§10.8 System Function Algebra and Block Diagram Representation

10.8.1 System Functions for Interconnections of LTI Systems

The system function algebra for analyzing discrete-time block diagrams such as series, parallel, and feedback interconnections is exactly the same as that for the corresponding continuous-time systems in Section 9.8.1. For example, the system function for the cascade of two discrete-time LTI systems is the product of the system functions for the individual systems in the cascade. Also, consider the feedback interconnection of two systems, as shown in Figure 10.17. It is relatively involved to determine the difference equation or impulse response for the overall system working directly in the time domain. However, with the systems and sequences expressed in terms of their z-transforms, the analysis involves only algebraic equations. The specific equations for the interconnection of Figure 10.17 exactly parallel eqs. (9.159)–(9.163), with the final result that the overall system function for the feedback system of Figure 10.17 is

$$\begin{cases} Y(z) = H_1(z)E(z) \\ E(z) = X(z) - H_2(z)Y(z) \end{cases} \Rightarrow \quad \frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}. \tag{10.115}$$

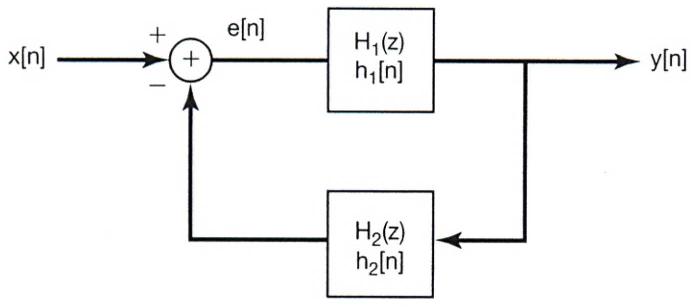


Figure 10.17 Feedback interconnection of two systems.

10.8.2 Block Diagram Representations for Causal LTI Systems Described by Difference Equations and Rational System Functions

Example 10.29

Suppose we now consider the causal LTI system with system function

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)(1 - 2z^{-1}). = \left[\frac{V(z)}{X(z)}\right]\left[\frac{Y(z)}{V(z)}\right] \tag{10.117}$$

As eq. (10.117) suggests, we can think of this system as the cascade of a system with system function $1/[1 - (1/4)z^{-1}]$ and one with system function $1 - 2z^{-1}$. We have illustrated the cascade in Figure 10.19(a), in which we have used the block diagram in Figure 10.18(a) to represent $1/[1 - (1/4)z^{-1}]$. We have also represented $1 - 2z^{-1}$ using a unit delay, an adder, and a coefficient multiplier. Using the time-shifting property, we then see that the input $v[n]$ and output $y[n]$ of the system with system function $1 - 2z^{-1}$

$$\Rightarrow \begin{cases} v[n] - \frac{1}{4}v[n-1] = x[n] \\ y[n] = v[n] - 2v[n-1] \end{cases}$$

are related by

$$y[n] = v[n] - 2v[n-1].$$

While the block diagram in Figure 10.19(a) is certainly a valid representation of the system in eq. (10.117), it has an inefficiency whose elimination leads to an alternative block-diagram representation. To see this, note that the input to both unit delay elements in Figure 10.19(a) is $v[n]$, so that the outputs of these elements are identical; i.e.,

$$w[n] = s[n] = v[n-1].$$

Consequently, we need not keep both of these delay elements, and we can simply use the output of one of them as the signal to be fed to both coefficient multipliers. The result is the block diagram representation in Figure 10.19(b). Since each unit delay element requires a memory register to store the preceding value of its input, the representation in Figure 10.19(b) requires less memory than that in Figure 10.19(a).

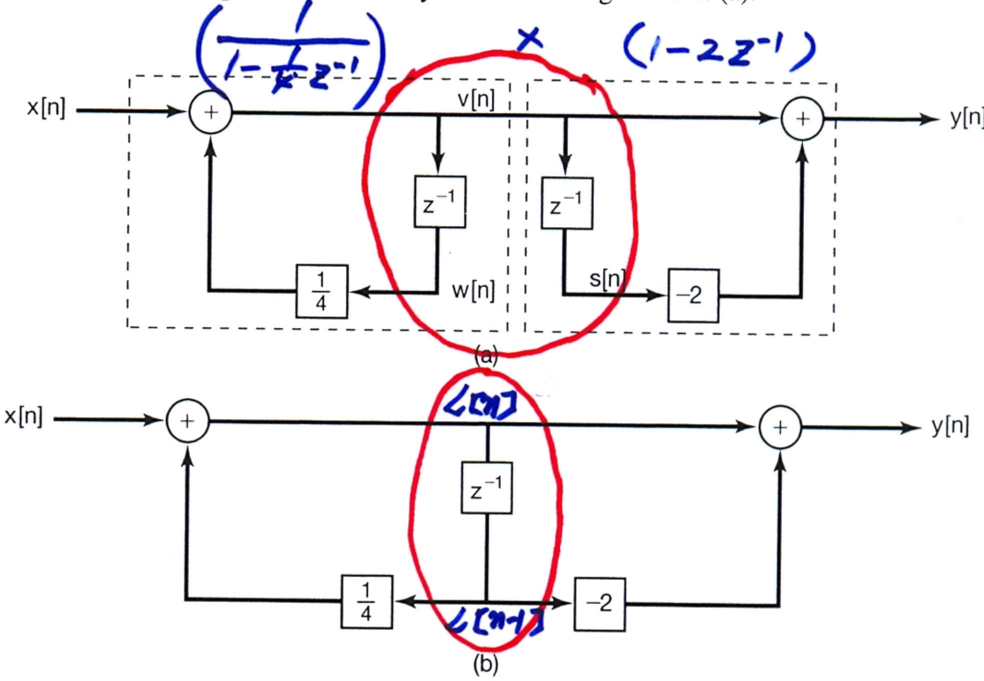


Figure 10.19 (a) Block-diagram representations for the system in Example 10.29; (b) equivalent block-diagram representation using only one unit delay element.

Example 10.30

Next, consider the second-order system function

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}, = \frac{Y(z)}{X(z)} \tag{10.118}$$

which is also described by the difference equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]. \tag{10.119}$$

Using the same ideas as in Example 10.28, we obtain the block-diagram representation for this system shown in Figure 10.20(a). Specifically, since the two system function blocks in this figure with system function z^{-1} are unit delays, we have

$$f[n] = y[n-1],$$
$$e[n] = f[n-1] = y[n-2],$$

so that eq. (10.119) can be rewritten as

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n],$$

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})} \times \frac{1}{(1-\frac{1}{4}z^{-1})}$$

$$H(z) = \frac{(\frac{2}{3})}{1+\frac{1}{2}z^{-1}} + \frac{(\frac{1}{3})}{1-\frac{1}{4}z^{-1}}$$

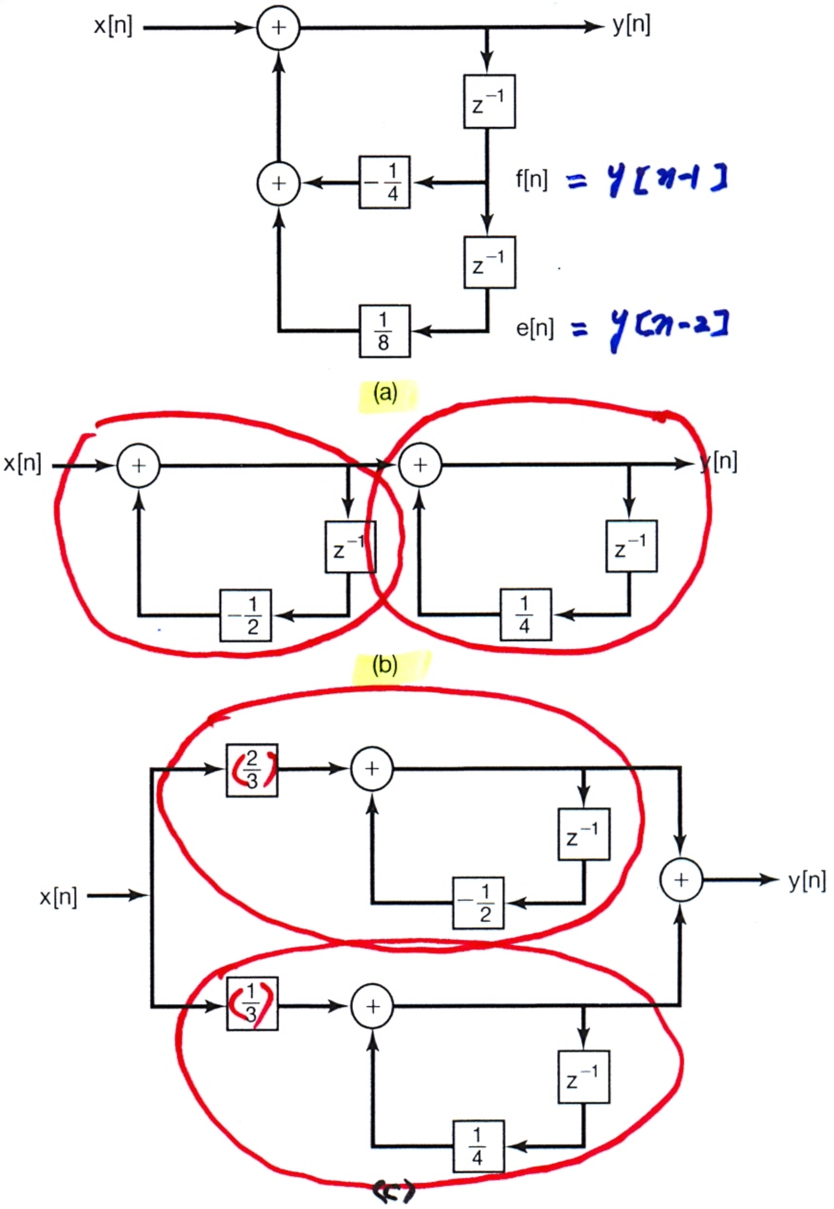


Figure 10.20 Block-diagram representations for the system in Example 10.30: (a) direct form; (b) cascade form; (c) parallel form.

< Example 10.31 >

$$H(z) = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$H(z) = \left(\frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}\right) \times \left(1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}\right)$$

$= \frac{\hat{Y}(z)}{X(z)}$, $= \frac{Y(z)}{\hat{Y}(z)}$

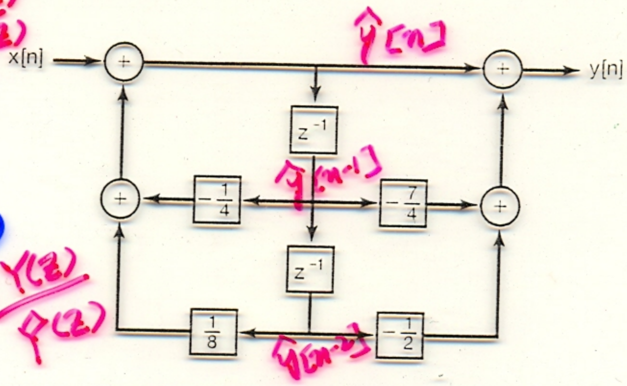
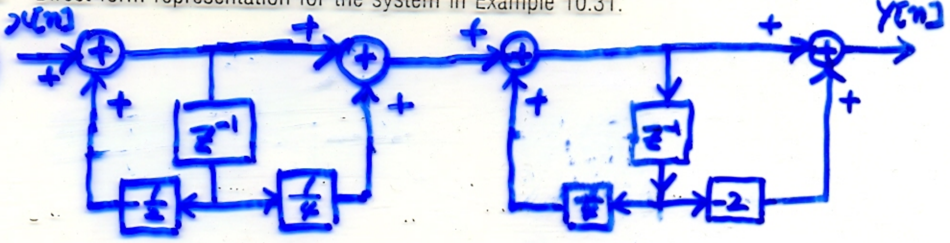
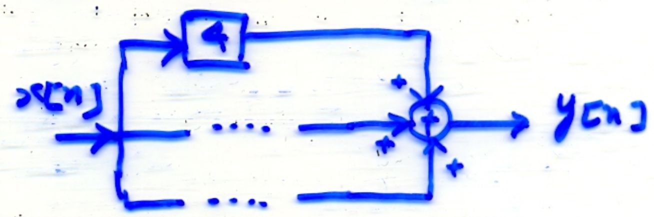


Figure 10.21 Direct-form representation for the system in Example 10.31.

$$H(z) = \left(\frac{1 + \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}}\right)$$



$$H(z) = 4 + \left(\frac{5/3}{1 + \frac{1}{2}z^{-1}}\right) - \left(\frac{14/3}{1 - \frac{1}{4}z^{-1}}\right)$$



block diagrams. Specifically, each block-diagram representation of a system can be translated directly into a computer algorithm for the implementation of the system. However, because the finite word length of a computer necessitates quantizing the coefficients in the block diagram and because there is numerical roundoff as the algorithm operates, each of these representations will lead to an algorithm that only approximates the behavior of the original system. Moreover, the errors in each of these approximations will be somewhat different. Because of these differences, considerable effort has been put into examining the relative merits of the various block-diagram representations in terms of their accuracy and sensitivity to quantization effects. For discussions of this subject, the reader may turn to the references on digital signal processing in the bibliography at the end of the book.

10.9 THE UNILATERAL z-TRANSFORM

The form of the z-transform considered thus far in this chapter is often referred to as the *bilateral z-transform.* As was the case with the Laplace transform, there is an alternative form, referred to as the *unilateral z-transform,* that is particularly useful in analyzing causal systems specified by linear constant-coefficient difference equations with nonzero initial conditions (i.e., systems that are not initially at rest). In this section, we introduce the unilateral z-transform and illustrate some of its properties and uses, paralleling our discussion of the unilateral Laplace transform in Section 9.9.

(cf.) The bilateral z-transform is not useful to find the solution (or the output response) but useful for finding the transferfunction or the system impulse response of noncausal systems.

The unilateral z-transform of a sequence $x[n]$ is defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}. \tag{10.125}$$

As in previous chapters, we adopt a convenient shorthand notation for a signal and its unilateral z-transform:

$$x[n] \stackrel{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z) = \mathcal{UZ}\{x[n]\}. \tag{10.126}$$

The unilateral z-transform differs from the bilateral transform in that the summation is carried out only over nonnegative values of n, whether or not $x[n]$ is zero for $n < 0$. Thus the unilateral z-transform of $x[n]$ can be thought of as the bilateral transform of $x[n]u[n]$ (i.e., $x[n]$ multiplied by a unit step). In particular, then, for any sequence that is zero for $n < 0$, the unilateral and bilateral z-transforms will be identical. Referring to the discussion of regions of convergence in Section 10.2, we also see that, since $x[n]u[n]$ is always a right-sided sequence, the region of convergence of $\mathcal{X}(z)$ is always the exterior of a circle.

Because of the close connection between bilateral and unilateral z-transforms, the calculation of unilateral transforms proceeds much as for bilateral transforms, with the caveat that we must take care to limit the range of summation in the transform to $n \geq 0$. Similarly, the calculation of inverse unilateral transforms is basically the same as for bilateral transforms, once we take into account the fact that the ROC for a unilateral transform is *always* the exterior of a circle.

10.9.1 Examples of Unilateral z-Transforms and Inverse Transforms

Example 10.32

Consider the signal

$$x[n] = a^n u[n]. \tag{10.127}$$

Since $x[n] = 0, n < 0$, the unilateral and bilateral transforms are equal for this example, and thus, in particular,

$$\mathcal{X}(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|. \tag{10.128}$$

Example 10.33

Let

$$x[n] = a^{n+1}u[n+1]. \tag{10.129}$$

In this case the unilateral and bilateral transforms are *not* equal, since $x[-1] = 1 \neq 0$. The bilateral transform is obtained from Example 10.1 and the time-shifting property set forth in Section 10.5.2. Specifically,

$x[n+1] \stackrel{Z}{\longleftrightarrow} z X(z) \Rightarrow$

$$X(z) = \frac{z}{1 - az^{-1}}, \qquad |z| > |a|. \tag{10.130}$$

In contrast, the unilateral transform is

$x[n] = a^{n+1}u[n+1] \Rightarrow$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n+1}z^{-n},$$

or

$$\mathcal{X}(z) = \frac{a}{1 - az^{-1}}, \quad |z| > |a|. \tag{10.131}$$

Example 10.34

Consider the unilateral z-transform

$$\mathcal{X}(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}. \tag{10.132}$$

In Example 10.9, we considered the inverse transform for a bilateral z-transform $X(z)$ of the same form as in eq. (10.132) and for several different ROCs. In the case of the unilateral transform, the ROC must be the exterior of the circle of radius equal to the largest magnitude of the poles of $\mathcal{X}(z)$—in this instance, all points z with $|z| > 1/3$. We can then invert the unilateral transform exactly as in Example 10.9, yielding

By (10.135),

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] \quad \text{for} \quad n \geq 0. \tag{10.133}$$

☆ In eq. (10.133), we have emphasized the fact that inverse unilateral z-transforms provide us with information about $x[n]$ *only* for $n \geq 0$.

Another approach to inverse transforms introduced in Section 10.3, namely, identifying the inverse transforms from the coefficients in the power-series expansion of the z-transform, also can be used for unilateral transforms. However, in the unilateral case, a constraint which must be satisfied is that, as a consequence of eq. (10.125), the power-series expansion for the transform cannot contain terms with positive powers of z. For instance, in Example 10.13 we performed long division on the bilateral transform

$$X(z) = \frac{1}{1 - az^{-1}} \tag{10.134}$$

in two ways, corresponding to the two possible ROCs for $X(z)$. Only one of these choices, namely, that corresponding to the ROC $|z| > |a|$, led to a series expansion without positive powers of z, i.e.,

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \ldots, \tag{10.135}$$

$\Rightarrow x[n] = a^n u[n]$

and this is the *only* choice for the expansion if eq. (10.134) represents a unilateral transform.

Note that the requirement that $\mathcal{X}(z)$ have a power-series expansion with no terms with positive powers of z implies that not every function of z can *be* a unilateral z-transform. In particular, if we consider a rational function of z written as a ratio of polynomials in z (not in z^{-1}), i.e.,

$x[n]$: causal signal

$$X(z) = \frac{p(z)}{q(z)}, \tag{10.136}$$

then for this to be a unilateral transform (with the appropriately chosen ROC as the exterior of a circle), the degree of the numerator must be no bigger than the degree of the denominator.

Example 10.35

A simple example illustrating the preceding point is given by the rational function in eq. (10.130), which we can write as a ratio of polynomials in z:

(i) $x[n] = -a^{n+1} u[n-2]$ if $|z| < |a|$

(ii) $x[n] = a^{n+1} u[n+1]$ if $|z| > |a|$

$$\xleftrightarrow{\;Z\;} \quad \frac{z^2}{z-a}. \tag{10.137}$$

There are two possible bilateral transforms that can be associated with this function, namely those corresponding to the two possible ROCs, $|z| < |a|$ and $|z| > |a|$. The choice $|z| > |a|$ corresponds to a right-sided sequence, but *not* to a signal that is zero for all $n < 0$, since its inverse transform, which is given by eq. (10.129), is nonzero for $n = -1$.

More generally, if we associate eq. (10.136) with the bilateral transform with the ROC that is the exterior of the circle with radius given by the magnitude of the largest root of $q(z)$, then the inverse transform will certainly be right sided. However, for it to be zero for *all* $n < 0$, it must also be the case that degree($p(z)$) $\leq$ degree($q(z)$).

10.9.2 Properties of the Unilateral z-Transform

The unilateral z-transform has many important properties, some of which are identical to their bilateral counterparts and several of which differ in significant ways. Table 10.3 summarizes these properties. Note that we have not included a column explicitly identifying the ROC for the unilateral z-transform for each signal, since the ROC of any unilateral z-transform is always the exterior of a circle. For example, the ROC for a rational unilateral z-transform is always outside the outermost pole.

By contrasting this table with the corresponding Table 10.1 for bilateral z-transforms, we can gain considerable insight into the nature of the unilateral transform. In particular, several properties—namely, linearity, scaling in the z-domain, time expansion, conjugation, and differentiation in the z-domain—are identical to their bilateral counterparts, as is the initial-value theorem stated in Section 10.5.9, which is fundamentally a unilateral transform property, since it requires $x[n] = 0$ for $n < 0$. One bilateral property, namely, the time-reversal property set forth in Section 10.5.4, obviously has no meaningful counterpart for the unilateral transform, while the remaining properties differ in important ways between the bilateral and unilateral cases.

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$z^{-1}\mathcal{X}(z) + x[-1]$
non causal $\Leftarrow$ Time advance	$x[n+1]$	$z\mathcal{X}(z) - zx[0]$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$
	$z_0^n x[n]$	$\mathcal{X}(z/z_0)$
	$a^n x[n]$	$\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \end{cases}$ for any m	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})\mathcal{X}(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1-z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

Let us examine the difference in the convolution property first. Table 10.3 states that if $x_1[n] = x_2[n] = 0$ for all $n < 0$, then

$$x_1[n] * x_2[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}_1(z)\mathcal{X}_2(z). \tag{10.138}$$

Since in this case the unilateral and bilateral transforms are identical for each of these signals, eq. (10.138) follows from the bilateral convolution property. Thus, the system analysis and system function algebra developed and used in this chapter apply without change to unilateral transforms, as long as we are considering causal LTI systems (for which the system function is both the bilateral *and* the unilateral transform of the impulse response) with inputs that are identically zero for $n < 0$. An example of such application is to the accumulation or summation property in Table 10.3. Specifically, if $x[n] = 0$ for $n < 0$, then

$$\sum_{k=0}^{n} x[k] = x[n] * u[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z)\mathcal{U}(z) = \mathcal{X}(z)\frac{1}{1-z^{-1}}. \tag{10.139}$$

$$= \sum_{k=0}^{\infty} x[k]\, u[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k]$$

As a second example, consider the following:

$x[n] \le 0,\ n \le 0$
$\Downarrow$
$y[n] \le 0,\ n \le 0$
if BIBO stable
$\Downarrow$
$Y(z) = H(z)X(z)$

Example 10.36

Consider the causal LTI system described by the difference equation

$$y[n] + 3y[n-1] = x[n], \tag{10.140}$$

together with the condition of initial rest. The system function for this system is

$$\mathcal{H}(z) = \frac{1}{1+3z^{-1}}. \tag{10.141}$$

Suppose that the input to the system is $x[n] = \alpha u[n]$, where α is a given constant. In this case, the unilateral (and bilateral) z-transform of the output $y[n]$ is

$$\begin{aligned}\mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) &= \frac{\alpha}{(1+3z^{-1})(1-z^{-1})} \\ &= \frac{(3/4)\alpha}{1+3z^{-1}} + \frac{(1/4)\alpha}{1-z^{-1}}.\end{aligned} \tag{10.142}$$

Applying Example 10.32 to each term of eq. (10.142) yields

$$y[n] = \alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]. \tag{10.143}$$

An important point to note here is that the convolution property for unilateral z-transforms applies *only* if the signals $x_1[n]$ and $x_2[n]$ in eq. (10.138) are both identically zero for $n < 0$. While it is generally true that the bilateral transform of $x_1[n] * x_2[n]$ equals the product of the bilateral transforms of $x_1[n]$ and $x_2[n]$, the unilateral transform of $x_1[n] * x_2[n]$ in general does not equal the product of the unilateral transforms if $x_1[n]$ or $x_2[n]$ is nonzero for $n < 0$. This point is explored further in Problem 10.41.

Much of the importance of the unilateral z-transform lies in its application to analyzing causal systems and, in particular, systems characterized by linear constant-coefficient difference equations with possibly nonzero initial conditions. In Section 10.7 we saw how the bilateral transform—particularly the shifting property for bilateral z-transforms—could be used to analyze and compute solutions for LTI systems characterized by such difference equations, together with the assumption of initial rest. As we will now see, the shifting property for unilateral transforms, which differs from its bilateral counterpart, plays an analogous role for initialized systems.

To develop the shifting property for the unilateral transform, consider the signal

$$y[n] = x[n-1]. \tag{10.144}$$

Then

$$\begin{aligned}\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-1]z^{-n} \\ &= x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} \\ &= x[-1] + \sum_{n=0}^{\infty} x[n]z^{-(n+1)},\end{aligned}$$

or

$$\mathcal{Y}(z) = x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n}, \tag{10.145}$$

so that

$$\mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z). \tag{10.146}$$

By repeated application of eq. (10.146), the unilateral transform of

$$w[n] = y[n-1] = x[n-2] \tag{10.147}$$

$$\Rightarrow \mathcal{W}(z) = \sum_{n=0}^{\infty} x[n-2]z^{-n} = x[-2] + x[-1]z^{-1} + \sum_{n=2}^{\infty} x[n-2]z^{-n}$$

$$\mathcal{W}(z) = x[-2] + x[-1]z^{-1} + z^{-2}\mathcal{X}(z). \tag{10.148}$$

Continuing this iterative procedure, we can also determine the unilateral transform of $x[n-m]$ for any positive value of m.

Eq. (10.146) is sometimes referred to as the time delay property, since $y[n]$ in eq. (10.144) is a delayed version of $x[n]$. There is also a time advance property for unilateral transforms that relates the transform of an advanced version of $x[n]$ to $\mathcal{X}(z)$. Specifically, as shown in Problem 10.60,

$$x[n+1] \overset{\mathcal{UZ}}{\longleftrightarrow} z\mathcal{X}(z) - zx[0]. \tag{10.149}$$

$$\left(\mathcal{UZ}\{x[n+1]\} = \sum_{n=0}^{\infty} x[n+1]z^{-n} = z\sum_{n=-1}^{\infty} x[n+1]z - zx[0] = z\mathcal{X}(z) - zx[0]\right)$$

10.9.3 Solving Difference Equations Using the Unilateral z-Transform

The following example illustrates the use of unilateral z-transforms and the time delay property to solve linear constant-coefficient difference equations with nonzero initial conditions:

Example 10.37 $y[n] + 3y[n-1] = x[n]$

Consider again the difference equation (10.140) with $x[n] = \alpha u[n]$ and with the initial condition

$$y[-1] = \beta. \tag{10.150}$$

Applying the unilateral transform to both sides of eq. (10.140) and using the linearity and time delay properties, we obtain

$$\mathcal{Y}(z) + 3\beta + 3z^{-1}\mathcal{Y}(z) = \frac{\alpha}{1 - z^{-1}}. \tag{10.151}$$

Solving for $\mathcal{Y}(z)$ yields

$$\mathcal{Y}(z) = -\frac{3\beta}{1 + 3z^{-1}} + \frac{\alpha}{(1 + 3z^{-1})(1 - z^{-1})}. \tag{10.152}$$

$$\Rightarrow y[n] = \begin{cases} \left(\frac{3\alpha}{4} - 3\beta\right)(-3)^n + \frac{\alpha}{4}, & n \geq 0 \\ \beta, & n = 0 \end{cases}$$

HW #10

5(c), 12(b), 16(c), 24, 34, 37, 46, 51, 58, 65