Chapter 10. The Z - Transform

where

$$\text{if} \quad H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}. \quad \text{is well defined} \quad (10.2)$$

For $z = e^{j\omega}$ with ω real (i.e., with $|z| = 1$), the summation in eq. (10.2) corresponds to the discrete-time Fourier transform of $h[n]$. More generally, when $|z|$ is not restricted to unity, the summation is referred to as the *z-transform* of $h[n]$.

The z-transform of a general discrete-time signal $x[n]$ is defined as[1]

$$\boxed{X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]z^{-n},} \tag{10.3}$$

where z is a complex variable. For convenience, the z-transform of $x[n]$ will sometimes be denoted as $\mathcal{Z}\{x[n]\}$ and the relationship between $x[n]$ and its z-transform indicated as

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z). \tag{10.4}$$

In Chapter 9, we considered a number of important relationships between the Laplace transform and the Fourier transform for continuous-time signals. In a similar, but not identical, way, there are a number of important relationships between the z-transform and the discrete-time Fourier transform. To explore these relationships, we express the complex variable z in polar form as

$$z = re^{j\omega}, \tag{10.5}$$

with r as the magnitude of z and ω as the angle of z. In terms of r and ω, eq. (10.3) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n},$$

or equivalently,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-j\omega n}. \tag{10.6}$$

From eq. (10.6), we see that $X(re^{j\omega})$ is the Fourier transform of the sequence $x[n]$ multiplied by a real exponential r^{-n}; that is,

$$X(re^{j\omega}) = \mathfrak{F}\{x[n]r^{-n}\}. \tag{10.7}$$

The exponential weighting r^{-n} may be decaying or growing with increasing n, depending on whether r is greater than or less than unity. We note in particular that, for $r = 1$, or

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[1] The z-transform defined in eq. (10.3) is often referred to as the *bilateral* z-transform, to distinguish it from the *unilateral* z-transform, which we develop in Section 10.9. The bilateral z-transform involves a summation from $-\infty$ to $+\infty$, while the unilateral transform has a form similar to eq. (10.3), but with summation limits from 0 to $+\infty$. Since we are mostly concerned with the bilateral z-transform, we will refer to $X(z)$ as defined in eq. (10.3) simply as the z-transform, except in Section 10.9, in which we use the words "unilateral" and "bilateral" to avoid ambiguity.

(10.6) $\Downarrow$ $r=1$

Hence, z-transform is more general than F-transform

equivalently, $|z| = 1$, eq. (10.3) reduces to the Fourier transform; that is,

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) = \mathfrak{F}\{x[n]\}. \tag{10.8}$$

The relationship between the z-transform and Fourier transform for discrete-time signals parallels closely the corresponding discussion in Section 9.1 for continuous-time signals, but with some important differences. In the continuous-time case, the Laplace transform reduces to the Fourier transform when the real part of the transform variable is zero. Interpreted in terms of the s-plane, this means that the Laplace transform reduces to the Fourier transform on the imaginary axis (i.e., for $s = j\omega$). In contrast, the z-transform reduces to the Fourier transform when the magnitude of the transform variable z is *unity* (i.e, for $z = e^{j\omega}$). Thus, the z-transform reduces to the Fourier transform on the contour in the complex z-plane corresponding to a circle with a radius of unity, as indicated in Figure 10.1. This circle in the z-plane is referred to as the *unit circle* and plays a role in the discussion of the z-transform similar to the role of the imaginary axis in the s-plane for the Laplace transform.

$$ROC \triangleq \{z \triangleq e^{j\omega} \in \mathbb{C} : \exists X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \ (5.9)\}$$

for F-transform

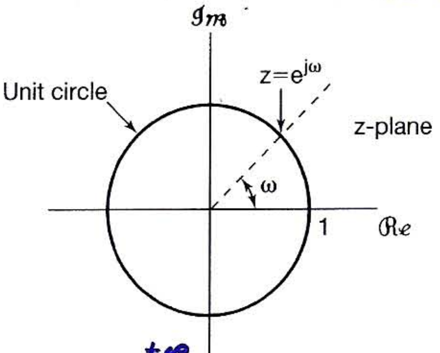

Figure 10.1 Complex z-plane. The z-transform reduces to the Fourier transform for values of z on the unit circle.

$$ROC \triangleq \{z \in \mathbb{C} : \exists X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} \ (10.3)\}$$

for Z-transform

From eq. (10.7), we see that, for convergence of the z-transform, we require that the Fourier transform of $x[n]r^{-n}$ converge. For any specific sequence $x[n]$, we would expect this convergence for some values of r and not for others. In general, the z-transform of a sequence has associated with it a range of values of z for which $X(z)$ converges. As with the Laplace transform, this range of values is referred to as the *region of convergence* (ROC). If the ROC includes the unit circle, then the Fourier transform also converges. To illustrate the z-transform and the associated region of convergence, let us consider several examples.

Example 10.1

Consider the signal $x[n] = a^n u[n]$. Then, from eq. (10.3),

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. Consequently, the region of convergence is the range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$.

Remark It can be shown using properties 4, 5, 6 in § 10.2 that

$$ROC = \overline{ROC} \triangleq \{z = re^{j\theta} \in \mathbb{C} : \sum_{n=-\infty}^{\infty} |x[n]|\, r^{-n} < \infty\} (10.6)\}$$

Then

(cf.) Example 5.1

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|. \tag{10.9}$$

Thus, the z-transform for this signal is well-defined for any value of a, with an ROC determined by the magnitude of a according to eq. (10.9). For example, for $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1.$$

We see that the z-transform in eq. (10.9) is a rational function. Consequently, just as with rational Laplace transforms, the z-transform can be characterized by its zeros (the roots of the numerator polynomial) and its poles (the roots of the denominator polynomial). For this example, there is one zero, at $z = 0$, and one pole, at $z = a$. The pole-zero plot and the region of convergence for Example 10.1 are shown in Figure 10.2 for a value of a between 0 and 1. For $|a| > 1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier transform of $a^n u[n]$ does not converge.

$ROC = \{z \in \mathbb{C} : |z| > |a|\}$

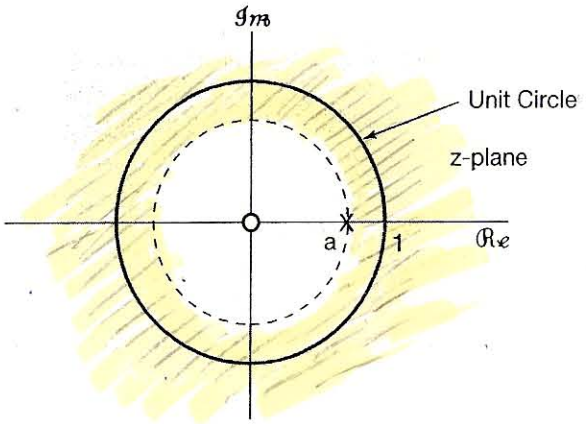

Figure 10.2 Pole-zero plot and region of convergence for Example 10.1 for $0 < a < 1$.

Example 10.2

Now let $x[n] = -a^n u[-n-1]$. Then

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{+\infty} a^n u[-n-1]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n}z^n = 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n. \end{aligned} \tag{10.10}$$

If $|a^{-1}z| < 1$, or equivalently, $|z| < |a|$, the sum in eq. (10.10) converges and

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| < |a|. \tag{10.11}$$

The pole-zero plot and region of convergence for this example are shown in Figure 10.3 for a value of a between 0 and 1.

$\mathrm{RoC} = \{ z \in \mathbb{C} : |z| < |a| \}$

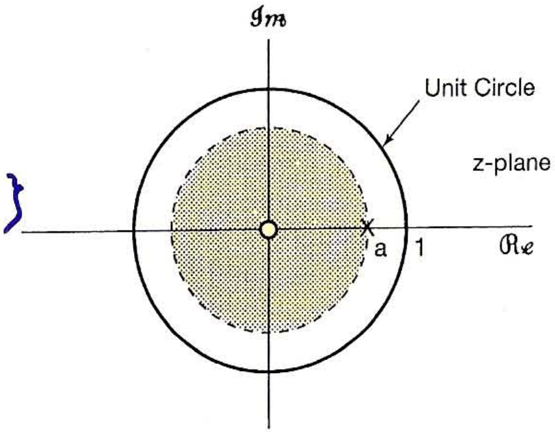

Figure 10.3 Pole-zero plot and region of convergence for Example 10.2 for $0 < a < 1$.

Comparing eqs. (10.9) and (10.11), and Figures 10.2 and 10.3, we see that the algebraic expression for $X(z)$ and the corresponding pole-zero plot are identical in Examples 10.1 and 10.2, and the *z*-transforms differ only in their regions of convergence. Thus, as with the Laplace transform, specification of the *z*-transform requires both the algebraic expression and the region of convergence. Also, in both examples, the sequences were exponentials and the resulting *z*-transforms were rational. In fact, as further suggested by the following examples, $X(z)$ will be rational whenever $x[n]$ is a linear combination of real or complex exponentials:

Example 10.3

Let us consider a signal that is the sum of two real exponentials:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]. \tag{10.12}$$

The *z*-transform is then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{+\infty} \left\{ 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \right\} z^{-n} \\ &= 7 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} \\ &= 7 \sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n - 6 \sum_{n=0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n \qquad (10.13) \\ &= \frac{7}{1 - \frac{1}{3}z^{-1}} - \frac{6}{1 - \frac{1}{2}z^{-1}} = \frac{1 - \frac{3}{2}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})} \qquad (10.14) \\ &= \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}. \qquad (10.15) \end{aligned}$$

For convergence of $X(z)$, both sums in eq. (10.13) must converge, which requires that both $|(1/3)z^{-1}| < 1$ and $|(1/2)z^{-1}| < 1$, or equivalently, $|z| > 1/3$ and $|z| > 1/2$. Thus, the region of convergence is $|z| > 1/2$.

when appropriate, we will use that form in our discussion. However, reference to the poles and zeros is always in terms of the roots of the numerator and denominator expressed as polynomials in z. Also, it is sometimes convenient to refer to $X(z)$, written as a ratio of polynomials in z, as having poles at infinity if the degree of the numerator exceeds the degree of the denominator or zeros at infinity if the numerator is of smaller degree than the denominator.

10.2 THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

In Chapter 9, we saw that there were specific properties of the region of convergence of the Laplace transform for different classes of signals and that understanding these properties led to further insights about the transform. In a similar manner, we explore a number of properties of the region of convergence for the z-transform. Each of the following properties and its justification closely parallel the corresponding property in Section 9.2.

(?)

Property 1: The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

<Proof> First, prove properties 4, 5, 6. Then, this is obvious..

not correct arguments

This property is illustrated in Figure 10.6 and follows from the fact that the ROC consists of those values of $z = re^{j\omega}$ for which $x[n]r^{-n}$ has a Fourier transform that converges. That is, the ROC of the z-transform of $x[n]$ consists of the values of z for which $x[n]r^{-n}$ is absolutely summable:[2]

$$\sum_{n=-\infty}^{+\infty} |x[n]|r^{-n} < \infty. \tag{10.21}$$

$$\boxed{\text{Property } 1' : \text{ROC} = \overline{\text{ROC}}}$$

<Proof> See Remark, p. 743.

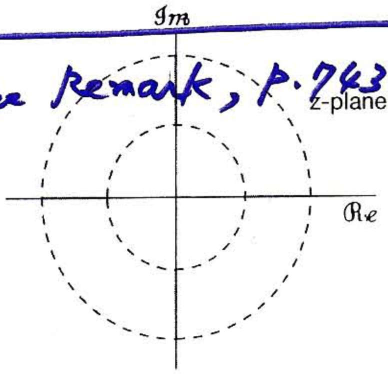

Figure 10.6 ROC as a ring in the z-plane. In some cases, the inner boundary can extend inward to the origin, in which case the ROC becomes a disc. In other cases, the outer boundary can extend outward to infinity.

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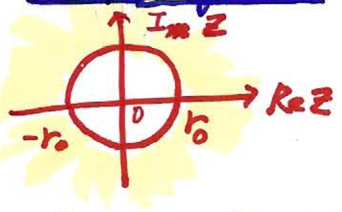
[2] For a thorough treatment of the mathematical properties of z-transforms, see R.V. Churchill and J.W. Brown, *Complex Variables and Applications* (5th ed.) (New York: McGraw–Hill, 1990), and E. I. Jury, *Theory and Application of the z-Transform Method* (Malabar, FL: R. E. Krieger Pub. Co., 1982).

Property 4: If $x[n]$ is a right-sided seq. and

$z_0 \in ROC$, then



$$\{z \in \mathbb{C} : r_0 < |z| < \infty\} \subset ROC \text{ where } r_0 \triangleq |z_0|$$

<Proof> By the def. of ROC, $\sum_{n=N_1}^{\infty} x[n] z_0^{-n}$ converges.

Then, $|x[n]| r_0^{-n} \to 0$ as $n \to \infty$. Hence, $\exists\, M > 0 \ni$

$$|x[n] z_0^{-n}| < M, \quad \forall\, n = N_1, (N_1+1), \dots$$

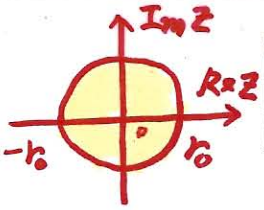
Let $|z| > r_0$. Then,

$$r_0/|z| < 1 \qquad (*1)$$

On the other hand,

$$|x[n] z^{-n}| = |x[n] z_0^{-n} (z/z_0)^{-n}| \le M (r_0/|z|)^n \qquad (*2)$$

By (*1), $\sum_{n=N_1}^{\infty} M (r_0/|z|)^n$ converges. This with (*2) implies that $\sum_{n=N_1}^{\infty} |x[n]|\, |z|^{-n}$ converges. Finally, if $N_1 < 0$, $\infty \notin ROC$ □



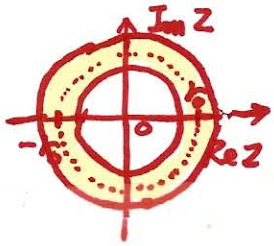
Replace by $z_0 \in ROC$

Property 5: If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, ✗ then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

Again, this property closely parallels the corresponding property for Laplace transforms, and the proof of it and its basis in intuition are similar to the proof and intuition for Property 4. In general, for left-sided sequences, from eq. (10.3), the summation for the z-transform will be of the form

$$X(z) = \sum_{n=-\infty}^{N_2} x[n]z^{-n}, \tag{10.27}$$

where N_2 may be positive or negative. If N_2 is positive, then eq. (10.27) includes negative powers of z, which become unbounded as $|z| \to 0$. Consequently, for left-sided sequences, the ROC will not in general include $z = 0$. However, if $N_2 \le 0$ (so that $x[n] = 0$ for all $n > 0$), the ROC will include $z = 0$.



Replace by $z_0 \in ROC$

Property 6: If $x[n]$ is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$. ✗

Thus, convergence is dependent only on $r = |z|$ and not on ω. Consequently, if a specific value of z is in the ROC, then all values of z on the same circle (i.e., with the same magnitude) will be in the ROC. This by itself guarantees that the ROC will consist of concentric rings. As we will see when we discuss Property 6, the ROC must in fact consist of only a single ring. In some cases the inner boundary of the ROC may extend inward to the origin, and in some cases the outer boundary may extend outward to infinity.

Property 2: The ROC does not contain any poles.

As with the Laplace transform, this property is simply a consequence of the fact that at a pole $X(z)$ is infinite and therefore, by definition, does not converge.

Property 3: If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

A finite-duration sequence has only a finite number of nonzero values, extending, say, from $n = N_1$ to $n = N_2$, where N_1 and N_2 are finite. Thus, the z-transform is the sum of a finite number of terms; that is,

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}. \tag{10.22}$$

For z not equal to zero or infinity, each term in the sum will be finite, and consequently, $X(z)$ will converge. If N_1 is negative and N_2 positive, so that $x[n]$ has nonzero values both for $n < 0$ and $n > 0$, then the summation includes terms with both positive powers of z and negative powers of z. As $|z| \to 0$, terms involving negative powers of z become unbounded, and as $|z| \to \infty$, terms involving positive powers of z become unbounded. Consequently, for N_1 negative and N_2 positive, the ROC does not include $z = 0$ or $z = \infty$. If N_1 is zero or positive, there are only negative powers of z in eq. (10.22), and consequently, the ROC includes $z = \infty$. If N_2 is zero or negative, there are only positive powers of z in eq. (10.22), and consequently, the ROC includes $z = 0$.

Example 10.5

Consider the unit impulse signal $\delta[n]$. Its z-transform is given by

$$\delta[n] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n]z^{-n} = 1, \tag{10.23}$$

with an ROC consisting of the entire z-plane, including $z = 0$ and $z = \infty$. On the other hand, consider the delayed unit impulse $\delta[n-1]$, for which

$$\delta[n-1] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{+\infty} \delta[n-1]z^{-n} = z^{-1}. \tag{10.24}$$

: well-defined except at $z=0$

$$\delta[n+1] \overset{z}{\longleftrightarrow} z$$

: well-defined for all finite values.

<Example 10.6>

$$x[n] = \begin{cases} a^n, & 0 \le n \le (N-1), \quad a > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \{1-(a z^{-1})^N\}/\{1-a z^{-1}\} = (z^N - a^N)/(z-a) z^{N-1}$$

Since $x[n]$ is of finite length, it follows from Property 3 that the ROC includes the entire z-plane except possibly the origin and/or infinity. In fact, from our discussion of Property 3, since $x[n]$ is zero for $n < 0$, the ROC will extend to infinity. However, since $x[n]$ is nonzero for some positive values of n, the ROC will not include the origin. This is evident from eq. (10.28), from which we see that there is a pole of order $N-1$ at $z = 0$. The N roots of the numerator polynomial are at

$(re^{j\theta})^N = a^N$ $\Longrightarrow$

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1. \tag{10.29}$$

The root for $k = 0$ cancels the pole at $z = a$. Consequently, there are no poles other than at the origin. The remaining zeros are at

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1. \tag{10.30}$$

The pole-zero pattern is shown in Figure 10.9.

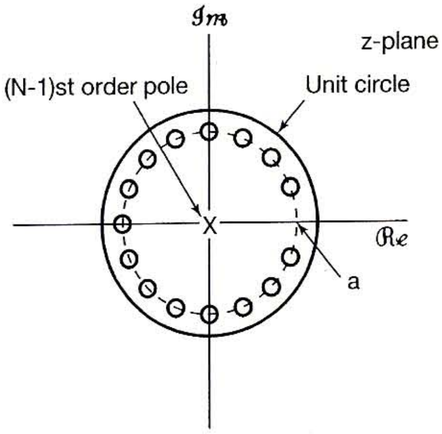


Figure 10.9 Pole-zero pattern for Example 10.6 with $N = 16$ and $0 < a < 1$. The region of convergence for this example consists of all values of z except $z = 0$.

Example 10.7

Let

$$x[n] = b^{|n|}, \quad b > 0. \tag{10.31}$$

This two-sided sequence is illustrated in Figure 10.10, for both $b < 1$ and $b > 1$. The z-transform for the sequence can be obtained by expressing it as the sum of a right-sided and a left-sided sequence. We have

$$x[n] = b^n u[n] + b^{-n} u[-n-1]. \tag{10.32}$$

From Example 10.1,

$$b^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - bz^{-1}}, \quad |z| > b, \tag{10.33}$$

and from Example 10.2,

$$b^{-n} u[-n-1] \overset{z}{\longleftrightarrow} \frac{-1}{1 - b^{-1}z^{-1}}, \quad |z| < \frac{1}{b}. \tag{10.34}$$

$$\Rightarrow X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}, \quad b < |z| < \frac{1}{b}$$

$(b < 1)$

In discussing the Laplace transform in Chapter 9, we remarked that for a rational Laplace transform, the ROC is always bounded by poles or infinity. We observe that in the foregoing examples a similar statement applies to the z-transform, and in fact, this is always true:

> **Property 7:** If the z-transform $X(z)$ of $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

Combining Property 7 with Properties 4 and 5, we have

$X(z)=\sum_{n=0}^{\infty} x[n] z^{-n}$

> **Property 8:** If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the ROC is the region in the z-plane outside the outermost pole—i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is causal (i.e., if it is right sided and equal to 0 for $n < 0$), then the ROC also includes $z = \infty$.

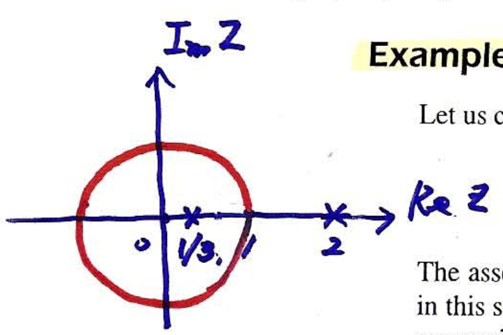
Thus, for right-sided sequences with rational transforms, the poles are all closer to the origin than is any point in the ROC.

$X(z)=\sum_{n=-\infty}^{0} x[n] z^{-n}$

> **Property 9:** If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the ROC is the region in the z-plane inside the innermost nonzero pole—i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if $x[n]$ is anticausal (i.e., if it is left sided and equal to 0 for $n > 0$), then the ROC also includes $z = 0$.

Thus, for left-sided sequences, the poles of $X(z)$ other than any at $z = 0$ are farther from the origin than is any point in the ROC.

For a given pole-zero pattern, or equivalently, a given rational algebraic expression $X(z)$, there are a limited number of different ROCs that are consistent with the preceding properties. To illustrate how different ROCs can be associated with the same pole-zero pattern, we present the following example, which closely parallels Example 9.8.

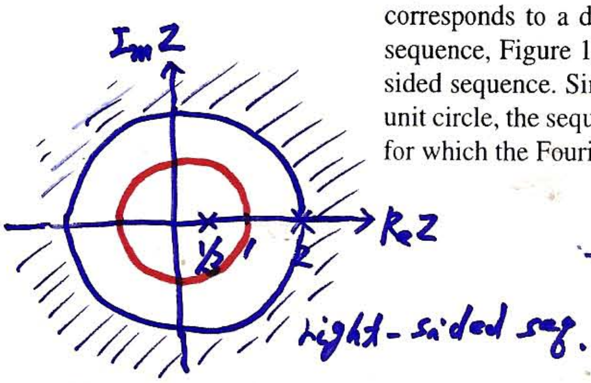


Example 10.8

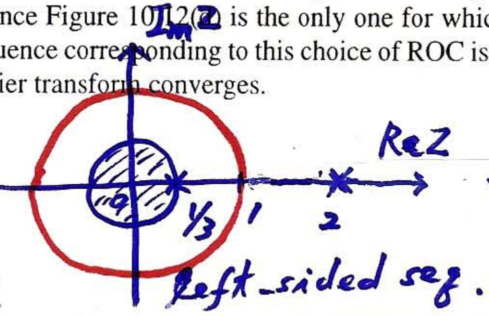
Let us consider all of the possible ROCs that can be connected with the function

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})}. \tag{10.37}$$

The associated pole-zero pattern is shown in Figure 10.12(a). Based on our discussion in this section, there are three possible ROCs that can be associated with this algebraic expression for the z-transform. These ROCs are indicated in Fi corresponds to a different sequence. Figure 10.12(b) is associ sequence, Figure 10.12(c) with a left-sided sequence, and Figure 10.12(d) sided sequence. Since Figure 10.12(d) is the only one for which the ROC includes the unit circle, the sequence corresponding to this choice of ROC is the only one of the three for which the Fourier transform converges.

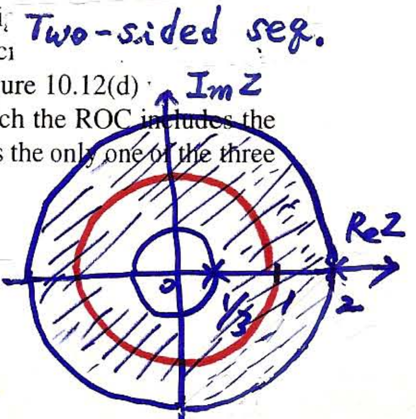


Right-sided seq.



Left-sided seq.

Two-sided seq.



§ 10.3 The Inverse Z- Transform

Recall that

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$ is well defined for all $z \in ROC$.

That is,

$$X(re^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$ is well defined for all ω

if $r \in \{ r \geq 0 : \exists \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} \}$. (*1)

By (5.13), P.366 and (5.8), (5.9), P.361,

$$\mathcal{F}\{ x[n] r^{-n} \} = X(re^{j\omega})$$

and $\mathcal{F}^{-1}\{ X(re^{j\omega}) \}$ is uniquely given by $x[n] r^{-n}$.

That is,

$$x[n] r^{-n} = \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) e^{j\omega n} \, d\omega$$

$$\Rightarrow x[n] = \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) (re^{j\omega})^{n} \, d\omega$$

$$\overset{z = re^{j\omega}}{\boxed{=}} \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz, \qquad (10.*1)$$

where $r > 0$ is chosen to satisfy (*1).

Note that the formal evaluation for the inverse z-transform (10.*1) requires the use of contour integration in the complex plane. Due to the uniqueness of $\mathcal{Z}^{-1}$, there are, however, a number of alternative efficient procedures.

See Example 10.9 — 10.14, P. 758 – 763

Example 10.9 : $X(z) = (3 - \frac{5}{6}z^{-1})/(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})$,

$|z| > 1/3$

Partial-fraction Expansion

this example, the partial-fraction expansion, expressed in polynomials in z^{-1}, is

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}. \quad , |z| > 1/3 \tag{10.43}$$

Thus, $x[n]$ is the sum of two terms, one with z-transform $1/[1-(1/4)z^{-1}]$ and the other with z-transform $2/[1-(1/3)z^{-1}]$. In order to determine the inverse z-transform of each of these individual terms, we must specify the ROC associated with each. Since the ROC for $X(z)$ is outside the outermost pole, the ROC for each individual term in eq. (10.43) must also be outside the pole associated with that term. That is, the ROC for each term consists of all points with magnitude greater than the magnitude of the corresponding pole. Thus,

$$x[n] = x_1[n] + x_2[n], \tag{10.44}$$

where

$$x_1[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}, \tag{10.45}$$

$$x_2[n] \overset{z}{\longleftrightarrow} \frac{2}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}. \tag{10.46}$$

From Example 10.1, we can identify by inspection that

$$x_1[n] = \left(\frac{1}{4}\right)^n u[n] \tag{10.47}$$

and

$$x_2[n] = 2\left(\frac{1}{3}\right)^n u[n], \tag{10.48}$$

and thus,

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]. \tag{10.49}$$

Example 10.10

Now let us consider the same algebraic expression for $X(z)$ as in eq. (10.42), but with the ROC for $X(z)$ as $1/4 < |z| < 1/3$. Equation (10.43) is still a valid partial-fraction expansion of the algebraic expression for $X(z)$, but the ROC associated with the individual terms will change. In particular, since the ROC for $X(z)$ is outside the pole at $z = 1/4$, the ROC corresponding to this term in eq. (10.43) is also outside the pole and consists of all points with magnitude greater than 1/4, as it did in the previous example. However, since in this example the ROC for $X(z)$ is inside the pole at $z = 1/3$, that is, since the points in the ROC all have magnitude less than 1/3, the ROC corresponding to this term must also lie inside this pole. Thus, the z-transform pairs for the individual components in eq. (10.44) are

$$x_1[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| > \frac{1}{4}, \tag{10.50}$$

and

$$x_2[n] \xleftrightarrow{z} \frac{2}{1-\frac{1}{3}z^{-1}}, \quad |z| < \tfrac{1}{3}. \tag{10.51}$$

The signal $x_1[n]$ remains as in eq. (10.47), while from Example 10.2, we can identify

$$x_2[n] = -2\left(\frac{1}{3}\right)^n u[-n-1], \tag{10.52}$$

so that

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]. \tag{10.53}$$

Example 10.11

Finally, consider $X(z)$ as in eq. (10.42), but now with the ROC $|z| < 1/4$. In this case the ROC is inside both poles, i.e., the points in the ROC all have magnitude smaller than either of the poles at $z = 1/3$ or $z = 1/4$. Consequently the ROC for each term in the partial-fraction expansion in eq. (10.43) must also lie inside the corresponding pole. As a result, the z-transform pair for $x_1[n]$ is given by

$$x_1[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{4}z^{-1}}, \quad |z| < \frac{1}{4}, \tag{10.54}$$

while the z-transform pair for $x_2[n]$ is given by eq. (10.51). Applying the result of Example 10.2 to eq. (10.54), we find that

$$x_1[n] = -\left(\frac{1}{4}\right)^n u[-n-1],$$

so that

$$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1].$$

The foregoing examples illustrate the basic procedure of using partial-fraction expansions to determine inverse z-transforms. As with the corresponding method for the Laplace transform, the procedure relies on expressing the z-transform as a linear combination of simpler terms. The inverse transform of each term can then be obtained by inspection. In particular, suppose that the partial-fraction expansion of $X(z)$ is of the form

$$X(z) = \sum_{i=1}^{m} \frac{A_i}{1-a_i z^{-1}}, \tag{10.55}$$

so that the inverse transform of $X(z)$ equals the sum of the inverse transforms of the individual terms in the equation. If the ROC of $X(z)$ is outside the pole at $z = a_i$, the inverse transform of the corresponding term in eq. (10.55) is $A_i a_i^n u[n]$. On the other hand, if the ROC of $X(z)$ is inside the pole at $z = a_i$, the inverse transform of this term is $-A_i a_i^n u[-n-1]$. In general, the partial-fraction expansion of a rational transform may include terms in

Example 10.13 : $X(z) = \dfrac{1}{1-az^{-1}}$, $|z|>|a|$

Power-series Expansion

or

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \ldots. \tag{10.58}$$

The series expansion of eq. (10.58) converges, since $|z| > |a|$, or equivalently, $|az^{-1}| < 1$. Comparing this equation with the definition of the z-transform in equation (10.3), we see, by matching terms in powers of z, that $x[n] = 0, n < 0$; $x[0] = 1$; $x[1] = a$; $x[2] = a^2$; and in general, $x[n] = a^n u[n]$, which is consistent with Example 10.1.

$$X(z) = \sum_{n=-\infty}^{\infty} \boxed{x[n]}\, z^{-n}$$

If, instead, the ROC of $X(z)$ is specified as $|z| < |a|$ or, equivalently, $|az^{-1}| > 1$, then the power-series expansion for $1/(1 - az^{-1})$ in eq. (10.58) does not converge. However, we can obtain a convergent power series by long division again:

$$\begin{array}{r} -a^{-1}z - a^{-2}z^2 - \cdots \\ -az^{-1}+1\,\overline{\big)\;1 \qquad\qquad} \\ 1 - a^{-1}z \\ \hline a^{-1}z \end{array},$$

or

$$\frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - \ldots. \tag{10.59}$$

In this case, then, $x[n] = 0$, $n \geq 0$; and $x[-1] = -a^{-1}$, $x[-2] = -a^{-2}, \ldots$; that is, $x[n] = -a^n u[-n-1]$. This is consistent with Example 10.2.

The power-series expansion method for obtaining the inverse z-transform is particularly useful for nonrational z-transforms, which we illustrate with the following example:

Example 10.14

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|. \tag{10.60}$$

With $|z| > |a|$, or, equivalently, $|az^{-1}| < 1$, eq. (10.60) can be expanded in a power series using the Taylor's series expansion

$$\log(1+v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}v^n}{n}, \quad |v| < 1. \tag{10.61}$$

Applying this to eq. (10.60), we have

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n z^{-n}}{n}, \tag{10.62}$$

from which we can identify

$$x[n] = \begin{cases} (-1)^{n+1}\dfrac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}, \tag{10.63}$$

$$= \frac{-(-a)^n}{n} u[n-1]$$

§ 10.4 Geometric Evaluation of the Fourier Transform from the Pole-Zero Plot

§ 10.4.1 First-order systems

$h[n] = a^n u[n]$, $a > 0$

$\Rightarrow H(z) = \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$, $|z| > a$

$\Rightarrow H(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}} = \dfrac{V_1}{V_2}$

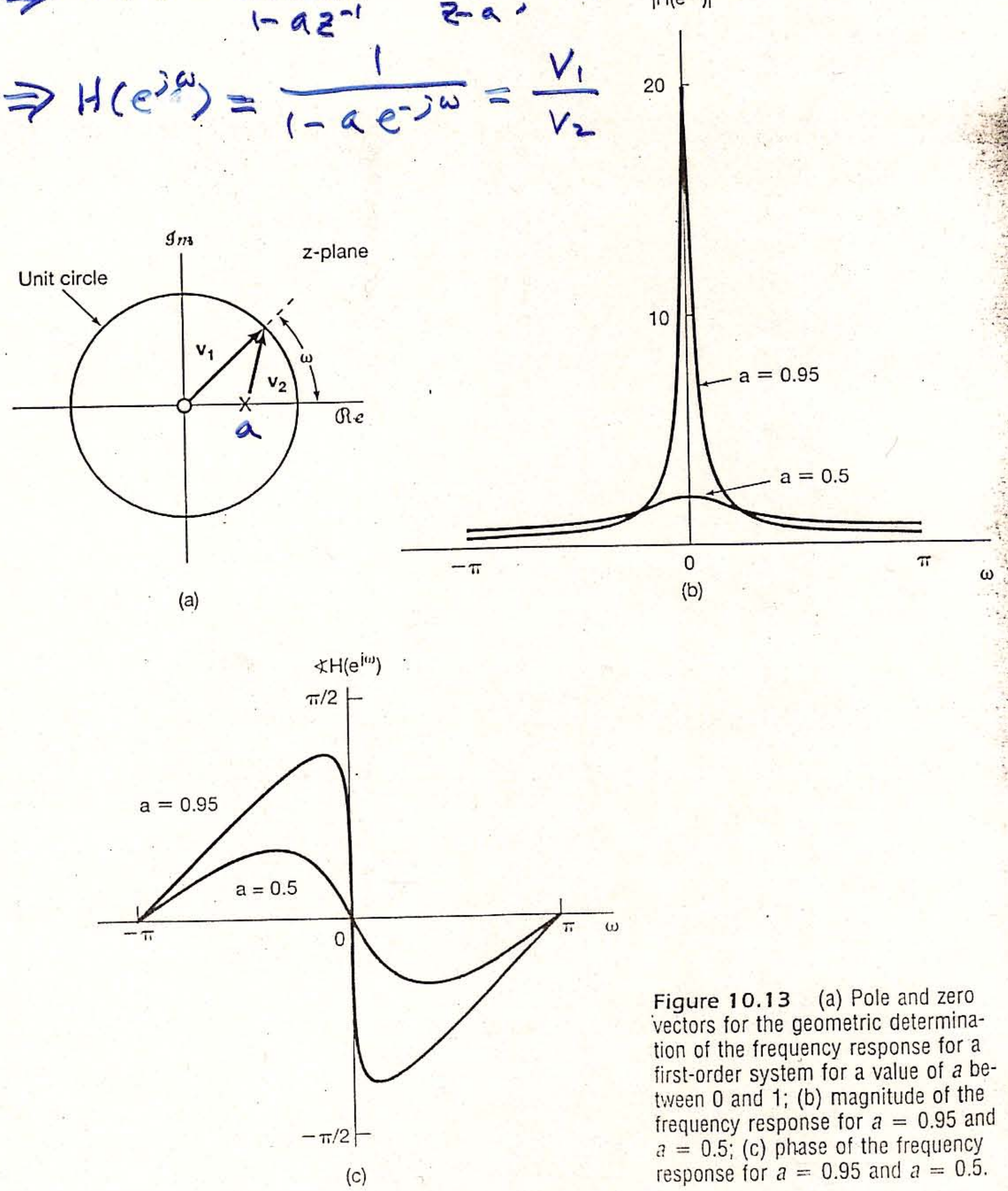


Figure 10.13 (a) Pole and zero vectors for the geometric determination of the frequency response for a first-order system for a value of a between 0 and 1; (b) magnitude of the frequency response for $a = 0.95$ and $a = 0.5$; (c) phase of the frequency response for $a = 0.95$ and $a = 0.5$.

the zero at the origin to the unit circle has a constant length of unity and thus has no effect on the magnitude of $H(e^{j\omega})$. The phase contributed to $H(e^{j\omega})$ by the zero is the angle of the zero vector with respect to the real axis, which we see is equal to ω. For $0 < a < 1$, the pole vector has minimum length at $\omega = 0$ and monotonically increases in length as ω increases from zero to π. Thus, the magnitude of the frequency response will be

§10.4.2 Second-Order Systems

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n] \qquad (6.57)$$

$$\Rightarrow H(z) = \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}} = \frac{1}{(1 - z_1 z^{-1})(1 - z_2 z^{-1})},$$

where $z_1 = re^{j\theta}$, $z_2 = re^{-j\theta}$

$$\Rightarrow H(e^{j\omega}) = \frac{e^{j\omega}}{(e^{j\omega} - z_1)(e^{j\omega} - z_2)} = \frac{V_1}{V_2 V_3}$$

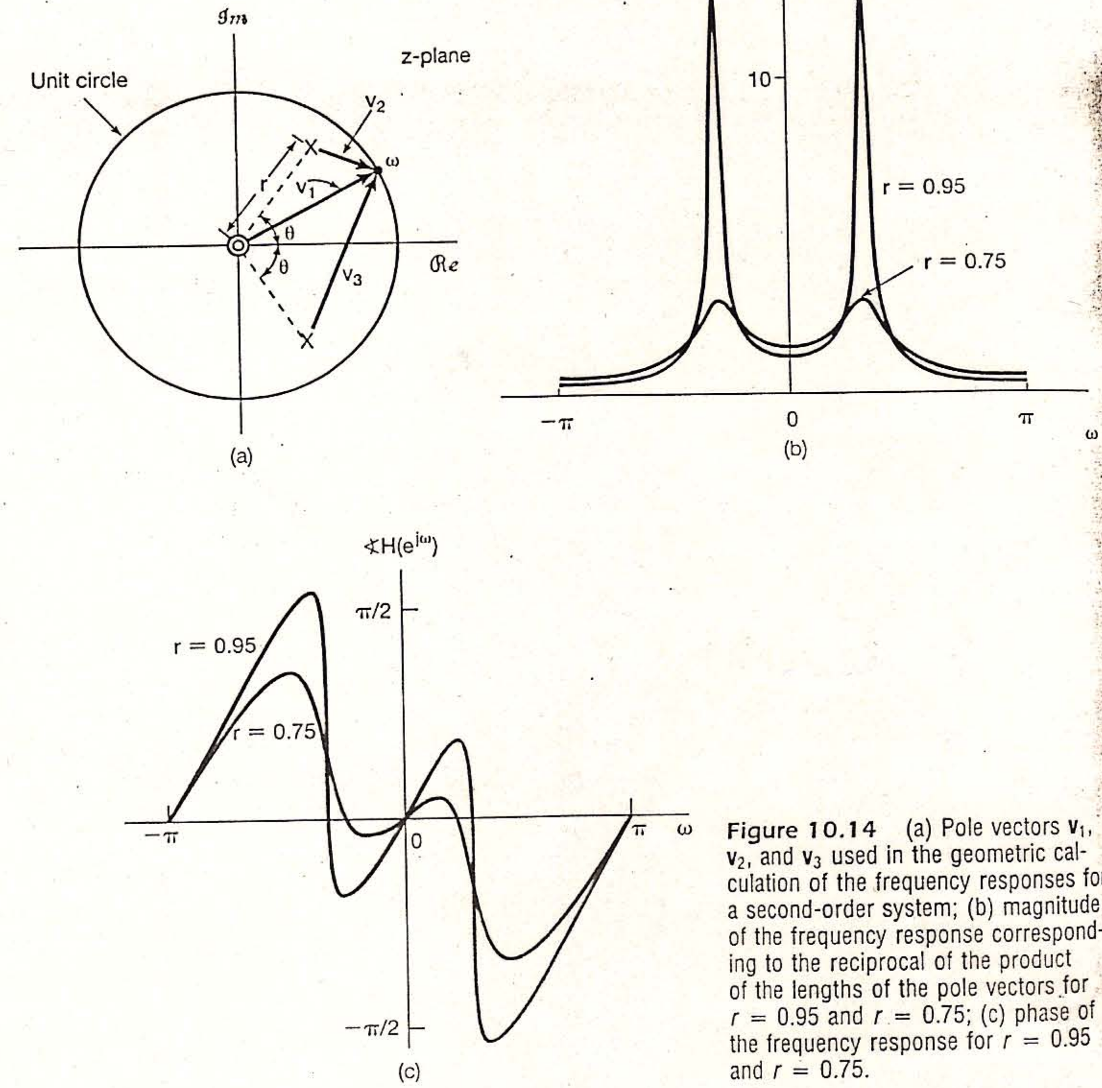


Figure 10.14 (a) Pole vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ used in the geometric calculation of the frequency responses for a second-order system; (b) magnitude of the frequency response corresponding to the reciprocal of the product of the lengths of the pole vectors for $r = 0.95$ and $r = 0.75$; (c) phase of the frequency response for $r = 0.95$ and $r = 0.75$.

along the unit circle from $\omega = 0$ toward $\omega = \pi$, the length of the vector $\mathbf{v}_2$ first decreases and then increases, with a minimum length in the vicinity of the pole location, at $\omega = \theta$. This is consistent with the fact that the magnitude of the frequency response peaks for ω near θ when the length of the vector $\mathbf{v}_2$ is small. Based on the behavior of the pole vectors, it is also evident that as r increases toward unity, the minimum length of the pole vectors will decrease, causing the frequency response to peak more sharply with increasing r. Also, for r near unity, the angle of the vector $\mathbf{v}_2$ changes sharply for ω in the vicinity of θ. Furthermore, from the form of the impulse response [eq. (10.67) and Figure 6.29] or the

§10.5 Properties of The z-Transformation

§10.5.1 Linearity

$$a x_1[n] + b\, x_2[n] \xleftrightarrow{Z} a X_1(z) + b X_2(z), \quad \forall z \in R_1 \cap R_2$$

§10.5.2 Time Shifting

$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z), \quad \forall z \in R$$

<Proof>

$$Z\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, z^{-n} = \sum_{\bar{n}=-\infty}^{\infty} x[\bar{n}]\, z^{-(\bar{n}+n_0)} \quad \square$$

§10.5.3 Scaling in the Z-domain

$$z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right), \quad \forall z \in |z_0| R$$

<Proof>

$$Z\{z_0^n x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z_0^n z^{-n} \quad \square$$

An Important Special Case

$$e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(e^{-j\omega_0} z)$$

§10.5.4 Time Reversal

$$x[n] \xleftrightarrow{Z} X\left(\frac{1}{z}\right), \quad \forall z \in \mathbb{C}, \ \underset{\sim\sim\sim}{\frac{1}{z} \in R}$$

<Proof>

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]\, z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\, \hat{z}^{-n}, \ \forall \hat{z} \triangleq \frac{1}{z} \in R \quad \square$$

§10.5.5 Time Expansion

$$x_{(k)}[n] \xleftrightarrow{Z} X(z^k), \quad \forall z \in \mathbb{C}, \ \underset{\sim\sim\sim}{z^k \in R}$$

<Proof>

$$Z\{x_{(k)}[n]\} = \sum_{n=km} x[n/k]\, z^{-n} = \sum_{m=-\infty}^{\infty} x[m]\, \hat{z}^{-m}, \ \forall \hat{z} \triangleq z^k \in R$$

$\square$

§ 10.5.6 Conjugation

$$x^*[n] \xleftrightarrow{\quad Z \quad} X^*(z^*) \ , \ \forall z \in R$$

<Proof>

$$z \in R \Rightarrow z^* \in R \quad \text{(See Fig. 10.8)}$$ □

§ 10.5.7 The Convolution Property

$$x_1[n] * x_2[n] \xleftrightarrow{\quad Z \quad} X_1(z) X_2(z) \ , \ \forall z \in R_1 \cap R_2$$

<Proof>

$$\mathcal{Z}\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} \Big\{ \sum_{k=-\infty}^{\infty} x_1[k] \, x_2[n-k] \Big\} z^{-n}$$

$$= \sum_{n} \Big\{ \sum_{k} x_1[k] z^{-k} \Big\} x_2[n-k] \, z^{-(n-k)}$$

$$= \sum_{k} x_1[k] z^{-k} \Big\{ \sum_{n} x_2[n-k] \, z^{-(n-k)} \Big\}$$

$$= \sum_{k} x_1[k] z^{-k} X_2(z) = X_1(z) X_2(z), \ \forall z \in R_1 \cap R_2$$ □

Remark (Finite Series)

Let $X_1(z) \triangleq \sum_{n=0}^{N_1} x_1[n] z^{-n}$, $X_2(z) \triangleq \sum_{n=0}^{N_2} x_2[n] z^{-n}$, $M \triangleq N_1 + N_2$

$$\Rightarrow Y(z) = \sum_{k=0}^{M} y[k] z^{-k} = \sum_{k=0}^{M} \Big\{ \sum_{m=-\infty}^{\infty} x_1[m] \, x_2[k-m] \Big\} z^{-k}$$

$$= \sum_{k=0}^{M} \sum_{m=-\infty}^{\infty} x_1[m] z^{-m} \, x_2[k-m] \, z^{-(k-m)}$$

$$= \sum_{m=0}^{N_1} x_1[m] z^{-m} \Big\{ \sum_{k=m}^{M} x_2[k-m] \, z^{-(k-m)} \Big\}$$

$$= \sum_{m=0}^{N_1} x_1[m] z^{-m} \sum_{k=m}^{N_2+m} x_2[k-m] \, z^{-(k-m)}$$ □

$$= X_1(z) \, X_2(z)$$

$$(\text{cf.})\ \mathcal{F}\{u[n]\} = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$\Rightarrow W(e^{j\omega}) = X(e^{j\omega})\,\mathcal{F}\{u[n]\}$$

$$= \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

That is, $y[n]$ is the first difference of the sequence $x[n]$. Since the first-difference operation is commonly thought of as a discrete-time counterpart to differentiation, eq. (10.83) can be thought of as the z-transform counterpart of the Laplace transform differentiation property presented in Section 9.5.7.

Example 10.16

$$x[n] = a^n u[n] \quad \updownarrow z \quad X(z) = \frac{1}{1-az^{-1}},\ |z| > a$$

Suppose we now consider the inverse of first differencing, namely, accumulation or summation. Specifically, let $w[n]$ be the running sum of $x[n]$:

$$w[n] = \sum_{k=-\infty}^{n} x[k] = u[n] * x[n]. \tag{10.85}$$

Then, using eq. (10.81) together with the z-transform of the unit step in Example 10.1, we see that

$$\boxed{w[n] = \sum_{k=-\infty}^{n} x[k] \overset{z}{\longleftrightarrow} \frac{1}{1-z^{-1}}X(z),} \tag{10.86}$$

with ROC including at least the intersection of R with $|z| > 1$. Eq. (10.86) is the discrete-time z-transform counterpart of the integration property in Section 9.5.9.

10.5.8 Differentiation in the z-Domain

If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R,$$

then

$$\boxed{nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad \text{with ROC} = R.} \tag{10.87}$$

This property follows in a straightforward manner by differentiating both sides of the expression for the z-transform given in eq. (10.3). As an example of the use of this property, let us apply it to determining the inverse z-transform considered in Example 10.14.

Example 10.17

If

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|, \tag{10.88}$$

then

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|. \tag{10.89}$$

By differentiating, we have converted the z-transform to a rational expression. The inverse z-transform of the right-hand side of eq. (10.89) can be obtained by using Example 10.1 together with the time-shifting property, eq. (10.72), set forth in Section 10.5.2.

<Proof of (10.87)> $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n},\ \forall z \in R$

$$\Rightarrow \frac{dX}{dz} = -\sum_{n=-\infty}^{\infty} n\,x[n]\,z^{-n},\ \forall\, \underline{z \in R\ (\text{power series})}$$

Specifically, from Example 10.1 and the linearity property,

$$a(-a)^n u[n] \overset{z}{\longleftrightarrow} \frac{a}{1+az^{-1}}, \quad |z| > |a|. \tag{10.90}$$

Combining this with the time-shifting property yields

$$a(-a)^{n-1} u[n-1] \overset{z}{\longleftrightarrow} \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|.$$

Consequently,

$$x[n] = \frac{-(-a)^n}{n} u[n-1]. \tag{10.91}$$

Example 10.18

As another example of the use of the differentiation property, consider determining the inverse z-transform for

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|. \tag{10.92}$$

From Example 10.1,

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| > |a|, \tag{10.93}$$

and hence,

$$na^n u[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|. \tag{10.94}$$

10.5.9 The Initial-Value Theorem

If $x[n] = 0, n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z). \tag{10.95}$$

This property follows by considering the limit of each term individually in the expression for the z-transform, with $x[n]$ zero for $n < 0$. With this constraint,

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$, whereas for $n = 0$, $z^{-n} = 1$. Thus, eq. (10.95) follows.

As one consequence of the initial-value theorem, for a causal sequence, if $x[0]$ is finite, then $\lim_{z\to\infty} X(z)$ is finite. Consequently, with $X(z)$ expressed as a ratio of polynomials in z, the order of the numerator polynomial cannot be greater than the order of the denominator polynomial; or, equivalently, the number of finite zeros of $X(z)$ cannot be

10.5.9' The Final Value Theorem

If $x[n]$ is causal, and all poles of $(1-z^{-1})X(z)$ are inside $|z|<1$, then

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (1-z^{-1}) X(z) \tag{10.95'}$$ □