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§4.3.4′ Differentiation

23. Derive equation (1) at the end of this section. *Hint:* Write $e^{-a|t|} = H(t)e^{-at} + H(-t)e^{at}$. To evaluate $\mathscr{F}\{H(-t)e^{at}\}$, notice that, if $f(t) = H(t)e^{-at}$, $H(-t)e^{at} = f(-t)$, and use the time reversal property of the Fourier transform.

24. Derive equation (2). *Hint:* Let $F(\omega) = \mathscr{F}\{e^{-at^2}\}$ and compute $F'(\omega) = -i\int_{-\infty}^{\infty} te^{-at^2}e^{-i\omega t}\,dt$. Integrate by parts using $u = e^{-i\omega t}$ and $dv = te^{-at^2}\,dt$. Use the fact that a is positive to obtain the differential equation $F'(\omega) = -(\omega/2a)F(\omega)$. Solve this separable differential equation and use the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ to determine the constant of integration.

17.10 Additional Properties of the Fourier Transform

∗ The proof of (4.31) in p.306 is not rigorous.

In this section, we will derive some of the operational formulas needed to use the Fourier transform to solve differential equations. As usual, $f^{(n)}$ denotes the nth derivative of f, with the convention that $f^{(0)} = f$.

THEOREM 17.18 Differentiation in the Time Variable

Let n be a positive integer. Suppose that $f^{(n)}$ is piecewise continuous on every interval $[-a, a]$ and that $\int_{-\infty}^{\infty} |f^{(n-1)}(t)|\,dt$ converges. Assume that

$$\lim_{t\to\infty} f^{(k)}(t) = \lim_{t\to-\infty} f^{(k)}(t) = 0$$

for $k = 0, 1, 2, \ldots, n-1$. Finally, let $\mathscr{F}\{f(t)\} = F(\omega)$. Then

$$\mathscr{F}\{f^{(n)}(t)\} = (i\omega)^n F(\omega).$$

In particular,

$$\mathscr{F}\{f'(t)\} = i\omega F(\omega) \quad\text{and}\quad \mathscr{F}\{f''(t)\} = -\omega^2 F(\omega).$$

Proof We will prove the theorem for the case $n = 1$; the general result can be proved by mathematical induction. The hypotheses of the theorem ensure existence of the Fourier transform of f'. Integrate by parts to get

$$\mathscr{F}\{f'(t)\} = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt \qquad [dv = f'(t)\,dt,\ u = e^{-i\omega t}]$$

$\lim_{a\to+\infty,\ b\to+\infty}$

$$= f(t)e^{-i\omega t}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}\,dt. \quad \exists$$

Since $f^{(0)} = f$, we have by assumption that

$$\lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) = 0. \qquad \left(\Leftarrow \exists \int_{-\infty}^{\infty} |f(t)|\,dt\right)$$

Further, $|e^{-i\omega t}| = |\cos(\omega t) - i\sin(\omega t)| = 1$ for all real ω and t. Thus,

$$f(t)e^{-i\omega t}\Big]_{-\infty}^{\infty} = 0,$$

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and the integration by parts yields

$$\mathscr{F}\{f'(t)\} = i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = i\omega F(\omega),$$

as was to be proved. ∎

If f has a jump discontinuity, as often occurs in applications, additional terms must be added to the formula in this theorem. The next theorem specifies what these terms are if we are dealing with just the first derivative f'.

THEOREM 17.19 Differentiation in the Time Variable for Discontinuous f

$f \in L_1(-\infty, \infty)$
f has both left and right derivatives (i.e. f' is piecewise continuous)

Suppose that f satisfies the hypotheses of Theorem 17.10 and that $f(t) \to 0$ as $t \to \infty$ and also as $t \to -\infty$. Suppose that f is continuous except at finitely many points t_k for $k = 1, 2, \ldots, M$ and that f has jump discontinuities at these points. Let

$$a_k = f(t_k+) - f(t_k-)$$

(this is the size of the jump at t_k). Finally, let $\mathscr{F}\{f(t)\} = F(\omega)$. Then

$$\exists \quad \mathscr{F}\{f'(t)\} = i\omega F(\omega) - \sum_{k=1}^{M} a_k e^{-i\omega t_k}.$$

Proof We will suppose that f has just one jump discontinuity, at t_0, with $a = f(t_0+) - f(t_0-)$. In the case in which f has more discontinuities, the argument simply involves more of the kind of calculation we will now do. Integrate by parts to get

$$\begin{aligned}
\mathscr{F}\{f'(t)\} &= \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt = \int_{-\infty}^{t_0} f'(t)e^{-i\omega t}\,dt + \int_{t_0}^{\infty} f'(t)e^{-i\omega t}\,dt \\
&= f(t)e^{-i\omega t}\Big]_{-\infty}^{t_0} + i\omega \int_{-\infty}^{t_0} f(t)e^{-i\omega t}\,dt \\
&\quad + f(t)e^{-i\omega t}\Big]_{t_0}^{\infty} + i\omega \int_{t_0}^{\infty} f(t)e^{-i\omega t}\,dt \\
&= \lim_{t\to t_0-} f(t)e^{-i\omega t} + i\omega \int_{-\infty}^{t_0} f(t)e^{-i\omega t}\,dt \\
&\quad - \lim_{t\to t_0+} f(t)e^{-i\omega t} + i\omega \int_{t_0}^{\infty} f(t)e^{-i\omega t}\,dt \\
&= f(t_0-)e^{-i\omega t_0} - f(t_0+)e^{-i\omega t_0} + i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\
&= -[f(t_0+) - f(t_0-)]e^{-i\omega t_0} + i\omega F(\omega) \\
&= i\omega F(\omega) - ae^{-i\omega t_0}. \quad ∎
\end{aligned}$$

EXAMPLE 17.26

Solve the first order differential equation

$$y' - 4y = H(t)e^{-4t}, \qquad -\infty < t < \infty.$$

§ 4.3.6. Duality

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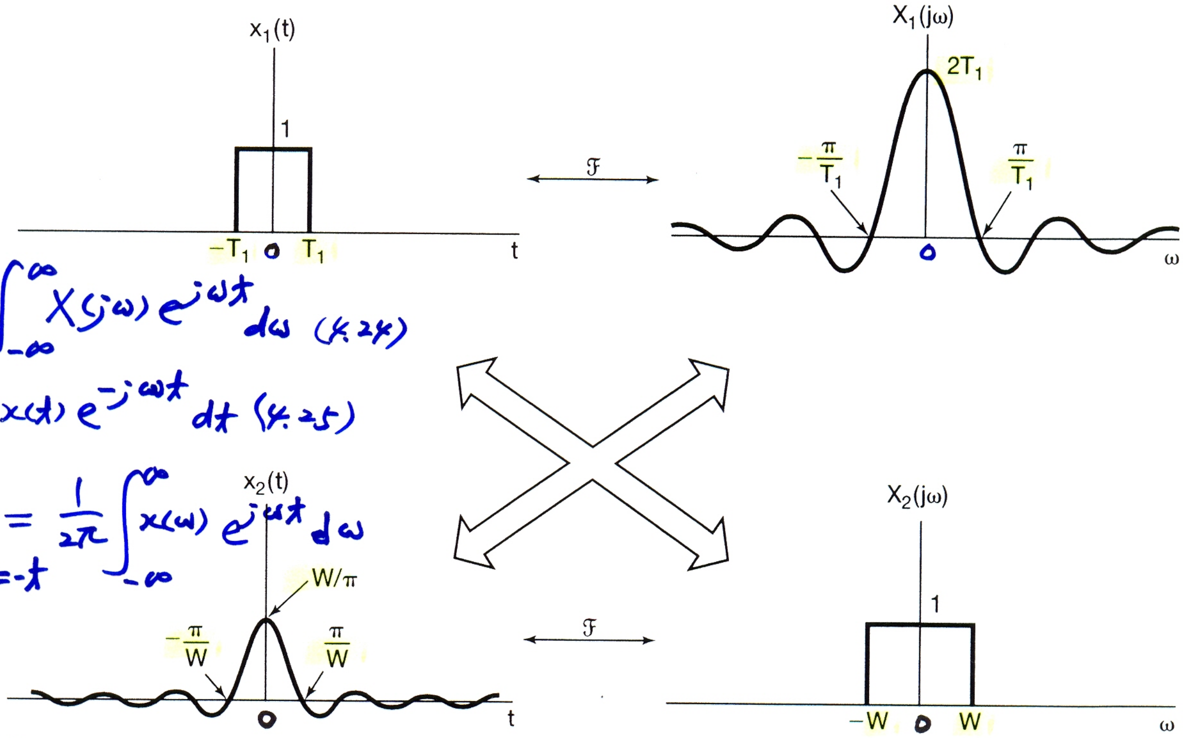
$$\begin{cases} x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega & (4.24) \\ X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt & (4.25) \end{cases}$$

$$\Rightarrow \frac{1}{2\pi} X(j\omega)\Big|_{\omega=-t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\, e^{j\omega t}\, d\omega$$



$$\Rightarrow \boxed{\frac{1}{2\pi} X(j\omega)\Big|_{\omega=-t}} \xrightarrow{\mathcal{F}} \boxed{x(t)\Big|_{t=\omega}} \quad (*)$$

Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

The two Fourier transform pairs and the relationship between them are depicted in Figure 4.17.

The symmetry exhibited by these two examples extends to Fourier transforms in general. Specifically, because of the symmetry between eqs. (4.24) and (4.25), for any transform pair, there is a dual pair with the time and frequency variables interchanged. This is best illustrated through an example.

Example 4.13 (Method I)

Let us consider using duality to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

In Example 4.2 we encountered a Fourier transform pair in which the Fourier transform, as a function of ω, had a form similar to that of the signal $x(t)$. Specifically, suppose we consider a signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

Then, from Example 4.2,

$$x(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

Method II: $e^{-|t|} \xrightarrow{\mathcal{F}} \frac{2}{1+\omega^2} \xRightarrow{\text{by } (*)} \frac{1}{2\pi}\frac{2}{1+t^2} \xrightarrow{\mathcal{F}} e^{-|\omega|}$

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The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega.$$

Now, interchanging the names of the variables t and ω, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt. \tag{4.38}$$

The right-hand side of eq. (4.38) is the Fourier transform analysis equation for $2/(1+t^2)$, and thus, we conclude that

$$\mathfrak{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}.$$

The duality property can also be used to determine or to suggest other properties of Fourier transforms. Specifically, if there are characteristics of a function of time that have implications with regard to the Fourier transform, then the same characteristics associated with a function of frequency will have *dual* implications in the time domain. For example, in Section 4.3.4, we saw that differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain. From the preceding discussion, we might then suspect that multiplication by jt in the time domain corresponds roughly to differentiation in the frequency domain. To determine the precise form of this dual property, we can proceed in a fashion exactly analogous to that used in Section 4.3.4. Thus, if we differentiate the analysis equation (4.25) with respect to ω, we obtain

$$\boxed{\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt.} \tag{4.39}$$

$\Downarrow$

That is,

$\boxed{\frac{dx}{dt} \xrightarrow{\mathcal{F}} j\omega X(j\omega)}$ $\xRightarrow{\text{by (*), P.64}}$

$$\boxed{-jtx(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.} \tag{4.40}$$

Similarly, we can derive the dual properties of eqs. (4.27) and (4.32):

$\boxed{x(t-t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)}$ $\xRightarrow{\text{by (*), P.64}}$

$$\boxed{e^{j\omega_0 t}x(t) \overset{\mathfrak{F}}{\longleftrightarrow} X(j(\omega-\omega_0))} \tag{4.41}$$

(proof, P.65-1)

and

$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)}$ $\Rightarrow$ by (*), P.64

$$\boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \overset{\mathfrak{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.} \tag{4.42}$$

(proof, P.65-1)

$$\boxed{e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))} \quad (4.41)$$

(Proof using duality)

Recall : $x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$ (4.29)

By (*), p. 64,

$$\frac{1}{2\pi} X(-jt)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} x(\omega-\omega_0) \quad (1)$$

If we let $\hat{x}(t) = \frac{1}{2\pi} X(-jt)$, we see from (**):

$$\mathcal{F}[\hat{x}(t)] = \hat{X}(j\omega) = x(\omega) \quad (2)$$

By (1) & (2),

$$\hat{x}(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \hat{X}(j(\omega-\omega_0))$$

$\square$

$$\boxed{-\frac{1}{jt} x(t) + \pi x(0)\,\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} X(j\eta)\, d\eta} \quad (4.42)$$

(Proof using duality)

Recall : $\int_{-\infty}^{t} x(\eta)\, d\eta \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega)$ (4.32)

Hence, $\int_{-\infty}^{t} x(\eta)\, d\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\frac{1}{j\omega} X(j\omega) + \pi X(0)\,\delta(\omega)\right\} e^{j\omega t}\, d\omega$

$$\overset{\substack{(t\to\omega)\\ \omega\to -t}}{\Longrightarrow} \int_{-\infty}^{\omega} x(\eta)\, d\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\frac{1}{-jt} X(-jt) + \pi X(0)\,\delta(-t)\right\} e^{-j\omega t}\, dt$$

Let $\hat{x}(t) = \frac{1}{2\pi} X(-jt)$. Then, $\mathcal{F}[\hat{x}(t)] = \hat{X}(j\omega) = x(\omega)$

$\Rightarrow$ (4.42) $\square$

Theorem (Interchanging the integration order)

Let X, Y be two subintervals of R and let $k : X \times Y \to R$ be <u>continuous</u> and <u>bounded</u> on $X \times Y$. Then, The two integrals $\int_X f(x) \int_Y g(y)\, k(x,y)\, dy\, dx$ and $\int_Y g(y) \int_X f(x)\, k(x,y)\, dx\, dy$ exist and they are equal, if $f \in L(X)$ and $g \in L(Y)$ $\square$

Theorem (continuous function)

Let $f : X \to Y$ be continuous at $x \in X$.

$\Longleftrightarrow$ for any sequence $\{x_k\}_1^\infty$ converging to x,

$$\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$$

$\square$

Theorem (Improper Integral)

Suppose that f is Riemann-integrable on $[a, b]$ for every $b \geq a$ and that $\exists\ M > 0 \ni$

$$\int_a^b |f(x)|\, dx \leq M(a) \text{ for every } b \geq a$$

Then, f and $|f|$ are improper Riemann-integrable on $(-\infty, +\infty)$.

$\square$

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4.3.7 Parseval's Relation

If $x(t)$ and $X(j\omega)$ are a Fourier transform pair, then

If $x \in L_2(-\infty, \infty)$, $X \in L_2(-j\infty, j\infty)$

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.} \tag{4.43}$$

This expression, referred to as Parseval's relation, follows from direct application of the Fourier transform. Specifically,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt.$$

Careful

Reversing the order of integration gives

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]d\omega.$$

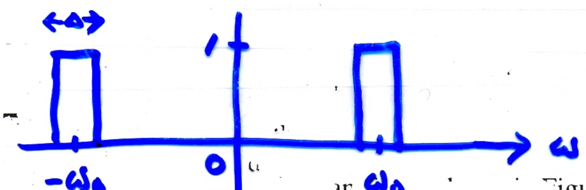
The bracketed term is simply the Fourier transform of $x(t)$; thus,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

The term on the left-hand side of eq. (4.43) is the total energy in the signal $x(t)$. Parseval's relation says that this total energy may be determined either by computing the energy per unit time ($|x(t)|^2$) and integrating over all time or by computing the energy per unit frequency ($|X(j\omega)|^2/2\pi$) and integrating over all frequencies. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$. (See also Problem 4.45.) Note that Parseval's relation for finite-energy signals is the direct counterpart of Parseval's relation for periodic signals (eq. 3.67), which states that the average *power* of a periodic signal equals the sum of the average powers of its individual harmonic components, which in turn are equal to the squared magnitudes of the Fourier series coefficients.

Real signal

$$x(t) \rightarrow \boxed{H(j\omega)} \rightarrow y(t)$$



$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(j\omega)|^2 = \frac{1}{2\pi}\int_{-\omega_0-\frac{\Delta}{2}}^{-\omega_0+\frac{\Delta}{2}} |X(j\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_0-\frac{\Delta}{2}}^{\omega_0+\frac{\Delta}{2}} |X(j\omega)|^2 d\omega$$

$$\simeq \frac{1}{2\pi}|X(-j\omega_0)|^2\Delta + \frac{1}{2\pi}|X(j\omega_0)|^2\Delta \quad \text{if } \Delta \ll 1.$$

$$= \frac{1}{\pi}|X(j\omega_0)|^2\Delta$$

§4.4 Convolution property

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Since $y(t)$ and its Fourier transform $Y(j\omega)$ are related by

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega, \tag{4.50}$$

we can identify $Y(j\omega)$ from eq. (4.49), yielding

$$Y(j\omega) = X(j\omega)H(j\omega). \tag{4.51}$$

As a more formal derivation, we consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \tag{4.52}$$

We desire $Y(j\omega)$, which is

well defined if $x, h \in L_1(-\infty,\infty)$

$$Y(j\omega) = \mathfrak{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt. \tag{4.53}$$

Interchanging the order of integration and noting that $x(\tau)$ does not depend on t, we have

valid if $h \in C^1$ and is bounded on $(-\infty,\infty)$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau. \tag{4.54}$$

By the time-shift property, eq. (4.27), the bracketed term is $e^{-j\omega\tau}H(j\omega)$. Substituting this into eq. (4.54) yields

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau. \tag{4.55}$$

Recall:

$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = F(s)$

Hence, if $f(t)$ is causal, $F(j\omega) = F(s)\big|_{s=j\omega}$

The integral is $X(j\omega)$, and hence,

$$\Rightarrow \quad Y(j\omega) = H(j\omega)X(j\omega). \quad \text{(from } Y(s) = H(s)X(s)\text{)}$$

That is,

$$\boxed{y(t) = h(t) * x(t) \overset{\mathfrak{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega).} \tag{4.56}$$

Equation (4.56) is of major importance in signal and system analysis. As expressed in this equation, the Fourier transform maps the convolution of two signals into the product of their Fourier transforms. $H(j\omega)$, the Fourier transform of the impulse response, is the frequency response as defined in eq. (3.121) and captures the change in complex amplitude of the Fourier transform of the input at each frequency ω. For example, in frequency-selective filtering we may want to have $H(j\omega) \approx 1$ over one range of frequencies, so that the frequency components in this band experience little or no attenuation or change due to the system, while over another range of frequencies we may want to have $H(j\omega) \approx 0$, so that components in this range are eliminated or significantly attenuated.

The frequency response $H(j\omega)$ plays as important a role in the analysis of LTI systems as does its inverse transform, the unit impulse response. For one thing, since $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$. In addition, many of the properties of LTI systems can be conveniently interpreted in terms of $H(j\omega)$. For example, in Section 2.3, we saw that the impulse response of the cascade of two LTI systems is the convolution of the impulse responses of the individual systems and that the overall impulse response does not depend on the order in which the systems are cascaded. Using eq. (4.56), we can rephrase this in terms of frequency responses. As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses. From this observation, it is then clear that the overall frequency response does not depend on the order of the cascade.

$h_2(t)*(h_1(t)*x(t))$
$= (h_2(t)*h_1(t))*x(t)$
$= (h_1(t)*h_2(t))*x(t)$
$= h_1(t)*(h_2(t)*x(t))$

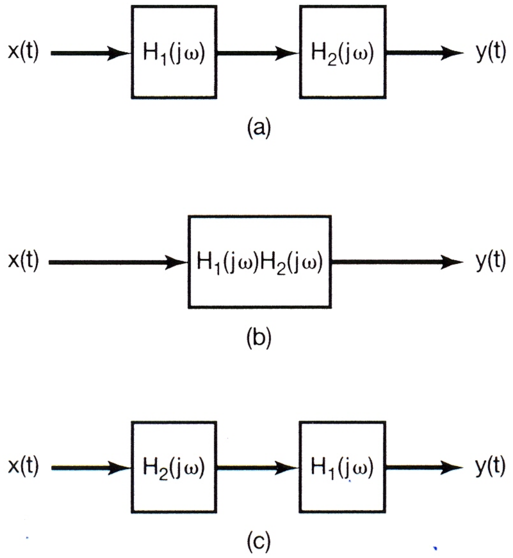

Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

As discussed in Section 4.1.2, convergence of the Fourier transform is guaranteed only under certain conditions, and consequently, the frequency response cannot be defined for every LTI system. If, however, an LTI system is stable, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty. \tag{4.57}$$

Equation (4.57) is one of the three Dirichlet conditions that together guarantee the existence of the Fourier transform $H(j\omega)$ of $h(t)$. Thus, assuming that $h(t)$ satisfies the other two conditions, as essentially all signals of physical or practical significance do, we see that a stable LTI system has a frequency response $H(j\omega)$.

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose impulse responses possess Fourier transforms. In order to use transform techniques to examine unstable LTI systems we will develop a generalization of