§ 5.6 Tables of Fourier Transform Properties and Basic Fourier Transform Pairs

- Table 5.1 & 5.2

§ 5.7 Duality

- No duality in the Discrete-Time Fourier <u>Transf.</u>

$$\begin{cases} x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega & (5.8) \\ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} & (5.9) \end{cases}$$

§ 5.7.1 Duality in the Discrete-Time Fourier <u>Series</u>

$$x[n] \xleftrightarrow{\mathcal{F}} a_k$$

If $x[n+N] = x[n]$, then $a_{k+N} = a_k$.

Suppose that two periodic sequences $f[n]$, $g[k]$ with period N are related as follows.

$$g[n] \xleftrightarrow{FS} f[k] \qquad (5.66)$$

Then,

$$f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(\frac{2\pi}{N})n} \qquad (5.65)$$

$$\xRightarrow[n \to -k]{k \to n} f[n] = \sum_{k=\langle N \rangle} \frac{1}{N} g[-k] e^{jk(\frac{2\pi}{N})n}$$

$$\Longrightarrow f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k] \qquad (5.67)$$

- useful in reducing the complexity of the calculation involved in determining Fourier series representation.

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \leq N_1 \\ 0, & N_1 < \|n\| \leq N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \|a\| < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \|n\| \leq N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \|\omega\| \leq W \\ 0, & W < \|\omega\| \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \|a\| < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

<Example 5.16'>

$$\overbrace{x[n-n_0]}^{g[n]} \xleftrightarrow{F.S} \overbrace{a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}}^{f[k]}$$

$$x[n]\, e^{-jnk\left(\frac{2\pi}{N}\right)n_0} \xleftrightarrow{F.S} a_{k+n_0}$$

<proof>

$$\begin{cases} f[n] = a_n e^{-jn\left(\frac{2\pi}{N}\right)n_0} \\ \frac{1}{N} g[-k] = \frac{1}{N} x[-k-n_0] \end{cases}$$

On the other hand,

$$x[n] \xleftrightarrow{F.S} a_k \Rightarrow \underbrace{a_n}_{x[n]} \xleftrightarrow{F.S} \underbrace{\frac{1}{N} x[-k]}_{a_k}$$

$$\Rightarrow \frac{1}{N} g[-k] = a_{k-n_0}$$

□

<Example 5.16''>

$$\overbrace{\sum_{r\in\langle N\rangle} x[r]\, y[n-r]}^{g[n]} \xleftrightarrow{F.S} \overbrace{N a_k b_k}^{f[k]}$$

$$x[n]\, y[n] \xleftrightarrow{F.S} \sum_{l\in\langle N\rangle} a_l b_{k-l} = \sum_{r\in\langle N\rangle} a_{-r} b_{k+r}$$

<proof>

$$f[n] = a_n b_n N, \quad \frac{1}{N} g[-k] = \frac{1}{N} \sum_{r\in\langle N\rangle} x[r]\, y[-k-r]$$

$$\Rightarrow a_n b_n \xrightarrow{F.S} \frac{1}{N^2} \sum_{r\in\langle N\rangle} x[r]\, y[-k-r]$$

On the other hand,

$$\underbrace{a_n}_{x[n]} \xleftrightarrow{F.S} \underbrace{\frac{1}{N} x[-k]}_{a_k}, \quad \underbrace{b_n}_{y[n]} \xleftrightarrow{F.S} \underbrace{\frac{1}{N} y[-k]}_{b_k}.$$

□

Example 5.17

The duality between the discrete-time Fourier transform synthesis equation and the continuous-time Fourier series analysis equation may be exploited to determine the discrete-time Fourier transform of the sequence

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

To use duality, we first must identify a continuous-time signal $g(t)$ with period $T = 2\pi$ and Fourier coefficients $a_k = x[k]$. From Example 3.5, we know that if $g(t)$ is a periodic square wave with period 2π (or, equivalently, with fundamental frequency $\omega_0 = 1$) and with

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases},$$

then the Fourier series coefficients of $g(t)$ are

$$a_k = \frac{\sin(kT_1)}{k\pi}.$$

Consequently, if we take $T_1 = \pi/2$, we will have $a_k = x[k]$. In this case the analysis equation for $g(t)$ is

$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$ (5.76) $\Rightarrow$

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{-jkt} dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1) e^{-jkt} dt.$$

Renaming k as n and t as ω, we have

$k \leftrightarrow n$, $t \leftrightarrow \omega$ $\Rightarrow$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1) e^{-jn\omega} d\omega. \tag{5.77}$$

Replacing n by $-n$ on both sides of eq. (5.77) and noting that the sinc function is even, we obtain

$n \leftrightarrow -n$ $\Rightarrow$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1) e^{jn\omega} d\omega.$$

duality ($k \leftrightarrow n$, $t \leftrightarrow \omega$)

The right-hand side of this equation has the form of the Fourier transform synthesis equation for $x[n]$, where

$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ (5.73) $\Rightarrow$

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| \le \pi \end{cases}.$$

In Table 5.3, we present a compact summary of the Fourier series and Fourier transform expressions for both continuous-time and discrete-time signals, and we also indicate the duality relationships that apply in each case.

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time			Discrete time		
	Time domain		**Frequency domain**	**Time domain**		**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ (5.75) continuous time periodic in time		$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$ (5.76) discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	duality	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
			duality			
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	duality	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ (5.73) discrete time aperiodic in time		$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ (5.74) continuous frequency periodic in frequency

(5.74) $\Uparrow$ (5.75)' ($k \leftrightarrow -n$, $t \leftrightarrow \omega$, $\omega_0 = 1$)

§5.8 Systems characterized by linear constant-coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (5.78)$$

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}) \quad \text{if } x, h \in l_1(-\infty,\infty)$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \qquad (5.80)$$

<Example 5.19>

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n] \qquad (5.84)$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}} \qquad (5.85)$$

$$= \frac{2}{\left(1 - \frac{1}{2} e^{-j\omega}\right)\left(1 - \frac{1}{4} e^{-j\omega}\right)}$$

$$= \frac{4}{1 - \frac{1}{2} e^{-j\omega}} - \frac{2}{1 - \frac{1}{4} e^{-j\omega}}$$

$$\Rightarrow \quad h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] \qquad (5.88) \quad \square$$

HW #5

5.13, 5.16, 5.19, 5.21(c), (k), 5.22 (a), (e), 5.29 (b), 5.43, 5.51, 5.53, 5.56 (a), (b)

5.52. **(a)** Let $h[n]$ be the impulse response of a real, causal, discrete-time LTI system. Show that the system is completely specified by the real part of its frequency response. (*Hint:* Show how $h[n]$ can be recovered from $\mathcal{E}v\{h[n]\}$. What is the Fourier transform of $\mathcal{E}v\{h[n]\}$?) This is the discrete-time counterpart of the *real-part sufficiency* property of causal LTI systems considered in Problem 4.47 for continuous-time systems.

(b) Let $h[n]$ be real and causal. If

$$\mathcal{R}e\{H(e^{j\omega})\} = 1 + \alpha\cos 2\omega (\alpha \text{ real}),$$

determine $h[n]$ and $H(e^{j\omega})$.

(c) Show that $h[n]$ can be completely recovered from knowledge of $\mathcal{I}m\{H(e^{j\omega})\}$ and $h[0]$.

(d) Find two real, causal LTI systems whose frequency responses have imaginary parts equal to $\sin\omega$.

EXTENSION PROBLEMS

5.53. One of the reasons for the tremendous growth in the use of discrete-time methods for the analysis and synthesis of signals and systems was the development of exceedingly efficient tools for performing Fourier analysis of discrete-time sequences. At the heart of these methods is a technique that is very closely allied with discrete-time Fourier analysis and that is ideally suited for use on a digital computer or for implementation in digital hardware. This technique is the *discrete Fourier transform (DFT)* for finite-duration signals.

Let $x[n]$ be a signal of finite duration; that is, there is an integer N_1 so that

$$x[n] = 0, \qquad \text{outside the interval } 0 \le n \le N_1 - 1$$

Furthermore, let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. We can construct a periodic signal $\tilde{x}[n]$ that is equal to $x[n]$ over one period. Specifically, let $N \ge N_1$ be a given integer, and let $\tilde{x}[n]$ be periodic with period N and such that

$$\tilde{x}[n] = x[n], \qquad 0 \le n \le N - 1$$

The Fourier series coefficients for $\tilde{x}[n]$ are given by

$$a_k = \frac{1}{N}\sum_{\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}$$

Choosing the interval of summation to be that over which $\tilde{x}[n] = x[n]$, we obtain

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n} \qquad \text{(P5.53–1)}$$

The set of coefficients defined by eq. (P5.53–1) comprise the DFT of $x[n]$. Specifically, the DFT of $x[n]$ is usually denoted by $\tilde{X}[k]$, and is defined as

$$\tilde{X}[k] = a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \qquad k = 0, 1, \ldots, N-1 \qquad \text{(P5.53–2)}$$

The importance of the DFT stems from several facts. First note that the original finite duration signal can be recovered from its DFT. Specifically, we have

$$x[n] = \sum_{k=0}^{N-1} \tilde{X}[k]e^{jk(2\pi/N)n}, \qquad n = 0, 1, \ldots, N-1 \qquad \text{(P5.53–3)}$$

Thus, the finite-duration signal can either be thought of as being specified by the finite set of nonzero values it assumes or by the finite set of values of $\tilde{X}[k]$ in its DFT. A second important feature of the DFT is that there is an extremely fast algorithm, called the *fast Fourier transform (FFT),* for its calculation (see Problem 5.54 for an introduction to this extremely important technique). Also, because of its close relationship to the discrete-time Fourier series and transform, the DFT inherits some of their important properties.

(a) Assume that $N \geq N_1$. Show that

$$\tilde{X}[k] = \frac{1}{N} X\left(e^{j(2\pi k/N)}\right)$$

where $\tilde{X}[k]$ is the DFT of $x[n]$. That is, the DFT corresponds to samples of $X(e^{j\omega})$ taken every $2\pi/N$. Equation (P5.53–3) leads us to conclude that $x[n]$ can be uniquely represented by these samples of $X(e^{j\omega})$.

(b) Let us consider samples of $X(e^{j\omega})$ taken every $2\pi/M$, where $M < N_1$. These samples correspond to more than one sequence of duration N_1. To illustrate this, consider the two signals $x_1[n]$ and $x_2[n]$ depicted in Figure P5.53. Show that if we choose $M = 4$, we have

$$X_1\left(e^{j(2\pi k/4)}\right) = X_2\left(e^{j(2\pi k/4)}\right)$$

for all values of k.

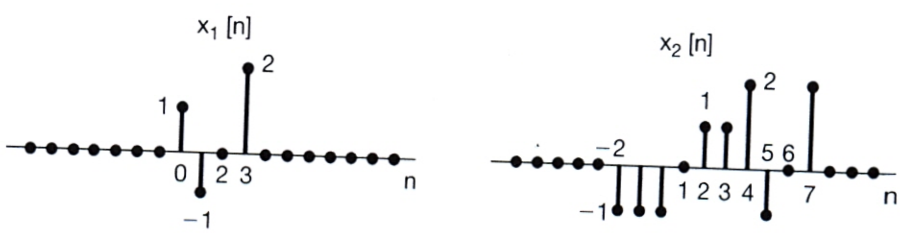


Fig P5.53

5.54. As indicated in Problem 5.53, there are many problems of practical importance in which one wishes to calculate the discrete Fourier transform (DFT) of discrete-time signals. Often, these signals are of quite long duration, and in such cases it is very important to use computationally efficient procedures. One of the reasons for the significant increase in the use of computerized techniques for the analysis of signals was the development of a very efficient technique known as the fast Fourier transform (FFT) algorithm for the calculation of the DFT of finite-duration sequences. In this problem, we develop the principle on which the FFT is based.

Let $x[n]$ be a signal that is 0 outside the interval $0 \leq n \leq N_1 - 1$. For $N \geq N_1$, the N-point DFT of $x[n]$ is given by

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \quad k = 0, 1, \ldots, N-1. \qquad \text{(P5.54–1)}$$

It is convenient to write eq. (P5.54–1) as

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]W_N^{nk}, \qquad \text{(P5.54–2)}$$

where

$$W_N = e^{-j2\pi/N}.$$

(a) One method for calculating $\tilde{X}[k]$ is by direct evaluation of eq. (P5.54–2). A useful measure of the complexity of such a computation is the total number of complex multiplications required. Show that the number of complex multiplications required to evaluate eq. (P5.54–2) directly, for $k = 0, 1, \ldots, N-1$, is N^2. Assume that $x[n]$ is complex and that the required values of W_N^{nk} have been precomputed and stored in a table. For simplicity, *do not* exploit the fact that, for certain values of n and k, W_N^{nk} is equal to ± 1 or $\pm j$ and hence does not, strictly speaking, require a full complex multiplication.

(b) Suppose that N is even. Let $f[n] = x[2n]$ represent the even-indexed samples of $x[n]$, and let $g[n] = x[2n+1]$ represent the odd-indexed samples.

(i) Show that $f[n]$ and $g[n]$ are zero outside the interval $0 \leq n \leq (N/2) - 1$.

(ii) Show that the N-point DFT $\tilde{X}[k]$ of $x[n]$ can be expressed as

$$\begin{aligned}\tilde{X}[k] &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk} + \frac{1}{N}W_N^k \sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk} \\ &= \frac{1}{2}\tilde{F}[k] + \frac{1}{2}W_N^k\tilde{G}[k], \quad k = 0, 1, \ldots, N-1, \qquad \text{(P5.54–3)}\end{aligned}$$

where

$$\tilde{F}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk},$$

$$\tilde{G}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk}.$$

(iii) Show that, for all k,

$$\tilde{F}\left[k + \frac{N}{2}\right] = \tilde{F}[k],$$

$$\tilde{G}\left[k + \frac{N}{2}\right] = \tilde{G}[k].$$

Note that $\tilde{F}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, and $\tilde{G}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, are the $(N/2)$-point DFTs of $f[n]$ and $g[n]$, respectively. Thus, eq. (P5.54–3) indicates that the length-N DFT of $x[n]$ can be calculated in terms of two DFTs of length $N/2$.

(iv) Determine the number of complex multiplications required to compute $\tilde{X}[k]$, $k = 0, 1, 2, \ldots, N - 1$, from eq. (P5.54–3) by first computing $\tilde{F}[k]$ and $\tilde{G}[k]$. [Make the same assumptions about multiplications as in part (a), and ignore the multiplications by the quantity 1/2 in eq. (P5.54–3).]

(c) If, like N, $N/2$ is even, then $f[n]$ and $g[n]$ can each be decomposed into sequences of even- and odd-indexed samples, and therefore, their DFTs can be computed using the same process as in eq. (P5.54–3). Furthermore, if N is an integer power of 2, we can continue to iterate the process, thus achieving significant savings in computation time. With this procedure, approximately how many complex multiplications are required for $N = 32, 256, 1{,}024$, and $4{,}096$? Compare this to the direct method of calculation in part (a).

5.55. In this problem we introduce the concept of *windowing*, which is of great importance both in the design of LTI systems and in the spectral analysis of signals. Windowing is the operation of taking a signal $x[n]$ and multiplying it by a finite-duration *window signal* $w[n]$. That is,

$$p[n] = x[n]w[n].$$

Note that $p[n]$ is also of finite duration.

The importance of windowing in spectral analysis stems from the fact that in numerous applications one wishes to compute the Fourier transform of a signal that has been measured. Since in practice we can measure a signal $x[n]$ only over a finite time interval (the *time window*), the actual signal available for spectral analysis is

$$p[n] = \begin{cases} x[n], & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases},$$

where $-M \leq n \leq M$ is the time window. Thus,

$$p[n] = x[n]w[n],$$

where $w[n]$ is the *rectangular window;* that is,

$$w[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}. \qquad \text{(P5.55–1)}$$

Windowing also plays a role in LTI system design. Specifically, for a variety of reasons (such as the potential utility of the FFT algorithm; see Problem P5.54), it is

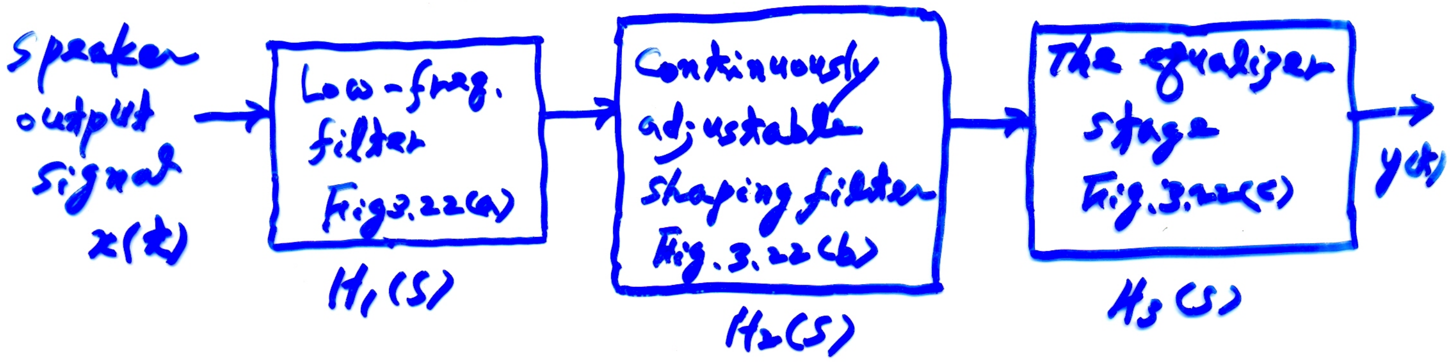
§ 3.9 Filtering (Chap. 5장에서 설명)

__Filtering__ : A process to change the relative amplitudes of the frequency components in a signal (frequency-shaping filters) or perhaps eliminate some frequency components entirely. (frequency-selective filters)

— Through the use of LTI systems

§ 3.9.1 Frequency-Shaping Filters

__Example 1__ : Equalizing filter to compensate for the freq. resp. characteristics of the speakers.



$$Y(s) = H_1(s)\, H_2(s)\, H_3(s)\, X(s)$$

$$\Rightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} a_k H_1(jk\omega) H_2(jk\omega) H_3(jk\omega) e^{jk\omega t}$$

if $x(t)$ is periodic with period T,

where $\omega \triangleq \dfrac{2\pi}{T}$.

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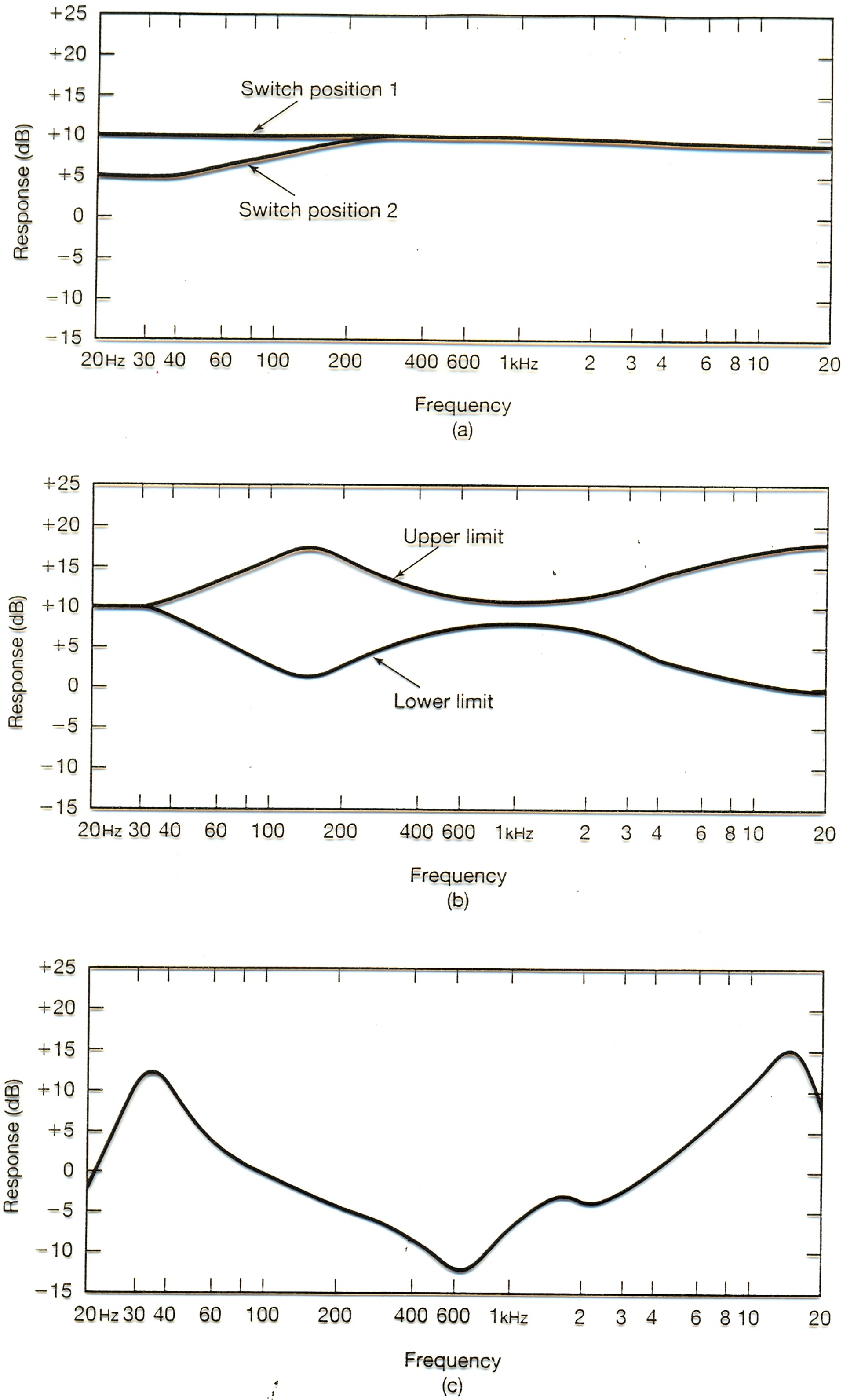


Figure 3.22 Magnitudes of the frequency responses of the equalizer circuits for one particular series of audio speakers, shown on a scale of $20 \log_{10} |H(j\omega)|$, which is referred to as a decibel (or dB) scale. (a) Low-frequency filter controlled by a two-position switch; (b) upper and lower frequency limits on a continuously adjustable shaping filter; (c) fixed frequency response of the equalizer stage.

Example 2: Differentiating filter to enhance edges in image (picture) processing.

$$g(x,y) = \left[|D_1 f(x,y)|^2 + |D_2 f(x,y)|^2 \right]^{\frac{1}{2}} \quad : \text{nonlinear system}$$

Periodic Expansion $f^*(x,y)$ of $f(x,y)$

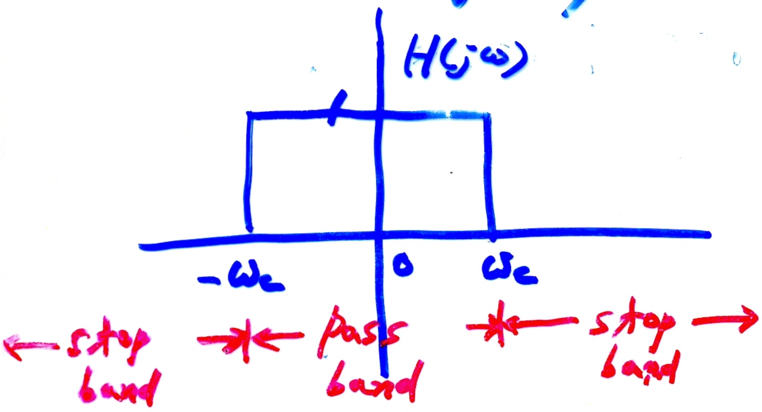
$$\begin{cases} f^*(x,y) = f(x,y) & \text{if } x \in [0, T_1],\ y \in [0, T_2] \\ f^*(x,y) = f(x+T_1, y+T_1), & \text{otherwise} \end{cases}$$

2-D Fourier series

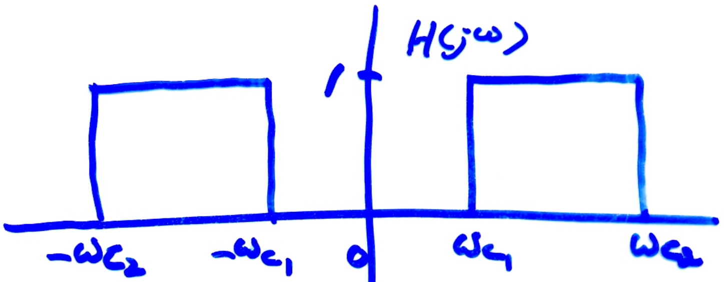
$$\begin{cases} f^*(x,y) = \displaystyle\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn}\, e^{j(m\omega_1 x + n\omega_2 y)} \\ a_{mn} = \dfrac{1}{T_1 T_2} \displaystyle\int_0^{T_1}\int_0^{T_2} f^*(x,y)\, e^{-j(m\omega_1 x + n\omega_2 y)}\, dx\, dy \end{cases}$$

– See Fig. 3.24 for effect of a diff. filter.

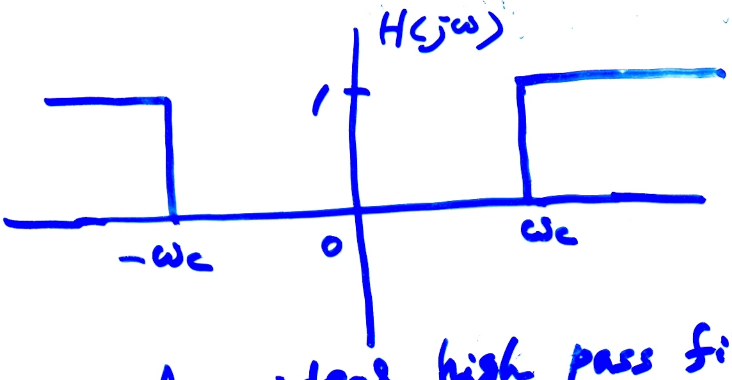
§ 3.9.2 Frequency-Selective Filters



An ideal low pass filter (LPF)



An ideal bandpass filter (BPF)



An ideal high pass filter (HPF)

- See Fig. 3.28 for Discrete-time ideal frequency-selective filters.

- Ideal filters are quite useful in describing idealized system configurations for a variety of applications. However, they are not realizable in practice and must be approximated.

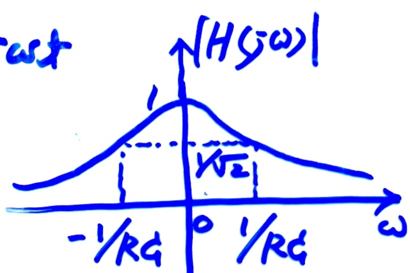
§3.10 Examples of Continuous-Time Filters

- The implementation of continuous-time and discrete-time freq.-selective filters through the use of differential and difference equations.

§3.10.1 A simple RC Low Pass Filter

$$RC\frac{dy}{dt} + y = x \iff X(s) \longrightarrow \boxed{\frac{1}{RCs+1}} \longrightarrow Y(s)$$

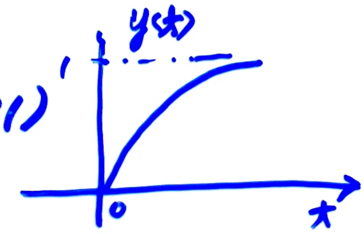
$$\Rightarrow \quad RC\,H(j\omega)j\omega e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$



$\Rightarrow$ ① $H(j\omega) = \dfrac{1}{1 + RCj\omega}$ (Fig. 3.30)

② Step input response

$$y(t) = (1 - e^{-t/RC})\,u(t) \quad \text{(Fig. 3.31)}$$

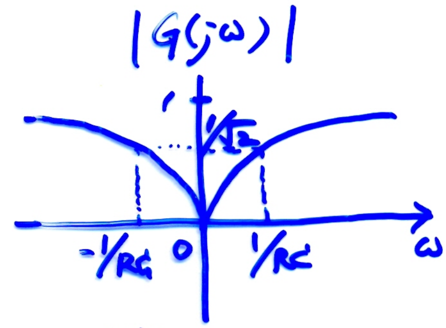


$\Rightarrow$ (?) Trade-Off between behaviour in the frequency domain and in the time domain.

$$RC \Rightarrow \infty \Rightarrow \begin{cases} \cdot \text{ sluggish in time} \\ \cdot \text{ reduced noise} \end{cases}$$

§3.10.2 A simple RC Highpass Filter

$$X(s) \longrightarrow \boxed{\frac{RCs}{RCs+1}} \longrightarrow Y(s)$$



① $G(j\omega) = \dfrac{j\omega RC}{1+j\omega RC}$ (Fig. 3.32)

② Step input response

$y(t) = e^{-t/RC}\, u(t)$ (Fig. 3.33)

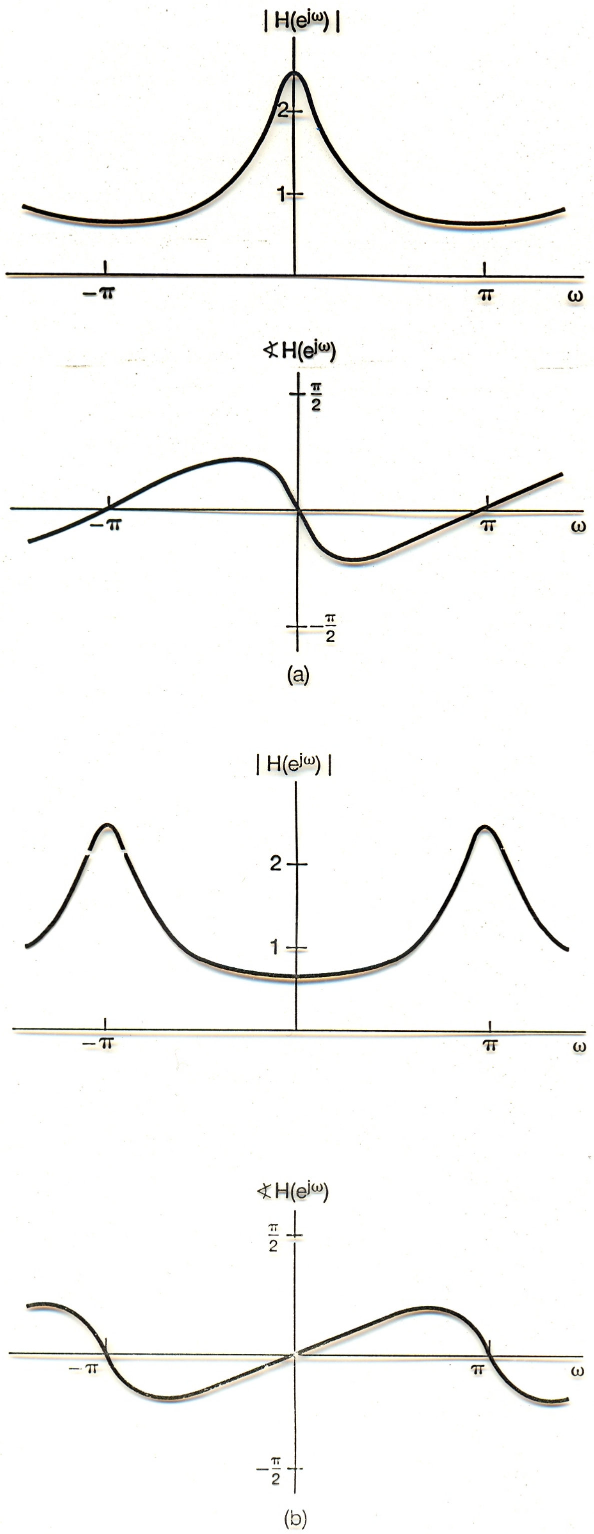
⇒(?) Trade-off exists : $RC \to \infty \Rightarrow$ { sluggish in time ; more components of low freq. }

Remark

Filters described by high-order diff. eq. offer considerably more flexibility in terms of their characteristics, allowing, for example, sharper passband-stopband transition or more control over the trade-offs between time response and frequency response.

§3.11 Examples of Discrete-Time Filters

- Filters described by difference equations are widely used in practice since they can be efficiently implemented in special/or general-purpose digital systems



a low pass filter if $0<a<1$

a highpass filter if $0>a>-1$

Figure 3.34 Frequency response of the first-order recursive discrete-time filter of eq. (3.151): (a) $a = 0.6$; (b) $a = -0.6$.

§3.11 First-order Recursive Discrete-Time Filters

$$y[n] - a\,y[n-1] = x[n]$$

$$\Rightarrow \quad H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

$\Rightarrow$ ① $H(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$

② step response $\quad s[n] = \dfrac{1-a^{n+1}}{1-a}\,u[n]$

$\Rightarrow$ $|a|$ controls the behaviour in the frequency and time domain.

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The summation in eq. (3.161) can be evaluated by performing calculations similar to those in Example 3.12, yielding

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \tag{3.162}$$

By adjusting the size, $N + M + 1$, of the averaging window we can vary the cutoff frequency. For example, the magnitude of $H(e^{j\omega})$ is shown in Figure 3.36 for $M+N+1 = 33$ and $M + N + 1 = 65$.

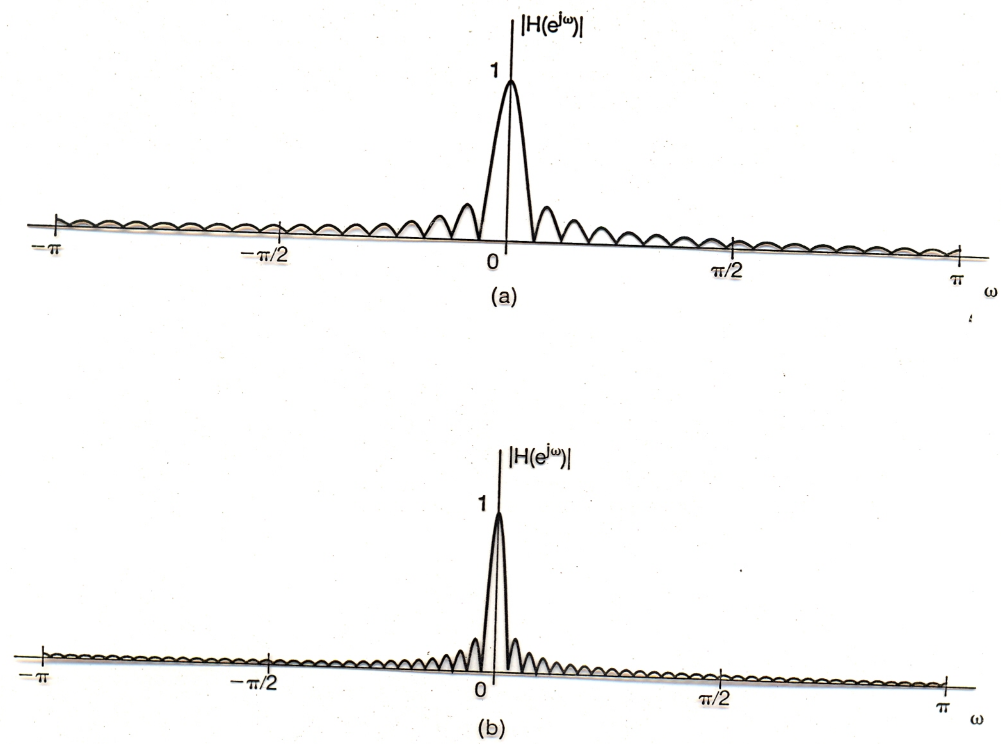


Figure 3.36 Magnitude of the frequency response for the lowpass moving-average filter of eq. (3.162): (a) $M = N = 16$; (b) $M = N = 32$.

§3.11.2 Nonrecursive Discrete-Time Filters

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k] \qquad (3.159)$$

Example 1 (A Moving average filter)

$$y[n] = \frac{1}{(N+M+1)} \sum_{k=-N}^{M} x[n-k] \qquad (3.160)$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{(N+M+1)} \sum_{k=-N}^{M} e^{-j\omega k}$$

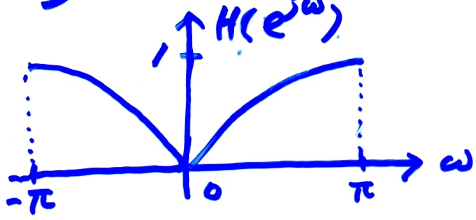
$= (3.162) \uparrow$

Example 2 A simple high pass filter

$$y[n] = \frac{x[n]-x[n-1]}{2} \qquad (3.163)$$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{2}[1-e^{-j\omega}] = j e^{j\omega/2} \sin(\omega/2) \qquad (3.164)$$

(Fig. 3.37)



Remark

- All FIR filters are BIBO stable.
- In off-line applications, causality is not a nec. constraint.
- In real-time processing, causality is essential.

§ 3.11.3 Digitization

Euler's Method

$$\dot{x}(kh) \cong \frac{x[k]-x[k-1]}{h} \quad \text{where } x[k] \triangleq x(kh)$$

<Example>

$$G(s) = \frac{Y(s)}{X(s)} = \frac{K(s+a)}{(s+b)}$$

$$\Rightarrow \dot{y} + by = K(\dot{x} + ax)$$

$$\Rightarrow \frac{y[k]-y[k-1]}{T} + by[k] = K\left\{\frac{x[k]-x[k-1]}{T} + a\,x[k]\right\}$$

Tustin's Approximation

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

<Example>

$$G(s) = 70\frac{s+2}{s+10} \rightarrow \frac{Y(z)}{X(z)} = \frac{56-49z^{-1}}{1-0.5z^{-1}} \xrightarrow[z^{-p}X(z)\rightarrow x[k-p]]{} \begin{aligned} &y[k]-0.223\,y[k-1] \\ &= 90\,x[k] - 58/2\,x[k-1]\end{aligned}$$