Lecture 6:

Comb resonator design (2)

-Intro. to Mechanics of Materials

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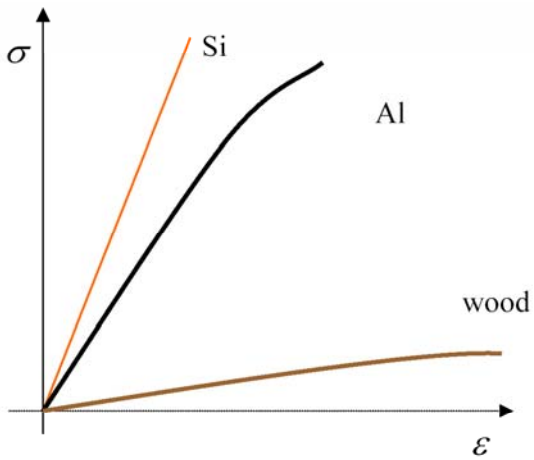
Stress

- ***Normal Stress:*** force applied to surface

$$\boxed{\sigma = F / A}$$

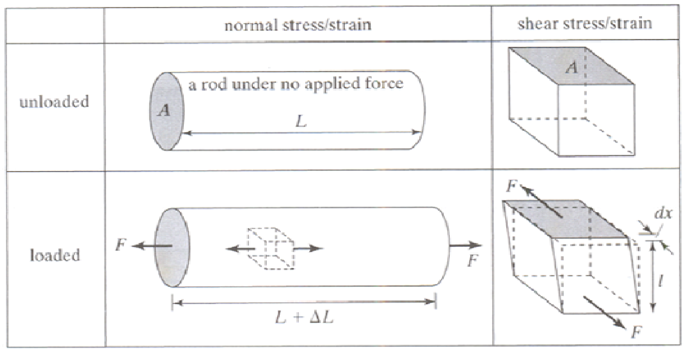
measured in N/m^2 or Pa,
compressive or tensile



- ***Shear Stress:*** force applied parallel to surface

$$\boxed{\tau = F / A}$$

measured in N/m^2 or Pa



Young's Modulus:

$$E = \sigma / \varepsilon$$

Hooke's Law:

$$K = F / \Delta l = EA / l$$

Strain

- ***Strain:*** ratio of deformation to length

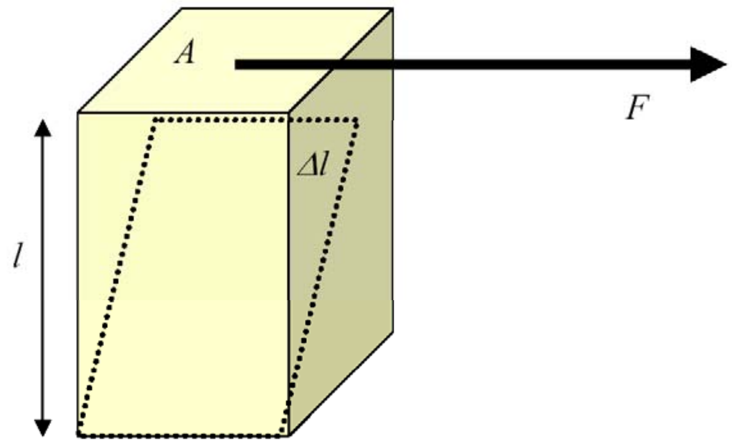
$$\boxed{\gamma = \Delta l / l}$$

- ***Shear Modulus***

$$\boxed{G = \tau / \gamma}$$

- ***Relation among:*** G, E, and ν

$$\boxed{G = \frac{E}{2(1+\nu)}}$$



Poisson's Ratio

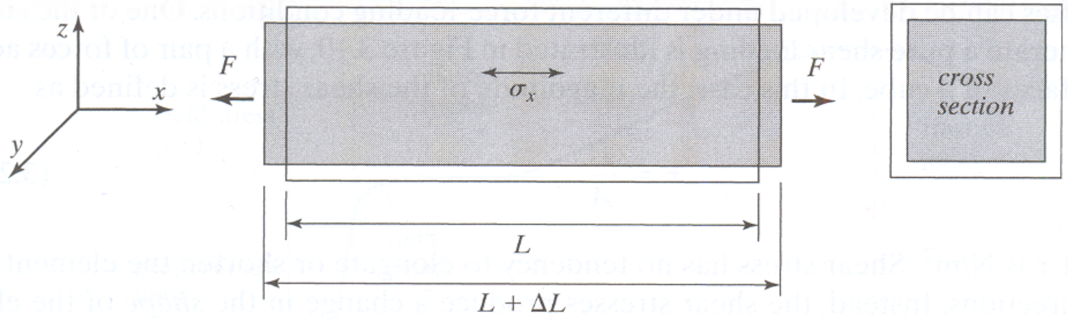
Tensile stress in x direction results in compressive stress in y and z direction (object becomes longer and thinner)

- ***Poisson's Ratio:***

$$\nu = \left| -\varepsilon_y / \varepsilon_x \right| = \left| -\varepsilon_z / \varepsilon_x \right|$$

$$= \left| -\text{transverse strain / longitudinal strain} \right|$$

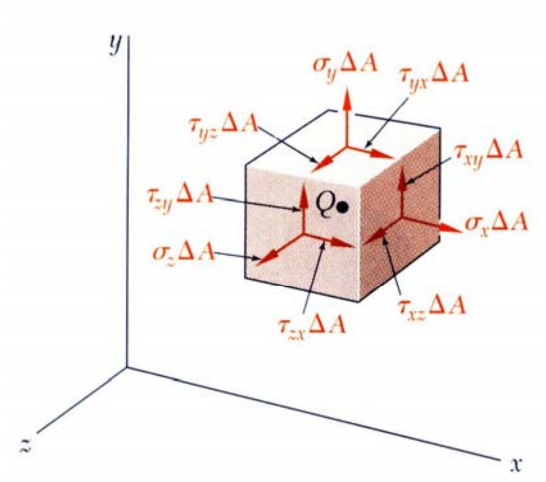
Metals: $\nu \approx 0.3$
Rubbers: $\nu \approx 0.5$
Cork: $\nu \approx 0$



State of stress

- **The combination of forces generated by the stresses must satisfy the conditions for equilibrium:**

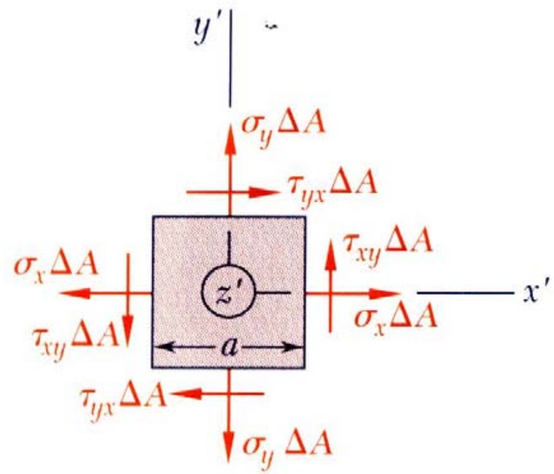
$$\sum F_x = \sum F_y = \sum F_z = 0$$
$$\sum M_x = \sum M_y = \sum M_z = 0$$



- **Consider the moments about the z axis:** $\sum M_z = 0 = (\tau_{xy} \Delta A) a - (\tau_{yx} \Delta A) a$

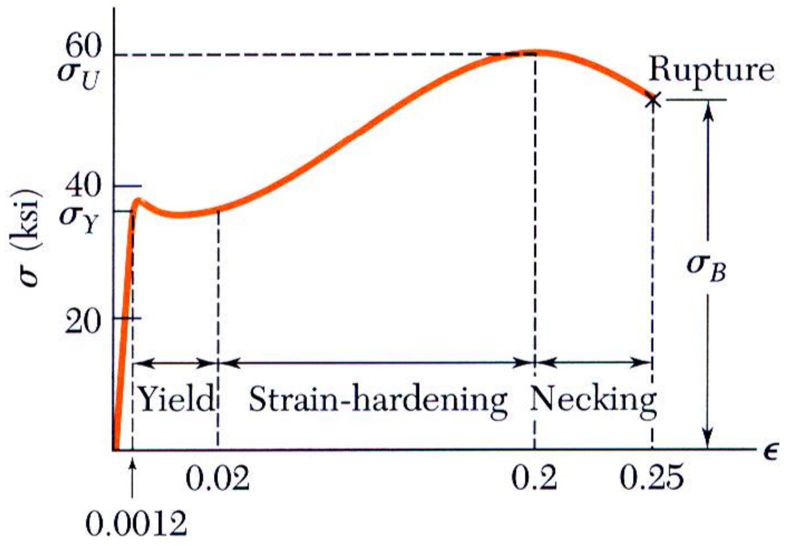
$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy} \text{ and } \tau_{yz} = \tau_{zy}$$

- **Only 6 components of stress are required to define the complete state of stress.**

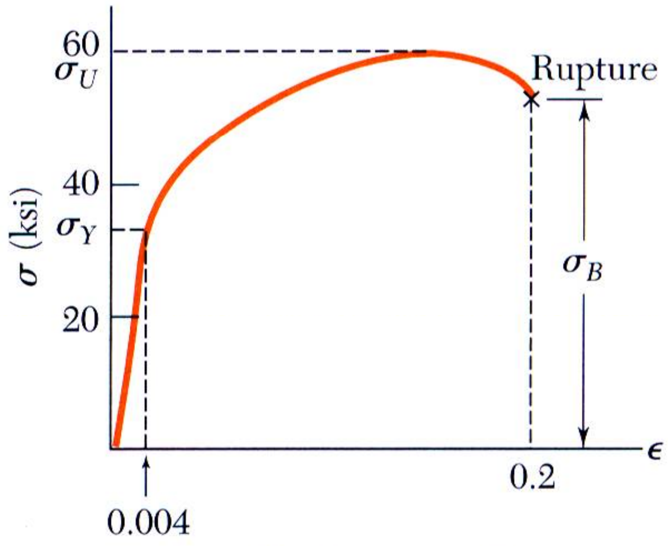


Stress and Strain Diagram

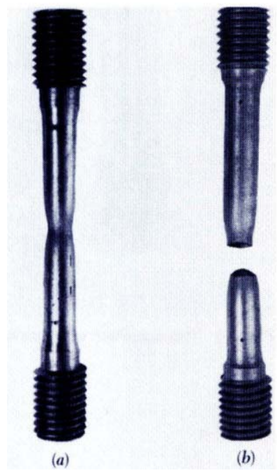
- **Ductile Materials**



(*a*) Low-carbon steel

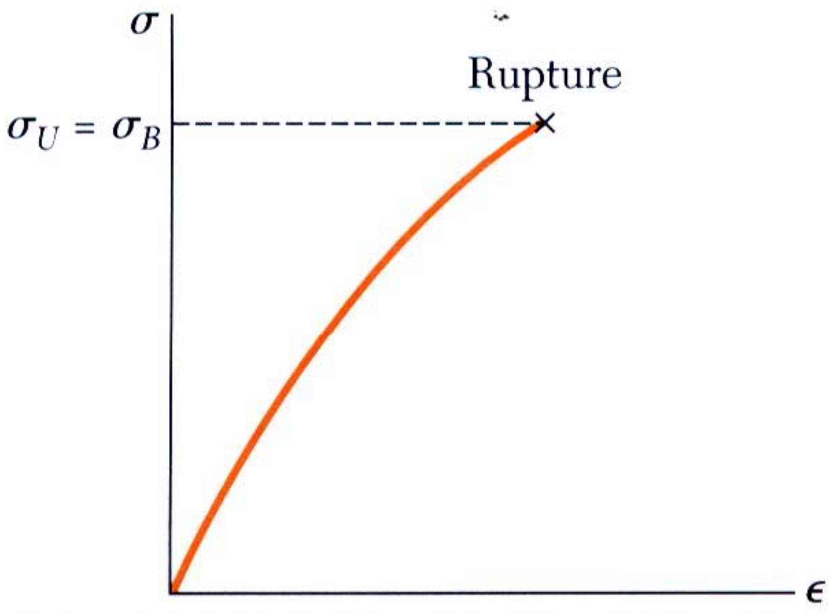


(*b*) Aluminum alloy

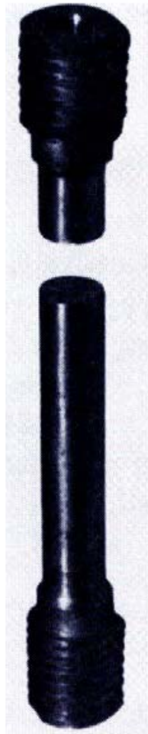


Stress and Strain Diagram (cont'd)

- **Brittle Materials**



Stress-strain diagram for a typical brittle material



Deformations Under Axial Loading

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

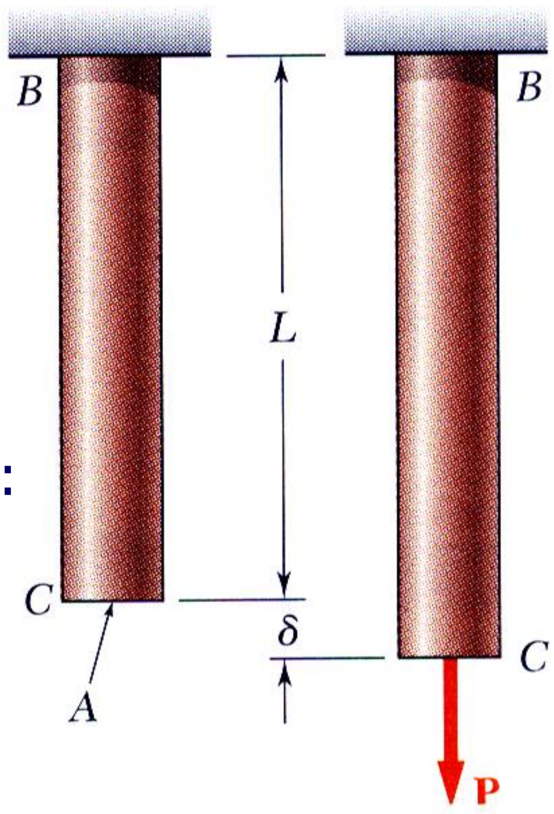
$$\varepsilon = \frac{\delta}{L}$$

- Equating and solving for the deformation:

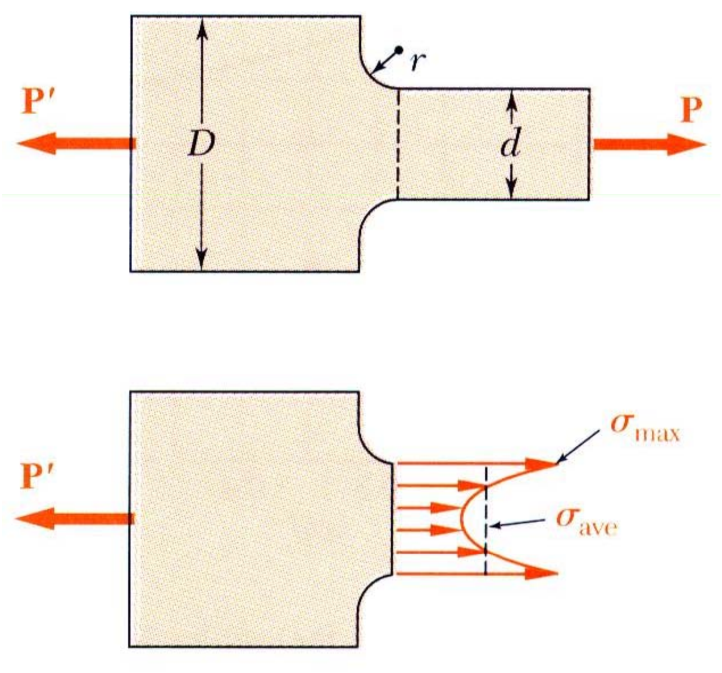
$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties:

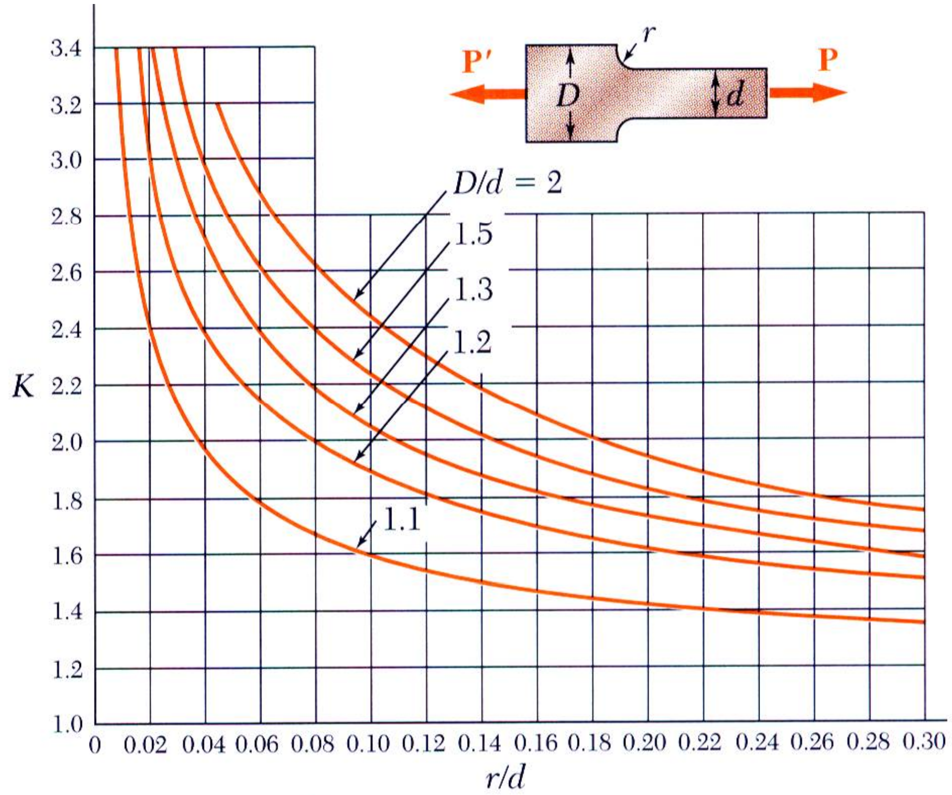
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$



Stress Concentration: Fillet



$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

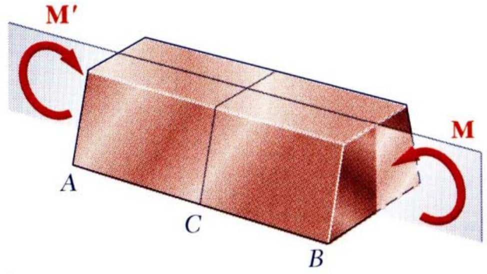


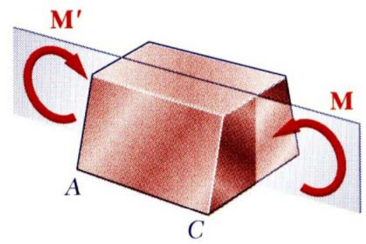
Flat bars with fillets

Symmetric Member in Pure Bending

- Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section bending moment

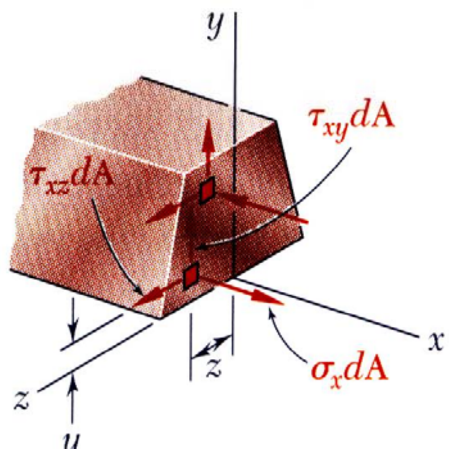




$$F_x = \int \sigma_x \, dA = 0$$

$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$



Strain Due to Bending

- Consider a beam segment of length L

After deformation, the length of the neutral surface remains L. At other sections,

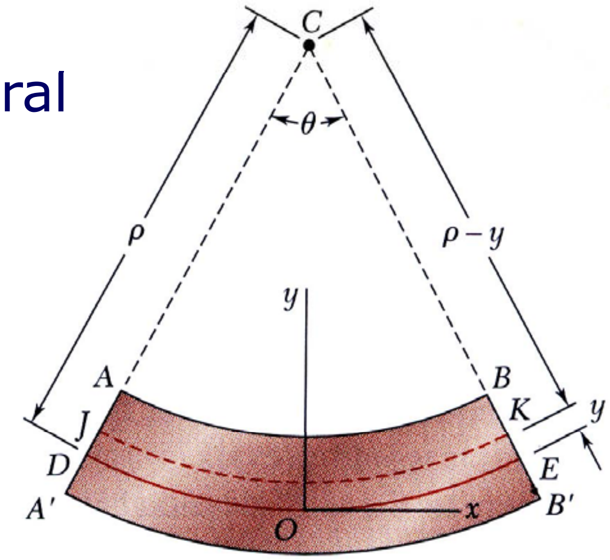
$$L' = (\rho - y)\theta$$

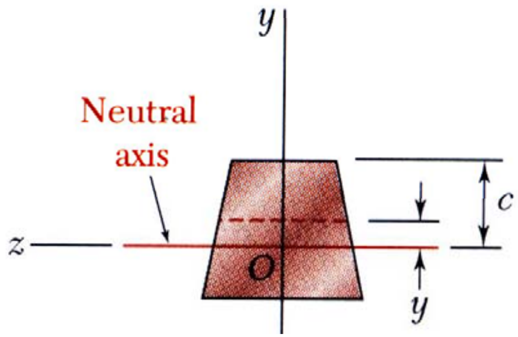
$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \qquad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$



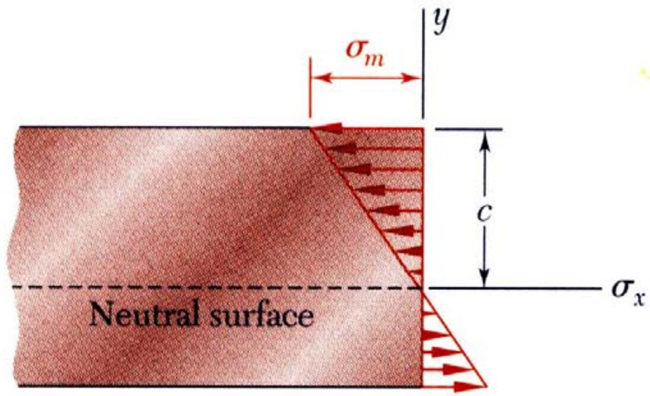


Stress Due to Bending

- For a linearly elastic material,

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

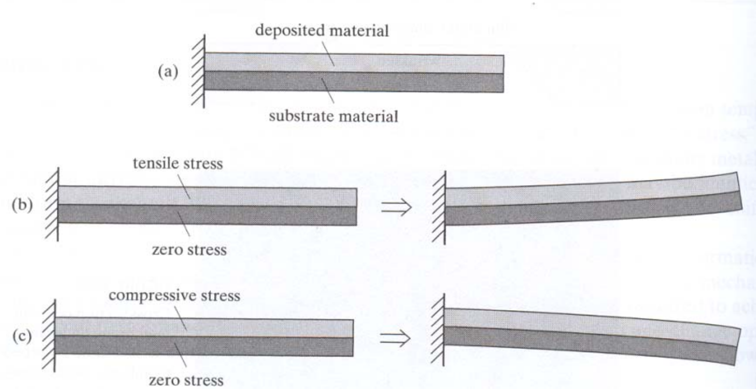


- For static equilibrium,

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



Stress Due to Bending (cont'd)

- For static equilibrium, *Bending Momentum* ***M***

$$M = \iint_A dF(h)h = \int_w \int_{h=-\frac{t}{2}}^{\frac{t}{2}} (\sigma(h)dA)h$$

$$M = \int_w \int_{h=-\frac{t}{2}}^{\frac{t}{2}} (\sigma_{max} \frac{h}{(\frac{t}{2})} dA)h = \frac{\sigma_{max}}{(\frac{t}{2})} \int_w \int_{h=-\frac{t}{2}}^{\frac{t}{2}} h^2 dA = \frac{\sigma_{max}}{(\frac{t}{2})} I$$

$$s_{max} = \frac{Mt}{2EI}$$

where σ_{max} : magnitude of stress

I : moment of inertia of the cross-section

h : height of a beam

t : thickness of a beam

s_{max} : maximum longitudinal strain

Deformations in a Transverse Cross Section

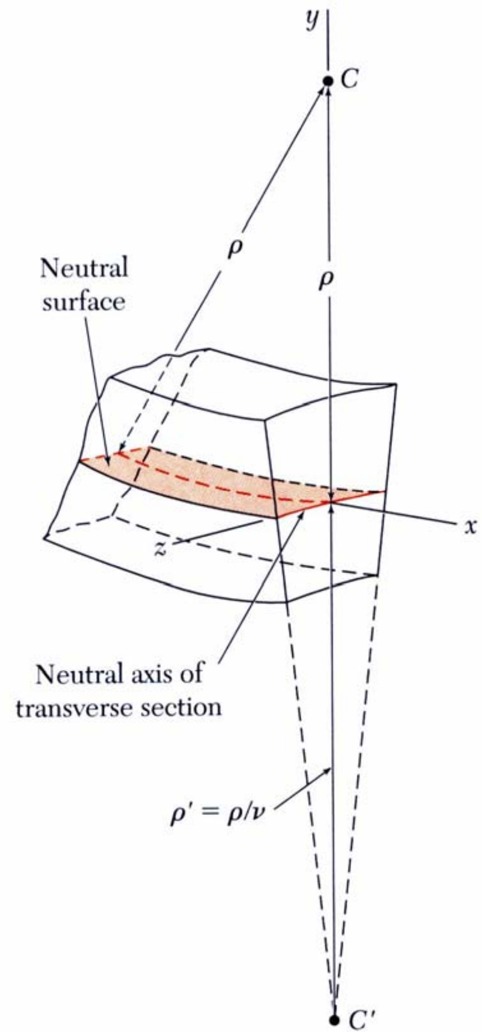
- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$= \frac{M}{EI}$$

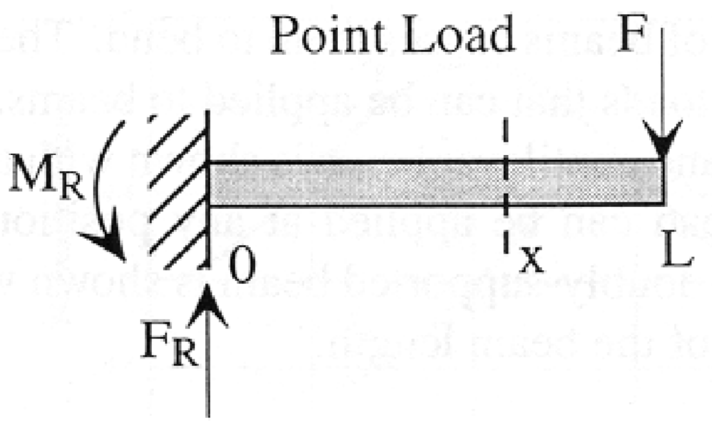
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$



Bending of Beams

- ***Reaction Forces and Moments***



- For equilibrium

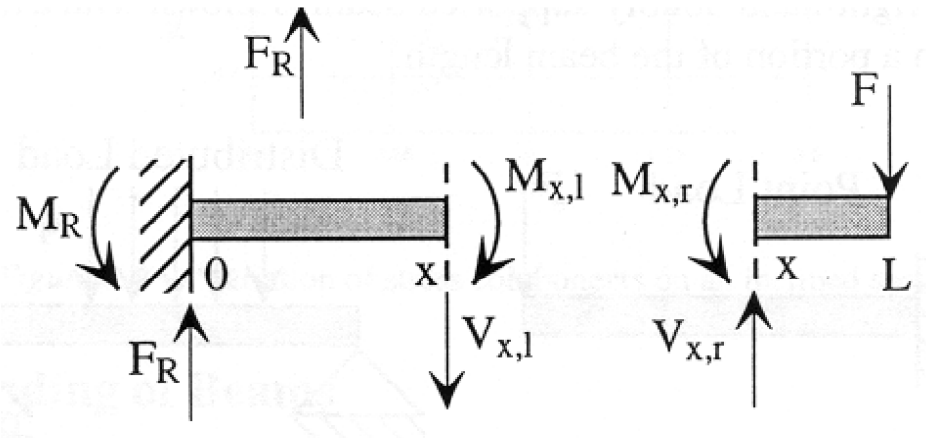
$$\sum F = 0 = F - F_R = 0, \text{ therefore } \underline{F_R = F}$$

$$\sum M_0 = 0 = -M_R + FL = 0, \text{ therefore } \underline{M_R = FL}$$

Bending of Beams (cont'd)

- ***Shear Forces and Moments***

  (at any point in the beam)



At every point along the beam equilibrium requires that,

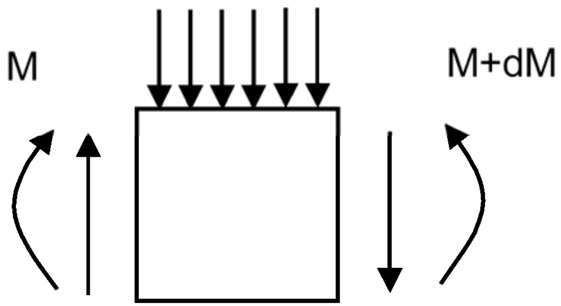
$$\sum F = 0 \text{ and } \sum M = 0$$

$$\sum F = 0 = -F + V(x) = 0 \rightarrow \underline{V = F}$$

$$\sum M_L = 0 = -M(x) + F(L - x) = 0 \rightarrow \underline{M(x) = -F(L - x)}$$

Bending of Beams – differential element

- ***Equilibrium of a fully loaded differential element:***



For equilibrium, $\sum F = 0$ and $\sum M = 0$

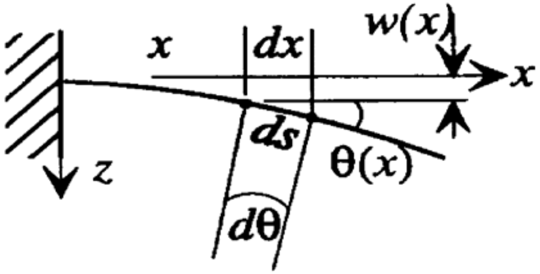
$$\sum F = 0 = q dx + (V + dV) - V = 0 \;\rightarrow\; \underline{q = \frac{(V + dV) - V}{dx} = -\frac{dV}{dx}}$$

$$\sum M = 0 = (M + dM) - M - (V + dV) dx - q dx \frac{dx}{2} = 0$$

$$\rightarrow \; \underline{V = \frac{(M + dM) - M}{dx} = \frac{dM}{dx}} \text{ (neglecting } q(dx^2) \text{ terms).}$$

Bending of Beams – differential element

- ***Approximation for radius of curvature:***



An increment of beam length *dx* is related to *ds* via

$$\cos(\theta) = \frac{dx}{ds}, \text{ for small } \theta \;\rightarrow\; dx \approx ds$$

The slope of the beam at any point is given by

$$\frac{dw}{dx} = \tan(\theta), \text{ for small } \theta \;\rightarrow\; \theta \approx \frac{dw}{dx}$$

For a given radius of curvature, ds is related to $d\theta$ via

$$ds = \rho d\theta, \text{ so for small } \theta \;\rightarrow\; \frac{d\theta}{dx} \approx \frac{1}{\rho} \approx \frac{d^2w}{dx^2}$$

Bending of Beams – differential element

- ***Basic Differential Equations for Beam Bending:***

For small $\theta \rightarrow \dfrac{d^2w}{dx^2} = \dfrac{1}{\rho}$

Now that we have a relationship between $w(x)$ and ρ we can express the moment and shear forces as a function of $w(x)$

Moments: $M = -\dfrac{d^2w}{dx^2}EI,$ now recall $V = \dfrac{dM}{dx}$

Shear: $V = -\dfrac{d^3w}{dx^3}EI,$ now recall $q = -\dfrac{dV}{dx}$

Uniform Load: $q = \dfrac{d^4w}{dx^4}EI$

Analysis of Cantilever Beam

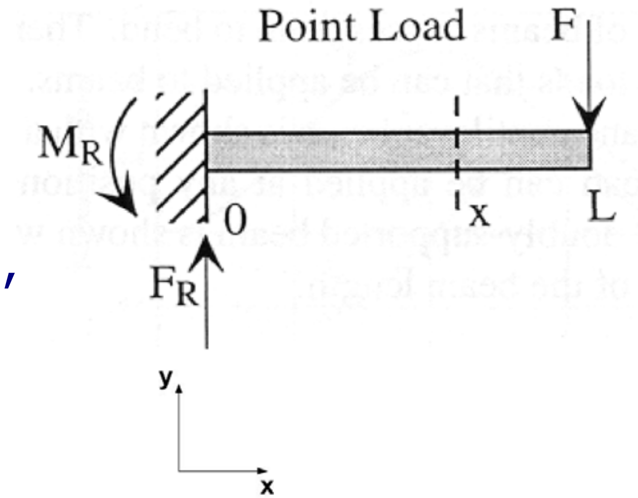
- ***Cantilever Beam with Point Load:***

$$M(x) = -F(L-x)$$

$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = \frac{F}{EI}(L-x)$$

Integrating the above equation twice, we have

$$w(x) = A + Bx + \frac{FL}{2EI}x^2 - \frac{F}{6EI}x^3$$



Boundary conditions:

$$w(0) = 0 \qquad \left.\frac{dw}{dx}\right|_{x=0} = 0$$

Analysis of Cantilever Beam (cont'd)

- ***Cantilever Beam with Point Load (cont'd):***

Using the boundary conditions,

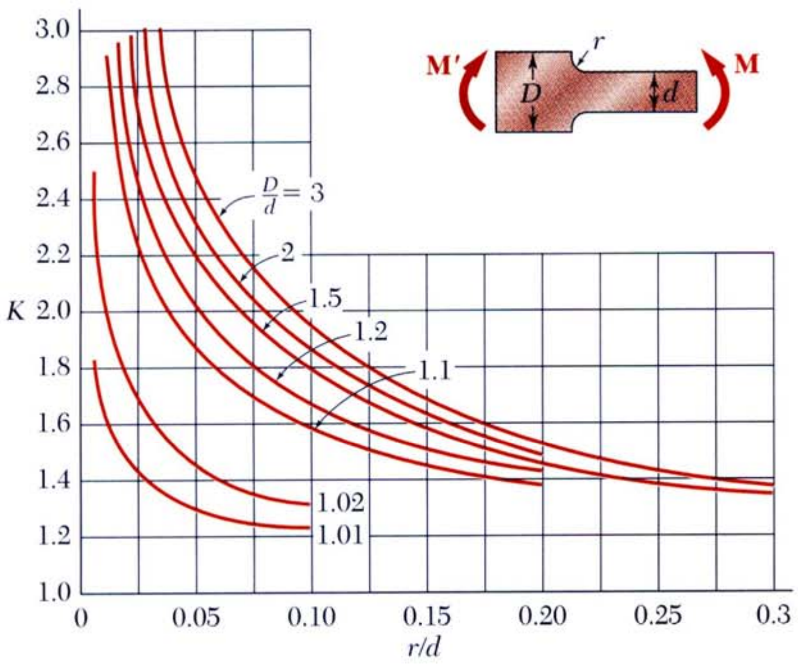
we obtain the beam deflection equation,

$$\boxed{w(x) = \frac{FLx^2}{2EI}(1 - \frac{x}{3L})}$$

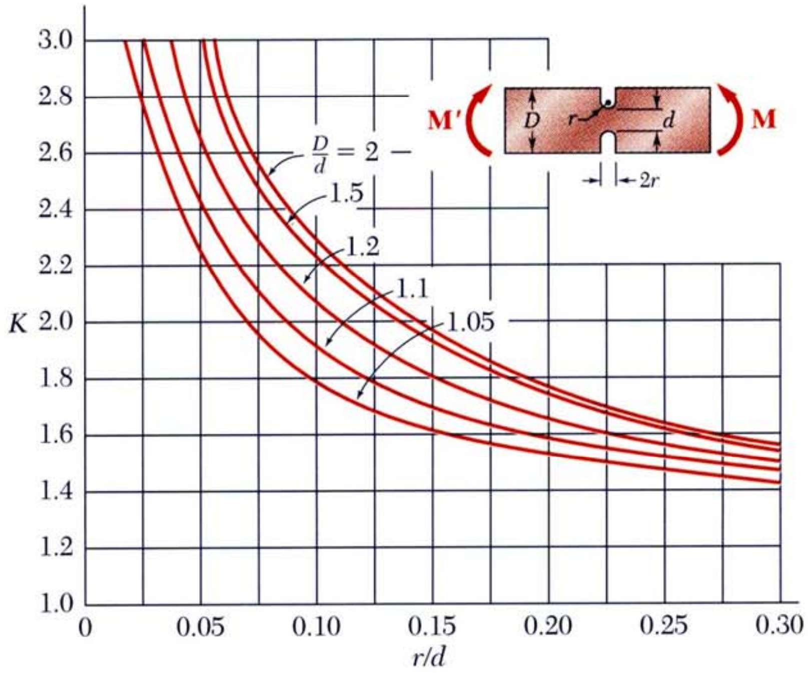
Maximum deflection : $\boxed{w(x) = \frac{FL^3}{3EI}}$

Spring constant : $\boxed{k = \frac{3EI}{L^3} = \frac{EWH^3}{4L^3}}$

Stress Concentration: Fillet



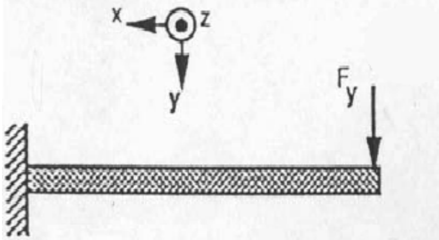
Flat bars with fillets



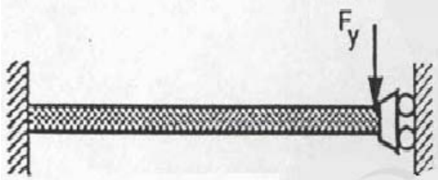
$$K = \frac{\sigma_{\text{max}}}{\sigma_{\text{ave}}}$$

Simple Beam Equations

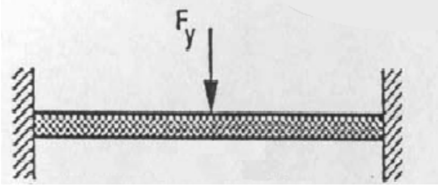
- ***Relation between Load and deflection (1)- concentrated load***



(a) cantilever beam



(b) guided-end beam



(c) fixed-fixed beam

	cantilever	guided-end	fixed-fixed
Elongation	$x = \dfrac{F_x L}{Ehw}$	$x = \dfrac{F_x L}{Ehw}$	$x = \dfrac{F_x L}{4Ehw}$
Deflection	$y = \dfrac{4F_y L^3}{Ehw^3}$	$y = \dfrac{F_y L^3}{Ehw^3}$	$y = \dfrac{1}{16}\dfrac{F_y L^3}{Ehw^3}$
	$z = \dfrac{4F_z L^3}{Ewh^3}$	$z = \dfrac{F_z L^3}{Ewh^3}$	$z = \dfrac{1}{16}\dfrac{F_z L^3}{Ewh^3}$

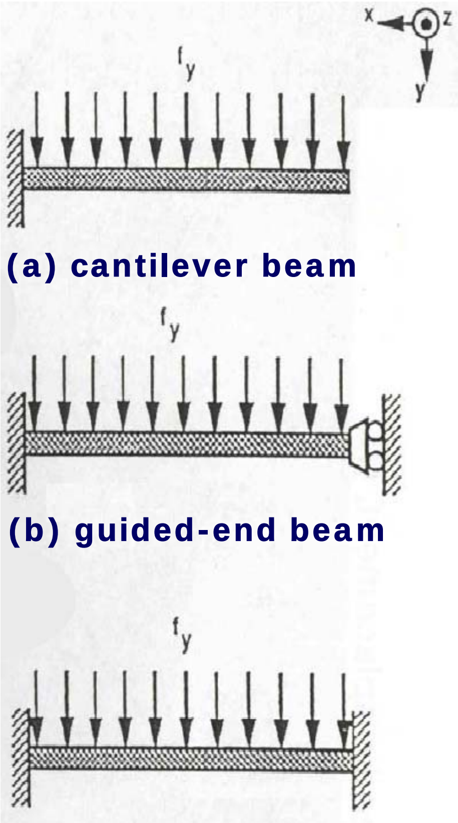
[notation] ***L: length of beam***

h: height of beam

w: width of beam

Simple Beam Equations (cont'd)

- ***Relation between Load and deflection (2)-Distributed load***



(a) cantilever beam

(b) guided-end beam

(c) fixed-fixed beam

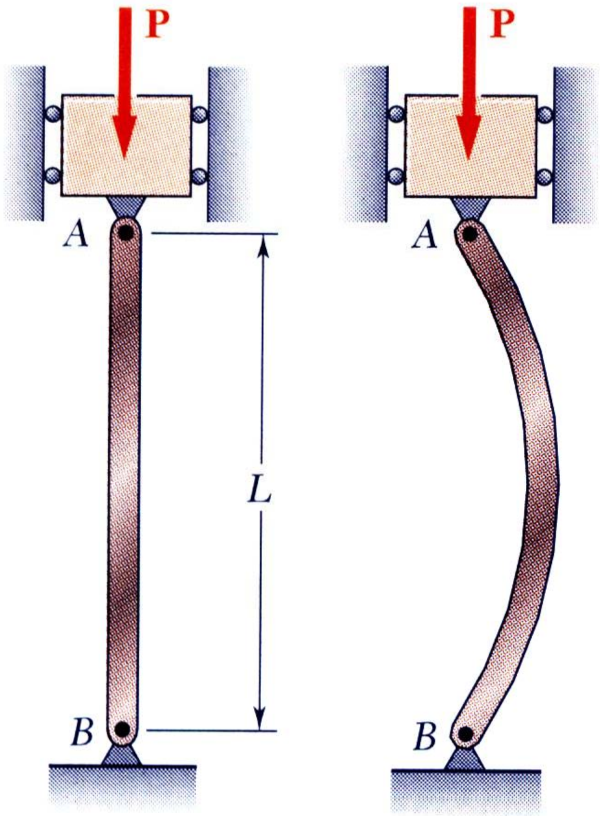
	cantilever	guided-end	fixed-fixed
Elongation	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{4E}$
Deflection	$y = \frac{3}{2}\frac{f_y L^4}{Ehw^3}$	$y = \frac{1}{2}\frac{f_y L^4}{Ehw^3}$	$y = \frac{1}{32}\frac{f_y L^4}{Ehw^3}$
	$z = \frac{3}{2}\frac{f_z L^4}{Ewh^3}$	$z = \frac{1}{2}\frac{f_z L^4}{Ewh^3}$	$z = \frac{1}{32}\frac{f_z L^4}{Ewh^3}$

[notation] ***L: length of beam***

h: height of beam

w: width of beam

Stability of Structures



- In the design of columns, cross-sectional area is selected such that
  - allowable stress is not exceeded

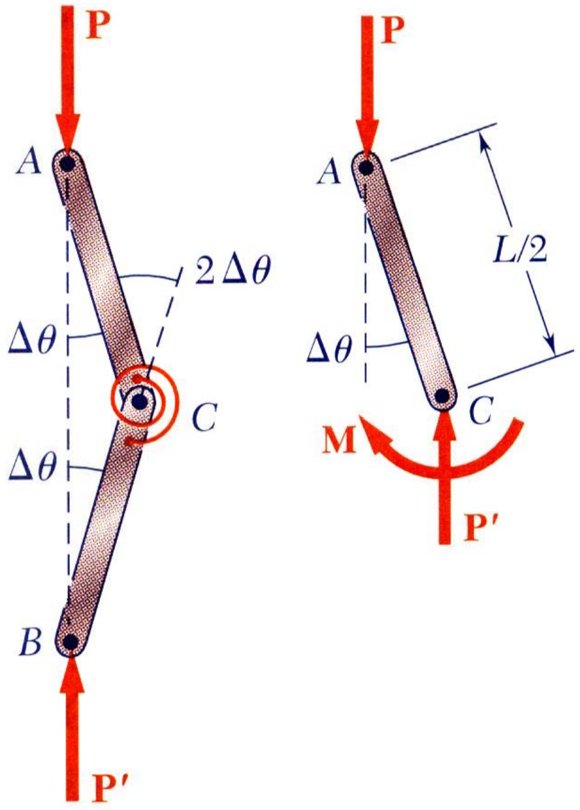
$$\sigma = \frac{P}{A} \leq \sigma_{all}$$

  - deformation falls within specifications

$$\delta = \frac{PL}{AE} \leq \delta_{spec}$$

- After these design calculations, may discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.

Stability of Structures (cont'd)



- Consider model with two rods and torsional spring. After a small perturbation,

$$K(2\Delta\theta) = \text{ restoring moment}$$

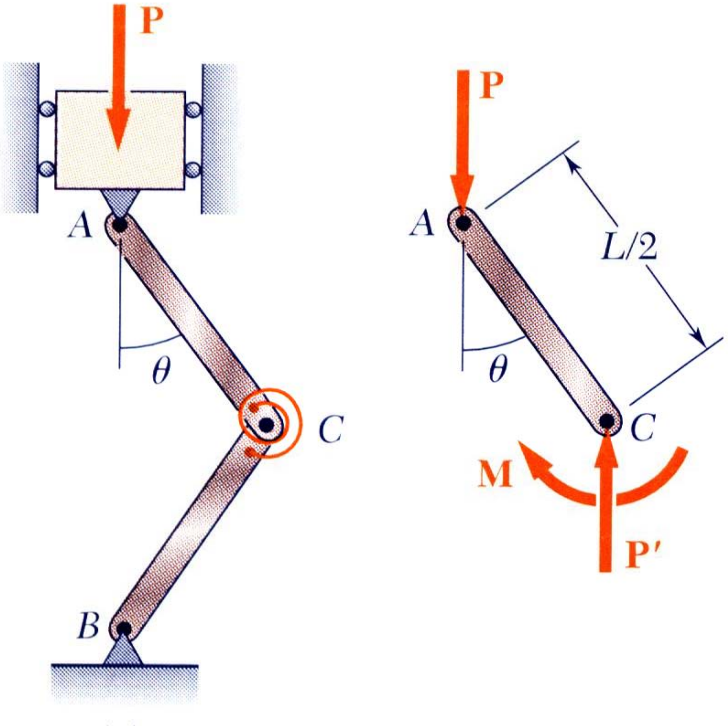
$$P\frac{L}{2}\sin\Delta\theta = P\frac{L}{2}\Delta\theta = \text{ destabilizing moment}$$

- Column is stable (tends to return to aligned orientation) if

$$P\frac{L}{2}\Delta\theta < K(2\Delta\theta)$$

$$P < P_{cr} = \frac{4K}{L}$$

Stability of Structures (cont'd)



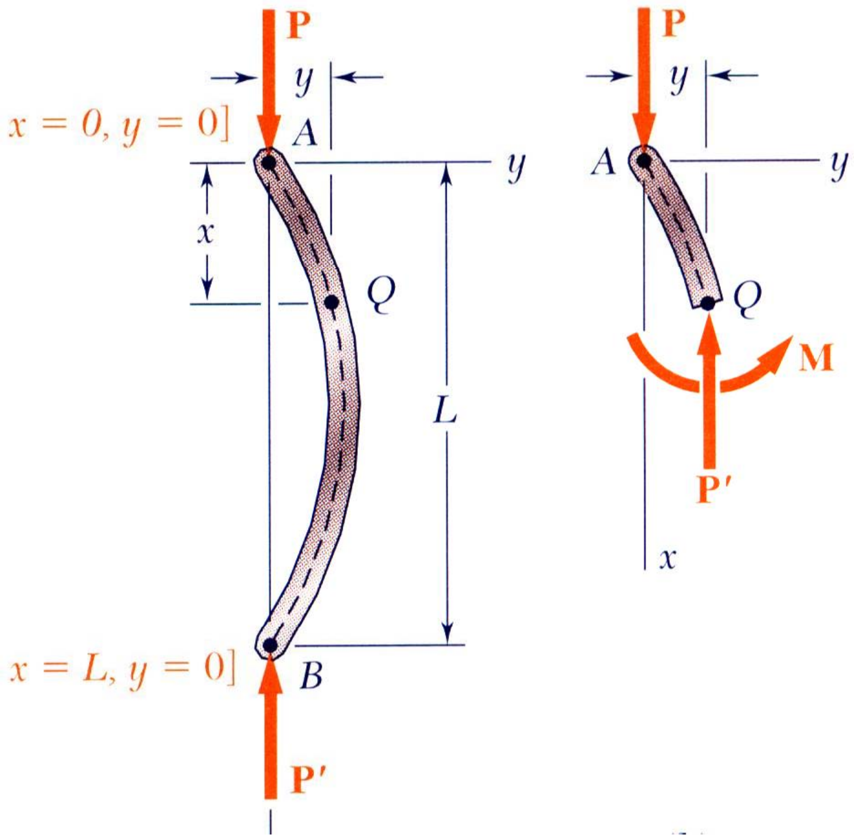
- Assume that a load P is applied. After a perturbation, the system settles to a new equilibrium configuration at a finite deflection angle.

$$P\frac{L}{2}\sin\theta = K\left(2\theta\right)$$

$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin\theta}$$

- Noting that $\sin\theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$.

Euler's Formula for Pin-Ended Beams for Buckling



- Consider an axially loaded beam. After a small perturbation, the system reaches an equilibrium configuration such that

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y \rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

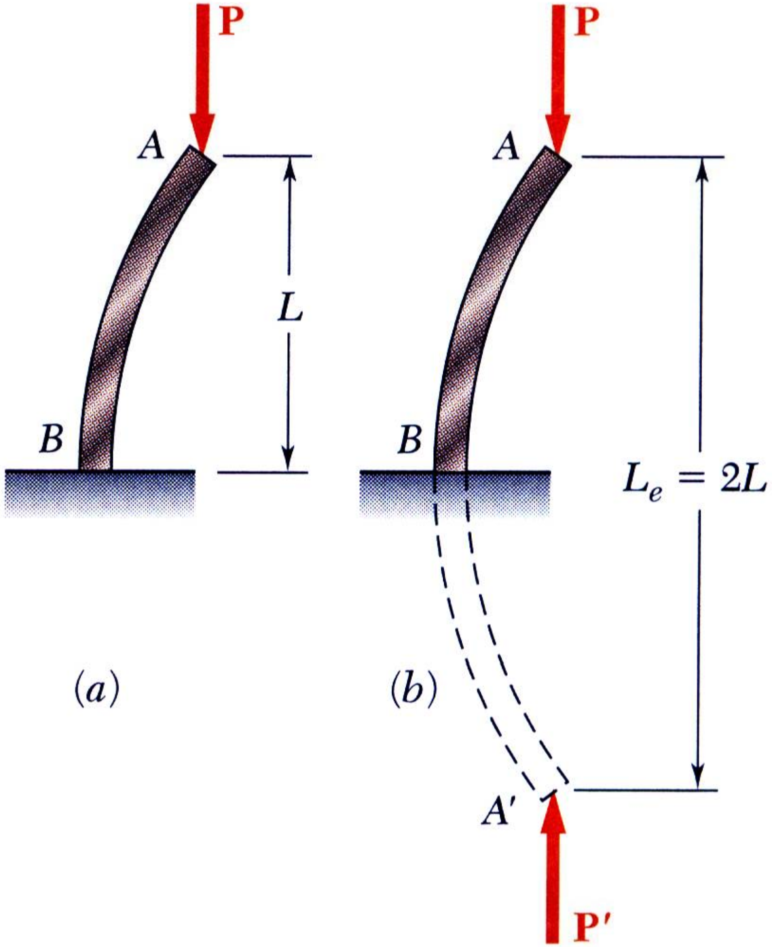
- Solution with assumed configuration can only be obtained if

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A} = \frac{\pi^2 E}{\left(L/r\right)^2}$$

where $r=\sqrt{I/A}$

Extension of Euler’s Formula



- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.
- The critical loading is calculated from Euler’s formula,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

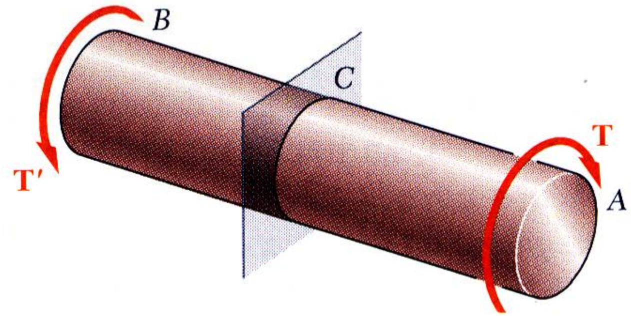
$$\sigma_{cr} = \frac{\pi^2 E}{\left(L_e/r\right)^2}$$

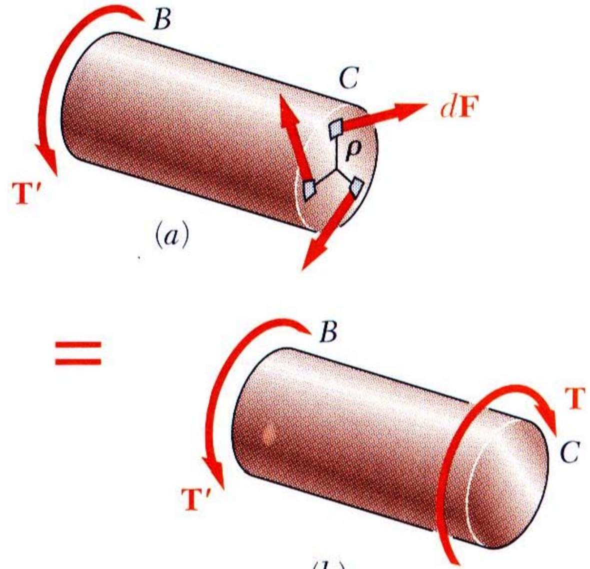
$$L_e = 2L = \text{ equivalent length}$$

Net Torque Due to Internal Stresses

- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque:

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$





(a)

(b)

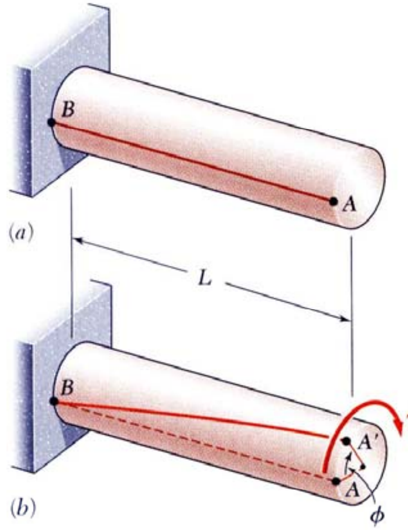
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

Shaft Deformations

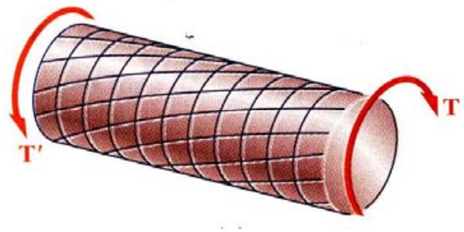
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length:

  $$\phi \propto T$$

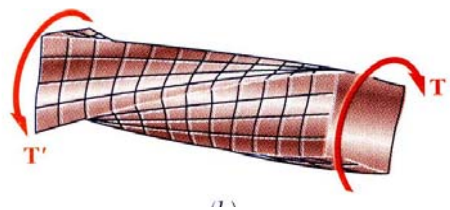
  $$\phi \propto L$$



- Cross-sections for hollow and solid circular shafts remain plain and <u>undistorted</u> because a circular shaft is axisymmetric.



- Cross-sections of noncircular (non-axisymmetric) shafts are <u>distorted</u> when subjected to torsion.



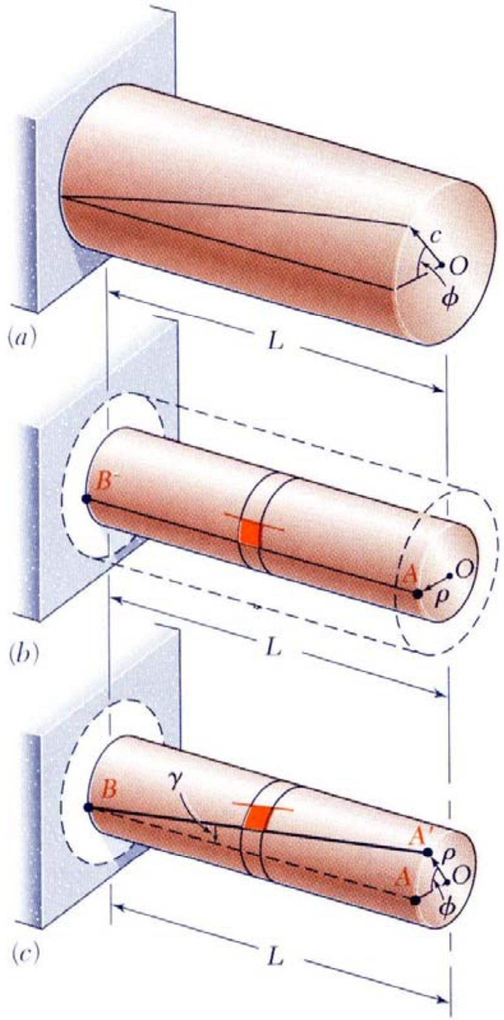
Shearing Strain

- Since the ends of the element remain planar, the shear strain is equal to angle of twist:

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$



Stresses in Elastic Range

- Multiplying the previous equation by the shear modulus,

  $$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

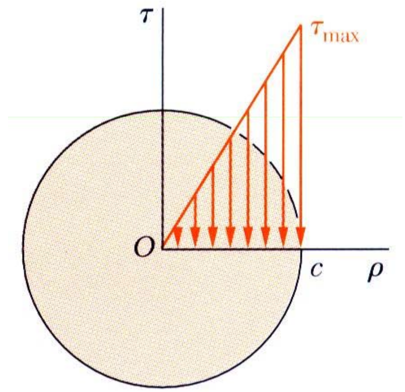
  From Hooke's Law, $\tau = G\gamma$, so

  $$\tau = \frac{\rho}{c}\tau_{max}$$

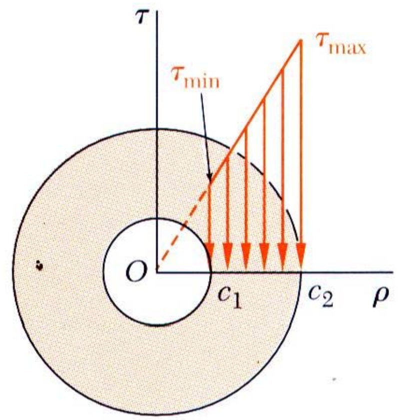
  The shearing stress varies linearly with the radial position in the section.

$$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$



$$J = \frac{1}{2}\pi c^4$$

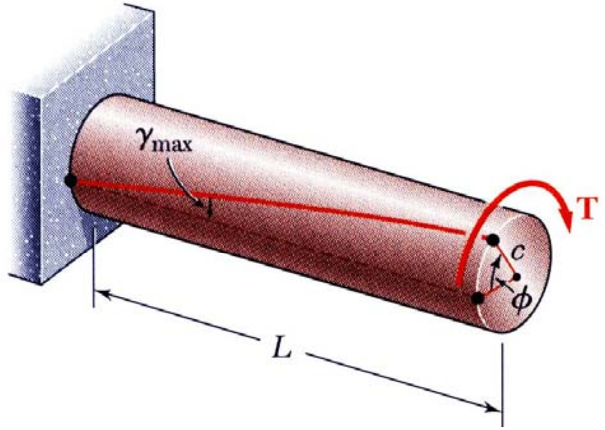


$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

Deformations Under Axial Loading

- The angle of twist and maximum shearing strain are related:

$$\gamma_{\max} = \frac{c\phi}{L}$$

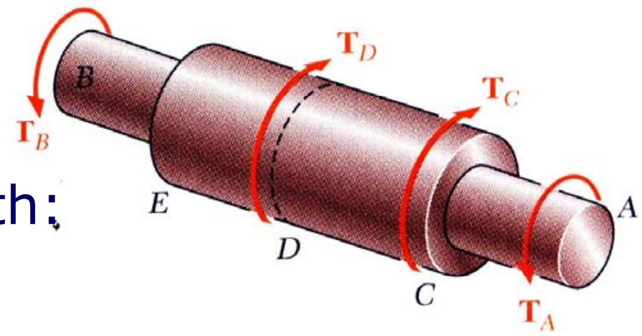


- The shearing strain and shear are related by Hooke's Law,

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG}$$

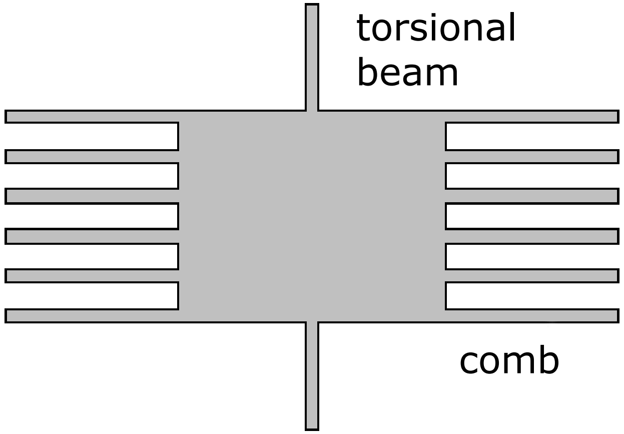
$$\phi = \frac{TL}{JG}$$

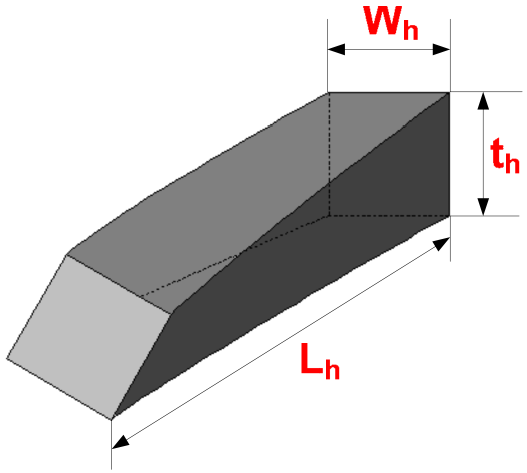
- With variations in the torsional loading and shaft cross-section along the length:



$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Torsion of a rectangular bar





Assume that the torsional beam is isotropic material

- Torsional stiffness (when, $t_h > W_h$)

$$k = \frac{2}{3} \cdot \frac{G}{L_h} t_h w_h^{\ 3} \cdot \left[1 - \frac{192}{\pi^5} \cdot \frac{w_h}{t_h} \cdot \left(\sum_n \frac{1}{n^5} \tanh\left(\frac{1}{2} n\pi \cdot \frac{t_h}{w_h} \right) \right) \right], \ n = 1,3,5 \ldots$$

[ref] S. P. Timoshenko and J. N. Goodier, “Theory of Elasticity,” McGraw-Hill, pp. 309 – 313, 1970.

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