## Superposition Property

$$a x_1(t) + b\, x_2(t) \longrightarrow a y_1(t) + b y_2(t)$$
$$a x_1[n] + b\, x_2[n] \longrightarrow a y_1[n] + b y_2[n]$$

in case of zero-state response (i.e. initially rests)

- Example 1.20, p.55 : $y[n] = 2\,x[n] + 3$
- Continuous-Time Systems :

$$\ddot{y}(t) + a\dot{y}(t) + b\,y(t) = x(t), \quad \dot{y}(0) = y(0) = 0$$

- Discrete-Time Systems :

$$y[n] + a\,y[n-1] + b\,y[n-2] = x[n], \quad y[-1] = y[-2] = 0$$

- The system is <u>linear</u> if the above property holds.

**HW#1**

3, 10, 11, 17, 18, 26, 27, 28, 30, 32, 34, 44

# Chapter 2. Linear TI Systems

## §2.1 Discrete-Time LTI Systems: The Convolution Sum

### The representation of Discrete-Time Signals in terms of Impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \qquad (2.2)$$

See Fig. 2.1, p. 76

Eq. (2.2) is called the "shifting property" of the discrete-time unit impulse.

## The convolution sum representation of LTI systems

$$\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n]$$

$\Downarrow$ Superposition property (zero state response)

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \longrightarrow \boxed{\text{Linear system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k][H\delta[n-k]]$$

if $H$ is continuous, i.e. BIBO stable

Let $h_k[n] \triangleq H\delta(n-k)$. Then, we can write

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]\,h_k[n] \xRightarrow{\text{Time-Invariant}} \boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \quad \text{where } h[n] \triangleq H\delta[n]}$$

(zero state response)

$$\Rightarrow \quad y[n] = x[n] * h[n] = h[n] * x[n] \quad \text{, commutative}$$

<Fact> An LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.

<Example 2.1> 12-1 , <Example 2.2> 12-2

### §2.2 Continuous-Time LTI systems: The Convolution Integral

## The convolution Integral Representation of LTI Systems

$$\delta_\Delta(t-k\Delta) \longrightarrow \boxed{\text{Linear System } H} \longrightarrow \underbrace{H\delta_\Delta(t-k\Delta)}_{\hat{h}_{k\Delta}(t)}$$

$\Downarrow$ Superposition property (zero state response)

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta \longrightarrow \boxed{\text{Linear System } H} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\underbrace{[H\delta_\Delta(t-k\Delta)]}_{\substack{\parallel \\ H\delta(t-k\Delta) \\ \text{as } \Delta\to 0}}\Delta$$

if $H$ is continuous, i.e. BIBO stable

$\Downarrow$

$$x(t) = \lim_{\Delta\to 0}\hat{x}(t) \longrightarrow \boxed{\text{Linear System } H} \longrightarrow y(t) = \lim_{\Delta\to 0}\hat{y}(t)$$

$$= \int_{-\infty}^{\infty} [H\delta(t-\tau)]\,x(\tau)\,d\tau$$

See Fig. 2.15, P. 95 (12-3).

Let $h_\tau(t) \triangleq H\delta(t-\tau)$, $-\infty < \tau < \infty$ for each $t \in (-\infty, \infty)$

Then, we can write

$$y(t) = \int_{-\infty}^{\infty} h_\tau(t)\, x(\tau)\, d\tau$$

$$\Downarrow \text{TI}$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau}$$ where $h(t) \triangleq H\delta(t)$

(zero state response)

$$\Downarrow$$

$$y(t) = x(t) * h(t) = h(t) * x(t)$$ : commutative

<Example 2.9> PP. 99 ~ 101 (13-1, 2)

## §2.3 Properties of Linear Time-Invariant Systems

**Properties of Convolution** (if $H_1$, $H_2$ are stable (BIBO))

- commutative
- distributive :

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

- Associative :

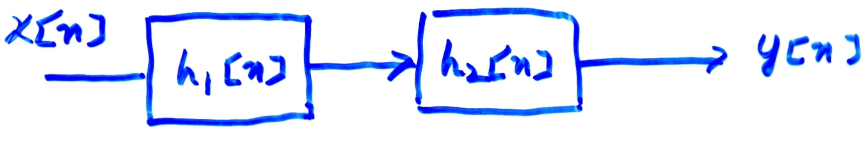
$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

(Prob 2.43 (a))

(distributive, associative) $\Longrightarrow$

interchangeable




(Fig. 2.25)

13-1



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\Delta$$

— : Time-Invariant

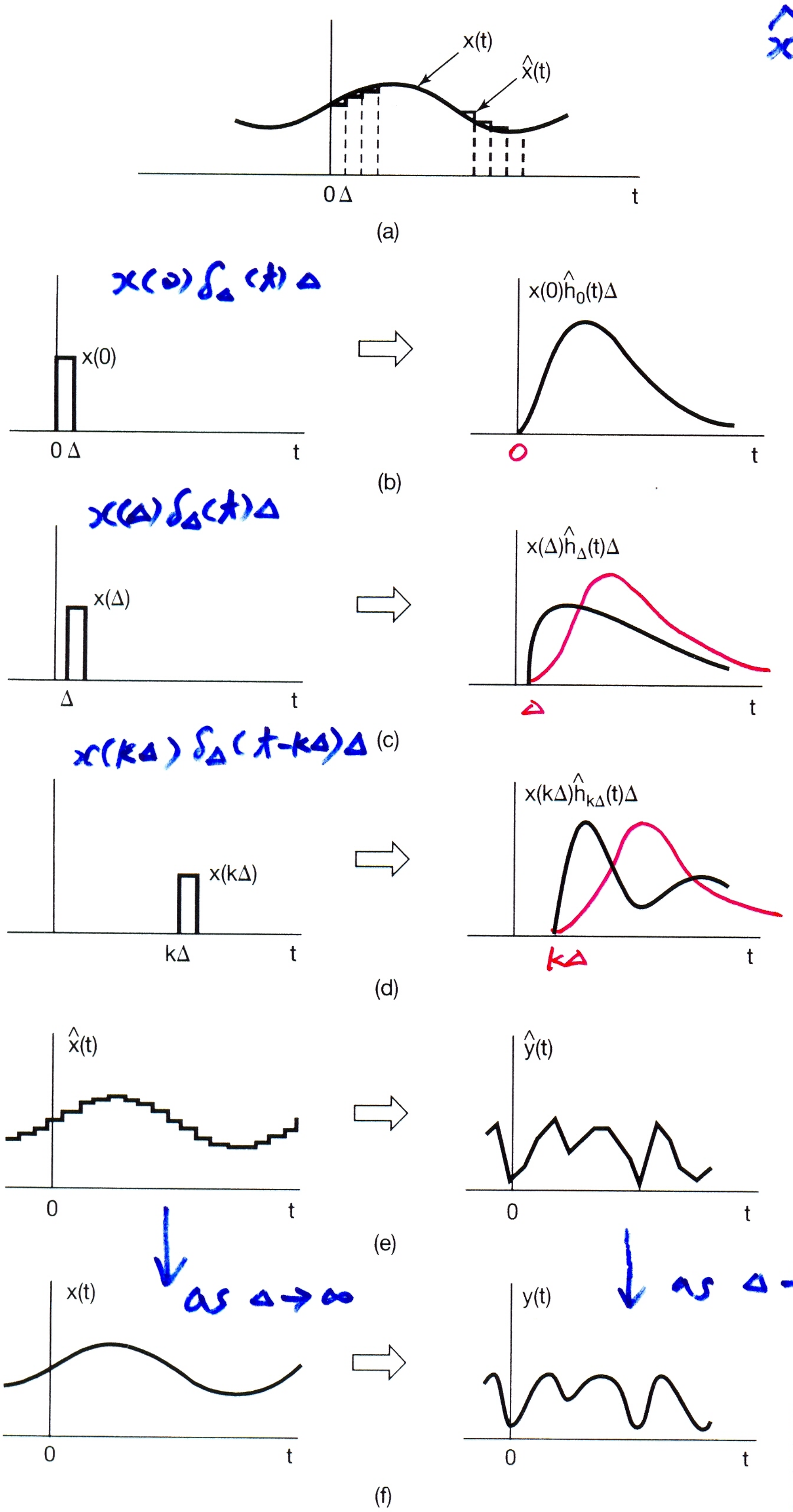



**Figure 2.15** Graphical interpretation of the response of a continuous-time linear system as expressed in eqs. (2.29) and (2.30).

# Time-Invariant Linear System :

$$(*)\begin{cases} x'(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

Let $\begin{cases} X(s) \triangleq \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} \mathcal{L}[x_1(t)] \\ \vdots \\ \mathcal{L}[x_n(t)] \end{bmatrix} \\ Y(s) \triangleq \mathcal{L}[y(t)], \quad U(s) \triangleq \mathcal{L}[u(t)] \end{cases}$

Then, from (*),

$$sX(s) - x(0) = \mathcal{L}[A\,x(t)] + \mathcal{L}[B\,u(t)]$$
$$= A\,X(s) + B\,U(s)$$

$$\Rightarrow X(s) = (sI_n - A)^{-1} x(0) + (sI_n - A)^{-1} B\,U(s)$$

$$\Rightarrow x(t) = \Phi(t)\,x(0) + (\Phi * B\,u)(t)$$

where $\Phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}] = e^{At}$

(Note that $\Phi'(t) = A\,\Phi(t)$, $\Phi(0) = I_n$)

$$\Rightarrow y(t) = C\,\Phi(t)\,x(0) + (C\,\Phi * B\,u)(t) + D\,u(t)$$
$$= C\,e^{At}\,x(0) + \int_0^t C\,e^{A(t-\tau)} B\,u(\tau)\,d\tau + D\,u(t)$$
$$= C\,e^{A(t-t_0)}\,x(t_0) + \int_{t_0}^t h(t-\tau)\,u(\tau)\,d\tau$$

where $h(t) \triangleq C\,e^{At} B + D\,\delta(t)$

# Invertibility of LTI Systems

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{\;y(t)\;} \boxed{h_1(t)} \longrightarrow w(t) = x(t)$$

$$\overset{?}{\Longrightarrow} \quad x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t)$$

That is, the continuous-time system (resp., the discrete-time system) is invertible if $\exists$ an inverse system $h_1$ $\ni$

$$h(t) * h_1(t) = \delta(t) \quad (\text{resp., } h[n] * h_1[n] = \delta[n])$$

<Example 2.11>

delay: $y(t) = x(t - t_0)$, $t_0 > 0$ $\Rightarrow$ $h(t) = \delta(t - t_0)$

$$\left( \text{since } \int_{-\infty}^{\infty} x(\tau)\,\delta(t - \tau - t_0)\,d\tau = x(t - t_0) \right)$$

Then,

$h_1(t) = \delta(t + t_0)$ since

$$w(t) = \int_{-\infty}^{\infty} y(\tau) h_1(t - \tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau - t_0)\,\delta(t - \tau + t_0)\,d\tau = x(t)$$

<Example 2.12>

$$h[n] = u[n] \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k] : \text{Summer}$$

$\Rightarrow$ The inverse system: a first difference equation

$$w[n] = y[n] - y[n-1] \qquad (2.74)$$

$$(h_1[n] = \delta[n] - \delta[n-1])$$

Since

$$w[n] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n].$$

<u>Casality for LTI systems</u> (ref. slide p.9) 15

<Fact> $h[n]=0,\ n<0 \Leftrightarrow$ causal

$(\Rightarrow)$ $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = \sum_{k\geq 0} h[k]\,\underline{x[n-k]}$

$(\Leftarrow)$ (Prove by way of contradiction)

If the claim is false, $\exists$ at least a positive integer $m \ni$

$h[-m] \neq 0$.

Choose now two discrete-time inputs $\bar{x}, \hat{x} \ni$

$\bar{x}[k] = \hat{x}[k] = 0$ for all $k$ but

$\hat{x}[k] = 1$ when $k = n+m$.

Clearly $\bar{x} \neq \hat{x}$ but

$\bar{x}[k] = \hat{x}[k],\ \forall\ k \leq n$

Then, we see that

$$\begin{cases} \bar{y}[n] = 0, \\ \hat{y}[n] = h[-m] \neq 0, \end{cases}$$

which implies that the system is not causal.

$\Rightarrow$ self-contradiction

$\square$

<Fact> $h(t)<0,\ t<0 \Leftrightarrow$ causal

$(\Rightarrow)$ easy

$(\Leftarrow)$ a little complicated. $\square$

$$\begin{cases} y[n] = \displaystyle\sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{n} x[k]\,h[n-k] \quad \text{if causal} \\[2em] y(t) = \displaystyle\int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau = \int_{-\infty}^{t} h(t-\tau)\,x(\tau)\,d\tau \quad \text{if causal} \end{cases}$$

## Stability for LTI systems (p. 10)

– A system is (BIBO) stable if the impulse response is absolutely stable, that is, iff

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \tag{2.36}$$

<Proof>

(⇐) $$|y[n]| \le \left| \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$\square$

(⇒) Prob. 2.49, p. 153

– A system is (BIBO) stable if the impulse response is absolutely integrable, that is, iff

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$$

<Proof>

(⇒) By way of contradiction; take $x(t) = \operatorname{sgn} h(t_1 - t)$ for a $t_1 > 0$. Then, $y(t_1) = \int_{-\infty}^{\infty} |h(\tau)|\,d\tau$

(⇐) $|y(t)| \le \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau \le B_x \int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$ $\square$

# §2.4 Causal LTI systems Described by Differential and Difference Equations

**The Nth-order linear constant-coefficient D.E.**

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = r(t), \quad t \ge t_0 \qquad (*)$$

Auxiliary Condition : $y^{(k)}(t_0)$, $k=0,1,\cdots,(N-1)$ given

- The system represented by $(*)$ is causal.

<proof>

Let $\begin{cases} \bar{r}(t) = \hat{r}(t), & \forall t \in [t_0, t_1] \\ \bar{r}(t) \ne \hat{r}(t), & \text{if } t > T \end{cases}$

Then,

$$\begin{cases} \sum_{k=0}^{N} a_k \bar{y}^{(k)}(t) = \bar{r}(t), & t \ge t_0 \\ \sum_{k=0}^{N} a_k \hat{y}^{(k)}(t) = \hat{r}(t), & t \ge t_0 \end{cases}$$

Let $\tilde{y}(t) \triangleq \bar{y}(t) - \hat{y}(t)$. Then,

$$\sum_{k=0}^{N} a_k \tilde{y}^{(k)}(t) = 0, \quad \forall t \in [t_0, t_1]$$

with $\tilde{y}^{(k)}(t) = 0$, $k=0,1,\cdots,(N-1)$

$\Rightarrow$ $\tilde{y}(t) = 0$ (i.e. $\bar{y}(t) = \hat{y}(t)$), $\forall t \in [t_0, t_1]$ by solution uniqueness

(Remarks) Strictly speaking,

- (2.109) in p.120 (i.e. $(*)$ with $r(t) = \sum_{k=0}^{M} b_k x^{(k)}(t)$) is <u>not</u> causal with respect to $x(t)$. (why?)
- $y(t) = \sum_{k=0}^{M} b_k z^{(k)}(t)$ and $\sum_{k=0}^{N} a_k z^{(k)}(t) = x(t)$ (2.109)' is causal if $N \ge M$ (why?) : $z^{(k)}(t_0)$, $k=0,1,\cdots,(N-1)$ given

# The Nth-order linear constant-coefficient D.E.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad \forall n \geq n_0 \qquad (2.113)$$

Auxiliary Cond. $y[n_0-1], y[n_0-2], \cdots, y_0[n_0-N]$ given

$\Rightarrow$ ① recursive equation if $N > 0$:

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\} \qquad (2.115)$$

② non recursive equation if $N = 0$:

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k] \qquad (2.116)$$

$$\Rightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \qquad (2.117)$$

$\Rightarrow$ ① The system specified by (2.116) is often called a <u>finite impulse response (FIR) system</u>.

② The system specified by (2.115) is commonly referred to as an <u>infinite impulse response</u> (<u>IIR</u>) system

③ causal if $M \geq 0$

## Block Diagram Representation of First-order Systems

Fig. 2.27 — Fig. 2.32

## The Condition of Initial Rest $\Rightarrow$ relaxed

- $y(t) = 0, \forall t \leq t_0$ if $x(t) = 0, \forall t \leq t_0$
- $y[n] = 0, \forall n < n_0$ if $x[n] = 0, \forall n < n_0$

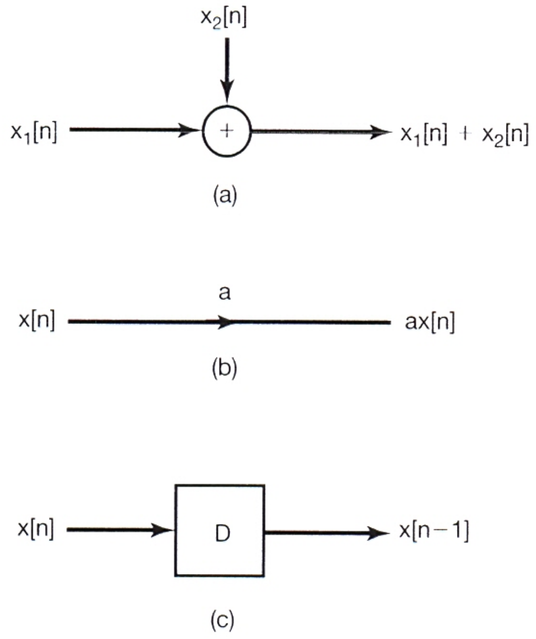



**Figure 2.27** Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

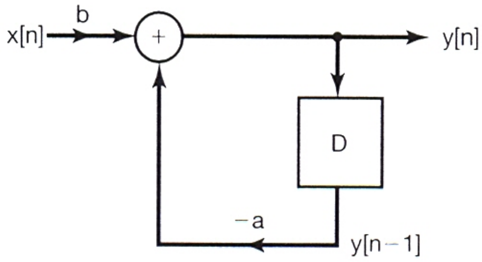



**Figure 2.28** Block diagram representation for the causal discrete-time system described by eq. (2.126).

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t). \tag{2.128}$$

As a first attempt at defining a block diagram representation for this system, let us rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t). \tag{2.129}$$

The right-hand side of this equation involves three basic operations: addition, multiplication by a coefficient, and differentiation. Therefore, if we define the three basic network elements indicated in Figure 2.29, we can consider representing eq. (2.129) as an interconnection of these basic elements in a manner analogous to that used for the discrete-time system described previously, resulting in the block diagram of Figure 2.30.

While the latter figure is a valid representation of the causal system described by eq. (2.128), it is not the representation that is most frequently used or the representation that leads directly to practical implementations, since differentiators are both difficult to implement and extremely sensitive to errors and noise. An alternative implementation that

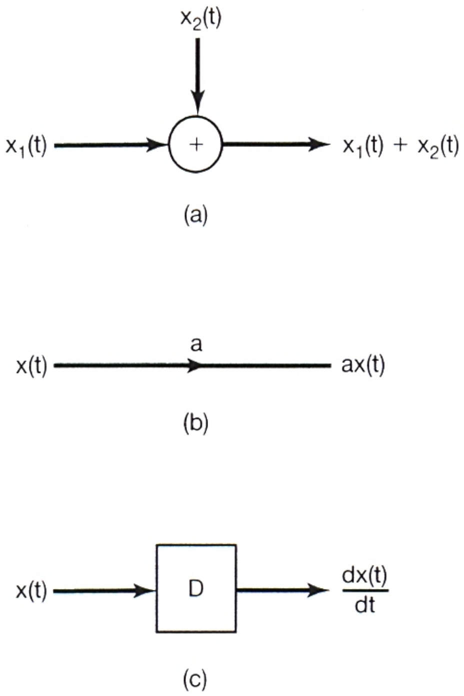



**Figure 2.29** One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

not causal in a strict sense

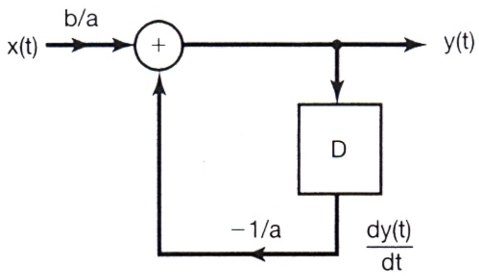



**Figure 2.30** Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

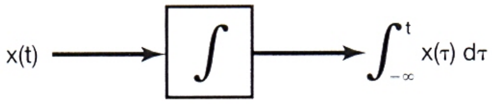



**Figure 2.31** Pictorial representation of an integrator.

a causal representation

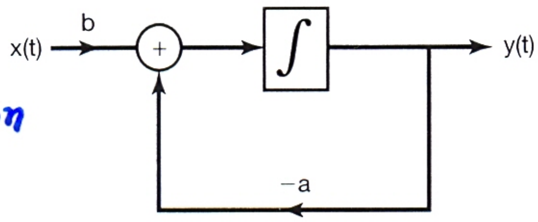



**Figure 2.32** Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

(Exercise)

Block diagram representation of (2.109)' in p.16 of the lecture note

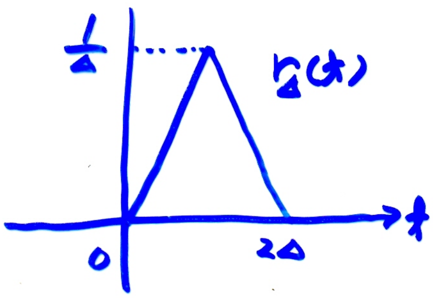
## §2.5 Singularity Functions

**The unit impulse as an idealized short pulse**

Let $r_\Delta(t) \triangleq \delta_\Delta(t) * \delta_\Delta(t)$

Then, $\lim\limits_{\Delta \to 0} r_\Delta(t) = \lim\limits_{\Delta \to 0} \delta_\Delta(t) * \lim\limits_{\Delta \to 0} \delta_\Delta(t) = \delta(t) * \delta(t)$

$= \delta(t)$ since $x(t) = x(t) * \delta(t)$




In fact, $\exists$ an unlimited number of very dissimilar-looking signals, all of which behave like an impulse in the limit.

The response of an LTI system to all of these signals is essentially identical, as long as the pulse is "short enough".

different for different systems. See Fig. 2.34, 2.35

**Defining the unit pulse through convolution**

– Definition of the unit impulse:

$$x(t) = x(t) * \delta(t) \quad \text{for any } x. \qquad (2.138)$$

– $(2.138) \Longleftrightarrow g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)\,d\tau$ for any $g$ (2.139)

<proof>

($\Leftarrow$) Let $g(\tau) \triangleq x(t-\tau)$. By (2.139),

$$x(t) = g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)\,d\tau$$

$$(\Longleftrightarrow)\quad g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{+\infty} g(\tau - t)\delta(\tau)\,d\tau$$

Letting $t=0$ gives (2.139).

- $f(t)\delta(t) = f(0)\delta(t)$ (2.142)

<proof>

$$(2.139) \Rightarrow \begin{cases} \int_{-\infty}^{\infty} g(\tau) f(\tau)\delta(\tau)\,d\tau = g(0)f(0) \\ \int_{-\infty}^{\infty} g(\tau) f(0)\delta(\tau)\,d\tau = g(0)f(0) \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \{f(\tau)\delta(\tau) - f(0)\delta(\tau)\}\, g(\tau)\,d\tau = 0 \text{ for any } g$$

$\Rightarrow$ (2.142)

## Unit Doublets and Other Singularity Functions

- Omitted

## Home Work #2

2.7, 2.11, 2.12, 2.19, 2.24, 2.31, 2.36,

2.43, 2.45(a), 2.48, 2.49, 2.58, 2.59