$\tau_{ij} = 2 \nu_T S_{ij}$

$\hat{V}$ L : l_m : zero-eq.

$\sqrt{k} \rightarrow \frac{dk}{dt} + \cdots$ 1-eq.

$\varepsilon \rightarrow \frac{d\varepsilon}{dt} + \cdots$

k-ε eq.
2-eq. model

⊙ Turbulent stress / flux - equation models

So far, local state of turb. is assumed to be characterized by one vel. scale, and the individual Reynolds stresses are related to this scale.

$\sqrt{k}$

→ transport of the individual stresses is not adequately assessed. $\overline{u_i u_j}$

→ transport eqs. for $\overline{u_i u_j}$

⊙ Reynolds-stress eqs.

$$\frac{\partial}{\partial t}\overline{u_i u_j} + \underbrace{U_\ell \frac{\partial}{\partial x_\ell}\overline{u_i u_j}}_{\text{conv.}} = \underbrace{-\frac{\partial}{\partial x_\ell}\overline{u_\ell u_i u_j} - \frac{1}{\rho}\left(\frac{\partial \overline{u_j p}}{\partial x_i} + \frac{\partial \overline{u_i p}}{\partial x_j}\right)}_{\text{diff.}}$$

$$\underbrace{-\overline{u_i u_\ell}\frac{\partial U_j}{\partial x_\ell} - \overline{u_j u_\ell}\frac{\partial U_i}{\partial x_\ell}}_{= P_{ij}\ \text{prod.}} \underbrace{- \beta\left(g_i \overline{u_j \phi} + g_j \overline{u_i \phi}\right)}_{\text{buoyancy prod.}} + \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{= \pi_{ij}\ \text{press-strain corr.}}$$

$$\underbrace{-2\nu\overline{\frac{\partial u_i}{\partial x_\ell}\frac{\partial u_j}{\partial x_\ell}}}_{= \varepsilon_{ij}\ \text{vis. dissipation}}$$

- at high Re, turb. is locally isotropic

$$\overline{\frac{\partial u_i}{\partial x_\ell}\frac{\partial u_j}{\partial x_\ell}} = 0 \quad \text{for } i \neq j \quad \Rightarrow \quad \varepsilon_{ij} = \frac{2}{3}\varepsilon\,\delta_{ij}$$

- press-strain corr. modelling : eliminate p via Poisson eq.

$$\pi_{ij,1} = -C_1 \frac{\varepsilon}{k}\left(\overline{u_i u_j} - \frac{2}{3}\delta_{ij} k\right)$$

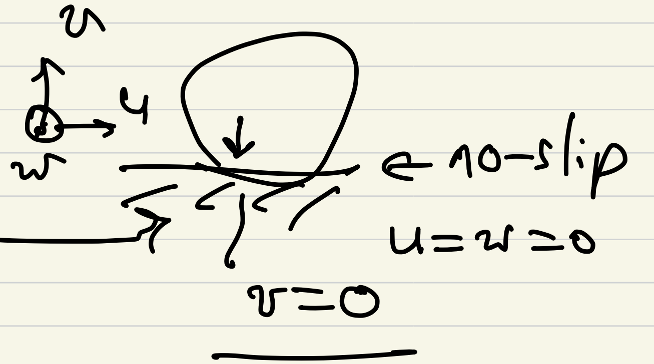
$$\pi_{ij,2} = -\gamma\left(P_{ij} - \frac{2}{3}\delta_{ij} P\right)$$

- diff. transport $= C_s \frac{\partial}{\partial x_\ell}\left(\frac{k}{\epsilon}\overline{u_k u_\ell}\frac{\partial \overline{u_i u_j}}{\partial x_k}\right)$
- $\varepsilon \rightarrow$ ε-eq : diffusion term in ε-eq

$$\rightarrow C_\varepsilon \frac{\partial}{\partial x_\ell}\left(\frac{k}{\epsilon}\overline{u_\ell u_k}\frac{\partial \varepsilon}{\partial x_k}\right)$$

- wall & free-surface effect

↳ low Re
wall-block effect
(damping)
↳ need corr.



⇒ Launder-Reece-Rodi model
(LRR)

good for walljet, turb-driven secondary flow
bad for roundjet, swirl, wake
not very popular these days.

- Algebraic stress/flux models

  $\overline{u_i u_j} \rightarrow$ 6 PDEs $\rightarrow$ too much

  neglect the rate of change & transport terms

  $\rightarrow$ sufficiently accurate in many cases

$$\overline{u_i u_j} = k\left[\frac{2}{3}\delta_{ij} + \frac{(1-\gamma)\left(\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{P}{\varepsilon}\right) + (1-C_3)\left(\frac{G_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{G}{\varepsilon}\right)}{C_1 + \frac{P+G}{\varepsilon} - 1}\right]$$

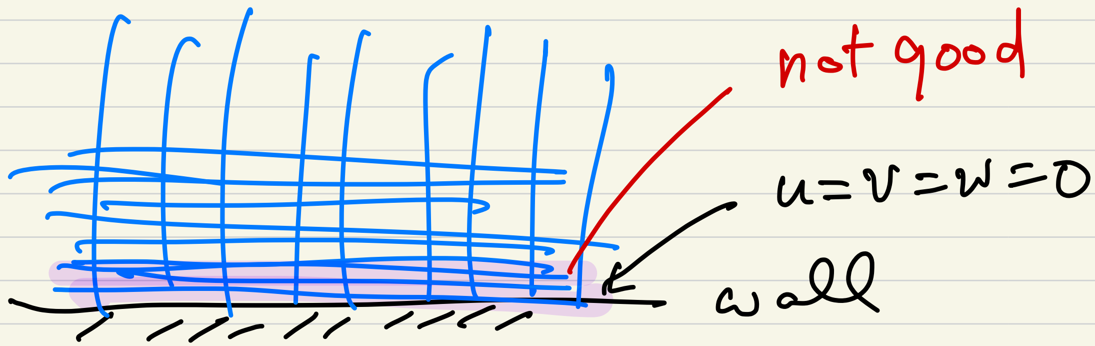
  use this eq. together with $k-\varepsilon$ eqs.

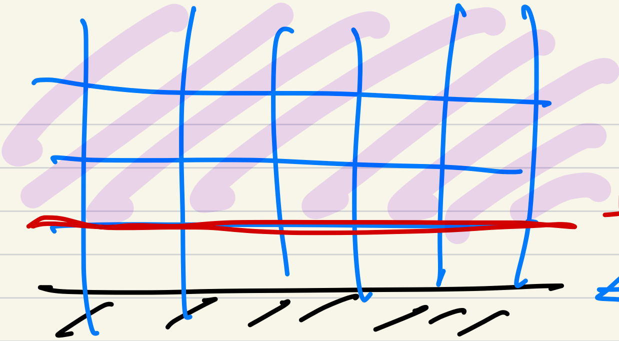
  ✓ ε-eq. has problems

- boundary conds.

  $k-\varepsilon$ model

  k-eq
  ε-eq.



↳ 

Other turbulence models

- Low Reynolds number $k-\epsilon$ model

  Launder – Sharma (1974)

- k – ω shear stress transport (SST) model

  Menter (AIAA J. 1994)

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho u_j k) = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta^* \rho \omega k + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_k \mu_t)\frac{\partial k}{\partial x_j}\right]$$

$$\frac{\partial}{\partial t}(\rho \omega) + \frac{\partial}{\partial x_j}(\rho u_j \omega) = \frac{\gamma}{\nu_t}\tau_{ij}\frac{\partial u_i}{\partial x_j} - \beta \rho \omega^2 + \frac{\partial}{\partial x_j}\left[(\mu + \sigma_\omega \mu_t)\frac{\partial \omega}{\partial x_j}\right]$$

$$+ 2(1-F_1)\rho \frac{\sigma_{\omega_2}}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}$$

$$\tau_{ij} = \mu_t \left[2 S_{ij} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] - \frac{2}{3} \rho k \delta_{ij}$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$F_1 = \tanh(\arg_1^4)$$

$$\mu_t = \rho a_1 k / \max(a_1 \omega, \Omega F_2)$$

$$\arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{\beta^* \omega d}, \frac{500\nu}{d^2 \omega} \right), \frac{4 \rho \sigma_{\omega 2} k}{CD_{k\omega} d^2} \right]$$

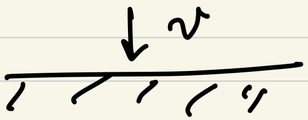
$$CD_{k\omega} = \max\left(2 \rho \sigma_{\omega 2} \frac{1}{\omega} \frac{\partial k}{\partial x_j} \frac{\partial \omega}{\partial x_j}, 10^{-20} \right)$$

- $k-\varepsilon-v^2-f$ model (V2F model)

  Durbin (1991)

blocking effect

Helmholtz eq.



wall

$$k-\varepsilon \text{ eqs} + \frac{\partial \overline{v^2}}{\partial t} - \text{eq} + f$$

$$\rho u_j \frac{\partial \overline{v^2}}{\partial x_j} = kf - \overline{v^2}\frac{\varepsilon}{k} + \frac{\partial}{\partial x_j}\left[(\mu + \frac{\mu_t}{\sigma_k})\frac{\partial \overline{v^2}}{\partial x_j}\right]$$

$$L^2 \nabla^2 f - f = (1-C_1)\left[\frac{2}{3} - \frac{\overline{v^2}}{k}\right]/T - C_2 \frac{P}{k}$$

. . . . . .

- Spalart-Allmaras model (1994) — 1-eq model
  quite successful for aerodynamic flows

✓ June 12 6:30 pm Final exam

✓ June 22 schedule TBA Term project presentation (zoom)