TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$			

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z-plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n+1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive powers of z. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

> A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.

If $H(z)$ is rational, then, from Property 8 in Section 10.2, for the system to be causal, the ROC must be outside the outermost pole and infinity must be in the ROC. Equivalently, the limit of $H(z)$ as $z \longrightarrow \infty$ must be finite. As we discussed in Section 10.5.9, this is equivalent to the numerator of $H(z)$ having degree no larger than the denominator when both are expressed as polynomials in z. That is:

> A discrete-time LTI system with rational system function $H(z)$ is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Example 10.20

Consider a system with system function whose algebraic expression is

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}.$$

Without even knowing the ROC for this system, we can conclude that the system is not causal, because the numerator of $H(z)$ is of higher order than the denominator.

Example 10.21

Consider a system with system function

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \quad |z| > 2 \tag{10.97}$$

Since the ROC for this system function is the exterior of a circle outside the outermost pole, we know that the impulse response is right-sided. To determine if the system is causal, we then need only check the other condition required for causality, namely that $H(z)$, when expressed as a ratio of polynomials in z, has numerator degree no larger than the denominator. For this example,

$$H(z) = \frac{2 - \frac{5}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{2z^2 - \frac{5}{2}z}{z^2 - \frac{5}{2}z + 1}, \tag{10.98}$$

so that the numerator and denominator of $H(z)$ are both of degree two, and consequently we can conclude that the system is causal. This can also be verified by calculating the inverse transform of $H(z)$. In particular, using transform pair 5 in Table 10.2, we find that the impulse response of this system is

$$h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right]u[n]. \tag{10.99}$$

Since $h[n] = 0$ for $n < 0$, we can confirm that the system is causal.

10.7.2 Stability

As we discussed in Section 2.3.7, the stability of a discrete-time LTI system is equivalent to its impulse response being absolutely summable. In this case the Fourier transform of $h[n]$

converges, and consequently, the ROC of $H(z)$ must include the unit circle. Summarizing, we obtain the following result:

> An LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$. $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| (1)^{-n} < \infty$

Example 10.22

$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}$

Consider again the system function in eq. (10.97). Since the associated ROC is the region $|z| > 2$, which does not include the unit circle, the system is not stable. This can also be seen by noting that the impulse response in eq. (10.99) is not absolutely summable. If, however, we consider a system whose system function has the same algebraic expression as in eq. (10.97) but whose ROC is the region $1/2 < |z| < 2$, then the ROC does contain the unit circle, so that the corresponding system is noncausal but stable. In this case, using transform pairs 5 and 6 from Table 10.2, we find that the corresponding impulse response is

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1], \tag{10.100}$$

which is absolutely summable.

Also, for the third possible choice of ROC associated with the algebraic expression for $H(z)$ in eq. (10.97), namely, $|z| < 1/2$, the corresponding system is neither causal (since the ROC is not outside the outermost pole) nor stable (since the ROC does not include the unit circle). This can also be seen from the impulse response, which (using transform pair 6 in Table 10.2) is

$$h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right] u[-n-1].$$

As Example 10.22 illustrates, it is perfectly possible for a system to be stable but not causal. However, if we focus on causal systems, stability can easily be checked by examining the locations of the poles. Specifically, for a causal system with rational system function, the ROC is outside the outermost pole. For this ROC to include the unit circle, $|z| = 1$, all of the poles of the system must be inside the unit circle. That is:

> A causal LTI system with rational system function $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the unit circle—i.e., they must all have magnitude smaller than 1.

Example 10.23

Consider a causal system with system function

$$H(z) = \frac{1}{1 - az^{-1}},$$

which has a pole at $z = a$. For this system to be stable, its pole must be inside the unit circle, i.e., we must have $|a| < 1$. This is consistent with the condition for the absolute summability of the corresponding impulse response $h[n] = a^n u[n]$.

§10.7.3. LTI systems characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad -\infty < n < \infty \qquad (10.105)$$

(*) The Nth-order Diff. Eq. in (10.105) represents a causal LTI system if n is assumed to be increasing.

Then taking z-transforms of both sides of eq. (10.105) and using the linearity and time-shifting properties, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

or

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k},$$

so that

deg. of num. ≤ deg. of denom. ⇐

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{10.106}$$

We note in particular that the system function for a system satisfying a linear constant-coefficient difference equation is always rational. Consistent with our previous example and with the related discussion for the Laplace transform, the difference equation by itself does not provide information about which ROC to associate with the algebraic expression $H(z)$. An additional constraint, such as the causality or stability of the system, however, serves to specify the region of convergence. For example, if we know in addition that the system is causal, the ROC will be outside the outermost pole. If the system is stable, the ROC must include the unit circle.

By (*), this naturally follows.

10.7.4 Examples Relating System Behavior to the System Function

As the previous subsections illustrate, many properties of discrete-time LTI systems can be directly related to the system function and its characteristics. In this section, we give several additional examples to show how z-transform properties can be used in analyzing systems.

Example 10.26

Suppose that we are given the following information about an LTI system:

1. If the input to the system is $x_1[n] = (1/6)^n u[n]$, then the output is

$$y_1[n] = \left[a\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n \right] u[n],$$

where a is a real number.

2. If $x_2[n] = (-1)^n$, then the output is $y_2[n] = \frac{7}{4}(-1)^n$. As we now show, from these two pieces of information, we can determine the system function $H(z)$ for this system, including the value of the number a, and can also immediately deduce a number of other properties of the system.

The z-transforms of the signals specified in the first piece of information are

It is implied that the system is initially rest at n=0

⇓

$$y[n] = \sum_{k=0}^{\infty} h[n-k]\, x[k]$$

⇓

$$Y(z) = H(z)\, X(z)$$

$$X_1(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}, \quad |z| > \frac{1}{6}, \tag{10.107}$$

$$\begin{aligned} Y_1(z) &= \frac{a}{1 - \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{3}z^{-1}} \\ &= \frac{(a+10) - (5 + \frac{a}{3})z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}. \end{aligned} \tag{10.108}$$

From eq. (10.96), it follows that the algebraic expression for the system function is

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{[(a+10) - (5 + \frac{a}{3})z^{-1}][1 - \frac{1}{6}z^{-1}]}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}. \tag{10.109}$$

$z^n \to$ $\boxed{\text{Discrete-Time LTI System}}$ $\to H(z)\, z^n$

Furthermore, we know that the response to $x_2[n] = (-1)^n$ must equal $(-1)^n$ multiplied by the system function $H(z)$ evaluated at $z = -1$. Thus from the second piece of information given, we see that

$$\frac{7}{4} = H(-1) = \frac{[(a+10) + 5 + \frac{a}{3}][\frac{7}{6}]}{(\frac{3}{2})(\frac{4}{3})}. \tag{10.110}$$

Solving eq. (10.110), we find that $a = -9$, so that

$$H(z) = \frac{(1 - 2z^{-1})(1 - \frac{1}{6}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}, \tag{10.111}$$

or

$$H(z) = \frac{1 - \frac{13}{6}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}, \tag{10.112}$$

or, finally,

$$H(z) = \frac{z^2 - \frac{13}{6}z + \frac{1}{3}}{z^2 - \frac{5}{6}z + \frac{1}{6}}. \tag{10.113}$$

Neglect by assuming $n \uparrow$

Also, from the convolution property, we know that the ROC of $Y_1(z)$ must include at least the intersections of the ROCs of $X_1(z)$ and $H(z)$. Examining the three possible ROCs for $H(z)$ (namely, $|z| < 1/3$, $1/3 < |z| < 1/2$, and $|z| > 1/2$), we find that the only choice that is consistent with the ROCs of $X_1(z)$ and $Y_1(z)$ is $|z| > 1/2$.

Since the ROC for the system includes the unit circle, we know that the system is stable. Furthermore, from eq. (10.113) with $H(z)$ viewed as a ratio of polynomials in z, the order of the numerator does not exceed that of the denominator, and thus we can conclude that the LTI system is causal. Also, using eqs. (10.112) and (10.106), we can write the difference equation that, together with the condition of initial rest, characterizes the system:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{13}{6}x[n-1] + \frac{1}{3}x[n-2].$$

$-\infty < n < \infty$

Example 10.27

Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$. Suppose it is known that $H(z)$ contains a pole at $z = 1/2$ and a zero somewhere on the unit circle. The precise number and locations of all of the other poles