)\cos\theta \frac{d\psi_o}{dr}$$

With Δ(a)=0, $$B_\theta(a) = B_{\theta o}(a)[1-(\frac{a}{R_o}+(\frac{d\Delta}{dr})_a)\cos\theta]$$

$$\frac{d\Delta}{dr} = \frac{1}{rB_{\theta o}^2}\frac{d}{dr}\left(\frac{r^3}{a^2}\beta_p B_{\theta a}^2 + \frac{1}{r}\int_0^r B_\theta^2 r dr\right) = \frac{2\mu_o}{rR_o B_{\theta o}^2}\frac{d}{dr}\left(r^2 p_o - \int_0^r (2p_o + \frac{B_{\theta o}^2}{2\mu_o}) r dr\right)$$

Using the definitions of β_p and l_i and taking $p_o(a)=0$, $$\left(\frac{d\Delta}{dr}\right)_a = -\frac{a}{R_o}\left(\beta_p + \frac{l_i}{2}\right)$$

$$B_\theta(a) = B_{\theta o}(a)\left(1 + \frac{a}{R_o}\Lambda\cos\theta\right) \qquad \Lambda = \beta_p + \frac{l_i}{2} - 1$$

The vacuum magnetic field must now be matched to this solution for $B_\theta(a)$.

$$\psi = \frac{\mu_o I_p}{2\pi} R_o\left(\ln\frac{8R_o}{r} - 2\right) - \frac{\mu_o I_p}{4\pi} r\left(\ln\frac{r}{a} + (\Lambda + \frac{1}{2})(1 - \frac{a^2}{r^2})\right)\cos\theta$$

The required vertical magnetic field for equilibrium is $$B_v = -\frac{\mu_o I_p}{4\pi R_o}\left(\ln\frac{8R_o}{a} + \Lambda - \frac{1}{2}\right)$$

Electric Fields

• **Three electric field components in a tokamak equilibrium**

: toroidal and poloidal (or parallel and perpendicular to the magnetic field) and radial (perpendicular to the magnetic surface)

• **Toroidal electric field**

: forms discharge by deriving a current in the outer region of the plasma. Then the current diffuses into the plasma.

• **ExB drift**

: inward drift E_z/B_θ balanced by the outward diffusive plasma flux.

$$v_r = 0 \qquad \vec{E} = \eta \vec{j}$$

• **Faraday's law in the final steady-state**

: constant toroidal voltage, $2\pi RE_\phi$, by the imposed flux change through the torus.

$$\nabla \times \vec{E} = 0 \qquad E_\phi = c / R \qquad \oint \vec{E}_p \cdot d\vec{s} = 0$$

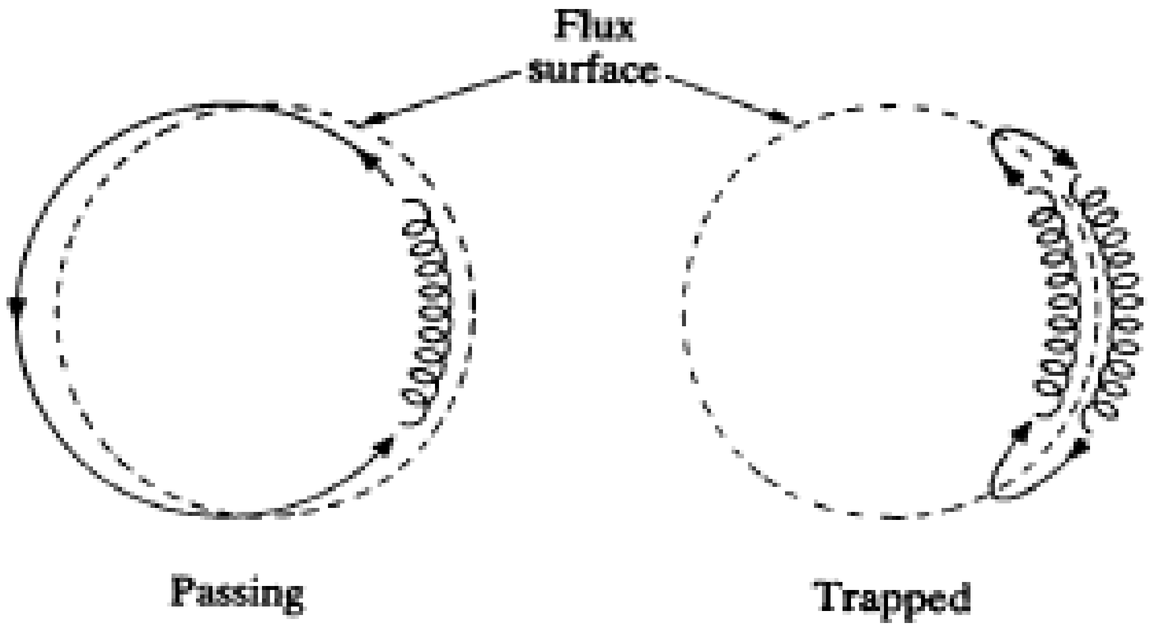
• **Force balance of each species in equilibrium**

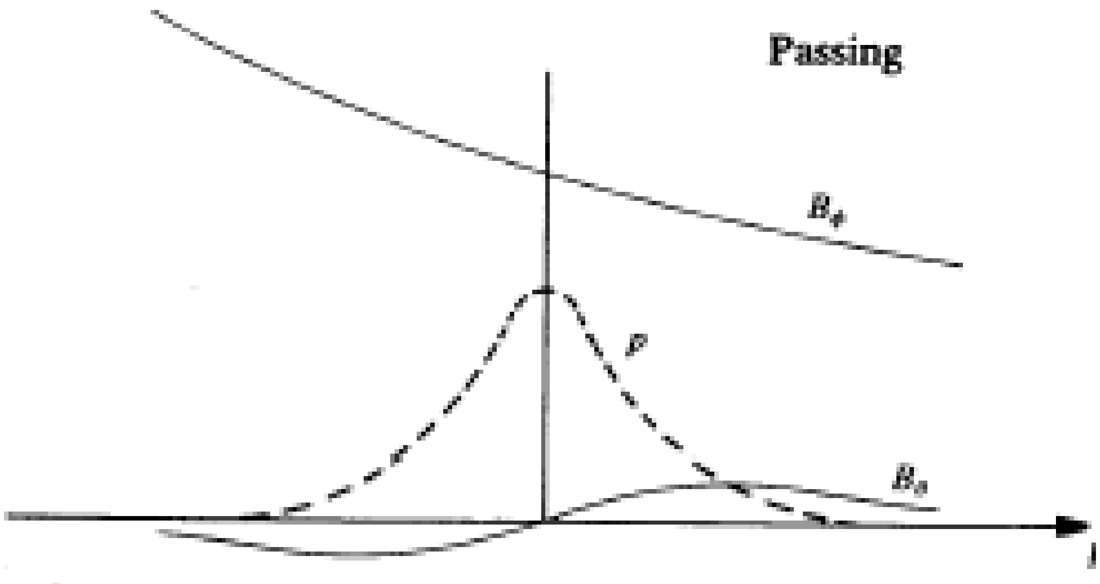
$$n_j e_j (E_n + (\vec{v}_j \times \vec{B})_n) = \frac{dp_j}{dx_n}$$

Particle Orbits in Tokamak Geometry

- Particle orbits
- Particle trapping
- Trapped particle orbits

Drift Surfaces for Passing and Trapped Particles





Particle Orbits

Change of mechanical momentum from the equation of motion

canonical momentum

$$m_j \frac{d}{dt}(Rv_\phi) = e_j R(\vec{v}\times\vec{B})_\phi = e_j R(\vec{v}\times(\nabla\phi\times\nabla\psi))_\phi$$

$$= -e_j \vec{v}\cdot\nabla\psi = -e_j \frac{d\psi}{dt} \longrightarrow \frac{d}{dt}(m_j R v_\phi + e_j\psi) = \frac{dp_\phi}{dt} = 0$$

for a small displacement *d* from the flux surface, $|\delta\psi| = |\nabla\psi| d$

$$d = \left| \frac{m_j}{e_j} \frac{\delta(Rv_\phi)}{\nabla\psi} \right| \qquad |\nabla\psi| = R_o B_\theta \qquad \delta(Rv_\phi) \le R_o v_\phi \longrightarrow d \le \left| \frac{m_j}{e_j}\frac{v_\phi}{B_\theta} \right| = \left| \frac{v_\phi}{\omega_{c\theta}} \right|$$

Drift surfaces of passing particles are determined by the poloidal rotation and vertical drift motions as following;

$$r\omega = (B_\theta / B) v_{//} \qquad v_d = \frac{m_j (v_{//}^2 + v_\perp^2/2)}{e_j R B_\phi}$$

$$\longrightarrow \left.\begin{aligned} \frac{dR}{dt} &= \omega z \\ \frac{dz}{dt} &= -\omega(R-R_c) + v_d \end{aligned}\right\} (R - R_c - \frac{v_d}{\omega})^2 + z^2 = const.$$

$$d = -\frac{v_d}{\omega} \cong -\frac{r}{R}\frac{v_{//}}{\omega_{c\theta}}$$

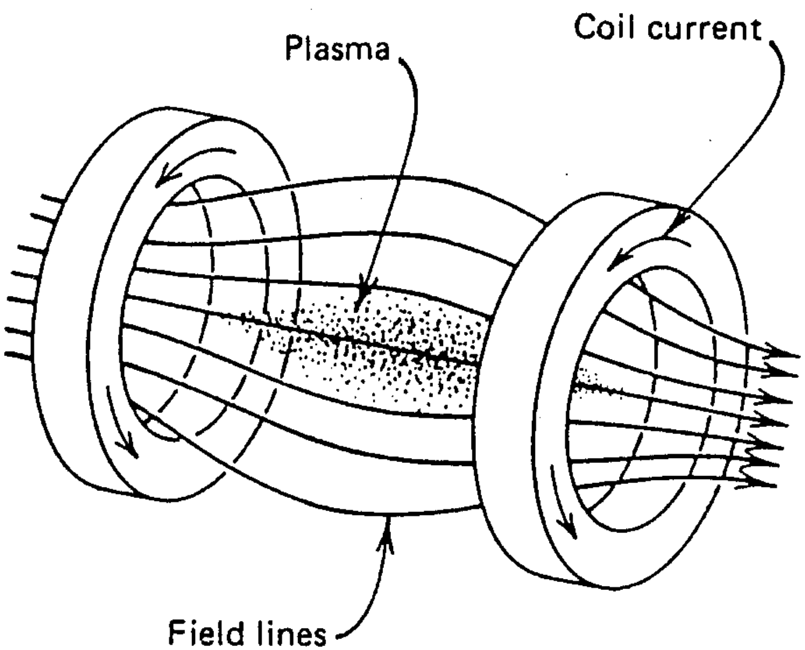
Particle Trapping in Mirror

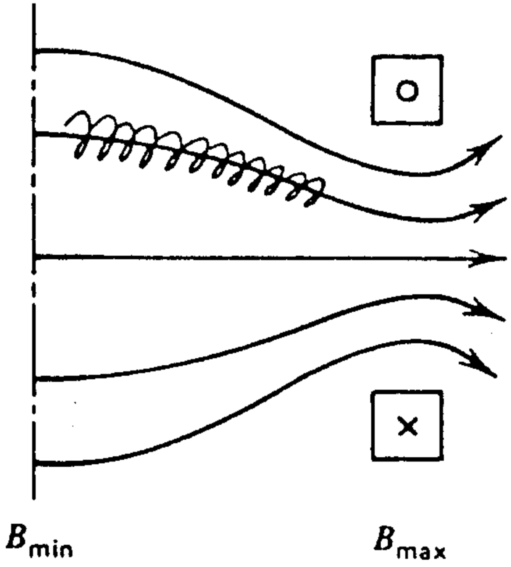
- Magnetic moment as an adiabatic invariant $\mu = \frac{1}{2} m v_\perp^2 / B$
- Trapping condition

$$\mu = \frac{1}{2} m v_\perp^2 / B = \frac{1}{2} m v_{\perp o}^2 / B_{\min}$$

$$B_b < B_{\max}$$

$$B_b / B_{\min} = 1 + v_{//o}^2 / v_{\perp o}^2$$





Simple magentic mirror

Particle Trapping in Tokamak

$B_b < B_{\max}$ gives

$$B = B_o \frac{R_o}{R} \qquad \frac{B_{\max}}{B_{\min}} = \frac{R_o + r}{R_o - r}$$

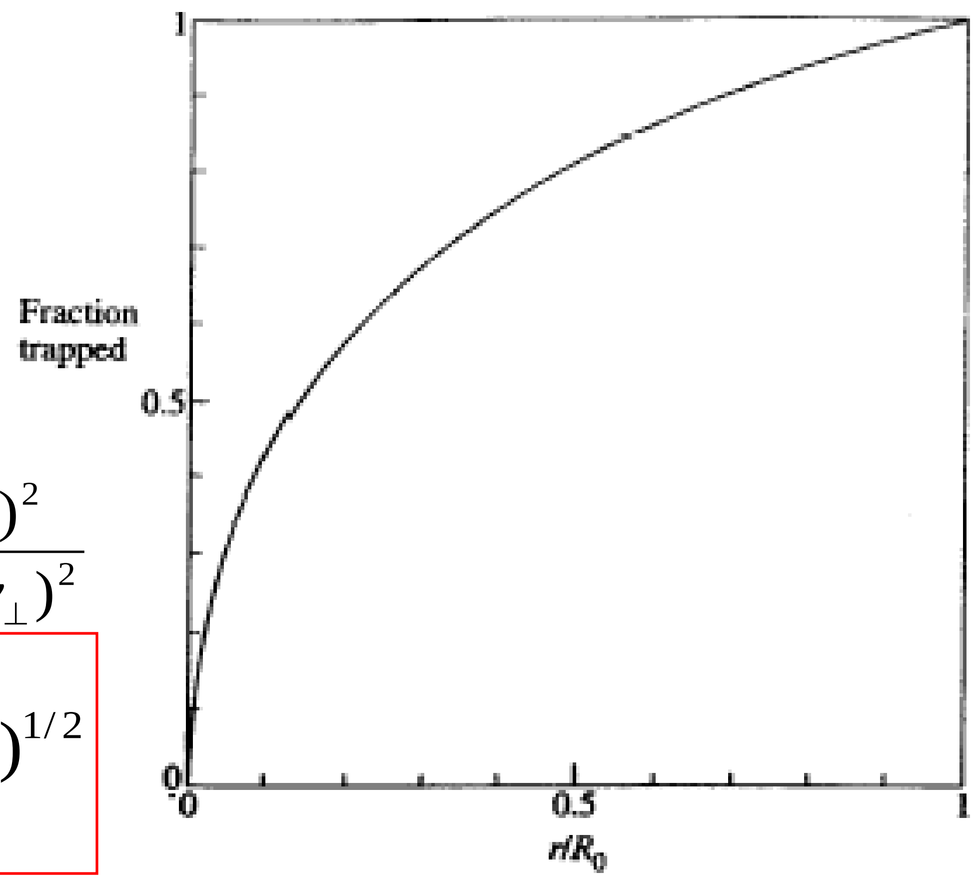
$$\left(\frac{v_{//o}}{v_{\perp o}}\right) < \left(\frac{2r}{R_o - r}\right)^{1/2}$$

For an isotropic Maxwellian distribution function,
the trapped particle fraction f

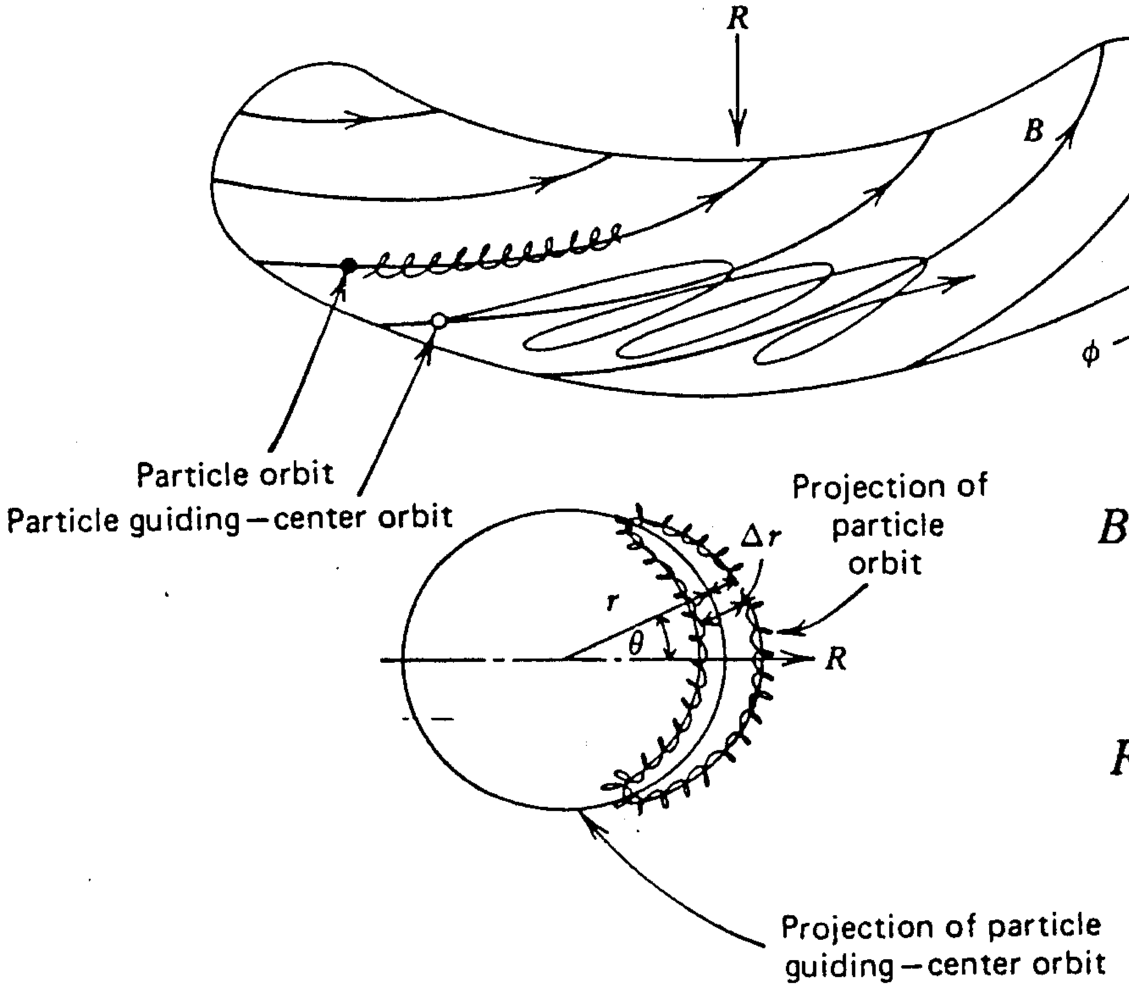
$$f = \left(\frac{v_{//o}}{v_o}\right)_{crit}$$

$$\left(\frac{v_{//}}{v}\right)^2 = \frac{(v_{//}/v_{\perp})^2}{1+(v_{//}/v_{\perp})^2}$$

$$\boxed{f = \left(\frac{2r}{R_o + r}\right)^{1/2} \approx (2\varepsilon)^{1/2}}$$



Trapped Particle Orbits in Tokamak



$$B \approx |B_0|\,(1 - \varepsilon \cos\theta)$$

$$R = R_0\,(1 + \varepsilon \cos\theta)$$

FIGURE 5.4. (*a*) Illustration of simple mirror trapping and particle reflection. (*b*) Mirror reflections produce complex orbits in a tokamak.

Trapped Particle Orbits in Tokamak

$$B = B_o \frac{R_o}{R} = \frac{B_o}{1 + (r/R_o \cos\theta)}$$ for large aspect-ratio

$$\approx B_o(1 - r/R_o \cos\theta) \approx B_o\{1 - r/R_o(1 - \theta^2/2)\}$$

For strongly trapped particles ($\theta << 1$), along the field line

$$\frac{dB}{ds} = \frac{rB_o}{R_o}\frac{d(\theta^2/2)}{ds} \qquad \frac{rd\theta}{ds} = \frac{B_\theta}{B} \longrightarrow \theta = \frac{B_\theta}{rB}s$$

From mirror force equation, $\vec{F} = m\frac{d\vec{v}}{dt} = -\mu\nabla_{//}B = -\frac{mv_\perp^2/2}{B}\nabla_{//}B$

$$\frac{d^2s}{dt^2} = -\frac{v_\perp^2/2}{B}\frac{dB}{ds} = -\frac{rB_o v_\perp^2/2}{BR_o}\frac{d(\theta^2/2)}{ds} = -\omega_b^2 s$$

, where bounce frequency with safety factor $q = rB_o/R_o B_\theta$

$$\boxed{\omega_b = \frac{v_\perp}{qR_o}\left(\frac{r}{2R_o}\right)^{1/2}}$$

Trapped Particle Orbits in Tokamak

Motion along the magnetic field line $s = s_b \sin \omega_b t$

since $\theta \propto s$ $\quad \theta = \theta_b \sin \omega_b t$

Turning point $\theta_b << 1$ $\quad B_b / B_{\min} = 1 + v_{//o}^2 / v_{\perp o}^2$

$$B_b / B_{\min} = 1 + \frac{r}{R_o}\frac{\theta_b^2}{2}$$

$$\theta_b = \frac{v_{//o}}{v_{\perp o}}\left(\frac{2R_o}{r}\right)^{1/2}$$

Drift surface by including the r-component of the vertical drift due to the toroidal magnetic field for v↗>>v//,

$$v_d = \frac{1}{2} m_j v_\perp^2 / e_j R B_\phi$$

$$\frac{dr}{dt} = v_d \sin\theta \cong v_d \theta$$

$$\frac{d\theta}{dt} = \omega_b \theta_b \cos \omega_b t \cong \omega_b \theta_b \left(1 - (\theta/\theta_b)^2\right)^{1/2}$$

$$\frac{dr}{d\theta} = \frac{v_d}{\omega_b \theta_b} \frac{\theta}{\left(1 - (\theta/\theta_b)^2\right)^{1/2}}$$

$$\boxed{\Delta r = \frac{\theta_b v_d}{\omega_b} = \frac{v_{//o}}{\omega_{c\theta}}}$$

Conditions of Collisionality

Detrapping time: collisions cause trapped particles to diffuse out of the trapping cone in atime proportional to the square of the trapping angle.

$$\tau_{\mathrm{det}rap} \cong \frac{2r}{R_o}\tau_{coll}$$

Collision time
for a large angle scatter

Condition for collisions to prevent trapping : detrapping time < bounce time

$$\tau_{\mathrm{det}rap} \cong \frac{2r}{R_o}\tau_{coll} < \frac{1}{\omega_b} = \frac{qR_o}{v_\perp}\left(\frac{2R_o}{r}\right)^{1/2}$$

$$\tau_e[\mathrm{sec}] = 1.09\times 10^{16}\frac{T_e^{3/2}[keV]}{n_i Z^2 \ln\Lambda}$$

or

$$\tau_i[\mathrm{sec}] = 6.60\times 10^{17}\frac{(m_i/m_p)^{1/2}T_i^{3/2}[keV]}{n_i Z^4 \ln\Lambda}$$

$$\boxed{\tau_{coll} \le \frac{qR_o}{\sqrt{2}v_\perp}\left(\frac{R_o}{r}\right)^{3/2}}$$

$$v_\perp = \sqrt{2}v_T$$