*Lecture 7:*

# Comb resonator design (3)
## -Intro. to Dynamics

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# Kinematics & Dynamics of particles

1) ***Kinematics***: a branch of dynamics that deals with the geometry of motion (i.e. acceleration, velocities, and displacement) apart from consideration of mass and forces.

$$\left.\begin{aligned} \vec{v} &= \frac{d\vec{s}}{dt} \\ \vec{a} &= \frac{d\vec{v}}{dt} \end{aligned}\right\} \; dt = \frac{d\vec{s}}{\vec{v}} = \frac{d\vec{v}}{\vec{a}}$$

$$v \cdot d\vec{v} = a \cdot d\vec{s}$$

$$\int_{v_1}^{v_2} v \cdot d\vec{v} = \int_{s_1}^{s_2} a \cdot d\vec{s}$$

# Kinematics & Dynamics of particles (cont'd)

2) ***Force momentum principle***:

$$\vec{f} = \frac{d\vec{p}}{dt} \qquad \vec{p} = mv \ (momentum)$$

3) ***Constitutive relations***:

$$\vec{p} = mv \ \text{(momentum)}$$

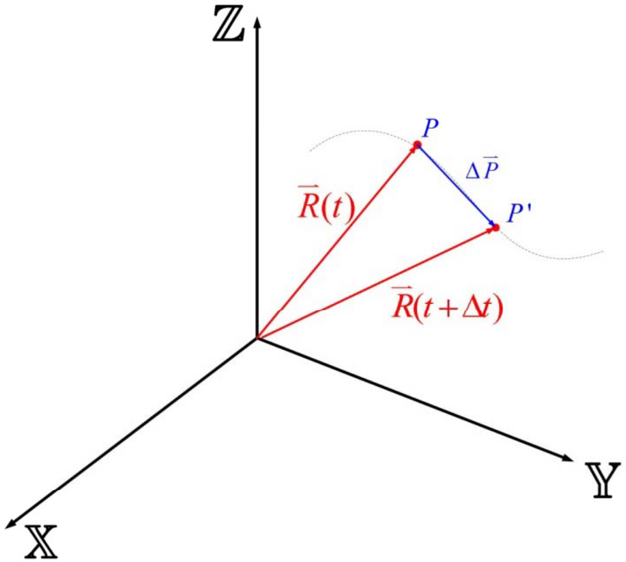
$$f_s = k_s$$

$$f_g = mg$$

$$f_f = \mu N$$

# Kinematics of particles

- ***Particle in moving space***




$$\Delta\vec{P} = \text{displacement vector}$$

$$\vec{R}(t) = \text{position vector}$$

$$\vec{v}(t) = \text{velocity vector} = \frac{d\vec{R}(t)}{dt}$$

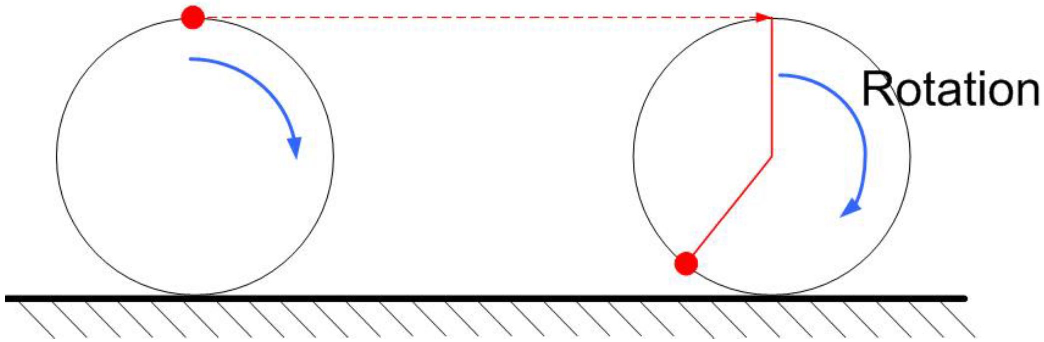
$$\vec{a}(t) = \text{acceleration vector} = \frac{d\vec{v}(t)}{dt}$$

# Kinematics of particles (cont'd)

- ***Inertial Reference Frame (IRF)***
  - $\mathbb{OXYZ}\ (\underline{U}_x,\ \underline{U}_y,\ \underline{U}_z)$
  - non-accelerating and non-rotating
  - an isolated particle maintains constant velocity
  - $\vec{f} = \dfrac{d\vec{p}}{dt}$ , $\vec{p} = momentum$

# Kinematics of particles (cont'd)

- ***Any motion can be described by the some of a single translation plus a single rotation.***

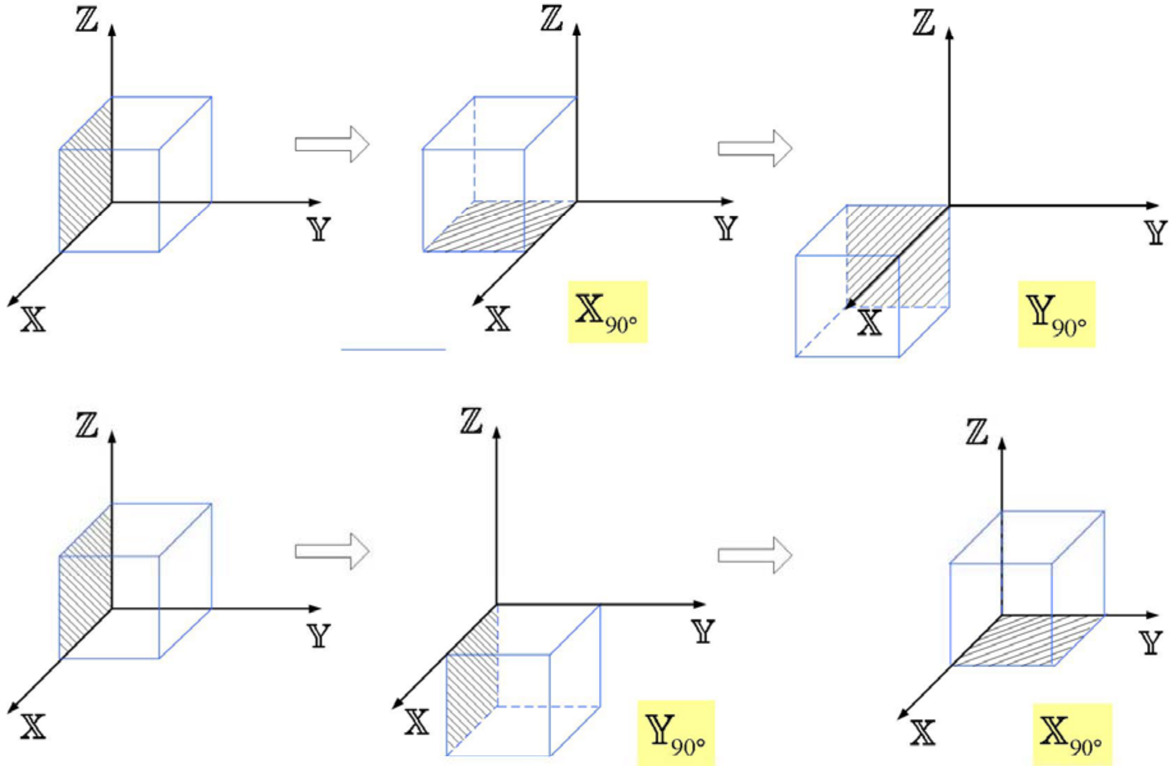


# Kinematics of particles (cont'd)

- ***Finite rotations are not vectors.***

  : Sequence is important.

$$\{\mathbb{X}\}_{90^\circ} + \{\mathbb{Y}\}_{90^\circ} \neq \{\mathbb{Y}\}_{90^\circ} + \{\mathbb{X}\}_{90^\circ}$$

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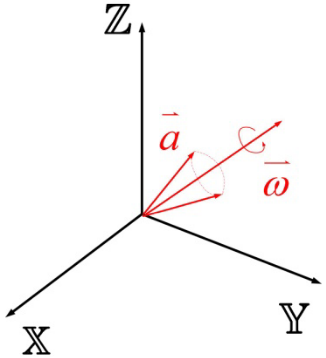
# Kinematics of particles (cont'd)

- ***Rate of change of rotating constant vector***




$$\frac{d\vec{a}}{dt} = \vec{\omega} \times \vec{a}$$

[ref] S. H. Crandall et al., "Dynamics of Mechanical and Electromechanical Systems", McGraw-Hill, pp. 51 – 53, 1970.

- ***Rate of change of rotating and changing vector***

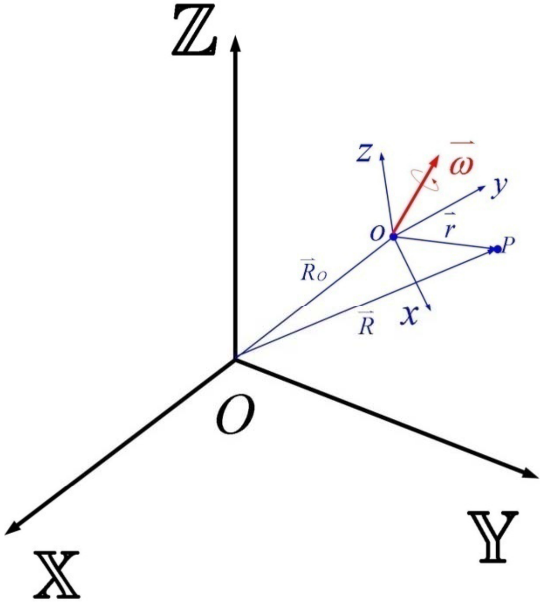
rotating frame, A

$$\frac{d\vec{A}}{dt} = \dot{A}_x\vec{u}_x + \dot{A}_y\vec{u}_y + \dot{A}_z\vec{u}_z + A_x\vec{\omega}\times\vec{u}_x + A_y\vec{\omega}\times\vec{u}_y + A_z\vec{\omega}\times\vec{u}_z$$

$$\frac{dA}{dt} = \left.\frac{\partial\vec{A}}{\partial t}\right)_{rel} + \vec{\omega}\times\vec{A} \quad \Rightarrow \quad \frac{d}{dt} = \left.\frac{\partial}{\partial t}\right)_{rel} + \vec{\omega}\times$$

# Kinematics of particles (cont'd)

- ***Kinematics using intermediate reference frame (irf)***




$$IRF = \mathbb{OXYZ}$$

$$irf = oxyz$$

# Kinematics of particles (cont'd)

- Motion of P defined in the irf (oxyz):

$$\vec{r}(t) = \text{position vector} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$$

Let $\left.\frac{\partial}{\partial t}\right)_{rel}$ be the time differentiation in oxyz.

velocity in oxyz :

$$\vec{v}_{rel} = \left.\frac{\partial \vec{r}}{\partial t}\right)_{rel} = \dot{x}\vec{u}_x + \dot{y}\vec{u}_y + \dot{z}\vec{u}_z$$

acceleration in oxyz :

$$\vec{a}_{rel} = \left.\frac{\partial^2 \vec{r}}{\partial t^2}\right)_{rel} = \ddot{x}\vec{u}_x + \ddot{y}\vec{u}_y + \ddot{z}\vec{u}_z$$

# Kinematics of particles (cont'd)

position of P in $\mathbb{OXYZ}$

$$\vec{R}(t) = \vec{R}_O(t) + \vec{r}(t)$$

velocity of P in $\mathbb{OXYZ}$

$$\vec{v}(t) = \frac{d\vec{R}}{dt} = \frac{d\vec{R}_O}{dt} + \frac{d\vec{r}}{dt} = \dot{\vec{R}}_O + \left[ \left( \frac{\partial}{\partial t} \right)_{rel} + \vec{\omega} \times \right] \vec{r} = \dot{\vec{R}}_O + \vec{v}_{rel} + \vec{\omega} \times \vec{r}$$

# Kinematics of particles (cont'd)

acceleration of P in $\mathbb{OXYZ}$

$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=\frac{d}{dt}(\dot{\vec{R}}_O)+\frac{d}{dt}(\vec{v}_{rel})+\frac{d}{dt}(\vec{\omega}\times\vec{r})$$

$$=\ddot{\vec{R}}_O+\left[\left.\frac{\partial}{\partial t}\right)_{rel}+\vec{\omega}\times\quad\right]\vec{v}_{rel}+\frac{d}{dt}(\vec{\omega})\times\vec{r}+\vec{\omega}\times\frac{d}{dt}(\vec{r})$$

$$=\ddot{\vec{R}}_O+\left.\frac{\partial\vec{v}_{rel}}{\partial t}\right)_{rel}+\vec{\omega}\times\vec{v}_{rel}+\dot{\vec{\omega}}\times\vec{r}+\vec{\omega}\times\left[\left.\frac{\partial}{\partial t}\right)_{rel}+\vec{\omega}\times\quad\right]\times\vec{r}$$

$$=\ddot{\vec{R}}_O+\vec{a}_{rel}+\vec{\omega}\times\vec{v}_{rel}+\dot{\vec{\omega}}\times\vec{r}+\vec{\omega}\times\left.\frac{\partial\vec{r}}{\partial t}\right)_{rel}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$=\ddot{\vec{R}}_O+\vec{a}_{rel}+2\cdot(\vec{\omega}\times\vec{v}_{rel})+\dot{\vec{\omega}}\times\vec{r}+\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

frame | Particle irf | Coriolis' accel. | Frame angular accel. | Centripital accel.

# Dynamics of particles

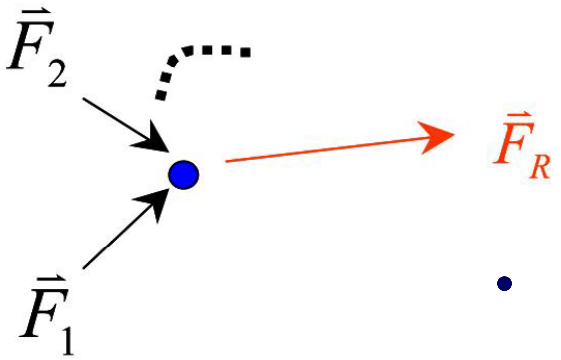
- Dynamics is based on kinematics and

  Newton's second law : $\vec{f} = \frac{d\vec{p}}{dt}$

- A reference system in which Newton's second law is vali d is called an inertial system

- All systems moving with constant and linear velocities a re inertial systems.

# Dynamics of particles (cont'd)




- When a particle is subjected to forces :

  $$\vec{F}_1, \vec{F}_2, \cdots$$

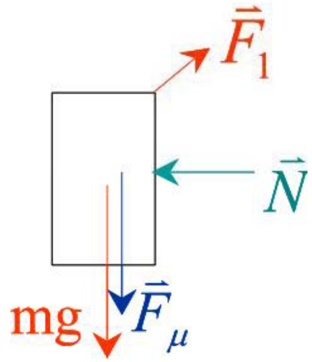
  $$\vec{F}_R = \sum_i \vec{F}_i = m\,\vec{a}$$

- We can separate :

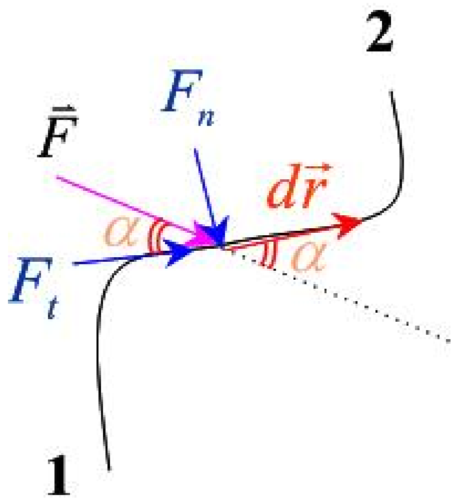
$$F_{Rx} = m\,a_x$$
$$F_{Ry} = m\,a_y$$
$$F_{Rz} = m\,a_z$$

[Equation of motion]

* Free body diagram :



# Work




- The work done by a force on a particle in a displacement of $d\vec{r}$

$$dU = \vec{F} \cdot d\vec{r} = F \cdot ds \cos\alpha \ , \ ds = |d\vec{r}|$$

if $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

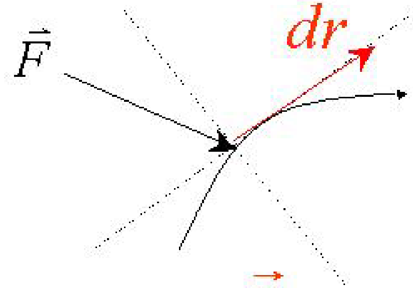
Then $\vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$

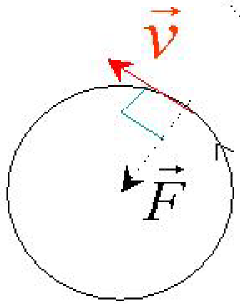
$\therefore$ The total work done : $U = \int \vec{F} \cdot d\vec{r}$

Now $dU = F_t \cdot ds \qquad (F \cos\alpha = F_t)$

$$= m a_t \cdot ds$$

# Kinetic energy




- Total work done by external force

$$W_{1-2} = \int_1^2 \vec{F} \cdot \overrightarrow{dr} = \int_1^2 F dr_t = \int_1^2 m a_t dr_t = \int_1^2 m \frac{dv_t}{dt} dr_t$$

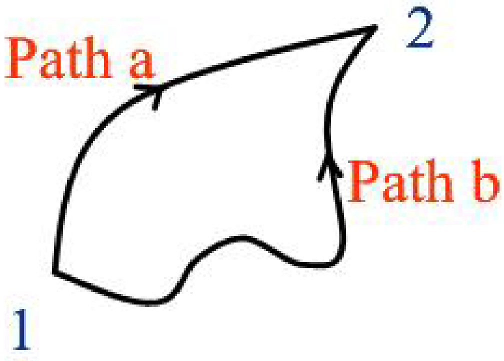
$$= \int_1^2 m \frac{dv}{dt} dr_t = \int_1^2 m \frac{dv}{dt} v dt = \int_1^2 m v dv$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = T_2 - T_1 = \Delta T$$

Total work done by external = change in K.E.

Power $P = \dfrac{dU}{dt} = \dfrac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$

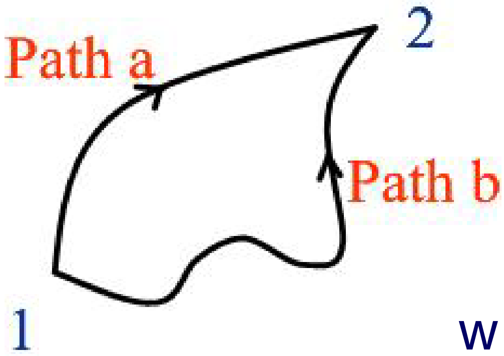
# Conservative force and potential energy




- A conservative force is a force having the ch -aracteristic that the work done by the force on the particle depends on the net change in position and not on the actual path followed by the particle.

- e.g.
  - gravitational force
  - elastic force
  - electrostatic force

# Conservative force and potential energy (cont'd)




- Work done to move P from 1 to 2 is independent of path

$$\int_{1(path\ a)}^{2} \vec{F} \cdot d\vec{r} = \int_{1(path\ b)}^{2} \vec{F} \cdot d\vec{r} = V - V_{ref}$$

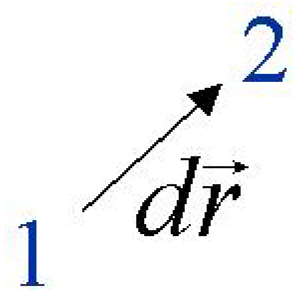
where $V$ is the potential depending on position $\vec{r}$
$\vec{F}$ is a conservative force

- The potential is written as

$$V(\vec{r}) = V_{ref} - \int_{S_o}^{S} \vec{F} \cdot d\vec{r}$$

at $S_0$, $V = V_{ref}$

# Conservative force and potential energy (cont'd)




when d$\vec{r}$ is very small,

$$dV = -\int_{S_o}^{S_2} \vec{F} \cdot d\vec{r} - \left(-\int_{S_o}^{S_1} \vec{F} \cdot d\vec{r}\right)$$

$$= -\int_{S_1}^{S_2} \vec{F} \cdot d\vec{r}$$

$$= -\vec{F} \cdot d\vec{r}$$

$$= -F_x dx - F_y dy - F_z dz$$

Since $V$ is a function of $\vec{r}$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$ (exact differential)

$$F_x = -\frac{\partial V}{\partial x} \; ; \; F_y = -\frac{\partial V}{\partial y} \; ; \; F_z = -\frac{\partial V}{\partial z}$$

# Work-energy equation

$$\Delta K.E. = \int \sum \vec{F} \cdot d\vec{r}$$

$$= \int \left( \sum \vec{F}^{(C)} + \sum \vec{F}^{(0)} \right) \cdot d\vec{r}$$

- $\sum \vec{F}^{(C)}$ - all conservative forces
- $\sum \vec{F}^{(0)}$ - non-conservative forces (frictional force), depend on paths

$$\int \sum \vec{F}^{(C)} \cdot d\vec{r} = -\Delta V$$

- If we specify

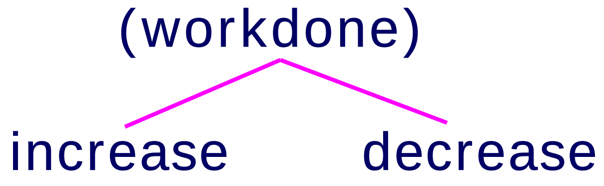
  $\int \sum \vec{F}^{(0)} \cdot d\vec{r} = W$, dissipative work

$$\therefore \Delta K.E. = -\Delta V + W$$

# Work-energy equation (cont'd)

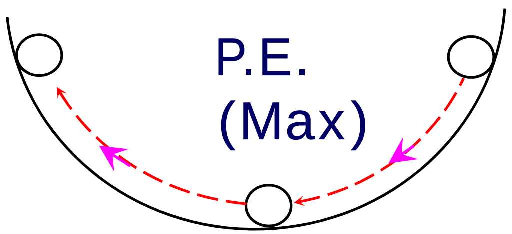
$$\therefore \Delta T + \Delta V = W$$

: work-energy eq.




If W = 0, then $\Delta T + \Delta V = 0$

The conservation of mechanical energy.



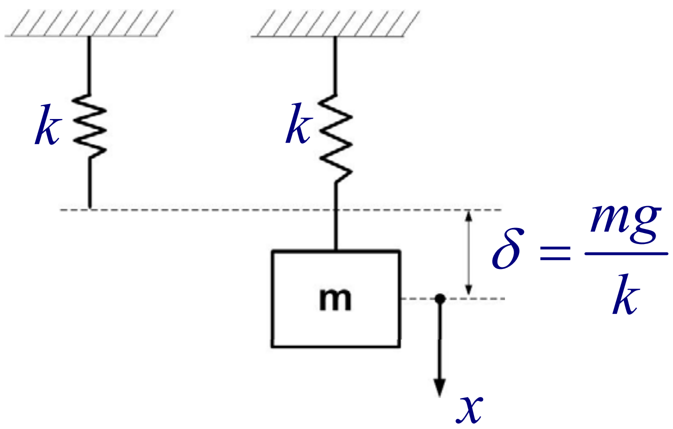

$$\therefore (T_2 - T_1) + (V_2 - V_1) = 0$$

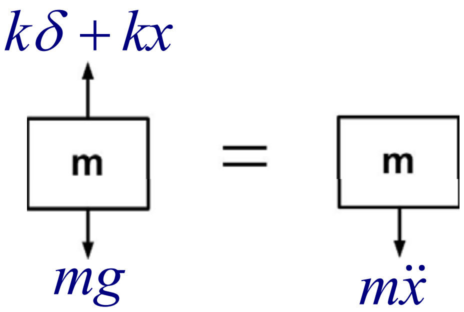
$$\therefore T_2 + V_2 = T_1 + V_1$$

# Free vibration of particle

- ***Free-vibration:*** absence of any imposed external forces
  - Undamped free vibration:

    (1) Translational motion




$k$: spring constant [N/m]

The equation of motion: $-k\delta - kx + mg = m\ddot{x} = F$

$$\Rightarrow \boxed{m\ddot{x} + kx = 0}$$

(simple harmonic motion)

# Free vibration of particle (cont'd)

or $\ddot{x} + \omega_n{}^2 x = 0$

where $\omega_n = \sqrt{\dfrac{k}{m}}$ $[rad\,/\,s]$

= natural frequency of the vibration

The solution: $x = A\cos(\omega_n t) + B\sin(\omega_n t)$

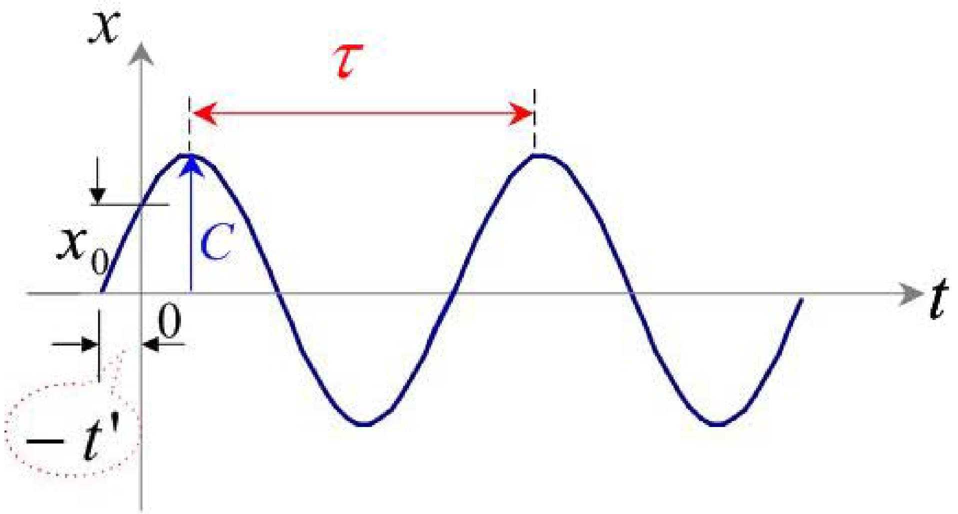
$= C\sin(\omega_n t + \psi)$

where $C$: amplitude

$\psi$: phase angle

Unknown factors are determined by initial conditions

# Free vibration of particle (cont'd)




$$x_0 = C \sin \psi$$

$$0 = C \sin(-\omega_n t' + \psi)$$

$$\Rightarrow \quad -\omega_n t' + \psi = 0$$

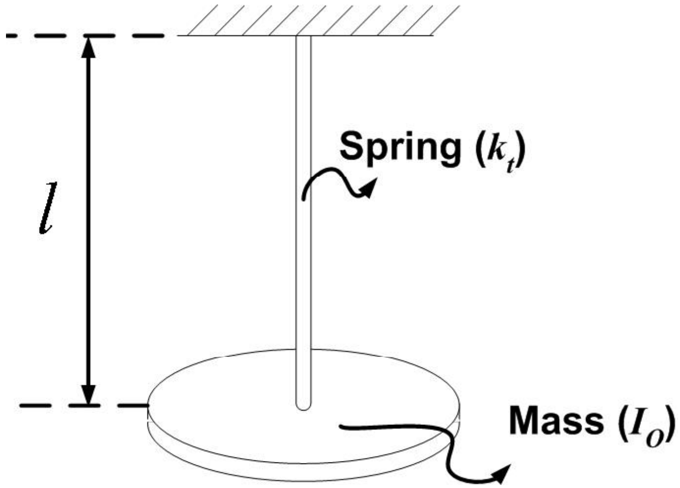
$$\therefore \psi = \omega_n t'$$

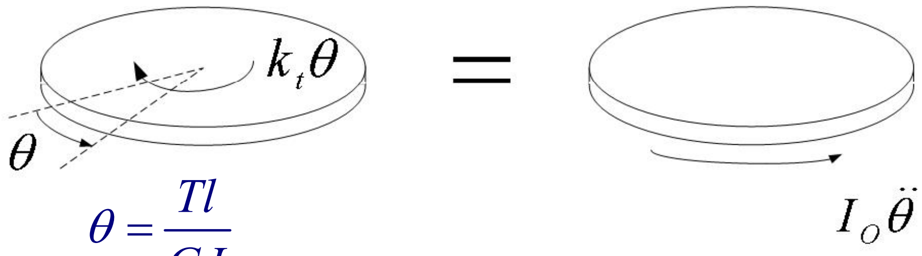
$$\tau = 2\pi / \omega_n = \text{period} = 1/f_n$$

where $f_n$ : natural frequency [Hz] $= \omega_n / 2\pi$

# Free vibration of particle (cont'd)

## (2) Torsional motion




$$\theta = \frac{Tl}{GJ}$$

$$k_t = GJ/l$$

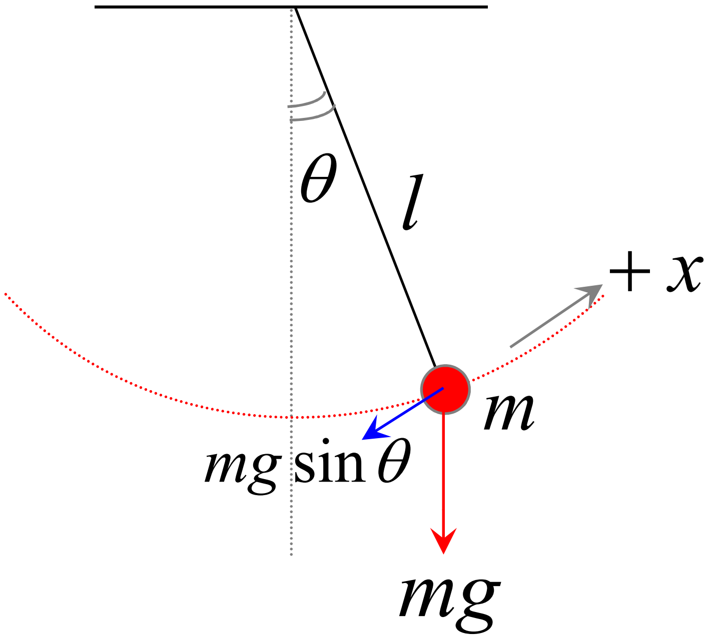
$J$: moment of inertia
$k_t$: torsional stiffness

The equation of motion: $I_o\ddot{\theta} + k_t\theta = 0$

$$\Rightarrow \omega_n = \sqrt{\frac{k_t}{I_O}} = \sqrt{\frac{GJ}{lI_O}}$$

# Free vibration of particle (cont'd)

## (3) Simple pendulum: (small oscillation)



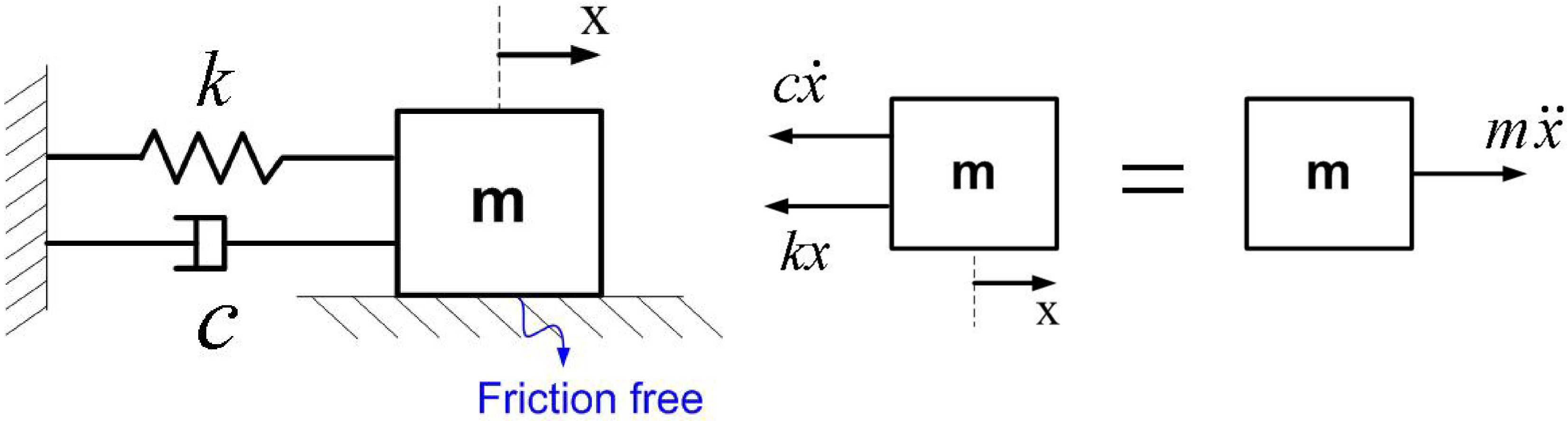

$$\sin\theta \approx \theta$$

$$F = -mg\sin\theta \approx -mg\theta$$
$$= -mgx/l = m\ddot{x}$$

$$\therefore \quad \ddot{x} + \frac{g}{l}x = 0$$

$$\therefore \quad \omega_n = \sqrt{\frac{g}{l}}$$

# Damped vibration of particle




The equation of motion : $\sum F = -kx - c\dot{x} = m\ddot{x}$

where $c$ = viscous damping constant [Ns/m]
(viscous damping coefficient)

Then $m\ddot{x} + c\dot{x} + kx = 0$

# Damped vibration of particle (cont'd)

Let the solution $x = ce^{\lambda t}$

Then $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n{}^2 = 0$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$c_{cr} = 2\sqrt{mk} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \text{ : damping coeff. } [N \cdot s / m]$$

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} \text{ : damping ratio [dimensionless]}$$

so $\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\dfrac{c^2 - 4mk}{4m^2}} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

# Damped vibration of particle (cont'd)

The general solution: $x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

There are three cases:

(1) $\zeta > 1$ or $c^2 - 4mk > 0$ (overdamping):

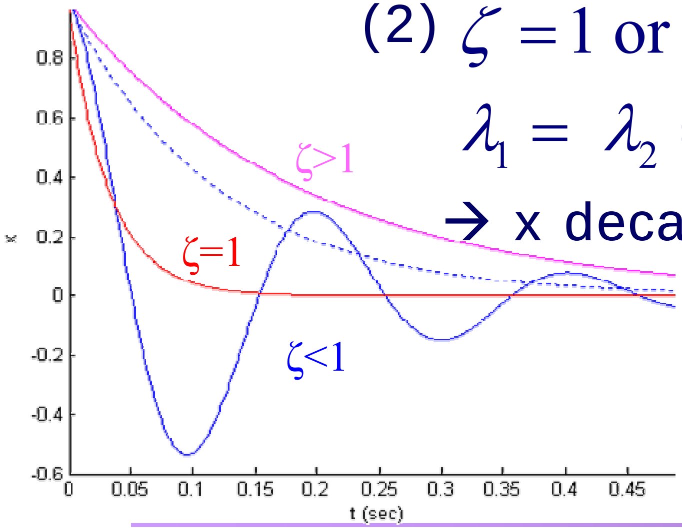
$\lambda_1,\ \lambda_2$ are distinct real numbers

→ x decays to zero. / No oscillation.

(2) $\zeta = 1$ or $c^2 - 4mk = 0$ (critical damping):

$\lambda_1 = \lambda_2 = -\omega_n$ : one real number

→ x decays to zero very fast. / No oscillation.



# Damped vibration of particle (cont'd)

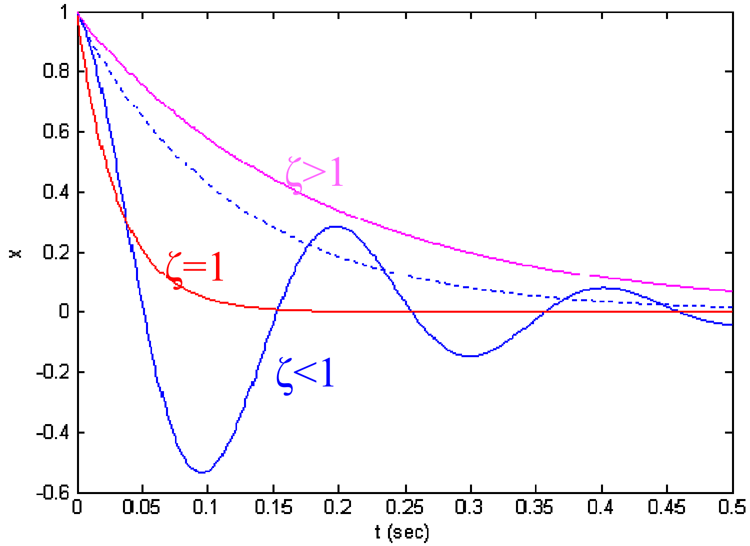
(3) $\zeta < 1$ or $c^2 - 4mk < 0$ (underdamping):

$$x = \left[ c_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + c_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right] \times e^{-\zeta\omega_n t}$$

$$= \left[ c_1 e^{i\omega_d t} + c_2 e^{-i\omega_d t} \right] e^{-\zeta\omega_n t}$$

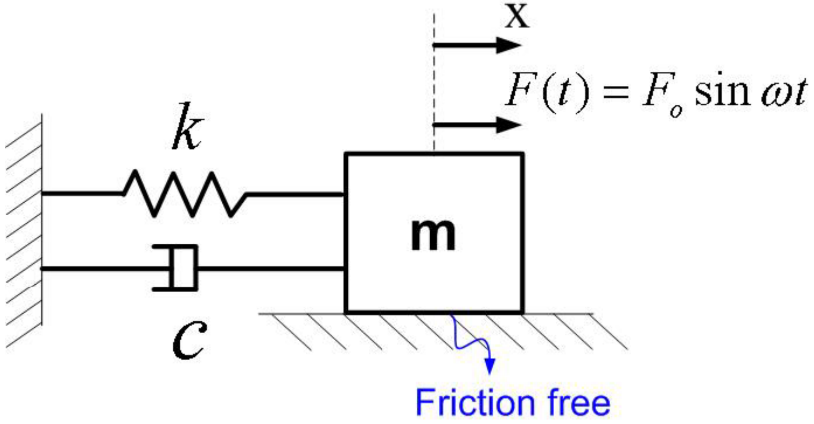
$$= C \cdot \sin\left(\omega_d t + \psi\right) e^{-\zeta\omega_n t}$$

$$= \left[ C \cdot e^{-\zeta\omega_n t} \right] \sin\left(\omega_d t + \psi\right)$$




Where $\omega_d = \omega_n \sqrt{1-\zeta^2}$ = damped natural frequency

# Forced vibration of particles




The equation of motion:

$$\sum F = -kx - c\dot{x} + F_o \sin \omega t = m\ddot{x}$$

$$\therefore \qquad m\ddot{x} + c\dot{x} + kx = F_o \sin \omega t$$

$$i.e. \qquad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{F_o}{m} \sin \omega t$$

# Forced vibration of particles (cont'd)

- Solution of damped forced vibration:

The equation of motion: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n{}^2 x = \dfrac{F_o}{m}\sin\omega t$

Solution is the sum of a complementary solution and a particular solution

$$x = x_c + x_p$$

$$x_c = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi)$$

dies down exponentially and is not important

Let : $x_p = X\sin(\omega t + \phi)$

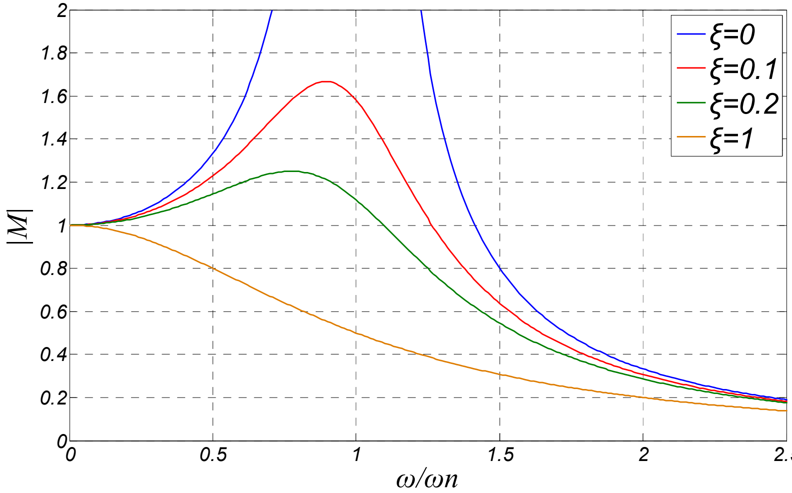
then $V(t) = X\omega\cos(\omega t + \phi) = V_o\cos(\omega t + \phi)$

# Forced vibration of particles (cont'd)

- The solution is: $$X = \frac{F_o / k}{\left\{ \left[ 1 - (\omega/\omega_n)^2 \right]^2 + \left[ 2\zeta\omega/\omega_n \right]^2 \right\}^{1/2}}$$

$$\tan\phi = \frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

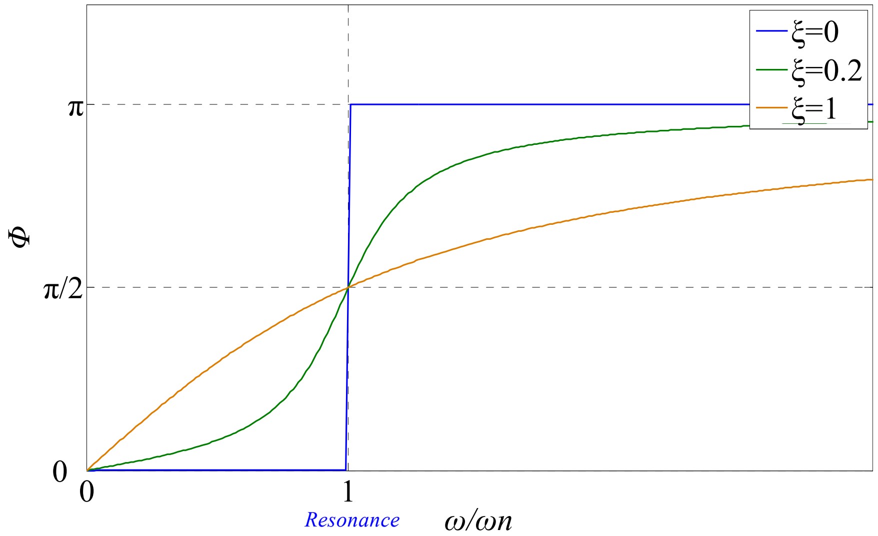
$$M = \frac{X}{F_o/k} = \frac{1}{\left\{ \left[ 1 - (\omega/\omega_n)^2 \right]^2 + \left[ 2\zeta\omega/\omega_n \right]^2 \right\}^{1/2}}$$




$$\frac{d}{d\omega}\{[1-(\frac{\omega}{\omega_n})^2]^2 + (\frac{2\zeta\omega}{\omega_n})^2\}^2 = 0$$

$$\omega_{\text{resonance}} = \sqrt{1\text{-}2\zeta^2}\,\omega_n$$

# Forced vibration of particles (cont'd)



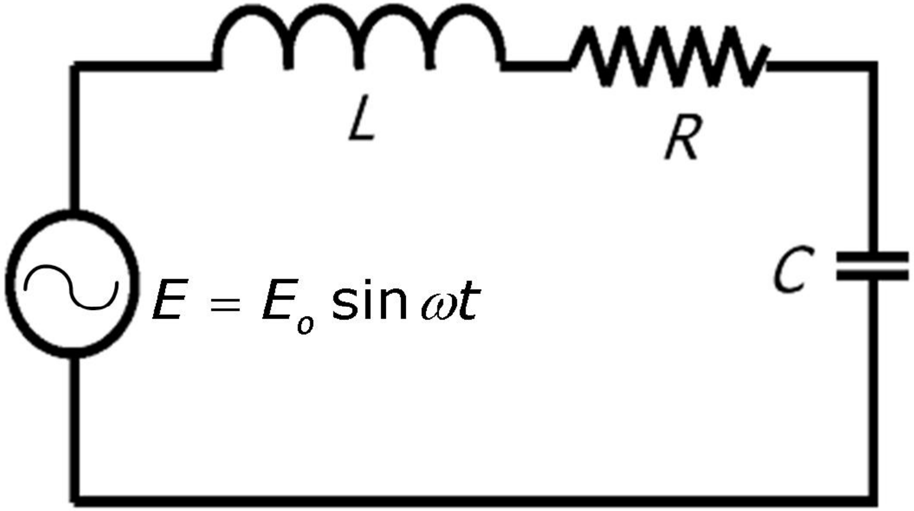

(1) $\omega$ is small, $\tan\phi > 0$, $\phi \rightarrow 0^{+}$, $x_p$ in phase with driving force

(2) $\omega$ is large, $\tan\phi < 0$, $\phi \rightarrow 0^{-}$, $\phi = \pi$, $x_p$ leads the driving force by $90^{o}$

(3) $\omega \rightarrow \omega_n^{-}$, $\tan\phi \rightarrow +\infty$, $\phi \rightarrow \pi/2^{(-)}$

$\omega \rightarrow \omega_n^{+}$, $\tan\phi \rightarrow -\infty$, $\phi \rightarrow \pi/2^{(+)}$

# Electric circuit analogy




q-charge

L-inductance

C-capacitance

R-resistance

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{q(t)}{C} = E$$

$$\dot{q} = i \ (current)$$

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