2008, 1st Electromagnetic field 1 Syllabus

교과목번호	420.202A	**강좌번호**	001	**교과목명**	electromagnetic field 1	**학점**	4

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Goal	Electromagnetic field 1 is important to learn electric engineering in which electromagnetic field is explained by equation. This course gives the comprehension of electric field made by electric charge and magnetic field made by current flow, and help to learn applied electric engineering.

Text	Text book : Field and Wave Electromagnetics by Cheng ADDISON WESLEY

Measurement	**출석**	**과제**	**중간1(4/2)**	**중간2(5/7)**	**기말(6/11)**	**합계**
	3%	7%	25%	25%	40%	100%
	비고	Total 3 test, Quiz after the end of each chapter.				

Attention	assistant : Tae-ho Lee (tolee@pllab.snu.ac.kr)

How to receive F	if a student is absent from 5 assignment, receive F. Cheating.

	week	Contents
Schedule	1st	Introduction / Electromagnetic Model, Vector Analysis (Chap.1, 2)
	2nd	Coulomb's Law, Gauss Law, Electric Potential (Chap.3)
	3rd	Conductors and Dielectrics, Electric Flux Density (Chap.3)
	4th	Boundary Conditions, Capacitance (Chap.3)
	5th	Electrostatic Energy(Chap.3)
	6th	Poisson Equation, Uniqueness Theorem, Boundary-Value Problems (Chap.4)
	7th	Current Density, Electromotive Force, Kirchhoff's Law(Chap.5)
	8th	Joule's Law, Resistance (Chap.5)
	9th	Magnetic flux density, Vector Magnetic Potential (Chap.6)
	10th	Biot-SavartLaw, Magnetic Dipole, Magnetization(Chap.6)
	11th	Magnetic Field Intensity, Magnetic Circuits, Magnetic materials, Boundary Conditions (Chap.6)
	12th	Inductance, Magnetic Energy, Magnetic Forces (Chap.6)
	13th	Faraday's Law, Maxwell's Equation (Chap.7)
	14th	Wave Equation, Quasi-Static Approximation (Chap.7)
	15th	Solutions to Wave Equation (Chap.7)

Chapter 2. Vector Analysis

2-1 Introduction [Text p.11]

Field *“A ftn which describes a physical quantity in space”*

Scalar Field (magnitude) : temperature, density, electric potential

Vector Field (magnitude and direction) : velocity, force

Electric field intensity ***E***, Electric displacement ***D***

Magnetic flux density ***B***, Magnetic field intensity ***H***

Representation : $\boxed{\vec{A},}\ \boldsymbol{A},\ \underrightarrow{A},\ \underset{\sim}{A},\ \cdots$

$$\vec{A} = \boxed{A_i}\,\hat{i} + A_j\,\boxed{\hat{j}} + A_h\,\hat{h}$$

i 방향의 성분 Unit vector

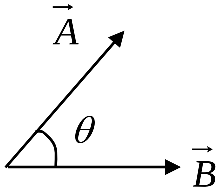
2-2 Vector Addision and Subtraction [Text p.12]

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$: Commutative law

$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$: Associative law

2-3 Products of Vectors [Text p.14]

2-3.1 Scalar (or dot, inner) product



$$\vec{A}\cdot\vec{B} \underset{=}{\Delta} AB\cos\theta$$

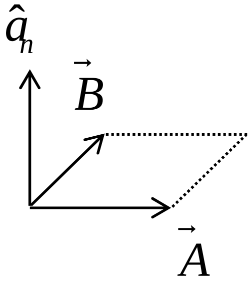
$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$: Commutative law

$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$: Distributive law

2-3.2 Vector (or cross, outer) product

$$\vec{A}\times\vec{B} \underset{=}{\Delta} \hat{a}_n \left|AB\sin\theta\right|$$

$\hat{a}_n$: unit vector perpendicular to $\vec{A}$ and $\vec{B}$
formed according to right-hand rule



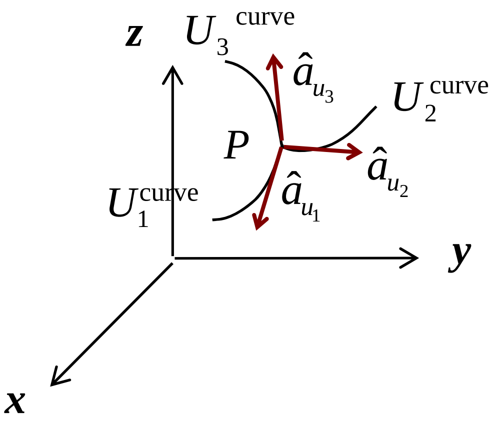
2-4 Orthogonal Coordinate Systems [Text p.20]

consider

$$\left.\begin{aligned} U_1(x,y,z) &= g_1 \\ U_2(x,y,z) &= g_2 \\ U_3(x,y,z) &= g_3 \end{aligned}\right\}$$

each represents a family of surfaces in space

Intersection of U_2, U_3 surfaces $\longrightarrow$ curve U_1
Intersection of U_3, U_1 surfaces $\longrightarrow$ curve U_2
Intersection of U_1, U_2 surfaces $\longrightarrow$ curve U_3



Three surfaces intersect at one pt $P(x_o,\ y_o,\ z_o)$

$\hat{a}_{u_1}$, $\hat{a}_{u_2}$, $\hat{a}_{u_3}$: Units vectors issuing from P target to u_1, u_2 , u_3 curve

(or orthogonal to q_1, q_2 , q_3 surfaces)

When three surfaces intersect one another orthogonally

→ orthogonal coordinate sys. $\hat{a}_{u_1} \times \hat{a}_{u_2} = \hat{a}_{u_3}$

$$\hat{a}_{u_1} \cdot \hat{a}_{u_2} = 0 , \qquad \hat{a}_{u_2} \cdot \hat{a}_{u_3} = 0 , \qquad \cdots$$

$$\hat{a}_{u_1} \cdot \hat{a}_{u_1} = 1$$

Let length elements $d\,l_i$ are related to Coordinate variables u_i by

$$dl_i = h_i \; du_i$$

Where h_i : metric coefficient

Then, the Length element

$$dl = \left[(dl_1)^2 + (dl_2)^2 + (dl_3)^2\right]^{\frac{1}{2}}$$
$$= \left[(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2\right]^{\frac{1}{2}}$$

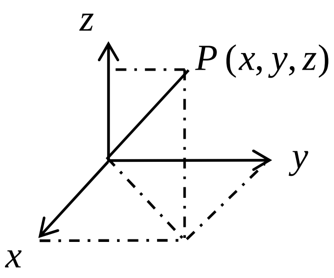
Volume element

$$dv = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$$

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$r \sin \theta$

2-4.1 Cartesian coord

$$(u_1, u_2, u_3) = (x, y, z)$$

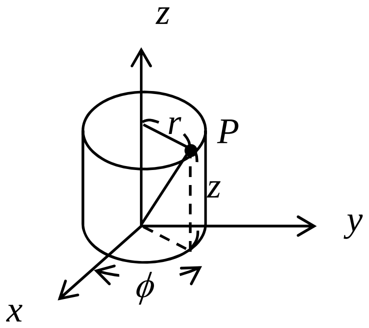


$$h_1 = 1, \qquad h_2 = 1, \qquad h_3 = 1$$

$$dv = dx\,dy\,dz$$

2-4.2 Cylindrical coord

$$(u_1, u_2, u_3) = (r, \phi, z)$$

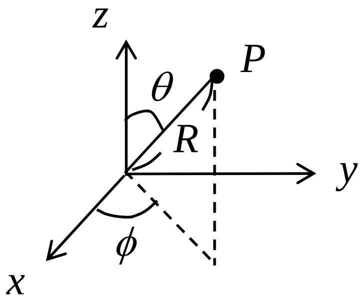


$$h_1 = 1, \qquad h_2 = r, \qquad h_3 = 1$$

$$dv = r dr d\phi dz$$

2-4.3 Spherical coord

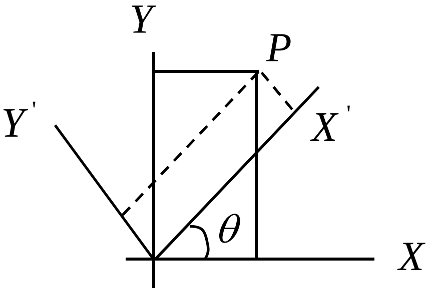
$$(u_1, u_2, u_3) = (R, \theta, \phi)$$



$$h_1 = 1, \qquad h_2 = R, \qquad h_3 = R\sin\theta$$

$$dv = R^2 \sin\theta\, dR\, d\theta\, d\phi$$

Invariance



Rotation of coordinate system
by an angle θ about Z-axis
coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and (x', y', z') of $\mathbf{P}$

$$\begin{cases} \mathbf{x}' = \mathbf{x}\cos\theta + \mathbf{y}\sin\theta \\ \mathbf{y}' = -\mathbf{x}\sin\theta + \mathbf{y}\cos\theta \\ \mathbf{z}' = \mathbf{z} \end{cases}$$

Symbolically $\mathbf{P}' = \mathbf{RP}$

where $\mathbf{P}' = (\mathbf{x}', \mathbf{y}', \mathbf{z}')$, $\mathbf{P} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ and

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Vectors

$$\begin{cases} \vec{\mathbf{P}} = \mathbf{x}\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}} + \mathbf{z}\hat{\mathbf{k}} \\ \vec{\mathbf{P}}' = \mathbf{x}'\hat{\mathbf{i}}' + \mathbf{y}'\hat{\mathbf{j}}' + \mathbf{z}'\hat{\mathbf{k}}' \end{cases}$$

$$\begin{cases} \vec{\mathbf{Q}} = \mathbf{u}\hat{\mathbf{i}} + \mathbf{v}\hat{\mathbf{j}} + \mathbf{w}\hat{\mathbf{k}} \\ \vec{\mathbf{Q}}' = \mathbf{u}'\hat{\mathbf{i}}' + \mathbf{v}'\hat{\mathbf{j}}' + \mathbf{w}'\hat{\mathbf{k}}' \end{cases}$$

Scalar product of $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$

$$\vec{P}\cdot\vec{Q} = xu + yv + zw$$

$$\begin{aligned} \vec{P}'\cdot\vec{Q}' &= x'u' + y'v' + z'w' \\ &= (x\cos\theta + y\sin\theta)(u\cos\theta + v\sin\theta) \\ &\quad + (-x\sin\theta + y\cos\theta)(-u\sin\theta + v\cos\theta) + zw \\ &= xu(\cos^2\theta + \sin^2\theta) + yv(\cos^2\theta + \sin^2\theta) + zw \\ &= \vec{P}\cdot\vec{Q} \end{aligned}$$

$\therefore \vec{\mathbf{P}}\cdot\vec{\mathbf{Q}}$ are independent of coordinate sys

: invariant (A quantity that is independent of coordinate system)

$\vec{\mathbf{P}}\times\vec{\mathbf{Q}}$, $\dfrac{\mathbf{d}\vec{\mathbf{p}}}{\mathbf{dt}}$, ∇, $\Phi(\mathbf{flux})$, $\nabla\cdot\vec{\mathbf{P}}$, are all invariant

2-5 Integrals Containing Vector Functions [Text p.37]

Line integral $$\int_c V d\vec{l} = \int_c V(x,y,z)\left[\hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz\right]$$

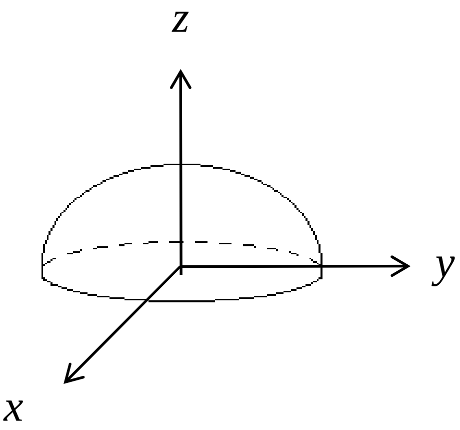
$$= \hat{a}_x \int_c V(x,y,z)dx + \hat{a}_y \int_c V(x,y,z)dy$$

$$+ \hat{a}_z \int_c V(x,y,z)dz$$

Surface integral $$\int_s \vec{A}\cdot d\vec{S}$$

Volume integral $$\int_v \vec{F} dv$$

Closed line or surface $\oint_c$ or $\oint_s$



(Ex) $\vec{A} = E\hat{z}$

1. $S \quad : \quad x^2 + y^2 + z^2 = 1$ 로 주어진 반구

2. $S \quad = \quad x^2 + y^2 = 1$ 로 주어진 원판

find $\int_s \vec{A}\cdot d\vec{S}$

2-6 Gradient of a Scalar Field [Text p.42]

grad $V \triangleq \nabla V \triangleq \hat{a}_n \dfrac{dV}{dn}$ $\quad\quad$ ∇ : del

where $\hat{a}_n$: direction of maximum rate of increase

Directional derivative

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\hat{a}_n \cdot \hat{a}_l = \nabla V \cdot \hat{a}_l$$

$$\begin{aligned} dV &= \nabla V \cdot d\vec{l} \\ &= \frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3 \\ &= \left(\hat{a}_{u_1}\frac{\partial V}{\partial l_1} + \hat{a}_{u_2}\frac{\partial V}{\partial l_2} + \hat{a}_{u_3}\frac{\partial V}{\partial l_3} \right) \cdot \underbrace{\left(\hat{a}_{u_1} dl_1 + \hat{a}_{u_2} dl_2 + \hat{a}_{u_3} dl_3 \right)}_{d\vec{l}} \end{aligned}$$

Thus

$$\nabla V = \hat{a}_{u_1}\frac{\partial V}{\partial l_1} + \hat{a}_{u_2}\frac{\partial V}{\partial l_2} + \hat{a}_{u_3}\frac{\partial V}{\partial l_3}$$

$$\boxed{\nabla V = \hat{a}_{u_1}\frac{\partial V}{h_1 \partial u_1} + \hat{a}_{u_2}\frac{\partial V}{h_2 \partial u_2} + \hat{a}_{u_3}\frac{\partial V}{h_3 \partial u_3}}$$

gradient의 성질

a. magnitude is the maximum rate of change with distance

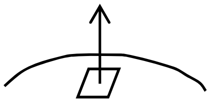
b. direction is that of the maximum rate of change

c. it points towards larger values of ftn.

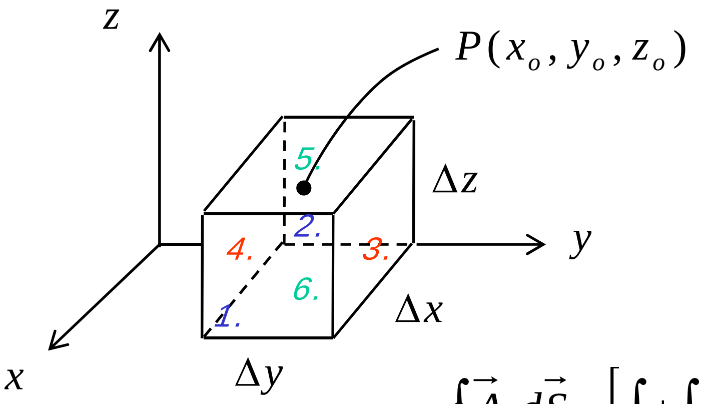
2-7 Divergence of a Vector Field [Text p.46]

Flux of $\vec{A}$ over a surface S

$$\Phi = \int_s \vec{A}\cdot d\vec{S} \longrightarrow$$ 미소 면적 성분, 수직 방향



$$div\ \vec{A} \overset{\Delta}{=} \lim_{\Delta v \to 0} \frac{\oint \vec{A}\cdot d\vec{S}}{\Delta v}$$ (net outward flux / vol)



$$\oint \vec{A}\cdot d\vec{S} = \left[\int_{1.} + \int_{2.} + \int_{3.} + \int_{4.} + \int_{5.} + \int_{6.}\ \right]\vec{A}\cdot d\vec{S}$$

for surface 1.

$$\begin{aligned}\vec{A}\cdot d\vec{S} &= \vec{A}\cdot(\Delta y \Delta z)\hat{q}_x \\ &= A_x(x_o + \frac{\Delta x}{2}, y_o, z_o)\Delta y \Delta z \\ &= \left\{ A_x(x_o, y_o, z_o) + \frac{\partial Ax}{\partial x}\Big]_{(x_o, y_o, z_o)} \frac{\Delta x}{2} + \cdots \right\}\Delta y \Delta z\end{aligned}$$

for surface 2.

$$\vec{A}\cdot d\vec{S} = -\left\{ A_x(x_o, y_o, z_o) + \frac{\partial Ax}{\partial x}\Big]_{(x_o, y_o, z_o)} (-\frac{\Delta l}{2}) + \cdots \right\}\Delta y \Delta z$$

$$\left[\int_{1.}+\int_{2.}\right]\vec{A}\cdot d\vec{S}=\frac{\partial Ax}{\partial x}\Delta x\Delta y\Delta z+\cdots\cdots$$

같은 방법으로,

$$\left[\int_{3.}+\int_{4.}\right]\vec{A}\cdot d\vec{S}=\frac{\partial Ay}{\partial y}\Delta x\Delta y\Delta z+\cdots\cdots$$

and

$$\left[\int_{5.}+\int_{6.}\right]\vec{A}\cdot d\vec{S}=\frac{\partial Az}{\partial z}\Delta x\Delta y\Delta z+\cdots\cdots$$

and

$$\oint \vec{A}\cdot d\vec{S}=\left(\frac{\partial Ax}{\partial x}+\frac{\partial Ay}{\partial y}+\frac{\partial Az}{\partial z}\right)\Delta x\Delta y\Delta z+\cdots\cdots$$

and

$$\boxed{\nabla\cdot\vec{A}\equiv div\ \vec{A}=\frac{\partial Ax}{\partial x}+\frac{\partial Ay}{\partial y}+\frac{\partial Az}{\partial z}}$$

For general orthogonal curvilinear coord

$$\boxed{\nabla\cdot\vec{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1)+\frac{\partial}{\partial u_2}(h_3h_1A_2)+\frac{\partial}{\partial u_3}(h_1h_2A_3)\right]}$$

2-8 Divergence Theorem **[Text p.50]**

For an arbitrary vol.

$$\sum_{j=1}^{N} (\nabla \cdot \vec{A})_j \, \Delta v_j = \sum_{j=1}^{N} \oint_{s_j} \vec{A} \cdot d\vec{S}$$

$$\boxed{\int_v \nabla \cdot \vec{A} \, dv = \oint_s \vec{A} \cdot d\vec{S}}$$

Divergence Theorem.

Nonzero divergence ⟶ source or sink of flow

$\nabla \cdot \vec{A}$: measure of strength of flow source

2-9 Curl of a vector field [Text p.54]

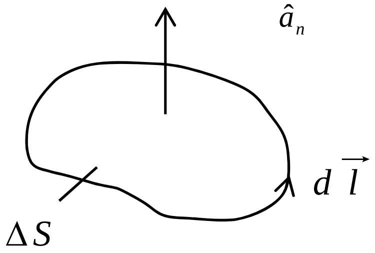
Circulation of $\vec{A}$ around contour $c \triangleq \oint_C \vec{A}\cdot d\vec{l}$

(Ex) 만일 $\vec{A}=\vec{F}$ (force)

Circulation of $\vec{A}=\oint_C \vec{F}\cdot d\vec{l}$ = work

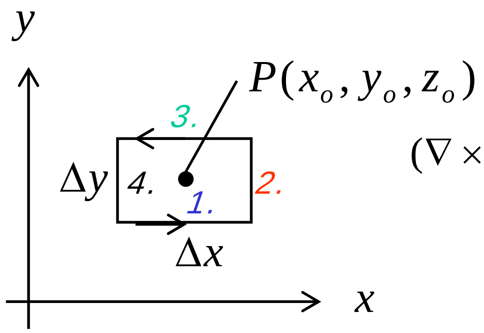
curl $\vec{A}=\nabla\times\vec{A}$

$$\triangleq \lim_{\Delta S\to 0}\frac{1}{\Delta S}\left[\hat{a}_n\oint_C \vec{A}\cdot d\vec{l}\right]_{\max}$$



$\hat{a}_n$: normal direction of area

right hand rule



$$(\nabla\times\vec{A})_z = \lim_{\Delta x\Delta y\to 0}\frac{1}{\Delta x\Delta y}\oint \vec{A}\cdot d\vec{l}$$

$$= \lim_{\Delta x\Delta y\to 0}\frac{1}{\Delta x\Delta y}\int_{1.\ 2.\ 3.\ 4.}\vec{A}\cdot d\vec{l}$$

along contour 1. $\oint_{1.}\vec{A}\cdot d\vec{l} = A_x(x_o, y_o-\frac{\Delta y}{2}, z_o)\Delta x$

$d\vec{l}=\Delta x\hat{a}_x$

$$=\left[A_x(x_o,y_o,z_o)+\frac{\partial A_x}{\partial y}\Big]_{x_o,y_o,z_o}\cdot\frac{(-\Delta y)}{2}+\cdots\cdots\right]\Delta x$$

along contour 2.

$$d\vec{l} = \Delta y \hat{a}_y \quad \int_{2.} \vec{A}\cdot d\vec{l} = A_y(x_o + \frac{\Delta x}{2}, y_o, z_o)\Delta y$$

$$= \left[A_y(x_o, y_o, z_o) + \frac{\partial Ay}{\partial x}\cdot\frac{\Delta x}{2} + \cdots\cdots \right]\Delta y$$

along contour 3.

$$d\vec{l} = -\Delta x \hat{a}_x \quad \int_{3.} \vec{A}\cdot d\vec{l} = \left[A_x + \frac{\partial Ax}{\partial y}\cdot\frac{\Delta y}{2} + \cdots\cdots \right](-\Delta x)$$

along contour 4.

$$d\vec{l} = -\Delta y \hat{a}_y \quad \int_{4.} \vec{A}\cdot d\vec{l} = \left[A_y + \frac{\partial Ay}{\partial x}\cdot\frac{(-\Delta x)}{2} + \cdots\cdots \right](-\Delta y)$$

$$\therefore \quad \oint \vec{A}\cdot d\vec{l} = -\frac{\partial A_x}{\partial y}\Delta x\Delta y + \frac{\partial A_y}{\partial x}\Delta x\Delta y$$

$$(\nabla \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

같은 방법으로,

$$\boxed{\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y}\right)\hat{a}_z}$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In general orthogonal curvilinear coord system

$$\nabla \times \vec{A} = \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1 \hat{a}_{u_1} & h_2 \hat{a}_{u_2} & h_3 \hat{a}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$\nabla \times \vec{A} = 0 \longrightarrow \vec{A}$ is a conservative field

2-10 Stoke's theorem [Text p.58]

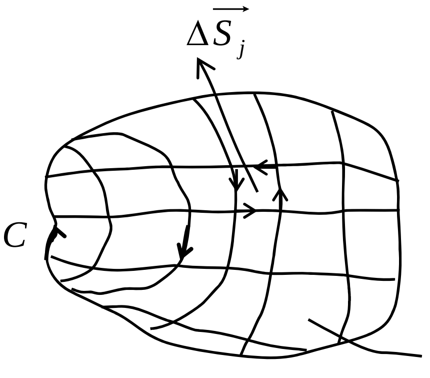
From the definition of curl, i.e.

$$\nabla \times \vec{A} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \left[\hat{a}_n \oint_C \vec{A} \cdot d\vec{l} \right]$$ For ΔS surrounded by C

$$(\nabla \times \vec{A}) \cdot \Delta S\ \hat{a}_n = \oint_C \vec{A} \cdot d\vec{l}$$

For an arbitrary surface S, subdivide it into many N

and add up all the differential areas



$$\lim_{\Delta S_j \to 0} \sum_{j=1}^{N} (\nabla \times \vec{A})\, j \cdot \Delta\vec{S}_j = \int_s \nabla \times \vec{A} \cdot d\vec{S}$$

$$= \lim_{\Delta S_j \to 0} \sum_{j=1}^{N} \left(\oint_{c_j} \vec{A} \cdot d\vec{l} \right) = \oint_c \vec{A} \cdot d\vec{l}$$

$$\therefore \quad \boxed{\int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l}}$$ Stoke's theorem

2-11 Two Null Identities [Text p.61]

2-11.1 Identity I

$$\nabla \times \left(\nabla \boldsymbol{V} \right) = 0$$

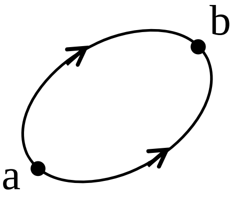
pf. By Stoke's theorem

$$\int_s \left[\nabla \times \left(\nabla V \right) \right] \cdot d\vec{a} = \oint \left(\nabla V \right) \cdot d\vec{l} = \oint dV = 0$$

← If $\nabla \times \vec{E} = 0$, 1. $\vec{E} = -\nabla V$

2. $\oint \vec{E} \cdot d\vec{l} = 0 \quad \rightarrow \quad \int_a^b \vec{E} \cdot d\vec{l}$

conservative field



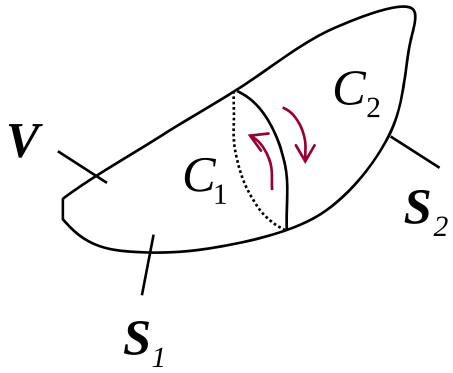
: depends only on end pts a and b

2-11.2 Identity II

$$\nabla \cdot \left(\nabla \times \vec{\boldsymbol{A}} \right) = 0$$

pf. By Divergence theorem

$$\int_v \nabla \cdot \left(\nabla \times \vec{A} \right) d\tau = \oint_s \left(\nabla \times \vec{A} \right) \cdot d\vec{a}$$



$$= \int_{S_1} \left(\nabla \times \vec{A} \right) \cdot d\vec{a} + \int_{S_2} \left(\nabla \times \vec{A} \right) \cdot d\vec{a}$$

$$= \oint_{c_1} \vec{A} \cdot d\vec{l} + \oint_{c_2} \vec{A} \cdot d\vec{l}$$

$$= 0$$

$\leftarrow$ If $\nabla \cdot \vec{B} = 0 \;\rightarrow\; \underset{\substack{\swarrow \\ \text{magnetic} \\ \text{flux density}}}{\vec{B}} = \underset{\substack{\searrow \\ \text{vector potential}}}{\nabla} \times \vec{A}$

Chap 3. Static Electric Fields

3-1 Introduction [Text p. 72]

Field : Spatial distribution of scalar and vector quantity
Electric field and magnetic field.

Electrostatics : 시간에 따라 변하지 않는 전하분포 혹은
전장과 관계된 전기 현상

Coulomb's law (1785)

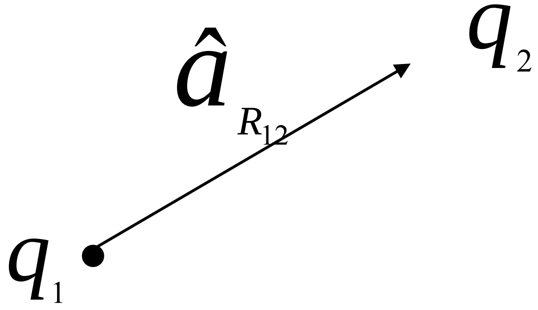
$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{R_{12}^2}\hat{a}_{R_{12}} \text{ (Newton)}$$

q_1, q_2 : coulomb

R_{12} : m

where $\varepsilon_o = 8.84 \times 10^{-12} (F/m)$ Permittivity of free space

Force exerted on q_2 by q_1



3-2 Fundamental Postulates of Electrostatics in Free Space [Text p. 74]

Electric field intensity

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \qquad (V/m)$$

Postulate 1. $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$ ρ : charge density (C/m^3)

Postulate 2. $\nabla \times \vec{E} = 0$

Integral form of two postulates

Postulate 1

$$\int_V \nabla \cdot E \, dv = \frac{1}{\varepsilon_o} \int_V \rho dv = \frac{Q}{\varepsilon_o}$$

Q : total charge in V

$$\boxed{\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\varepsilon_o}}$$

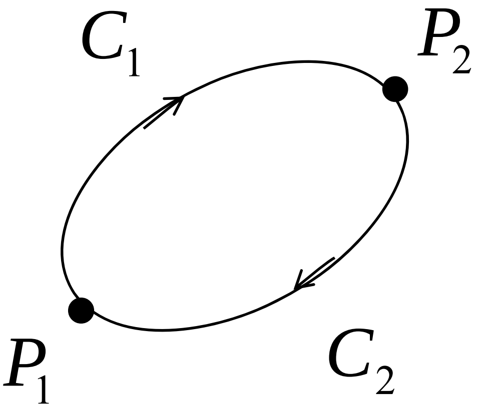
: Gauss law

Postulate 2.

$$\int_S \nabla \times \overrightarrow{E} \cdot d\overrightarrow{S} = \boxed{\oint_C \overrightarrow{E} \cdot d\overrightarrow{l} = 0}$$

since

$$\oint \overrightarrow{E} \cdot d\overrightarrow{l} = \int_{C_1} \overrightarrow{E} \cdot d\overrightarrow{l} + \int_{C_2} \overrightarrow{E} \cdot d\overrightarrow{l} = 0$$



$$\underset{C_1\text{을 따라}}{\int_{P_1}^{P_2} \overrightarrow{E} \cdot d\overrightarrow{l}} = \underset{C_2\text{를 따라}}{-\int_{P_2}^{P_1} \overrightarrow{E} \cdot d\overrightarrow{l}}$$

$$= \underset{C_2\text{를 따라}}{\int_{P_1}^{P_2} \overrightarrow{E} \cdot d\overrightarrow{l}}$$

line integral of $\overrightarrow{E}$ is independent of path

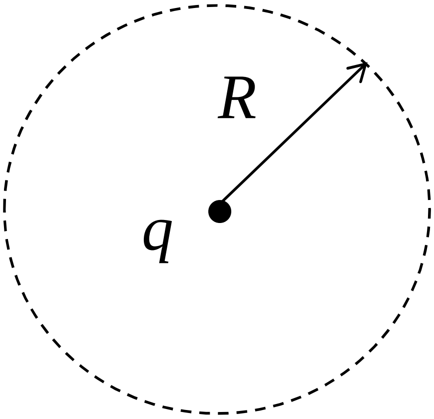
depends only on end points

conservation of energy

Electrostatic field : conservative field

3-3 Coulomb's law [Text p.77]

Electric field due to point charge applying Gauss law



$$\oint_S \vec{E}\cdot d\vec{S} = \oint_S (E_R\,\hat{a}_R)\cdot(\hat{a}_R\,dS)$$

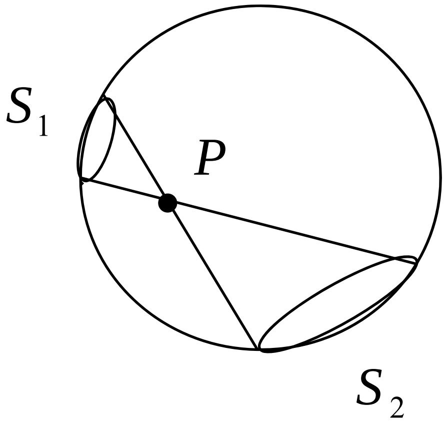
$$= E_R\oint_S dS$$

$$= E_R\;4\pi R^2$$

$$= q/\varepsilon_o$$

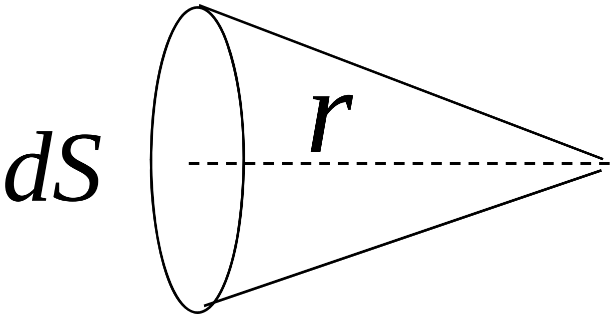
$$\vec{E} = \hat{a}_R\,E_R = \frac{q}{4\pi\varepsilon_o R^2}\hat{a}_R \quad V/m$$

(Ex 3-2) Electric field inside of a spherical shell with a total charge Q



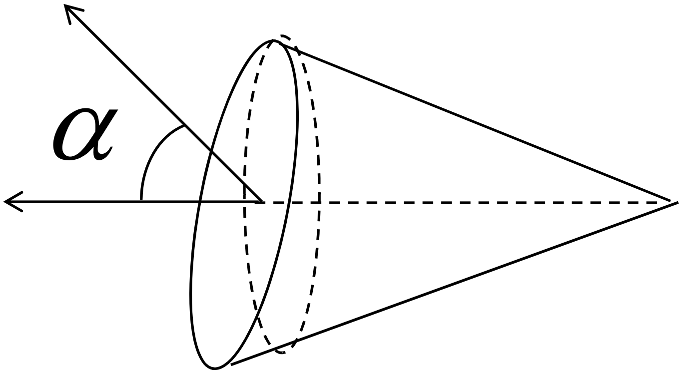
1. By Gauss law
2. Field at P due to S_1 and S_2

$$dE = \frac{\rho_S}{4\pi\varepsilon_o}\left(\frac{dS_1}{r_1^2} - \frac{dS_2}{r_2^2}\right) = 0$$



Solid angle

$$d\Omega = \frac{dS}{r^2}$$



Tilted surface

$$d\Omega = \frac{dS}{r^2} \cos \alpha$$

3-3.1 Electric field due to a system of discrete charges [Text p. 82]

Electric field : linear ftn of $\dfrac{q}{R}\hat{a}_R$

→ Principle of superposition

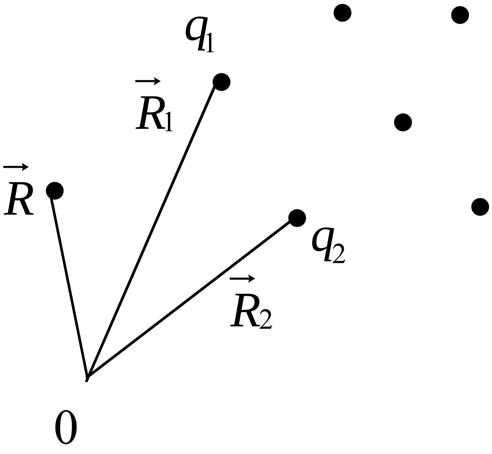
$$\overrightarrow{E}_{(R)} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{n}\frac{\left(\overrightarrow{R} - \overrightarrow{R}^{1}_{k}\right)}{\left|\overrightarrow{R} - \overrightarrow{R}^{1}_{k}\right|^{3}} q_k$$

Continuous distribution

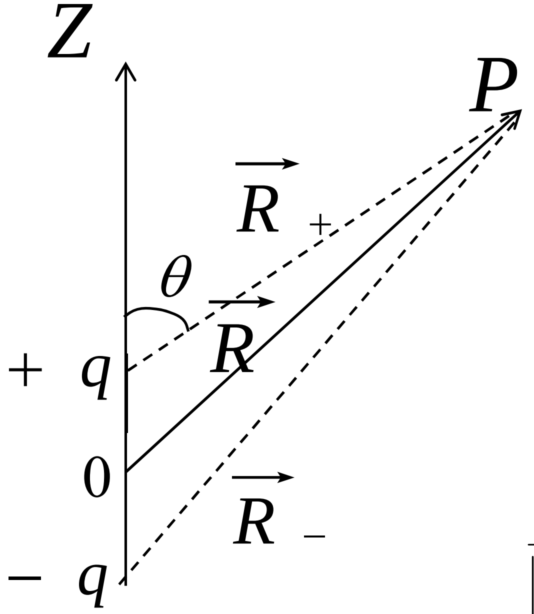
$$\overrightarrow{E} = \frac{1}{4\pi\varepsilon_o}\int_{v'}\hat{a}_R\frac{\rho}{R^2}dv'$$: Volume charge

$$\overrightarrow{E} = \frac{1}{4\pi\varepsilon_o}\int_{S'}\hat{a}_R\frac{\rho_S}{R^2}ds'$$: Surface

$$\overrightarrow{E} = \frac{1}{4\pi\varepsilon_o}\int_{L'}\hat{a}_R\frac{\rho_l}{R^2}dl'$$: Line



- Electric dipole : sys of charges consists of a pair of equal and opposite charges $+q$ and $-q$ separated by a small distance $d << R$



Electric field at P

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\left[\frac{\vec{R}-\frac{\vec{d}}{2}}{\left|\vec{R}-\frac{\vec{d}}{2}\right|^3} - \frac{\vec{R}+\frac{\vec{d}}{2}}{\left|\vec{R}+\frac{\vec{d}}{2}\right|^3}\right]$$

$$\frac{\vec{R}-\frac{\vec{d}}{2}}{\left|\vec{R}-\frac{\vec{d}}{2}\right|^3} = \frac{\vec{R}-\frac{\vec{d}}{2}}{\left(R^2 - \vec{R}\cdot\vec{d} + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

$$\simeq \frac{\vec{R}-\frac{\vec{d}}{2}}{R^3}\left(1+\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)$$

같은 방법으로,

$$\frac{\vec{R}+\frac{\vec{d}}{2}}{\left|\vec{R}+\frac{\vec{d}}{2}\right|^3} = \left(\vec{R}+\frac{\vec{d}}{2}\right)\left(1-\frac{3}{2}\frac{\vec{R}\cdot\vec{d}}{R^2}\right)/R^3$$

and $$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{1}{R^3}\left(-\vec{d}+3\frac{\vec{R}\cdot\vec{d}}{R^2}\vec{R}\right)$$

$\vec{P} = q\vec{d}$: Electric dipole moment

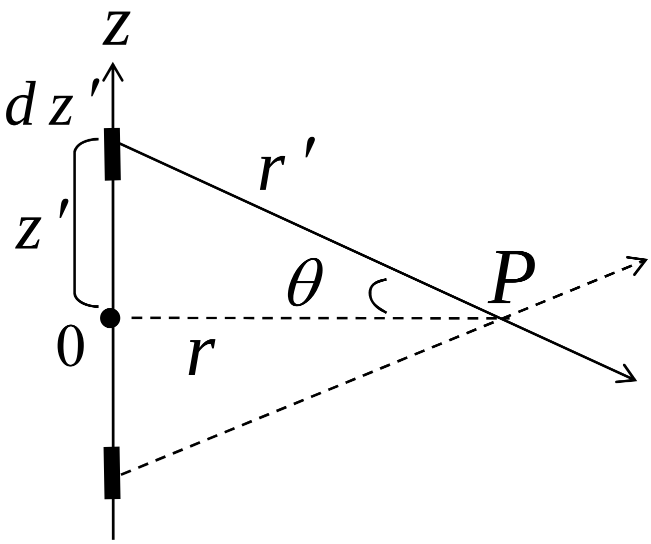
$\left|\vec{E}\right| \propto R^{-3}$

3-3.2 Electric field due to a continuous distribution of charge [Text p.84]

Electric field due to charge in differential volume dv' is

$$d\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{\rho dv'}{R^2}\hat{a}_R \text{ and } \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_o}\int_{V'}\hat{a}_R\frac{\rho}{R^2}dv'}$$

(Ex 3-4) Electric field due to an infinitely long, straight line charge (ρ_l)



(방법 1)

field at P due to dz'

$$d\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{\rho_l dz'}{r'^2}\cos\theta\ \hat{a}_r$$

($\hat{a}_z$ components cancel)

$$\overrightarrow{E} = \frac{1}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{\rho_l dz'}{r'^2}\cos\theta\ \hat{a}r$$

$$= \frac{\rho_l}{4\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{r}{(r^2+z'^2)^{\frac{3}{2}}}dz'\,\hat{a}_r$$

$$Let\ \ z' = r\tan\alpha \quad (-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2})$$

$$= \frac{\rho_l r}{4\pi\varepsilon_o}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r\sec^2\alpha}{[r^2(1+\tan^2\alpha)]^{3/2}}d\alpha\ \hat{a}_r$$

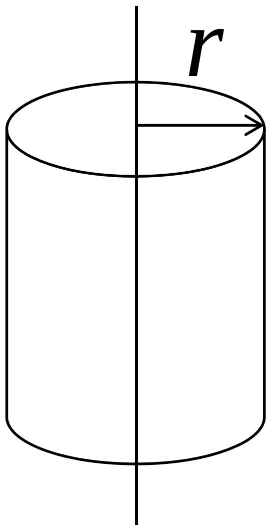
$$= \frac{\rho_l}{4\pi\varepsilon_o r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\alpha d\alpha\ \hat{a}_r$$

$$= \frac{\rho_l}{2\pi\varepsilon_o r}\hat{a}_r$$

3-4 Gauss's Law and Applications [Text p.87]

(Ex 3-5) Electric field due to an infinitely long, straight line charge (ρ_l) , **Use Gauss's law**

(방법 2) use gauss law



Since $\overrightarrow{E}$ is symmetric and has radial component only

$$\oint \overrightarrow{E} \cdot d\overrightarrow{S} = E_r\, 2\pi r L = \frac{\rho_l L}{\varepsilon_o}$$

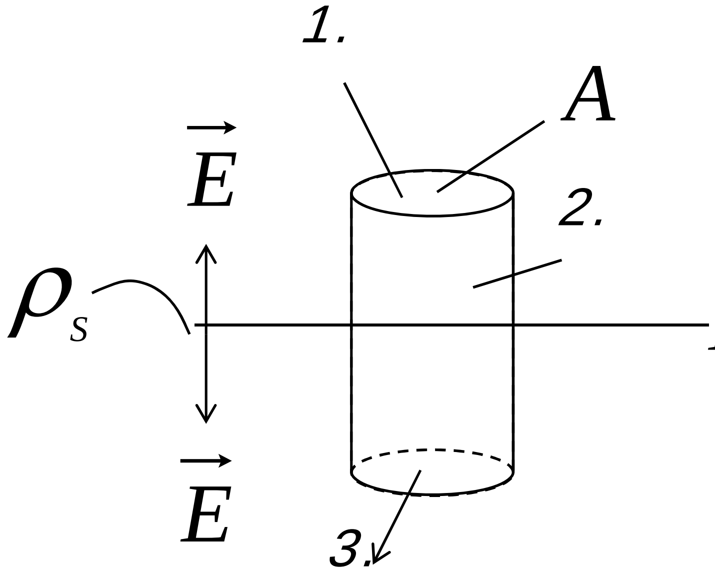
$$\overrightarrow{E} = \hat{a}_r\, E_r = \hat{a}_r\, \frac{\rho_l}{2\pi\varepsilon_o r}$$

(Ex 3-5) Electric field due to an infinite planar charge (ρ_S)

Infinite planar charge

⟶ Electric field is normal to surface

$$\oint \vec{E}\cdot d\vec{S} = \underset{1.}{\int} + \underset{2.}{\int} + \underset{3.}{\int} \quad \vec{E}\cdot d\vec{S}$$



$$= AE_z + AE_z + 0 = \frac{\rho_S A}{\varepsilon_o}$$

$$E_z = \frac{\rho_S}{2\varepsilon_o}$$

$$\vec{E} = \hat{a}_z \frac{\rho_S}{2\varepsilon_o}$$

조명 : 백열등 $E \propto \frac{1}{r^2}$

형광등 $E \propto \frac{1}{r}$

조명패널 E indep of distance

3-5 Electric potential [Text p.92]

since $\nabla \times \vec{E} = 0$ for a static field,

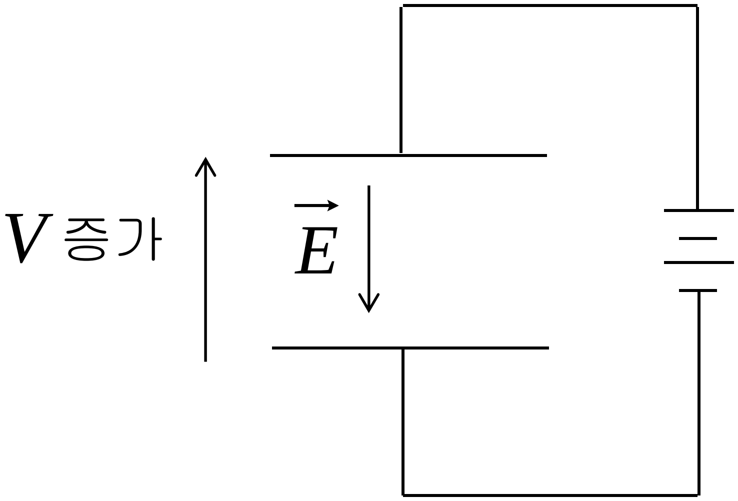
$\vec{E}$ can be represented as

$$\boxed{\vec{E} = -\nabla V}$$

where V : Electric potential

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\,\vec{l}}$$

- (-) sign : $\vec{E}$ 에 거슬러 갈 때 V 가 증가



- field line $\perp$ equipotential line

Work needed to move a point charge q in an electric field $\overrightarrow{E}$

Since force in q is $\overrightarrow{F} = qE$

$$W = -\int_{P_1}^{P_2} \overrightarrow{F} \cdot d\overrightarrow{l}$$

$$= -q\int_{P_1}^{P_2} \overrightarrow{E} \cdot d\overrightarrow{l}$$

$$= q\int_{P_1}^{P_2} \nabla V \cdot d\overrightarrow{l}$$

$$= q\int_{P_1}^{P_2} dV$$

$$= q\left(V_2 - V_1\right)$$

1. needed work = potential energy difference
2. = independent of path

3-5.1 Electric potential due to a charge distribution [Text p.94]

Electric potential due to a point charge

$$V = -\int_{\infty}^{R} \hat{a}_R \frac{q}{4\pi\varepsilon_o R^2} \cdot \hat{a}_R \, dR$$

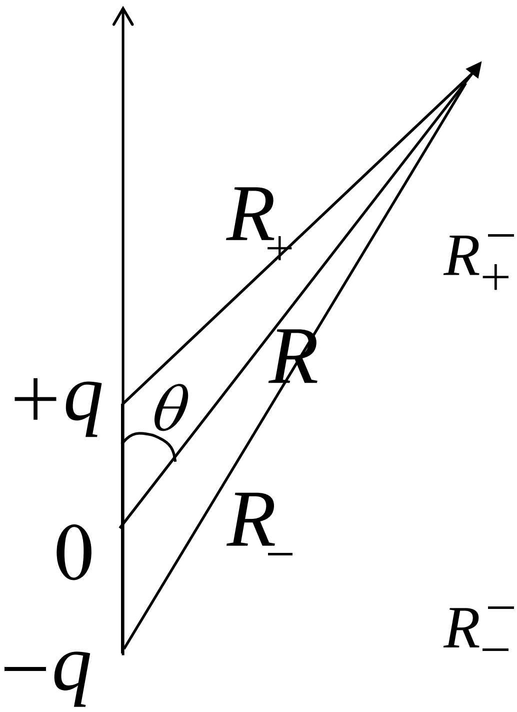
$$= \frac{q}{4\pi\varepsilon_o R}$$

(Reference is taken at $R = \infty$ for $V = 0$)

Electric potential due to many charges

$$V(\vec{R}) = \frac{1}{4\pi\varepsilon_o} \sum_{k=1}^{n} \frac{q_k}{\left|\vec{R} - \vec{R}_k\right|}$$

potential due to a dipole



$$V = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{R_+} - \frac{1}{R_-}\right)$$

$$R_+^{-1} = \left(R^2 + \frac{d^2}{4} - 2 \cdot R \cdot \frac{d}{2}\ \cos\theta\right)^{-\frac{1}{2}}$$

$$\simeq R^{-1}\left(1 + \frac{1}{2}\frac{d}{R}\cos\theta + \cdots\cdots\right)$$

$$R_-^{-1} \simeq R^{-1}\left(1 - \frac{1}{2}\frac{d}{R}\cos\theta + \cdots\cdots\right)$$

$$V = \frac{q}{4\pi\varepsilon_o}\frac{d}{R^2}\cos\ \theta$$

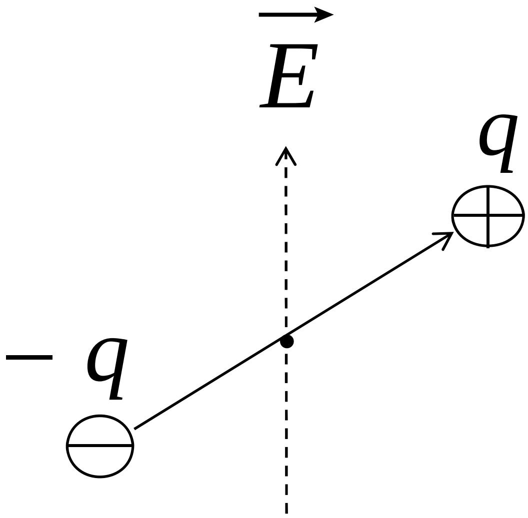
$$= \frac{\overrightarrow{P} \cdot \hat{a}_r}{4\pi\varepsilon_o R^2}$$

$$\overrightarrow{E} = -\nabla V = -\left(\frac{\partial}{\partial R}\hat{a}_r + \frac{1}{R}\frac{\partial}{\partial\theta}\hat{a}_\theta\right)$$

$$= \frac{P}{4\pi\varepsilon_o R^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

Electric dipole in an external field

$$\overrightarrow{F} = q\overrightarrow{E} + (-q)\overrightarrow{E} = 0$$



$$\overrightarrow{\tau} = \vec{r} \times \overrightarrow{F}$$

$$= \frac{\vec{d}}{2} \times q\overrightarrow{E} + -\frac{\vec{d}}{2} \times (-q\overrightarrow{E})$$

$$= q\vec{d} \times \overrightarrow{E} \qquad = \overrightarrow{P} \times \overrightarrow{E}$$

Potential energy $U = \int^{\theta} \vec{\tau} \cdot d\vec{\theta}$

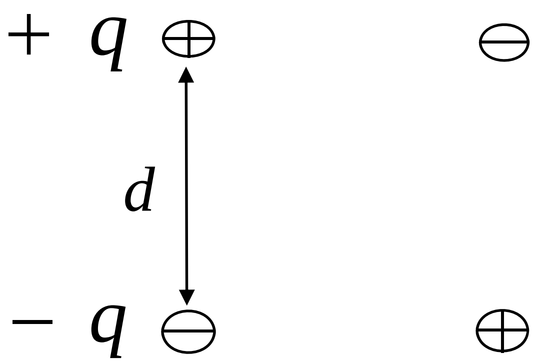
$$= \int^{\theta} \overrightarrow{P} \times \overrightarrow{E} \cdot d\vec{\theta}$$

$$= \int^{\theta} PE \sin\theta \, d\theta$$

$$= -PE\cos\theta$$

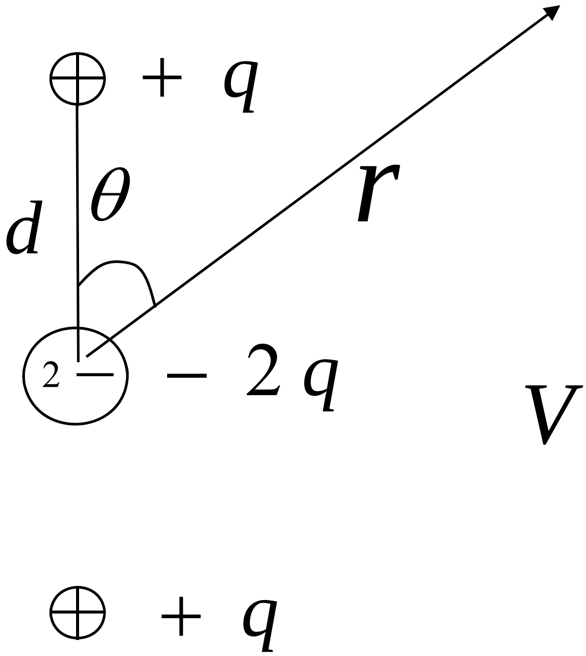
$$= -\overrightarrow{P} \cdot \overrightarrow{E}$$

Quadrapole



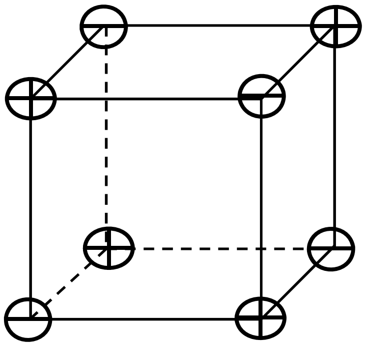
$$V = \frac{qd^2 \sin\theta \cos\theta}{2\pi\varepsilon_o r^3}$$

Linear Quadrapole



$$V = \frac{qd^2}{2\pi\varepsilon_o r^3} \frac{3\cos^2\theta - 1}{2}$$

Octopole



Distance factor for multipole

Configuration	Electric potential	Electric field
Single charge	r^{-1}	r^{-2}
Dipole	r^{-2}	r^{-3}
Quadrapole	r^{-3}	r^{-4}
Octopole	r^{-4}	r^{-5}

$2^l - poles$ $\quad$ $r^{-(l+1)}$ $\quad$ $r^{-(l+2)}$

l : number of independent displacement required to specify the configuration

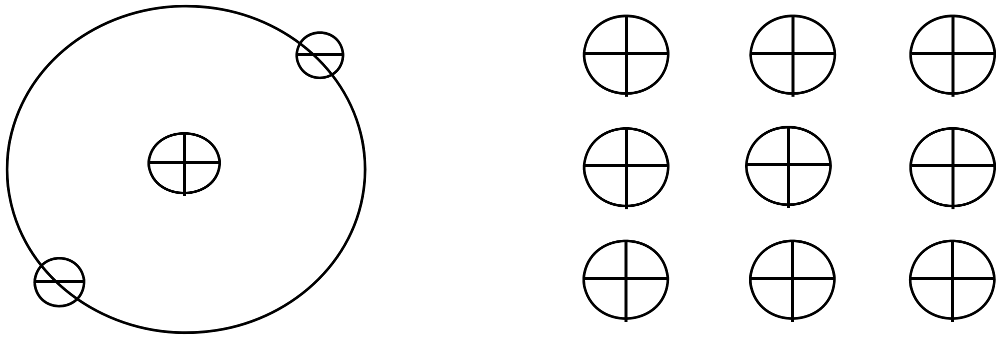
Electric potential due to a continuous distribution of charge

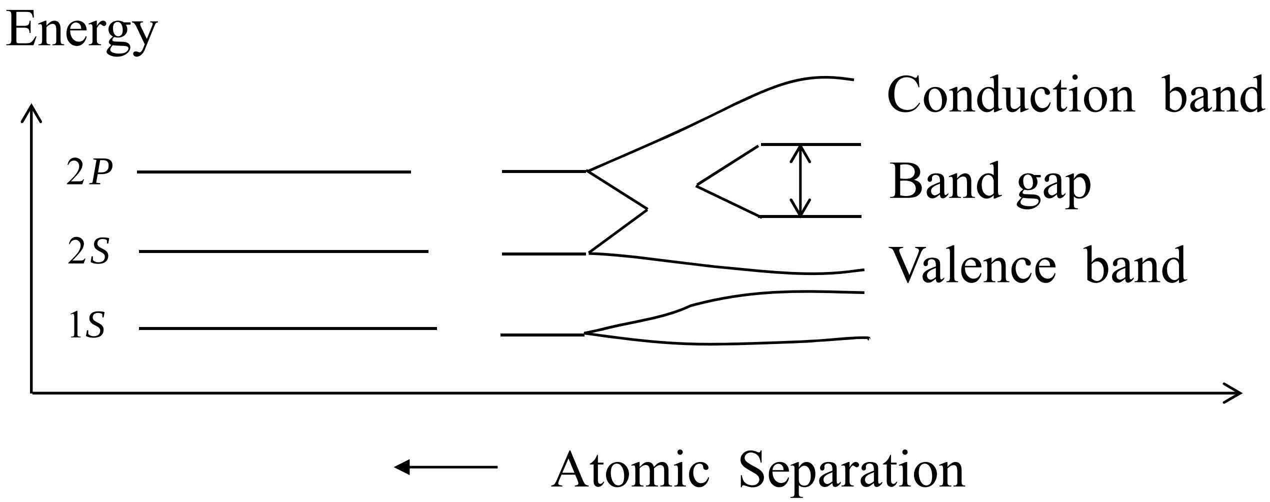
$$V = \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\rho}{R} dv'$$

3-6 Conductors in Static Electric Field

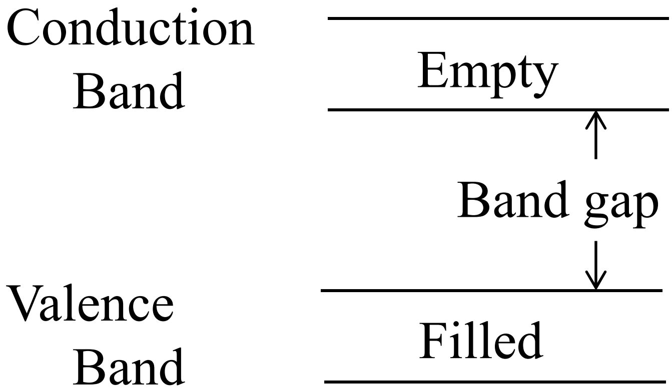
[Text p.100]

Individual atom atom in solid

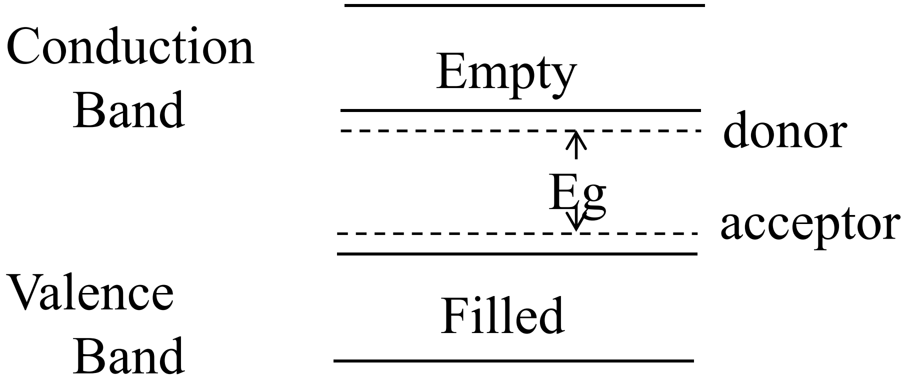




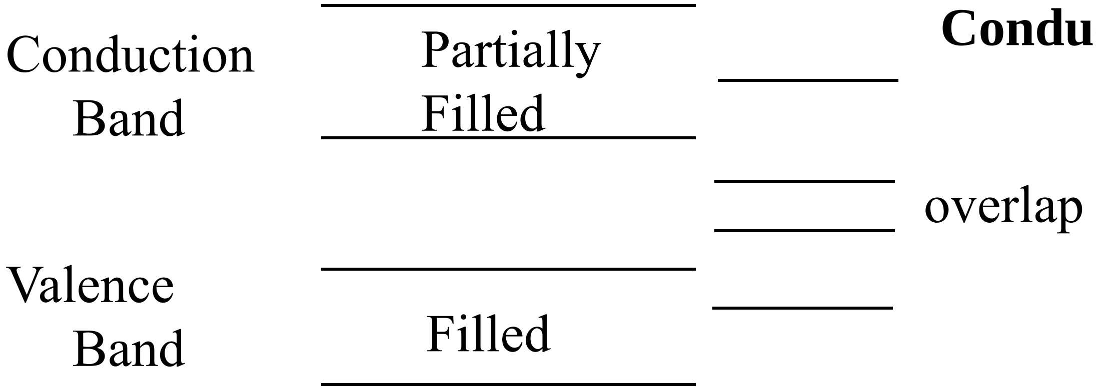
Insulator



Semiconductor



Conductor



Eg
- Diamond $5eV$
- Si $1.1eV$

$$\vec{j} = \sigma \vec{E}$$

σ : conductivity

Charge inside of conductor

⟶ $\vec{E}$

⟶ $\vec{F} = q\vec{E}$

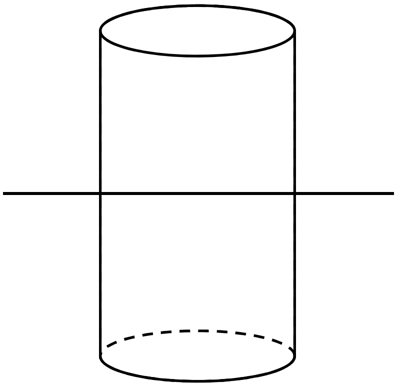
⟶ $\vec{E} = 0$, q ⟶ surface

⟶ No tangential component of $\vec{E}$ on surface

(if $\exists \; \vec{E}_{\tan}$, charge will move)

⟶ $\vec{E}$ is normal to surface

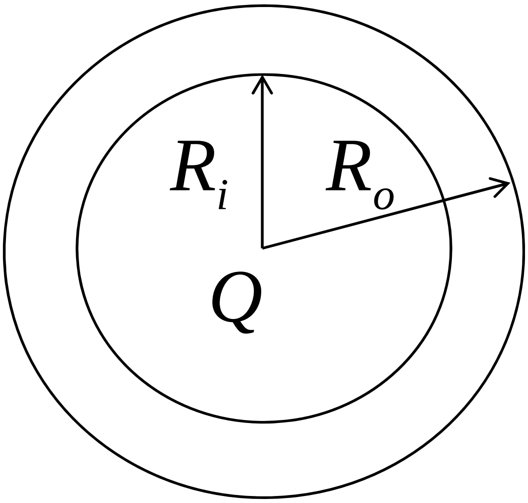
surface is an equipotential surface



$$\oint \vec{E} \cdot d\vec{S} = E_n \Delta S = \frac{\rho_S \Delta S}{\varepsilon_o}$$

$$E_n = \frac{\rho_S}{\varepsilon_o}$$ where ρ_S = surface charge density

(Ex 3-11)



For $R > R_o$

$$\int_S \overrightarrow{E} \cdot d\overrightarrow{S} = E \cdot 4\pi R^2 = \frac{Q}{\varepsilon_o}$$

$$\overrightarrow{E} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{r}$$

$$V = -\int_\infty^R \frac{QdR}{4\pi\varepsilon_o R^2} = \frac{Q}{4\pi\varepsilon_o R}$$

For $R_i < R < R_o$

$$E = 0$$

$$V = \frac{Q}{4\pi\varepsilon_o R_o}$$

For $R < R_i$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_o R^2}\hat{r}$$

$$V = \frac{Q}{4\pi\varepsilon_o R} + C$$

at $R = R_i$, $V = \frac{Q}{4\pi\varepsilon_o R_o}$ requires

$$\frac{Q}{4\pi\varepsilon_o R_i} + C = \frac{Q}{4\pi\varepsilon_o R_o}$$

$$C = \frac{Q}{4\pi\varepsilon_o R_o} - \frac{Q}{4\pi\varepsilon_o R_i}$$

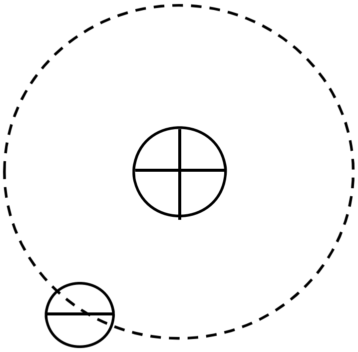
3-7 Dielectrics in Static Electric Field [Text p.105]

Dielectrics : insulation

Molecules in dielectrics

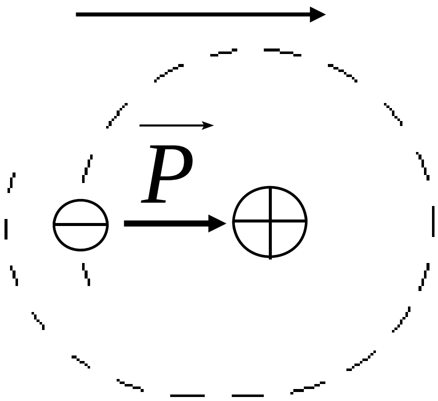
1. Electronic Polarization

$\vec{E} = 0$

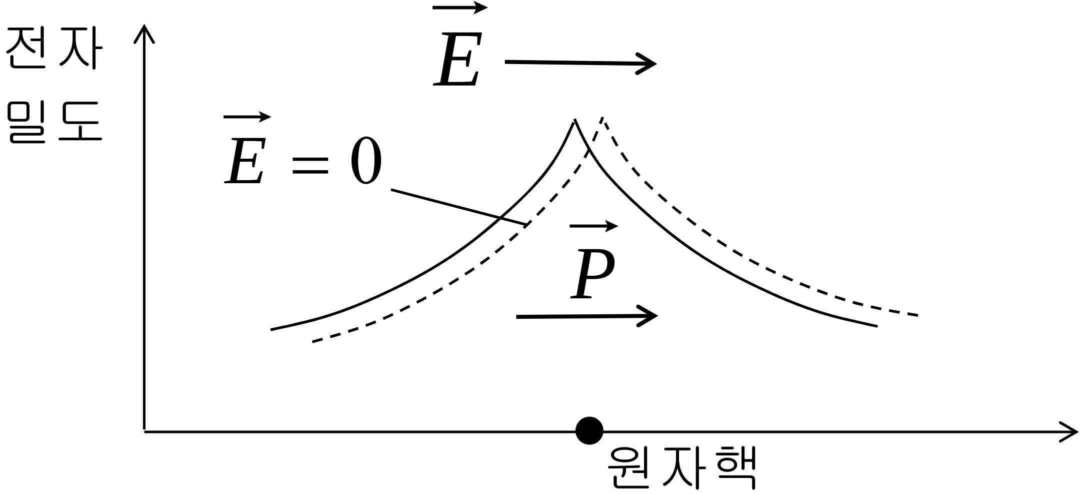


center of positive and negative charge coincide

$\vec{E} \neq 0$



center of positive and negative charge does not coincide and induces dipole moment



2. Ionic Polarization

이온결합 결정 ⟶ 정이온 부이온 역방향으로 이동

$\oplus \rightarrow \quad \leftarrow \ominus$ $\quad NaCl$

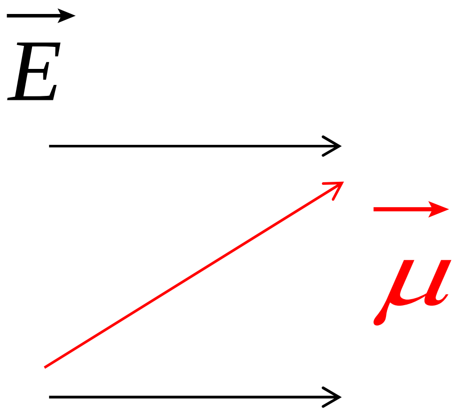
$\leftarrow \ominus \quad \oplus \rightarrow$

$\xrightarrow{\qquad} \quad \xrightarrow{\qquad}$

$\vec{E} \qquad \vec{P}$

3. Orientational Polarization

- Polar molecular : HCl permanent dipole moment
- Nonpolar molecular : O_2 , H_2



$\vec{\mu}$
- $\vec{E}$ 방향으로 정렬
- 열운동에 의한 교란

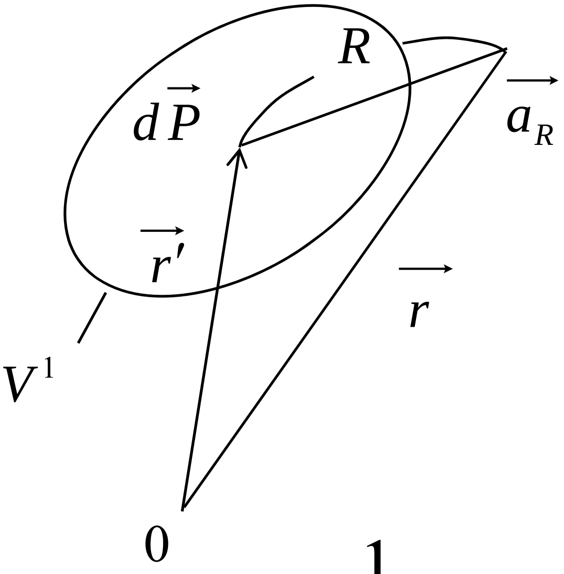
$$< \mu > = \frac{\mu^2 E}{3kT}$$

3-7.1 Equivalent charge distributions of polarized dielectrics [Text p.106]

Polarization vector

$$\overrightarrow{P} = \lim_{\Delta v \to 0} \frac{\Sigma \overrightarrow{P_k}}{\Delta v} \qquad (C / m^2)$$

: volume density of electric dipole moment potential due to a polarized dielectric



Potential due to $d\overrightarrow{P} = \overrightarrow{P} dv'$

$$dV = \frac{\overrightarrow{P} \cdot \hat{a}_R}{4\pi\varepsilon_o R^2} dv'$$

Then $V = \int_{V'} \frac{\overrightarrow{P} \cdot \hat{a}_R}{4\pi\varepsilon_o R^2} dv'$

Since $\frac{1}{R} = \frac{1}{\left|\overrightarrow{r} - \overrightarrow{r}'\right|}$

$$\nabla'\left(\frac{1}{R}\right) = \nabla' \frac{1}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|} = \frac{\overrightarrow{a_R}}{\left|\overrightarrow{r} - \overrightarrow{r'}\right|^2}$$

$$V = \int_{V'} \frac{1}{4\pi\varepsilon_o} \overrightarrow{P} \cdot \nabla' \left(\frac{1}{R} \right) dv'$$

$$\nabla' \cdot (f \overrightarrow{A}) = f\nabla' \cdot \overrightarrow{A} + \nabla' f \cdot \overrightarrow{A}$$

$$= \frac{1}{4\pi\varepsilon_o} \int \left[\nabla' \cdot \left(\frac{\overrightarrow{P}}{R} \right) - \frac{\nabla' \cdot \overrightarrow{P}}{R} \right] dv'$$

$$= \frac{1}{4\pi\varepsilon_o} \oint_{S'} \frac{\overrightarrow{P} \cdot \hat{a}'_n}{R} dS' + \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{-\nabla' \cdot \overrightarrow{P}}{R} dv'$$

$$= \frac{1}{4\pi\varepsilon_o} \oint_{S'} \frac{\rho_{PS}}{R} dS' + \frac{1}{4\pi\varepsilon_o} \int_{V'} \frac{\rho_P}{R} dv'$$

Where $\rho_{PS} = \overrightarrow{P} \cdot \hat{a}_n$: polarization (bound) surface charge density

$\rho_P = -\nabla \cdot \overrightarrow{P}$: polarization (bound) volume charge density

3-8 Electric Flux Density and Dielectric Constant [Text p.109]

Electric Field due to ρ in a dielectric

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_o} \rho_{total} = \frac{1}{\varepsilon_o} \left(\rho + \rho_P \right)$$

Since $\rho_P = - \nabla \cdot \vec{P}$,

$$\nabla \cdot (\varepsilon_o \vec{E} + \vec{P}) = \rho$$

$$\boxed{\nabla \cdot \vec{D} = \rho}$$

$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$: Electric flux density

Electric displacement

Integral form

$$\int_V \nabla \cdot \vec{D} \, dv = \oint_S \vec{D} \cdot d\vec{S} = Q$$

For a linear, isotropic dielectric

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

χ_e : electric susceptibility

Then

$$\overrightarrow{D} = \varepsilon_o (1 + \chi_e) \overrightarrow{E}$$

$$= \varepsilon_o \, \varepsilon_r \, \overrightarrow{E}$$

$$= \varepsilon \overrightarrow{E}$$

Where $\varepsilon_r = 1 + \chi_e$: relative permittivity
dielectric displacement

$\varepsilon = \varepsilon_o \varepsilon_r$: (absolute) permittvity

σ_b and σ_f

$$\vec{P} = \vec{D} - \varepsilon_o \vec{E} = \frac{\varepsilon_r - 1}{\varepsilon_r} \vec{D}$$

$$\nabla \cdot \vec{P} = \frac{\varepsilon_r - 1}{\varepsilon_r} \; \nabla \cdot D = \frac{\varepsilon_r - 1}{\varepsilon_r} \rho_f$$

$$-\rho_b = \frac{\varepsilon_r - 1}{\varepsilon_r} \rho_f$$

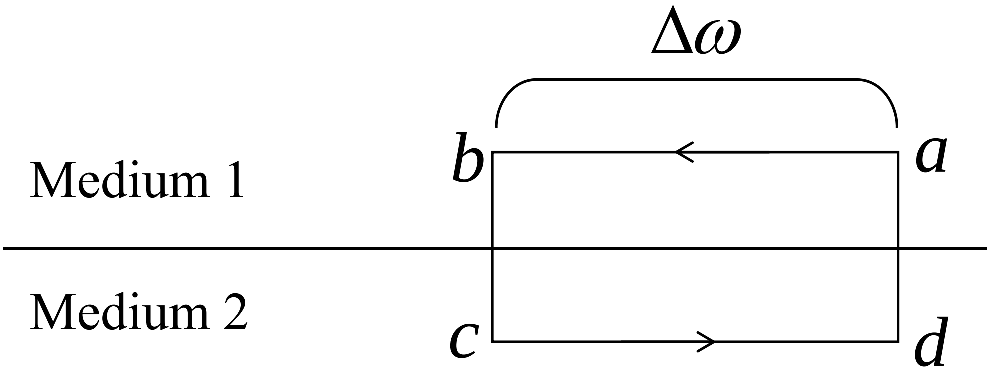
Total charge density

$$\rho_t = \rho_f + \rho_b = \rho_f - \nabla \cdot \vec{P} = \frac{\rho_f}{\varepsilon_r}$$

3-9 Boundary Condition for Electrostatic Fields

[Text p. 116]

charge of $\overrightarrow{E}$ and $\overrightarrow{D}$ across an interface between two media



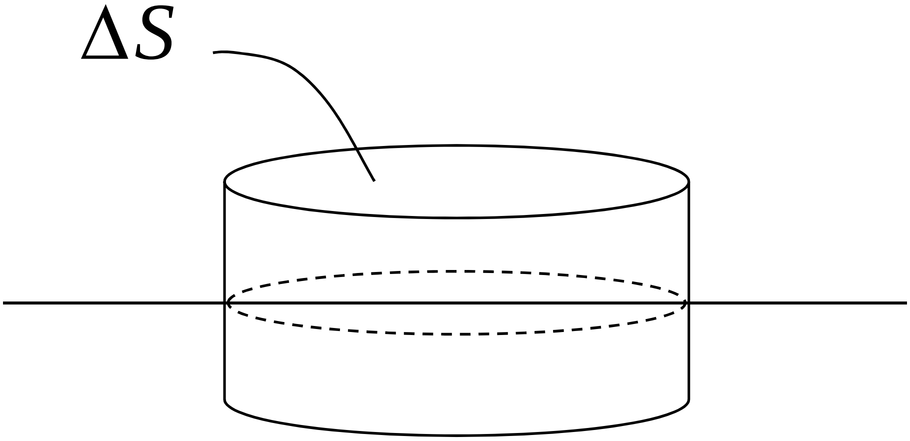
for path $abcda$,

$$\int \nabla \times \overrightarrow{E} \cdot d\overrightarrow{S}$$

$$= \oint \overrightarrow{E} \cdot d\,\overrightarrow{l}$$

$$= E_{1t}\Delta\omega - E_{2t}\Delta\omega = 0$$

$$\boxed{E_{1t} = E_{2t}}$$



$$\nabla \cdot \overrightarrow{D} = \rho$$

$$\int \nabla \cdot \overrightarrow{D} dv = \oint_S \overrightarrow{D} \cdot d\overrightarrow{S}$$

$$= \left(\overrightarrow{D_1} \cdot \hat{a}_{n2} + \overrightarrow{D_2} \cdot \hat{a}_{n1}\right)\Delta S$$

$$= \hat{a}_{n2} \cdot \left(\overrightarrow{D_1} - \overrightarrow{D_2}\right)\Delta S$$

$$= \rho_s \Delta S$$

$$\boxed{D_{1n} - D_{2n} = \rho_S}$$

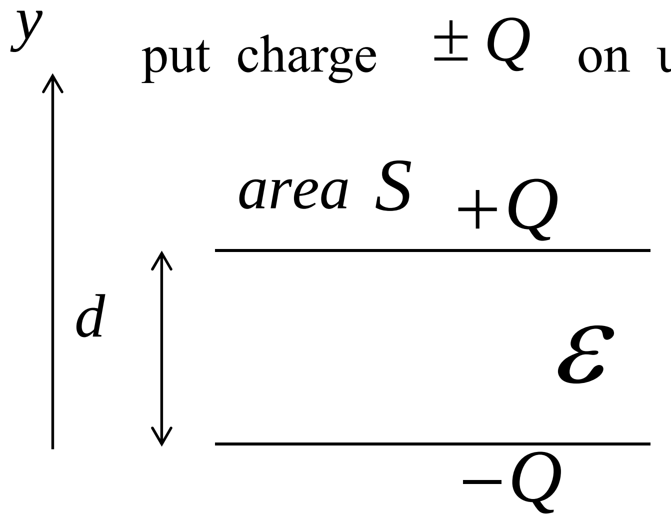
3-10 Capacitance and Capacitors [Text p.121]

Conductor in a Static Electric Field

$$C = \frac{Q}{V} \quad : \quad \text{capacitance}$$

(Ex 3-17) capacitance of a parallel-plate capacitor

put charge $\pm Q$ on upper and lower plate



$$\vec{E} = -\frac{Q}{\varepsilon S}\hat{y}$$

$$V_{12} = -\int_0^y \vec{E}\cdot d\vec{l} = \int_0^d \frac{Q}{\varepsilon S}dy = \frac{Q}{\varepsilon S}d$$

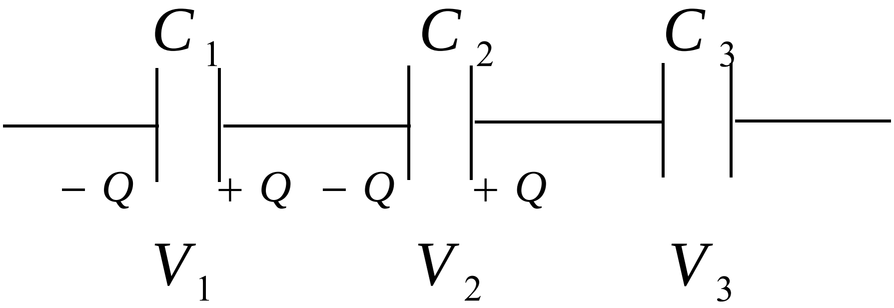
$$C = \frac{Q}{V_{12}} = \varepsilon\frac{S}{d}$$

(Ex 3-18) Cylindrical capacitor

(Ex 3-19) Spherical capacitor

3-10.1 Series and parallel connections of capacitors [Text p.126]

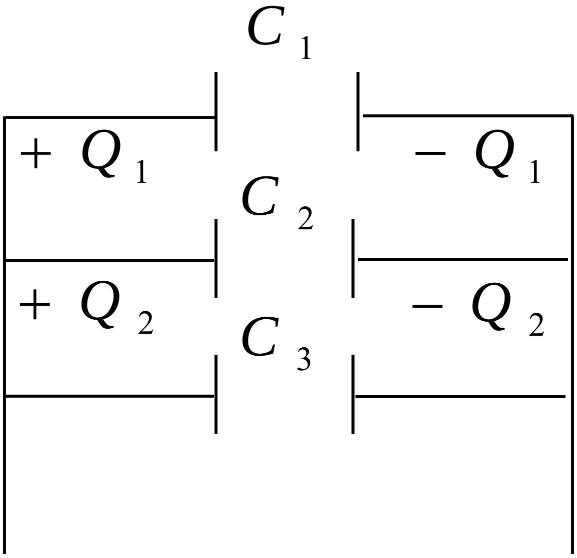
a. Series connection



$$V = V_1 + V_2 + V_3 \cdots\cdots = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \cdots\cdots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots\cdots$$

b. Parallel connection

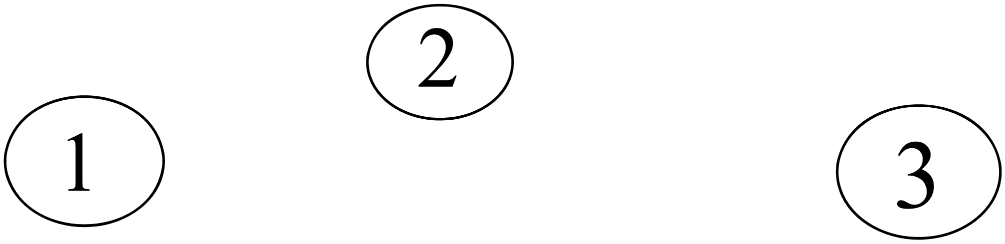


$$Q = Q_1 + Q_2 + \cdots$$

$$= C_1 V + C_2 V + C_3 V + \cdots$$

$$C = C_1 + C_2 + C_3 + \cdots$$

3-10.2 Capacitances in multi conductor sysem [Text p.129]



$$V_1 = P_{11}Q_1 + P_{12}Q_2 + \cdots\cdots \qquad + P_{1N}Q_N$$

$$\vdots$$

$$\vdots$$

$$V_N = P_{N1}Q_1 + P_{N2}Q_2 + \cdots\cdots \qquad + P_{NN}Q_N$$

$\Downarrow$ P_{ij} : Coefficients of potential

$$Q_1 = C_{11}V_1 + C_{12}V_2 + \cdots\cdots + C_{1N}V_N$$

$$\vdots$$

$$\vdots$$

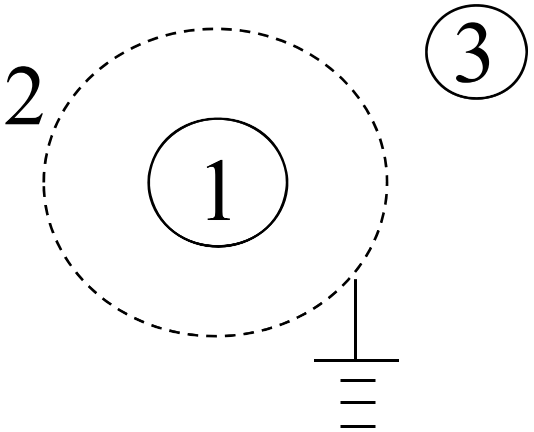
$$Q_N = C_{N1}V_1 + C_{N2}V_2 + \cdots\cdots + C_{NN}V_N$$

C_{ii} : Coefficients of capacitance

$C_{ij}\ (i \neq j)$: Coefficients of induction

3-10.3 Electrostatic Shielding [Text p.132]

Reduction of capacitive coupling between conductors ① and ③



⟶ enclose ① by
a ground conducting shell

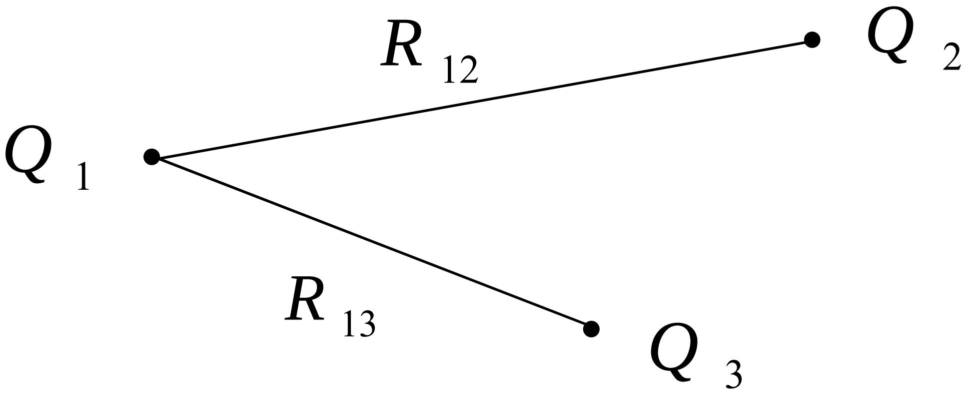
$$Q_1 = \left(C_{10} + C_{12} + C_{13}\right)V_1 - C_{12}V_2 - C_{13}V_3$$

With $V_2 = 0$,

$V_1 = V_2 = 0$ When $Q_1 = 0$ No field inside shell

$C_{13} = 0 \longrightarrow$ No change in Q_1 by change in V_3

3-11 Electrostatic Energy and Forces [Text p.133]



Bring charge Q_2 from ∞ to R_{12}

against the field of charge Q_1

work required is $W_2 = Q_2 V_2 = Q_2 \dfrac{Q_1}{4\pi\varepsilon_o R_{12}}$

Bring another charge Q_3 from ∞ to R_{13}

$$W_3 = Q_3 V_3 = Q_3 \left(\frac{Q_1}{4\pi\varepsilon_o R_{13}} + \frac{Q_2}{4\pi\varepsilon_o R_{12}} \right)$$

$$\vdots$$

Total potential energy

$$W = W_1 + W_2 + W_3 + \cdots\cdots$$

$$= \frac{Q_1}{4\pi\varepsilon_o}\left(0 + \frac{Q_2}{R_{12}} + \frac{Q_3}{R_{13}} + \cdots\cdots + \frac{Q_N}{R_{1N}}\right)$$

$$+ \frac{Q_2}{4\pi\varepsilon_o}\left(\qquad 0 + \frac{Q_3}{R_{23}} + \cdots\cdots + \frac{Q_N}{R_{2N}}\right) + \cdots\cdot$$

$$+ \frac{Q_N}{4\pi\varepsilon_o}\left(\qquad\qquad\qquad 0 \quad\right)$$

$$2W = \frac{Q_1}{4\pi\varepsilon_o}\left(0 + \frac{Q_2}{R_{12}} + \frac{Q_3}{R_{13}} + \cdots\cdots + \frac{Q_N}{R_{1N}}\right)$$

$$+ \frac{Q_2}{4\pi\varepsilon_o}\left(\frac{Q_1}{R_{21}} + 0 + \frac{Q_3}{R_{23}} + \cdots\cdots + \frac{Q_N}{R_{1N}}\right)$$

$$+ \cdots$$

$$= Q_1V_1 + Q_2V_2 + \cdots\cdots + Q_NV_N$$

$$\boxed{W = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k} \longrightarrow \boxed{\frac{1}{2}\int_V \rho V dv}$$

3-11.1 Electrostatic energy in terms of field quantities [Text p.137]

$$W_e = \frac{1}{2}\int \rho V dv$$

$$= \frac{1}{2}\int \left(\nabla \cdot \overrightarrow{D}\right) V dv$$

$$= \frac{1}{2}\int \left[\nabla \cdot \left(V \overrightarrow{D}\right) - \overrightarrow{D} \cdot \nabla V\right] dv$$

$$= \frac{1}{2}\oint_{S^1} V \overrightarrow{D} \cdot d\overrightarrow{S} + \frac{1}{2}\int \overrightarrow{D} \cdot \overrightarrow{E} dv$$

Since $V \propto R^{-1}$, $\overrightarrow{D} \propto R^{-2}$ and areas $S' \propto R^2$,

first term $\frac{1}{2}\oint_{S^1} V \overrightarrow{D} \cdot d\overrightarrow{S} \propto R^{-1}$ and

$\longrightarrow 0$ as $R \rightarrow \infty$

$$\boxed{W_e = \frac{1}{2}\int \overrightarrow{D} \cdot \overrightarrow{E} dv}$$

for a linear medium $\overrightarrow{D} = \varepsilon \overrightarrow{E}$ and

$$W_e = \frac{1}{2}\int \varepsilon E^2 dv \quad \text{or} \quad W_e = \frac{1}{2}\int \frac{D^2}{\varepsilon} dv$$

Electrostatic energy density

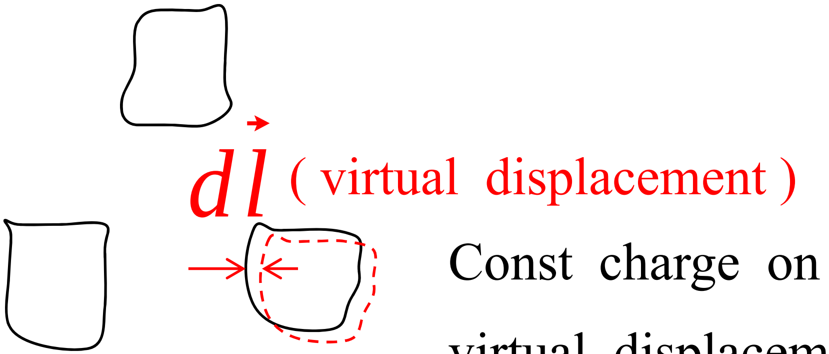
$$W_e = \frac{1}{2}\overrightarrow{D} \cdot \overrightarrow{E}$$

or $W_e = \frac{1}{2}\varepsilon E^2$

or $W_e = \frac{1}{2}\frac{D^2}{\varepsilon} \qquad (J / m^3)$

3-11.2 Electrostatic forces [Text p.140]

A. Sg of bodies with fixed charges



Const charge on each body

virtual displacement of one body

mechanical work done by system.

$$dW = \vec{F}_Q \cdot d\vec{l}$$

For an isolated system (Conservation of energy) $0 = dW + \underbrace{dW_e}_{\text{Change in electrostatic energy}}$

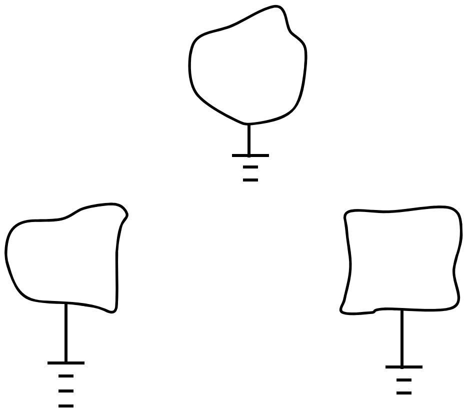
Thus

$$\vec{F}_Q \cdot d\vec{l} = -dW_e = -(\nabla W_e) \cdot d\vec{l}$$

and

$$\boxed{\vec{F}_Q = -\nabla W_e}$$

B. Sg of bodies with fixed potentials



Fixed potentials for each of bodies

virtual displacement of one body

Charge must be supplied to keep

The potential constant.

work done by the source | mechanical work done by system

$$dW_S = \Sigma V_k dQ_k \qquad dW = \vec{F}_V \cdot d\vec{l}$$

charge in electrostatic energy

$$dW_e = \frac{1}{2}\Sigma V_k dQ_k$$

conservation of energy for an isolated sys

$$dW_s = dW + dW_e$$

Thus $\vec{F}_V \cdot d\vec{l} = dW_e = \nabla W_e \cdot d\vec{l}$

$$\boxed{\vec{F}_V = \nabla W_e}$$

Chap 4. Solution of Electrostatic Problems

4-2 Poisson's and Laplace's Equations

[Text p.152]

since $$\nabla \cdot \vec{D} = \rho$$

using $$\vec{D} = \varepsilon \vec{E} = -\varepsilon \nabla V$$

$$\nabla \cdot (\varepsilon \nabla V) = -\rho$$

for a homogeneous medium

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$$

Poisson's eq.

$$\nabla^2 \equiv \nabla \cdot \nabla$$ Laplacian

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ in cartesian coord.

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$ in cylindrical coord.

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)$$
$$+ \frac{1}{R^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$ in spherical coord.

When $\rho = 0$ (no free charge)

$$\boxed{\nabla^2 V = 0}$$ Laplace's equation

4-3 Uniqueness of Electrostatic Solutions

[Text p.157]

uniqueness theorem :

A solution of Poisson's eq (or Laplace's eq) that satisfies the given boundary condition is a unique solution.

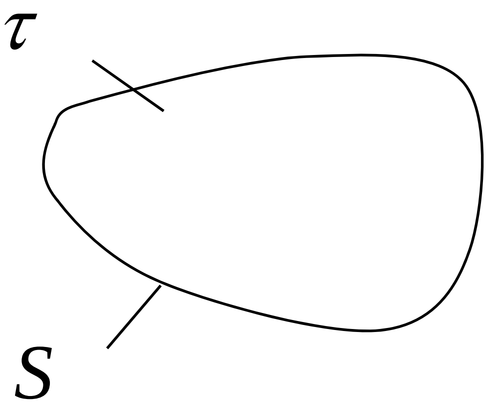
pf. Let V_1 and V_2 are two solutions of Poisson's equation.

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon}$$

$$\nabla^2 V_2 = -\frac{\rho}{\varepsilon}$$

Let $$V = V_1 - V_2$$

Then $$\nabla^2 V = 0$$



Consider,

$$\int_{\tau} \nabla \cdot (V\nabla V)dv = \int_{\tau}\left[\nabla V \cdot \nabla V + V\nabla^{2}V\right]dv$$

$$= \oint_{S}(V\nabla V)\cdot d\vec{S}$$

as $R \rightarrow \infty,$ surface integral $\rightarrow 0$

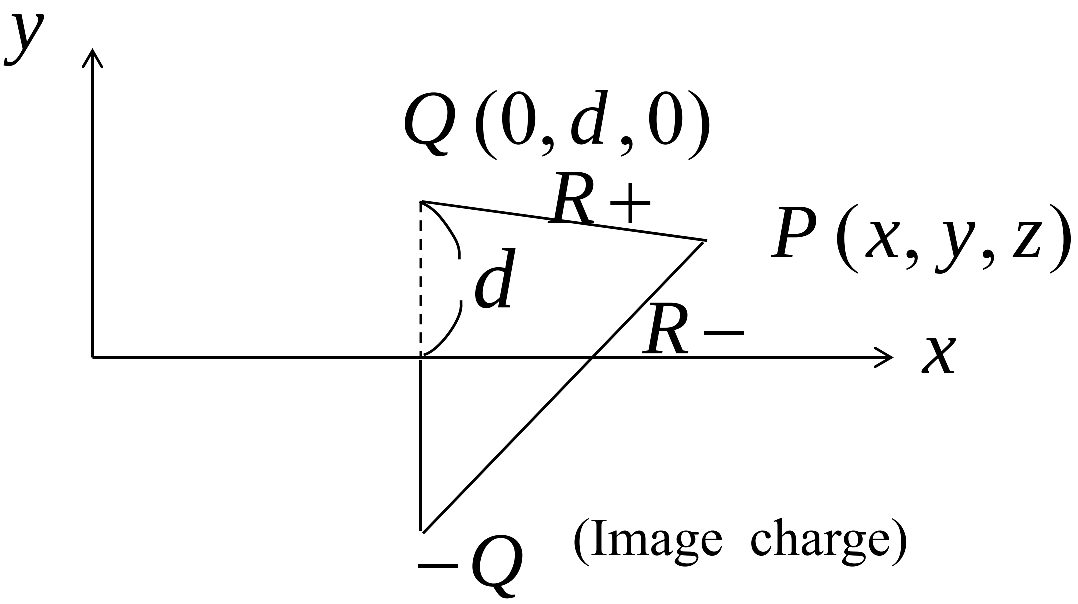
$$(V \propto R^{-1},\ \nabla V \propto R^{-2},\ S \rightarrow R^{2})$$

hence $\int \left|\nabla V\right|^{2} dv = 0$

and $\left|\nabla V\right| = 0 \ \rightarrow\ V = const$

4-4 Method of Images [Text p.159]

4-4.1 point charge and conducting planes [Text p.161]



Electric field intensity in $y > 0$ region.

$$\nabla^2 V = -\frac{Q\,\delta\left(\vec{r}-\vec{r}\,'\right)}{\varepsilon} \text{ where } \vec{r}\,' = (0, d, 0)$$

B.C. $V = 0$ at $y = 0$

$V = 0$ at $y \to \infty$

Consider an image charge $-Q$ at $(0,-d,0)$

$$V=\frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{R_+}-\frac{1}{R_-}\right)$$

where $R_+=\left[x^2+(y-d)^2+z^2\right]^{\frac{1}{2}}$

$$R_-=\left[x^2+(y+d)^2+z^2\right]^{\frac{1}{2}}$$

Electric field intensity in $y\geq 0$ region

$$\vec{E}=-\nabla V$$

Induced surface charge density

$$\sigma = \varepsilon_o E_n = \varepsilon_o E_y\,(x,0,z)$$

$$= \varepsilon_o \cdot \frac{Q}{4\pi\varepsilon_o}\left[\frac{y-d}{\left[x^2+(y-d)^2+z^2\right]^{\frac{3}{2}}} - \frac{y+d}{\left[x^2+(y+d)^2+z^2\right]^{\frac{3}{2}}}\right]_{x,0,z}$$

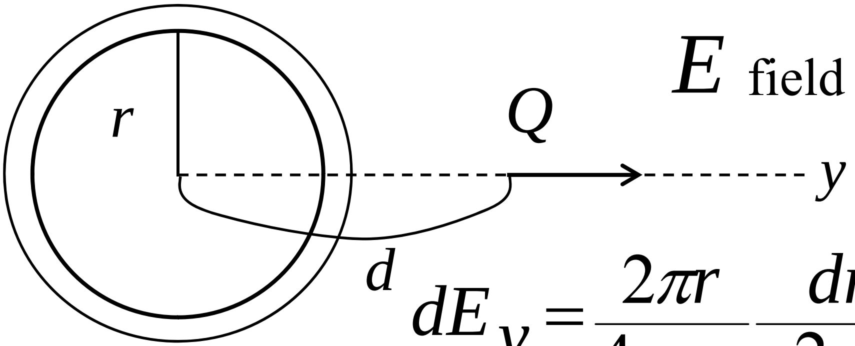
$$= -\frac{Qd}{2\pi}\frac{1}{(x^2+z^2+d^2)^{3/2}} = -\frac{Qd}{2\pi}\frac{1}{(r^2+d^2)^{3/2}}$$

Total induced charge

$$q = \int \sigma\, ds = -\frac{Qd}{2\pi} \int_0^\infty \frac{2\pi r dr}{\left(r^2+d^2\right)^{\frac{3}{2}}} = -Qd\left[\frac{-1}{\left(r^2+d^2\right)^{\frac{1}{2}}}\right]_0^\infty$$

$$= -Q$$

Force between Q and plane



E field by a charged ring

$$dE_y = \frac{2\pi r}{4\pi\varepsilon_o}\frac{dr\,\sigma}{r^2+d^2}\cdot\frac{d}{\left(r^2+d^2\right)^{1/2}}$$

Total field by the whole plane

$$E_y = \int dE_y$$

$$= \frac{d}{2\varepsilon_o}\int_0^\infty \frac{rdr}{\left(r^2+d^2\right)^{\frac{3}{2}}}\cdot\frac{-Qd}{2\pi}\frac{1}{\left(r^2+d^2\right)^{\frac{3}{2}}}$$

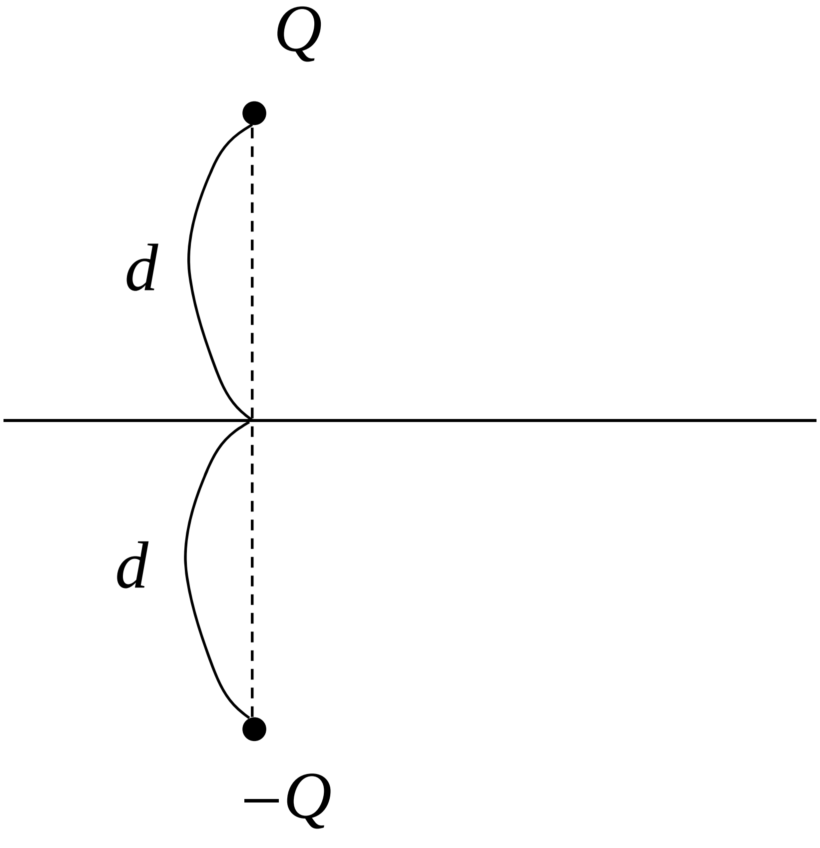
$$= -\frac{Qd^2}{4\pi\varepsilon_o}\int_0^\infty \frac{rdr}{\left(r^2+d^2\right)^3}$$

$$= -\frac{Qd^2}{4\pi\varepsilon_o}\cdot\left.\frac{-\frac{1}{4}}{\left(r^2+d^2\right)^2}\right]_0^\infty$$

$$= \frac{-Q}{16\pi\varepsilon_o d^2}$$

Total force

$$F = QE_y = -\frac{Q^2}{16\pi\varepsilon_o d^2} = -\frac{Q^2}{4\pi\varepsilon_o (2d)^2}$$



(Ex) Line charge and parallel conducting cylinder

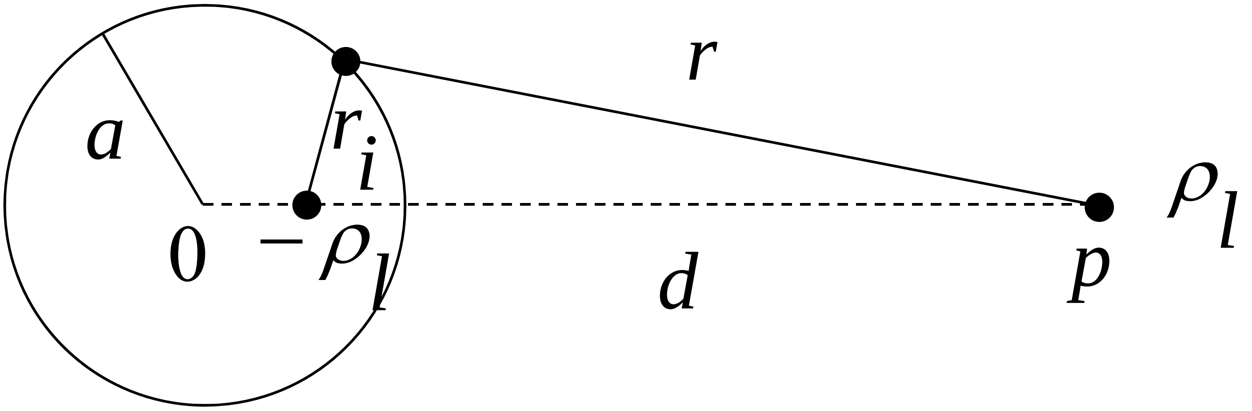
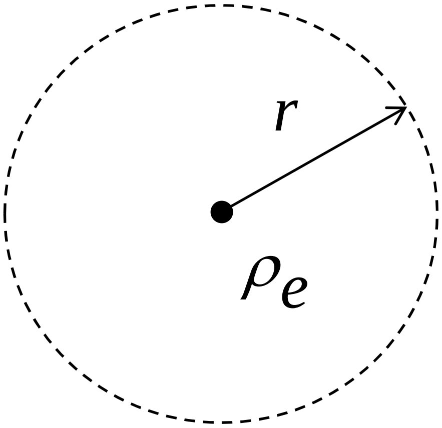


Image line charge : $-\rho_l$ between 0 and p

potential by a line charge



$$E = \frac{\rho_l}{2\pi r \varepsilon_o}$$

$$V = -\int_{r_o}^{r} E \, dr$$

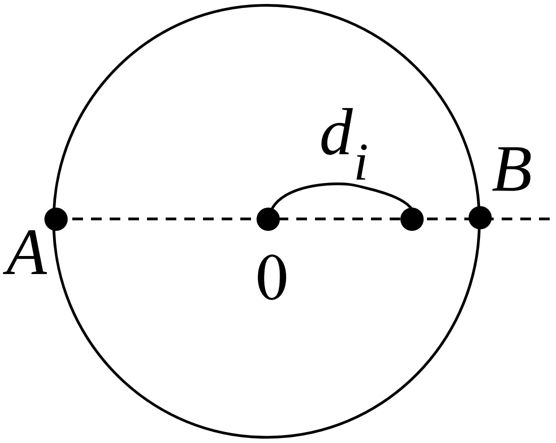
$$= -\frac{\rho_e}{2\pi\varepsilon_o} \ln \frac{r}{r_o}$$

Then potential on the cylinder surface

$$V_M = \frac{\rho_e}{2\pi\varepsilon_o}\left[\ln\frac{r_o}{r} - \ln\frac{r_o}{r_i}\right]$$

$$= \frac{\rho_e}{2\pi\varepsilon_o}\ln\frac{r_i}{r}$$

since it hold on the whole surface

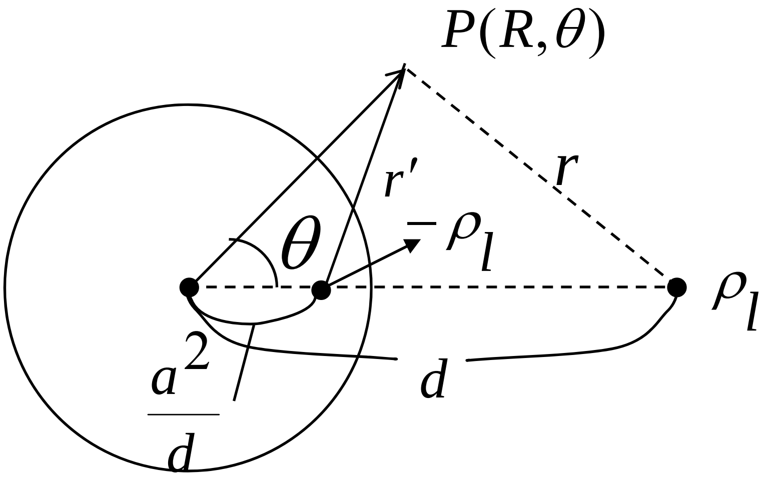


at $A, \quad r_i = a + d_i \quad , \qquad r = d + a$

at $B, \quad r_i = a - d_i \quad , \qquad r = d - a$

$$\frac{a + d_i}{d + a} = \frac{a - d_i}{d - a} \qquad d_i = \frac{a^2}{d}$$

Induced charge distribution on cylinder surface



$$r=\sqrt{d^2+R^2-2dR\cos\theta}$$

$$r'=\sqrt{\left(\frac{a^2}{d}\right)^2+R^2-2\cdot\frac{Ra^2}{d}\cos\theta}$$

$$V(R,\theta)=-\frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{\left(\frac{a^2}{d}\right)^2+R^2-2\cdot\frac{Ra^2}{d}\cos\theta}{d^2+R^2-2dR\cos\theta}\right)^{\frac{1}{2}}$$

$$\vec{E}(a,\theta)=-\frac{\rho_l}{4\pi\varepsilon_o}\hat{r}\left[\frac{2R-2\frac{a^2}{d}\cos\theta}{\left(\left(\frac{a^2}{d}\right)^2+R^2-2\cdot\frac{Ra^2}{d}\cos\theta\right)}-\frac{2R-2d\cos\theta}{\left(d^2+R^2-2dR\cos\theta\right)}\right]$$

$$+\frac{\rho_l}{4\pi\varepsilon_o}\frac{1}{R}\hat{\theta}\left[\frac{2\frac{Ra^2}{d}\sin\theta}{\left(\frac{a^2}{d}\right)^2+R^2-2\frac{Ra^2}{d}\cos\theta}-\frac{2dR\sin\theta}{d^2+R^2-2dR\cos\theta}\right]$$

$$=-\frac{\rho_l}{4\pi\varepsilon_o}\left[\hat{r}\,2\frac{\frac{d^2}{a^2}\left(a-\frac{a^2}{d}\cos\theta\right)-a+d\cos\theta}{d^2+a^2-2da\cos\theta}\right.$$

$$\left.+\frac{2\hat{\theta}}{a}\frac{\left(\cancel{\frac{d^2}{a^2}\cdot\frac{a^3}{d}\sin\theta-da}\sin\theta\right)}{d^2+a^2-2da\cos\theta}\right]$$

$$= -\frac{\rho_l}{2\pi\varepsilon_o} \frac{\frac{d^2}{a} - a}{d^2 + a^2 - 2da\cos\theta} \hat{r}$$

$$\sigma = +\varepsilon_o E_r\left(a, \theta\right) = -\frac{\rho_l}{2\pi} \frac{d^2 - a^2}{a\left(d^2 + a - 2da\cos\theta\right)}$$

Total induced charge / length

$$\int_0^{2\pi} \sigma\, a d\theta = -\frac{\rho_l (d^2 - a^2)}{2\pi} \int_0^{2\pi} \frac{d\theta}{d^2 + a^2 - 2da\cos\theta}$$

$$= -\frac{\rho_l (d^2 - a^2)}{2\pi} \cdot \frac{2\pi}{d^2 - a^2}$$

$$= -\rho_l$$

4-4.2 Line charge and parallel conducting cylinder [Text p.162]

- Potential by a line charge

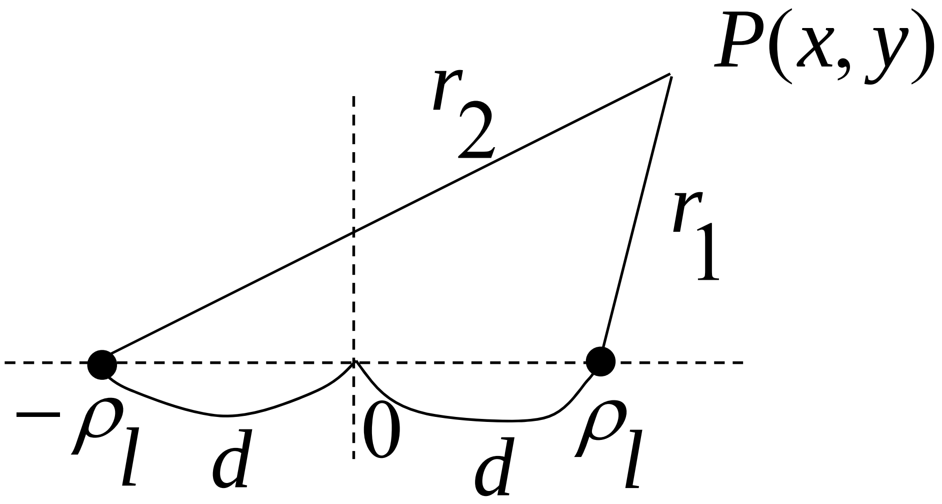
● ρ_l

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_o r}\hat{r}$$

$$V = -\int_{r_o}^{r} \vec{E} \cdot d\vec{r}$$

$$= -\frac{\rho_l}{2\pi\varepsilon_o} \ln \frac{r}{r_o}$$

Potential by a pair of line charges ρ_l & $-\rho_l$



$$V(x, y) = -\frac{\rho_l}{2\pi\varepsilon_o} \ln \frac{r_1}{r_2}$$

Equipotential surface ⟶ $\frac{r_1}{r_2} = const\ C$

1. when $C = 1$, equipotential surface
→ midplane

2. for an arbitrary C ,

since

$$r_1 = \left[\left(x-d \right)^2 + y^2 \right]^{\frac{1}{2}}$$

$$r_2 = \left[\left(x+d \right)^2 + y^2 \right]^{\frac{1}{2}}$$

$$\frac{r_1}{r_2} = \left[\frac{\left(x-d \right)^2 + y^2}{\left(x+d \right)^2 + y^2} \right]^{\frac{1}{2}} = C$$

$$\left(c^2-1\right)x^2+2dx\left(c^2+1\right)+\left(c^2-1\right)d^2+\left(c^2-1\right)y^2=0$$

$$\left(x+\frac{c^2+1}{c^2-1}d\right)^2+y^2=\left(\frac{c^2+1}{c^2-1}d\right)^2-d^2$$

$$=\frac{\left(c^2+1\right)^2-\left(c^2-1\right)^2}{\left(c^2-1\right)^2}d^2$$

$$=\frac{4c^2}{\left(c^2-1\right)^2}d^2$$

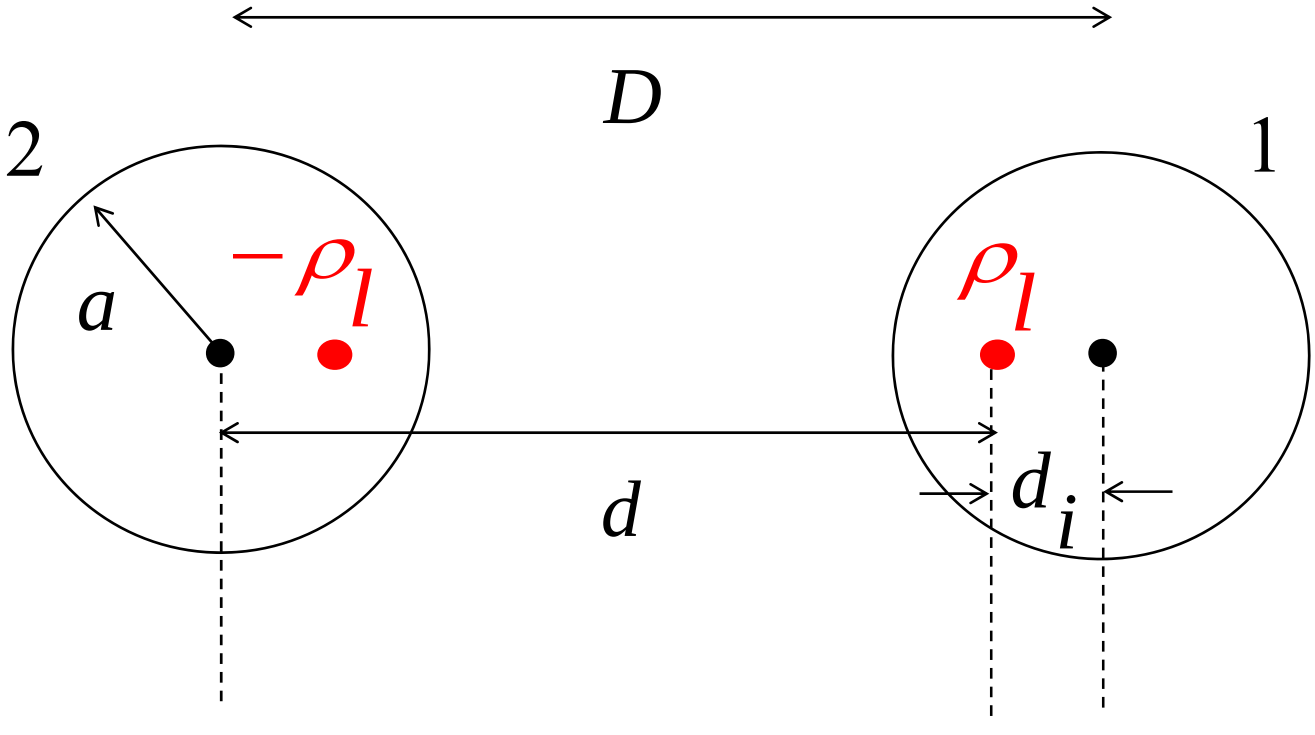
중심이 $\left(-\frac{c^2+1}{c^2-1}d\ ,\ \ 0\right)$

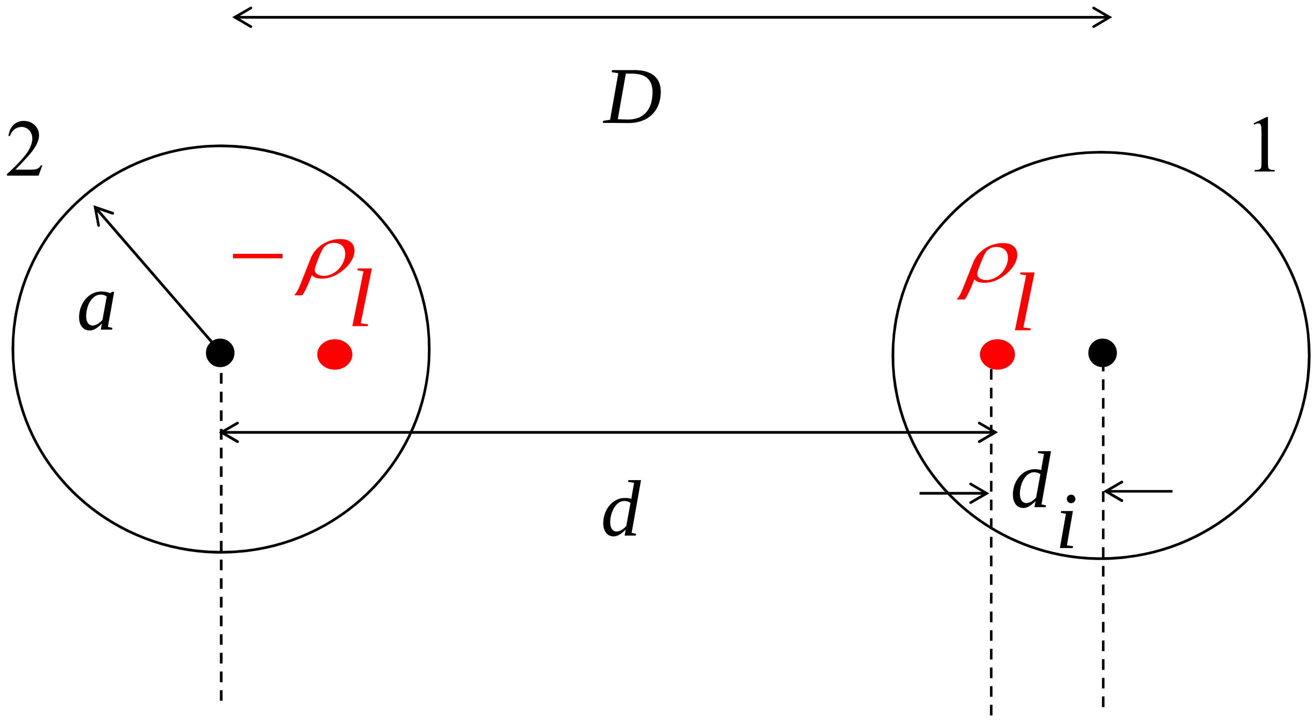
반경이 $\frac{2cd}{c^2-1}$ 인 원

(Ex) Long cylinder (ρ_l charge / length) near a conducting plane

⟶ Image charges : two line charges

(Ex) Capacitance between two long, parallel, circular conducting wires of radius a





Two equipotential surfaces are generated by

ρ_l and $-\rho_l$ separated by $D-2d_i = d-d_i$

$$V_2 = \frac{\rho_l}{2\pi\varepsilon_o}\ln\frac{a}{d}$$

$$V_1 = \frac{-\rho_l}{2\pi\varepsilon_o}\ln\frac{a}{d}$$

Capacitance $$c = \frac{\rho_l}{V_1 - V_2} = \frac{\pi\varepsilon_o}{\ln\left(d/a\right)}$$

since $$d = D - d_i = D - \frac{a^2}{d}$$

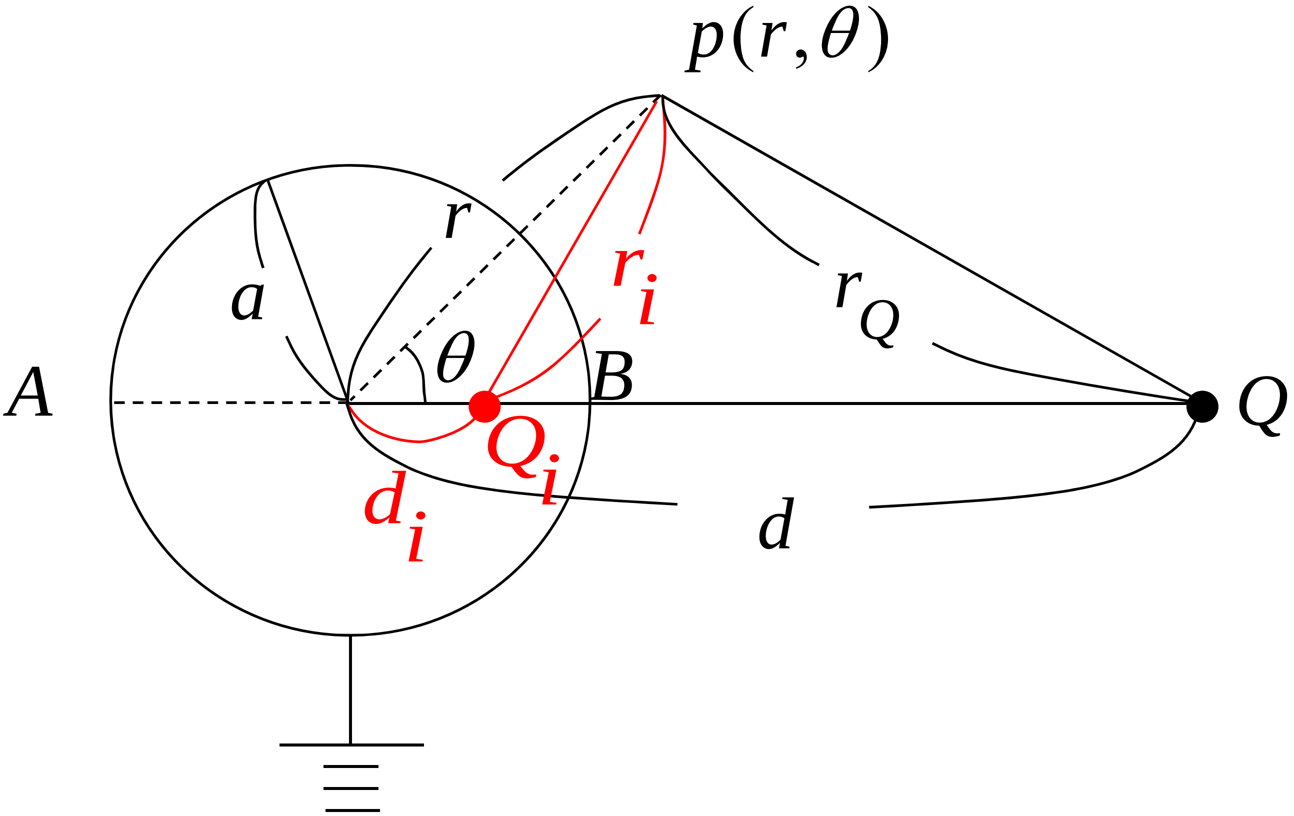
$$d = \frac{1}{2}\left(D + \sqrt{D^2 - 4a^2}\right)$$

$$c = \frac{\pi\varepsilon_o}{\ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right]}$$

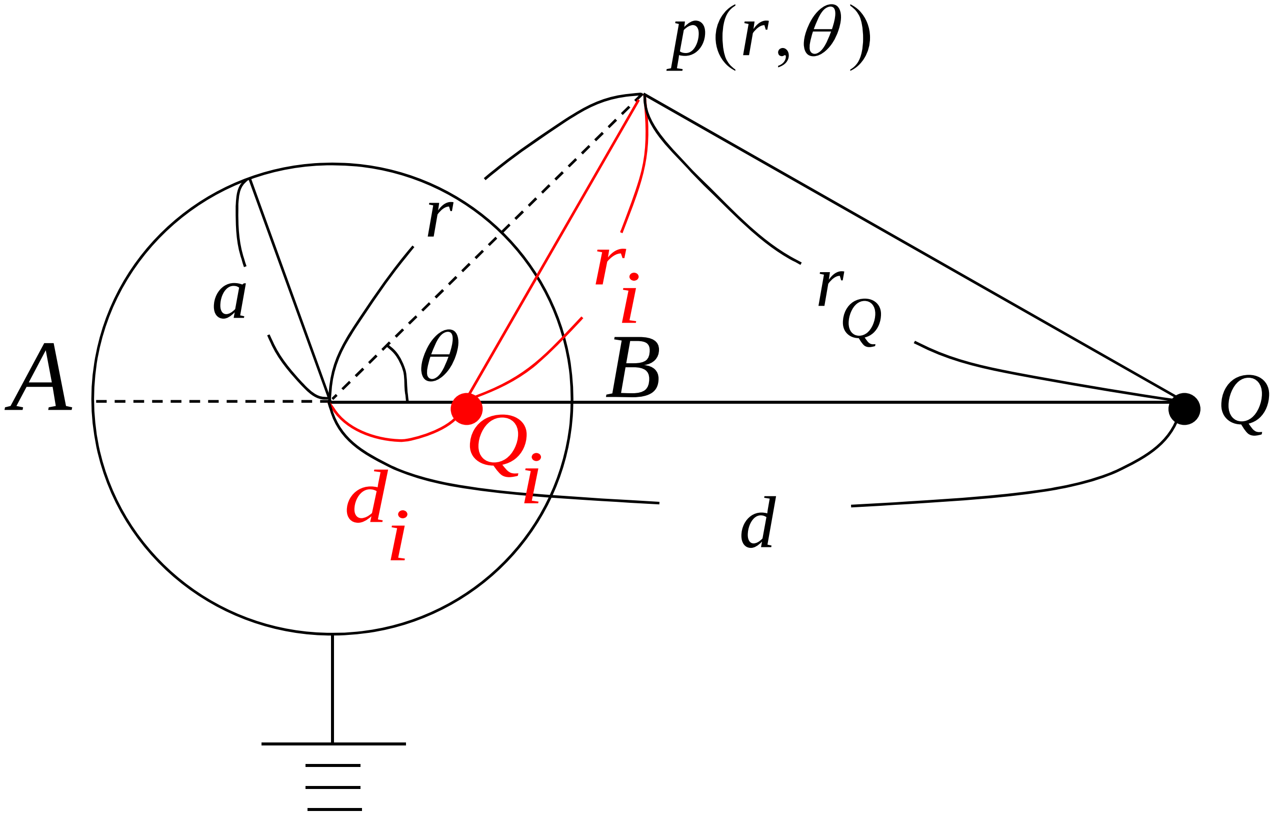
$$= \frac{\pi\varepsilon_o}{\cosh^{-1}\left(\frac{D}{2a}\right)}$$

4-4-3 Point charge and conducting sphere

(grounded)



$$V(r,\theta)=\frac{1}{4\pi\varepsilon_\circ}\left(\frac{Q}{r_Q}+\frac{Q_i}{r_i}\right)$$



on the surface

at A ,
$$0=\frac{Q}{a+d}+\frac{Q_i}{a+d_i} \qquad 1.$$

at B ,
$$0=\frac{Q}{d-a}+\frac{Q_i}{a-d_i} \qquad 2.$$

from 1, 2

$$Q_i = \frac{-a+d_i}{a+d}Q \ , \qquad 0 = \frac{Q}{d-a} + \frac{-1}{a-d_i}\cdot\frac{a+d_i}{a+d}Q$$

$$Q_i = -\frac{a+\frac{a^2}{d}}{a+d}Q = -\frac{a(\cancel{d+a})}{d(\cancel{a+d})}Q = -\frac{a}{d}Q$$

$$(d-a)(a+d_i) = (a-d_i)(a+d)$$

$$\cancel{ad} - a^2 + dd_i - \cancel{ad_i} = a^2 - \cancel{ad_i} + \cancel{ad} - dd_i$$

$$dd_i = a^2$$

$$d_i = \frac{a^2}{d}$$

Induced surface charge density

$$\sigma = \varepsilon_o E_r (r=a) = -\varepsilon_o \frac{\partial V}{\partial r}(r=a)$$

$$= -\varepsilon_o \cdot \frac{1}{4\pi\varepsilon_o} \frac{\partial}{\partial r}\left[\frac{Q}{\sqrt{r^2+d^2-2rd\cos\theta}} \right.$$

$$\left. + \frac{-\frac{a}{d}Q}{\sqrt{r^2+\left(\frac{a^2}{d}\right)^2 - 2r\cdot\frac{a^2}{d}\cos\theta}} \right]_{r=a}$$

$$=\frac{-Q}{4\pi}\left[\frac{-(r-d\cos\theta)}{\left(r^2+d^2-2rd\cos\theta\right)^{\frac{3}{2}}}+\frac{+\frac{a}{d}\left(r-\frac{a^2}{d}\cos\theta\right)}{\left(r^2+\left(\frac{a^2}{d}\right)^2-2r\cdot\frac{a^2}{d}\cos\theta\right)^{\frac{3}{2}}}\right]_{r=a}$$

$$=\frac{-Q}{4\pi}\frac{-a+d\cos\theta+\frac{a}{d}\left(a-\frac{a^2}{d}\cos\theta\right)\frac{d^3}{a^3}}{\left(a^2+d^2-2ad\cos\theta\right)^{\frac{3}{2}}}$$

$$=\frac{-Q}{4\pi}\frac{-a+\frac{d^2}{a}}{\left(a^2+d^2-2ad\cos\theta\right)^{\frac{3}{2}}}$$

$$=\frac{Q}{4\pi a}\frac{a^2-d^2}{\left(a^2+d^2-2ad\cos\theta\right)^{\frac{3}{2}}}$$

Total induced charge

$$Q_{Ind} = \int \sigma da = \int \sigma\, a^{2} \sin\theta d\theta d\phi$$

$$= \frac{Q\cancel{a^{2}}\left(a^{2}-d^{2}\right)}{4\pi\cancel{a}}\int_{0}^{\pi}\frac{2\pi\sin\theta d\theta}{\left(a^{2}+d^{2}-2ad\cos\theta\right)^{\frac{3}{2}}}$$

$$= \frac{Q\left(a^{2}-d^{2}\right)\cancel{a}}{2}\left.\frac{\frac{-1}{\cancel{a}d}}{\left(a^{2}+d^{2}-2ad\cos\theta\right)^{\frac{1}{2}}}\right]_{0}^{\pi}$$

$$=-\frac{Q(a^2-d^2)}{2d}\left(\frac{1}{\left(a^2+d^2+2ad\right)^{\frac{1}{2}}}-\frac{1}{\left(a^2+d^2-2ad\right)^{\frac{1}{2}}}\right)$$

$$=-\frac{Q(a^2-d^2)}{2d}\left(\frac{1}{a+d}-\frac{1}{d-a}\right)$$

$$=-\frac{Q(a^2-d^2)}{2d}\frac{-2a}{d^2-a^2}$$

$=-\frac{a}{d}Q$ ⟶ equal to image charge

Force between sphere and charge

$$F=\frac{1}{4\pi\varepsilon_{\circ}}\frac{Q\cdot\left(-\frac{a}{d}Q\right)}{\left(d-\frac{a^{2}}{d}\right)^{2}}$$

인 력

(problem 1)
point charge and insulated conducting sphere

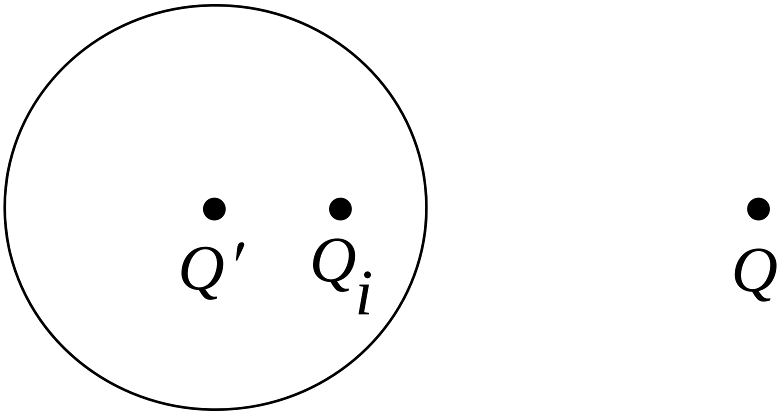
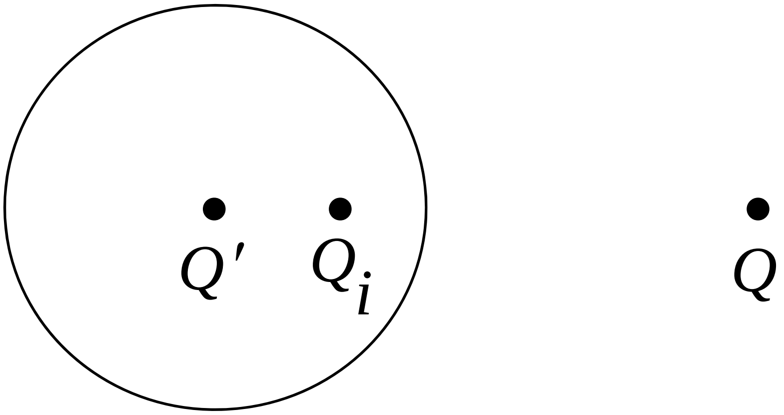


Image charge at $d_i = \frac{a^2}{d}$, $Q_i = -\frac{a}{d}Q$

$\longrightarrow$ Q and Q_i will make the surface

An equipotential surface



⟶ Surface will have a charge of

$$Q_i = -\frac{a}{d}Q$$

⟶ need to add $Q' = \frac{a}{d}Q$ to the surface to make surface neutral

⟶ Q' Will distribute evenly on the surface

⟶ equivalent to Q' at the center

(problem 2)

pt charge (Q) charged (Q') conducting sphere

Q, Image charge $Q_i = -\frac{a}{d}Q$ at $d_i = \frac{a^2}{d}$,

$Q'' = Q' - Q_i$ at the center

(problem 3)

pt charge near a conducting sphere at fixed potential V

Q, Image charge

$Q_i = -\frac{a}{d}Q$ at $d_i = -\frac{a^2}{d}$,

$Q' = 4\pi\varepsilon_0 Va$ at the center

4-5 Boundary Value Problems in Cartesian Coordinates [Text p.174]

Eqs $$-\nabla^2 V = 0$$

B.C.

1. V is given on boundary
   : Dirichlet coordinate
2. $\dfrac{\partial V}{\partial n}$ is given on boundary
   : Neumann coordinate
3. V and $\dfrac{\partial V}{\partial n}$ is given on boundary
   : Cauchy coordinate

⟶ over specification

V is given over some boundaries

$\dfrac{\partial V}{\partial n}$ is given over remaining boundaries

⟶ Mixed boundary-value problems

Laplace eq in Cartesian coord

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V(x,y,z)=0$$

assume $V(x,y,z)=X(x)\,Y(y)\,Z(z)$

: separation of variables

$$\frac{1}{X}\frac{d^2}{dx^2}X+\frac{1}{Y}\frac{d^2}{dy^2}Y+\frac{1}{Z}\frac{d^2}{dz^2}Z=0$$

$$\frac{1}{X}\frac{d^2}{dx^2}X=-\left(\frac{1}{Y}\frac{d^2}{dy^2}Y+\frac{1}{Z}\frac{d^2}{dz^2}Z\right)=-k_x{}^2$$

$$\frac{d^2}{dx^2}X+k_x{}^2X=0$$

Possible solution

$$k_x = 0 \qquad X = Ax + B$$

$$k_x = k \qquad X = A\sin kx + B\cos kx$$

$$k_x = jk \qquad X = A\sinh\, x + B\cosh\, x$$

Similar eqs for Y and Z

$$\frac{d^2}{dy^2}Y + k_y{}^2 Y = 0$$

$$\frac{d^2}{dz^2}Z + k_z{}^2 Z = 0$$

with $k_x{}^2 + k_y{}^2 + k_z{}^2 = 0$

4-6 Boundary Value Problems in Cylindrical Coordinates [Text p.183]

- Laplace's equation in cylindrical coordinate

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}V\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}V+\frac{\partial^2}{\partial z^2}V=0$$

assume $V(r,\phi,z)=R(r)\Phi(\phi)Z(z)$

$$\frac{1}{V}X(\ \ )\ :\ \frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}R\right)+\frac{1}{r^2}\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi+\frac{1}{Z}\frac{d^2}{dz^2}Z=0$$

$$Z\ eq\ :\ let\qquad \frac{1}{Z}\frac{d^2}{dz^2}Z = h^2$$

$$r^2\times\ :\ \frac{r}{R}\frac{d}{dr}\left(r\frac{d}{dr}R\right)+h^2r^2 = -\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi = n^2$$

$$\Phi\ eq\ :\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+n^2 = 0$$

$$R\ eq\ :\ \frac{d^2}{dr^2}R+\frac{1}{r}\frac{d}{dr}R+\left(h^2-\frac{n^2}{r^2}\right)R = 0$$

: Bessel eq

general sol :

$$R(r) = C_n J_n(hr) + D_n N_n(hr)$$

$J_n(hr)$: Bessel ftn of first kind of nth order with argument hr

$N_n(hr)$: Bessel ftn of second kind of nth order with argument hr

4-7 Boundary-Value Problems in Spherical Coordinates [Text p.188]

$$-\nabla^2 V$$

$$= -[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]V = 0$$

$$V(R,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\sin^2\theta\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right)$$

$$+\frac{1}{\Phi}\frac{d^2}{d\phi^2}\Phi = 0$$

$$\frac{d^2}{d\phi^2}\Phi + m^2\Phi = 0$$

$$\Phi = \begin{Bmatrix} e^{im\phi} \\ e^{-im\phi} \end{Bmatrix}$$

with $m = integer$

$\because \Phi$ must be single valued

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right)$$

$$-\frac{m^2}{\sin^2\theta} = 0$$

let $$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{d}{dr}R\right)=n^2$$

$$r^2\frac{d^2}{dr^2}R+2r\frac{d}{dr}R-n^2R=0$$

with $$R=A_k r^k+B_k r^{-(k+1)}$$

$$k(k-1)+2k-n^2=k^2+k-n^2=0$$

$$k(k+1)=n^2$$

$$(k+1)(k+2)-2(k+1)-n^2=k^2+k-n^2$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Theta\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0$$

with $x=\cos\theta$,

$$\frac{d}{dx}\left[(1-x^2)-\frac{d}{dx}\Theta\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0$$

: generalized Legendre equation

Sol : $P_l^m(\cos\theta)$: Associated Legendre ftn

성질 : 1. for given m, P_l^m forms an

orthogonal set in the index l

on $-1 < x < 1$, i.e.

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l}$$

2. $$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

•
•
•

Ref : "Modern Analysis"

by E.Whittaker and G.Watson

Cambridge University

for $m=0$, (azimuthal symmetric sol)

$$V(r,\theta,\phi)=R(r)\Theta(\theta)$$

R eq : $R(r)=A_l r^l + B_l r^{-(l+1)}$

Θ eq : $\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\Theta\right]+l(l+1)\Theta \quad =0$

: Legendre eq.

Sol : $\Theta = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$

$$a_{j+2} = \frac{(\alpha + j)(\alpha + j + 1) - l(l+1)}{(\alpha + j + 1)(\alpha + j + 2)} a_j$$

with if $a_o \neq 0, \qquad \alpha(\alpha - 1) = 0$

if $a_1 \neq 0, \qquad \alpha(\alpha + 1) = 0$

성질 : 1. $P_l(x) = \frac{1}{2l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$

: Rodrigues' formula

$P_o(x) = 1$

$P_1(x) = x$ Legendre polynomial

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

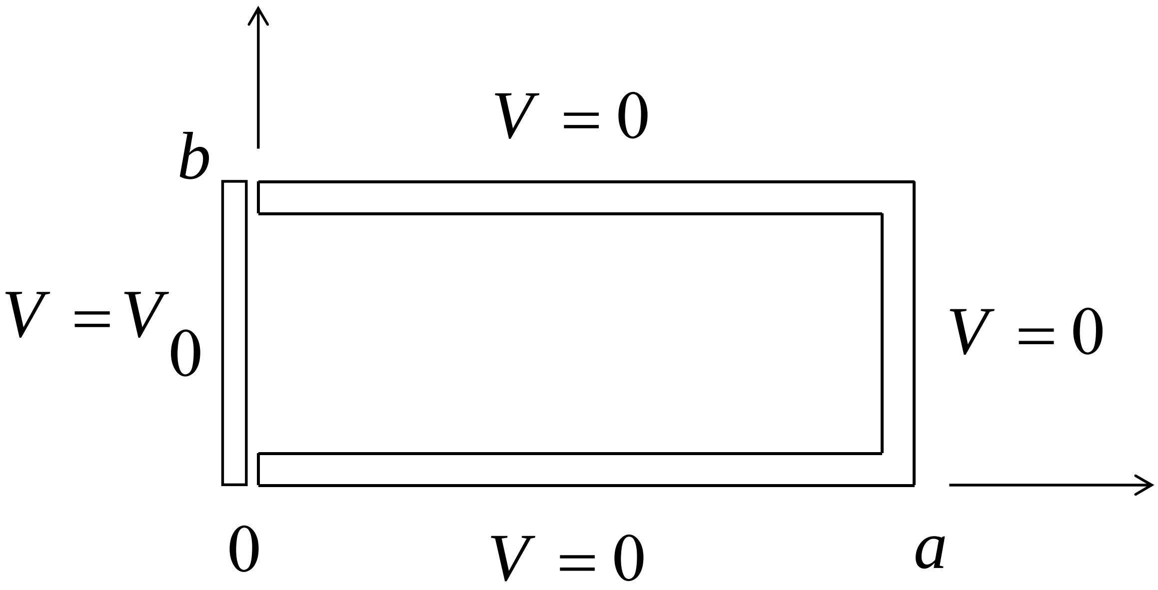
$\vdots$

2. P_l forms a complete, orthogonal set of ftns on $-1 \le x \le 1$

$$\int_{-1}^{1} P_{l'}(x) P_l(x) dx = \frac{2}{2l+1} \delta_{l'l}$$

$$\vdots$$

Example 4-7 Potential inside of an infinitely long, hollow enclosure [Text p. 181]



Since it is infinitely long along z-direction

$$V(x, y, z) = V(x, y)$$

let $$V(x,y) = X(x)\,Y(y)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x,y) = 0$$

$$\frac{1}{X}\frac{d^2}{dx^2}X = -\frac{1}{Y}\frac{d^2}{dy^2}Y \equiv k^2$$

Y eq : $$\frac{d^2}{dy^2}Y + k^2 Y = 0$$

$$Y = A\sin ky + B\cos ky$$

B.C. $V = 0$ for $y = 0 \longrightarrow B = 0$

$V = 0$ for $y = b \longrightarrow \sin kb = 0$

$$k = \frac{n}{b}\pi \quad , \quad n = 1,\ 2,\ 3,\ \cdots$$

X eq : $$\frac{d^2}{dx^2}X - k^2 X = 0$$

$$X = C\sinh kx + D\cosh kx$$

B.C. $V = 0$ for $x = a$

$$0 = C\sinh ka + D\cosh ka$$

$$D = -\frac{\sinh ka}{\cosh ka}C$$

$$X = C\left[\sinh kx - \frac{\sinh ka}{\cosh ka}\cosh kx\right]$$

$$= \frac{C}{\cosh ka}\sinh k(x-a)$$

Thus

$$V_n(x,y)=C_n\sinh\frac{n\pi}{b}(x-a)\sin\frac{n\pi}{b}y$$

for $n=1,\ 2,\ 3,\ \cdots$

B.C. $$V_o=\sum_{n=1}^{\infty}V_n(0,y)$$

$$=\sum_{n=1}^{\infty}C_n\sinh\left(-\frac{n\pi}{b}a\right)\sin\frac{n\pi}{b}y$$

since

$$\int_o^b \sin\frac{n\pi}{b}y\ \sin\frac{m\pi}{b}y\,dy$$

$$=\frac{1}{2}\int_o^b\left[\cos\frac{n-m}{b}\pi y-\cos\frac{n+m}{b}\pi y\right]dy$$

$$=\frac{1}{2}\left[\frac{b}{(n-m)\pi}\sin\frac{n-m}{b}\pi y-\frac{b}{(n+m)\pi}\sin\frac{n+m}{b}\pi y\right]_o^b$$

$$=\begin{cases}\dfrac{b}{2} & \text{for}\quad m=n\\[2ex] 0 & \text{otherwise}\end{cases}$$

$$\int_o^b V_o \sin\frac{m\pi}{b} y\, dy$$

$$= \int_o^b \sin\frac{m\pi}{b} y \sum_{n=1}^{\infty} C_n \sinh\left(-\frac{n\pi}{b} a\right) \sin\frac{n\pi}{b} y dy$$

$$V_o\left(-\frac{b}{m\pi}\cos\frac{m\pi}{b} y\right)\Big|_0^b = C_m \sinh\left(-\frac{m\pi}{b} a\right)\frac{b}{2}$$

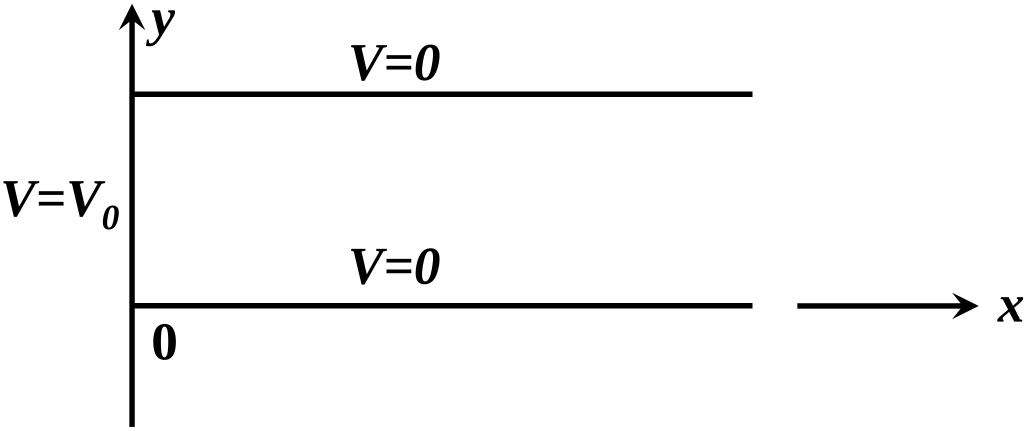
$$C_m = \frac{-\dfrac{b}{m\pi}(-2)}{-\dfrac{b}{2}\sinh\dfrac{m\pi}{b} a} V_o \quad , m = odd$$

$$= -\frac{4V_o}{m\pi \sinh\dfrac{m\pi}{b} a}$$

$$V(x,y)$$

$$= \frac{4V_o}{\pi} \sum_{n=1,3,5\cdots}^{\infty} \frac{\sinh \frac{n\pi}{b}(a-x)}{m \sinh \frac{n\pi}{b} a} \sin \frac{n\pi}{b} y$$

Example 4-6



B.C. : ① V(x, y, z) = V(x, y)

② V(0, y) = V_0

V(∞, y) = 0

③ V(x, 0) = 0

V(x, b) =0

Sol :
$$\frac{d^2}{dx^2}X + {k_x}^2 X = 0$$

$$\frac{d^2}{dy^2}Y + {k_y}^2 Y = 0$$

$$with \qquad {k_x}^2 + {k_y}^2 = 0$$

$$Then \qquad X(x) = Ae^{-kx}$$

$$Y(y) = B\sin ky$$

$$and \qquad V_n(x,y) = C_n e^{-kx}\sin ky$$

$$at \quad y = b, \quad V_n(x,y) = 0 = C_n e^{-kx}\sin kb$$

$$\sin kb = 0, \quad kb = n\pi$$

$$k = \frac{n\pi}{b}, \qquad n = 1,2,3,\cdots \quad /$$

$$V(x,y) = \sum C_n e^{-\frac{n\pi}{b}x}\sin\frac{n\pi}{b}y$$

$$at \quad x = 0,$$

$$V(0,y) = V_0 = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{b}y$$

$$\sum_{n=1}^{\infty} C_n \int_0^b \sin\frac{n\pi}{b}y\sin\frac{m\pi}{b}ydy = V_0\int_0^b \sin\frac{m\pi}{b}ydy$$

LHS: $$\sum_{n=1}^{\infty}\frac{C_n}{2}\int_0^b\left[\cos\frac{(n-m)}{b}\pi y-\cos\frac{(n+m)}{b}\pi y\right]dy$$

$$=\sum_{n=1}^{\infty}\frac{C_n}{2}\left[\frac{\sin\frac{(n-m)}{b}\pi y}{\frac{(n-m)}{b}\pi}-\frac{\sin\frac{(n+m)}{b}\pi y}{\frac{(n+m)}{b}\pi}\right]_0^b$$

$$=\begin{cases}\frac{C_n}{2}b & if \quad n=m\\ 0 & if \quad n\neq m\end{cases}$$

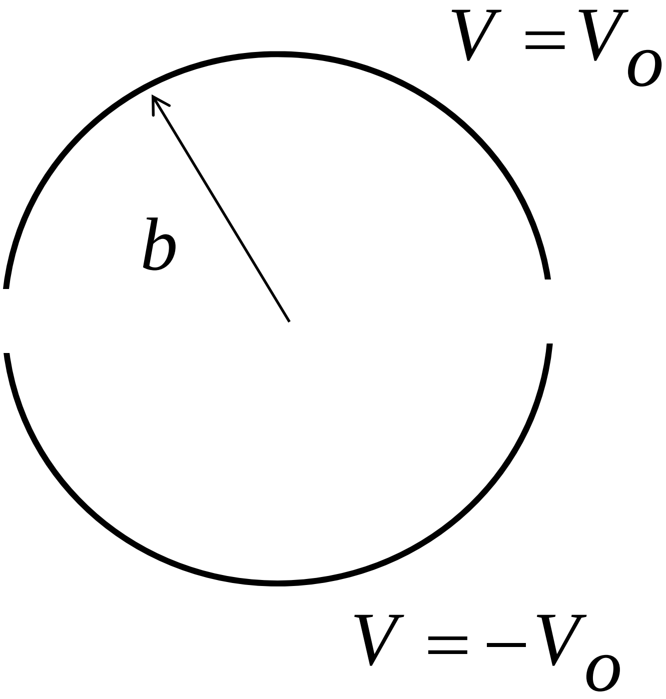
RHS: $$V_0\left[\frac{-\cos\frac{m\pi}{b}y}{\frac{m\pi}{b}}\right]_0^b=\begin{cases}\frac{2V_0b}{m\pi} & if \quad m \quad is \quad odd\\ 0 & if \quad m \quad is \quad even\end{cases}$$

$$C_n=\begin{cases}\frac{4V_0}{n\pi} & if \quad n \quad is \quad odd\\ 0 & if \quad n \quad is \quad even\end{cases}$$

$$\therefore V(x,y)=\sum_{n=1,3,5}^{\infty}\frac{4V_0}{n\pi}e^{-\frac{n\pi}{b}x}\sin\frac{n\pi}{b}y$$

Example

Infinitely long, conducting circular tube of radius b



B.C.

$$V(b,\phi) = \begin{cases} V_o = & for \quad 0 < \phi < \pi \\ -V_o = & for \quad \pi < \phi < 2\pi \end{cases}$$

Infinitely long $\longrightarrow$ $V(r,\phi,z) = V(r,\phi)$

Φ eq :
$$\frac{d^2}{d\phi^2}\Phi + n^2\Phi = 0$$

$$\longrightarrow \Phi = A\sin n\phi + B\cos n\phi$$

n must be an integer for single valued sol

r eq :
$$\frac{d^2}{dr^2}R + \frac{1}{r}\frac{d}{dr}R - \frac{n^2}{r^2}R = 0$$

$$R = C\,r^{n} + D\,r^{-n}$$

i) Inside

$D = 0 \quad \because \quad V$ must be finite everywhere

$B = 0 \quad \because \quad B.C.$ requires $V(r,\phi)$ is an odd ftn of ϕ

$$V(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \sin n\phi$$

$$V_o = \sum_{n=1}^{\infty} A_n b^n \sin n\phi$$

$$\int_0^{\pi} V_o \sin m\phi d\phi = \frac{A_m b^m}{2}\pi = -V_o \left.\frac{\cos m\phi}{m}\right]_0^{\pi}$$

$$A_n = \frac{4V_o}{n\pi b^n}$$ for odd n

$$\int_0^{\pi} \sin m\phi \sin n\phi \, d\phi$$

$$= \frac{1}{2}\int_0^{\pi} \left[\cos(m-n)\phi - \cos(m+n)\phi\right] d\phi$$

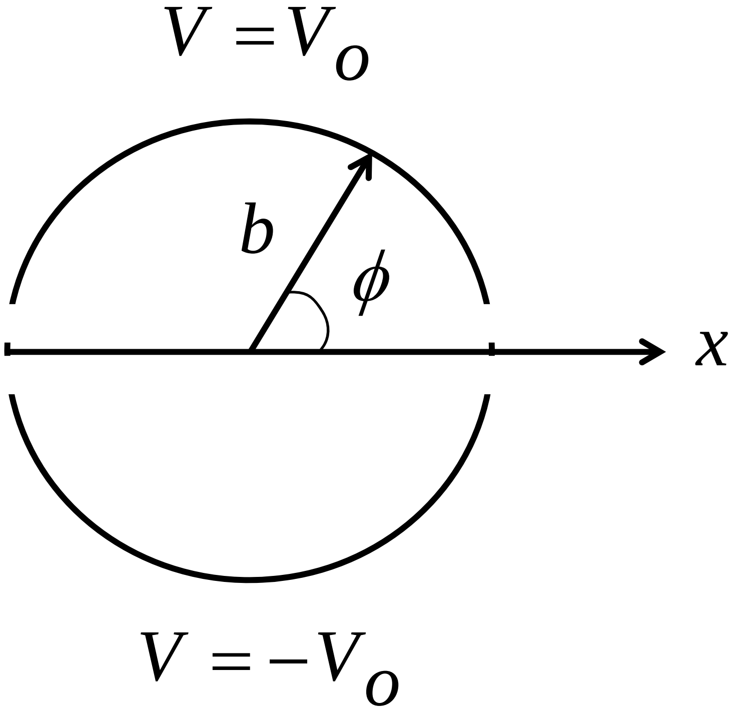
$$= \frac{1}{2}\left[\frac{\sin(m-n)\phi}{m-n}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\delta_{m,n}$$

$$V(r,\phi) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n}\left(\frac{r}{b}\right)^n \sin n\phi$$

for $r < b$

Example 4-9 [Text p.185]



- Infinitely long, thin, conducting circular tube split in two halves
- Potential distribution inside and outside

$$V = \begin{cases} V_o & for \quad 0 < \phi < \pi \\ -V_o & for \quad \pi < \phi < 2\pi \end{cases}$$

Infinitely long cylinder

$$V(r,\phi,z)=V(r,\phi)=\Sigma V_n(r,\phi)$$

$$V_n(r,\phi)=r^n(A_n \sin n\phi+B_n \cos n\phi)$$

$$+r^{-n}(A_n{}' \sin n\phi+B_n{}' \cos n\phi)$$

a) inside $(r<b)$

V must be finite inside of tube

$$\longrightarrow \quad A_n{}'=0=B_n{}' \qquad for \quad all \quad n$$

V must be an odd ftn of $\phi \longrightarrow B_n=0$

$$V(r,\phi)=\Sigma V_n(r,\phi)=\Sigma A_n r^n \sin n\phi$$

B.C. at $r=b$

$$V(b,\phi)=V_o=\Sigma A_n b^n \sin n\phi$$

$$for \qquad 0<\phi<\pi$$

$$V_o\int_0^{\pi}\sin m\phi\, d\phi = V_o\left[\frac{-\cos m\phi}{m}\right]_0^{\pi}=\frac{2V_o}{m}$$

$$=\Sigma A_n b^n \int_0^{\pi}\sin n\phi \sin m\phi d\phi$$

$$=\Sigma A_n b^n \int_0^{\pi}\frac{1}{2}\left[\cos(n-m)\phi-\cos(n+m)\phi\right]d\phi$$

$$=\Sigma A_n b^n\cdot\frac{1}{2}\left[\frac{\sin(n-m)\phi}{n-m}-\frac{\sin(n+m)\phi}{n+m}\right]_0^{\pi}$$

$$=A_m b^m\frac{\pi}{2}$$

$$A_m = \begin{cases} \dfrac{4V_o}{m\pi b^m} & for \quad odd \quad m \\ 0 & for \quad even \quad m \end{cases}$$

$$V(r,\phi) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n}\left(\frac{r}{b}\right)^n \sin n\phi$$

b) outside $(r > b)$

V must be finite outside of tube

$$\longrightarrow \quad A_n = 0 = B_n$$

V must be odd ftn of ϕ

$$\longrightarrow \quad B_n' = 0 \quad for \quad all \quad n$$

$$V(r,\phi) = \Sigma V_n(r,\phi) = \Sigma A_n' r^{-n} \sin n\phi$$

B.C. at $r = b$

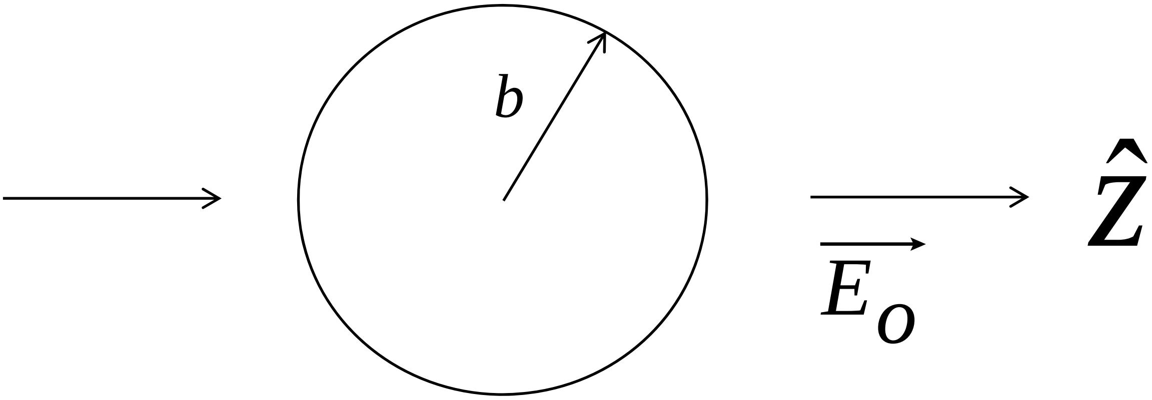
$$V(b,\phi) = \Sigma A_n' b^{-n} \sin n\phi$$

$$= \begin{cases} V_o & for \quad o < \phi < \pi \\ -V_o & for \quad \pi < \phi < 2\pi \end{cases}$$

$$A_n' = \begin{cases} \dfrac{4b^n V_o}{n\pi} & for \quad odd \;\; n \\ 0 & for \quad even \;\; n \end{cases}$$

$$V(r,\phi) = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n}\left(\frac{b}{r}\right)^n \sin n\phi$$

Example 4-10 conducting sphere in a uniform electric field $\overrightarrow{E_o} = E_o \hat{a}_z$ [Text p.190]



Boundary condition :

$$\begin{cases} V(b,\theta)=0 \\ V(r,\theta)=-E_o Z=-E_o r\cos\theta \end{cases}$$

$$for \qquad r >> b$$

Sol

$$V(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta)$$

Apply B.C. for r=b,

$$0 = \sum_{n=0}^{\infty} (A_n b^n + B_n b^{-(n+1)}) P_n(\cos\theta)$$

$$0 = \sum_{n=0}^{\infty} (A_n b^n + B_n b^{-(n+1)})$$

$$\cdot \int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta)$$

$$= (A_m b^m + B_m b^{-(m+1)}) \frac{2}{2m+1} \delta_{m,n}$$

$$B_n = -A_n b^{2n+1}$$

Since V must be finite for $r \to \infty$,

$A_n = 0$ for all n except $n = 1$

$$A_1 = -E_o$$

Thus $B_1 = E_o b^3$ and all other B_n' are zero

$$V(r,\theta) = (-E_o r + E_o b^3 r^{-2}) P_1(\cos\theta)$$

$$= -\left[1 - \left(\frac{b}{r}\right)^3\right] E_o r \cos\theta$$

Electric field

$$E_r = -\frac{\partial V}{\partial r} = E_o\left[1 + 2\left(\frac{b}{r}\right)^3\right]\cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_o\left[1 - \left(\frac{b}{r}\right)^3\right]\sin\theta$$

Induced Surface Charge Density

$$\sigma = \varepsilon_o\,E_r(r = b) = 3\varepsilon_o E_o\cos\theta$$

Chap 5. Steady Electric Current

5-1 Introduction [Text p.198]

Types of electric current caused by motion of free Charges :

1. Conduction current :
   Drift motion of conduction electron or ion

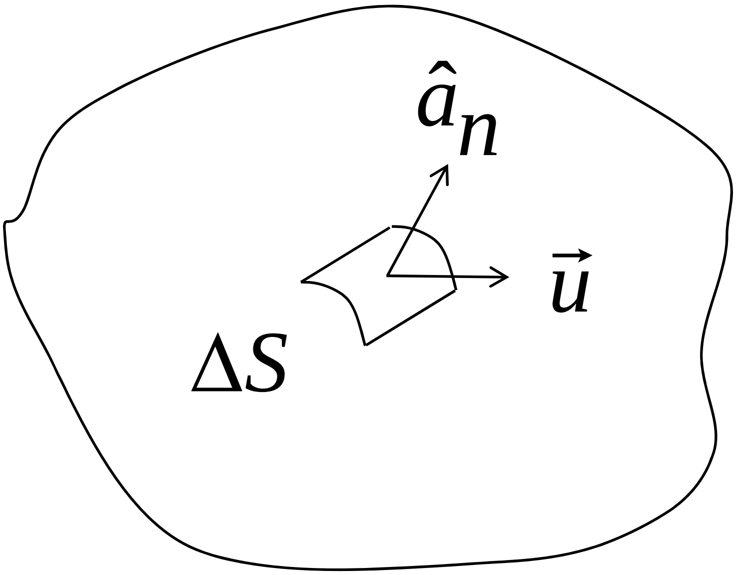
2. Electrolytic current :
   Migration of positive and negative ions

3. Convection current :
   Movement of positive or negative charged particles in vacuum or rarefied ionized gas

   plasma

5-2 Current Density and Ohm's law

[Text p.199]



Movement of charge carriers

q : Charge

N : No of charge carriers / vol

$\vec{u}$: Velocity

charge passing through ΔS in Δt

$$\Delta Q = (N\, q\, \vec{u}\, \Delta t)\cdot(\hat{a}_n \Delta S)$$

current $$\Delta I = \frac{\Delta Q}{\Delta t} = N\, q\, \vec{u}\cdot\hat{a}_n \Delta S$$

$$= \vec{J}\cdot\Delta\hat{S}$$

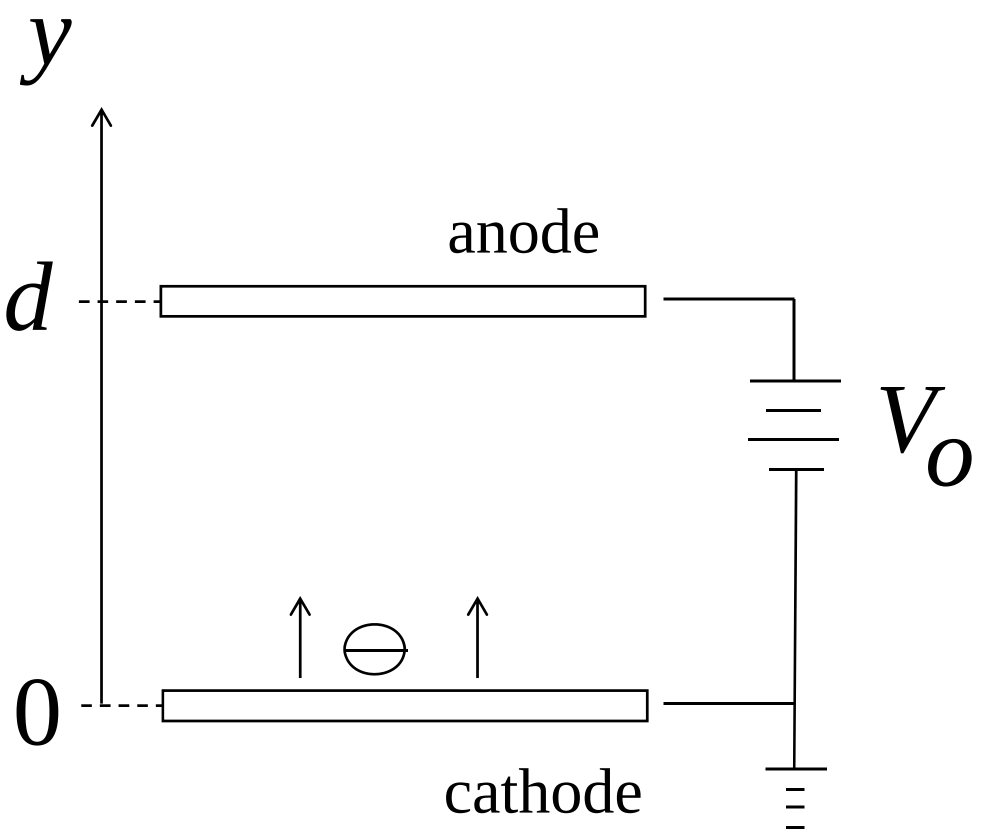
where

$$\vec{J} = Nq\vec{u}$$: convection current density

$$= \rho\vec{u}$$

$$\rho = Nq$$: charge density

Example 5-1 Child - Langmuir's space limited current [Text p.200]



Parallel conducting plate diode

Electron emission from cathode with zero velocity

Force on electron

$$F_y = m\frac{du}{dt} = -eE = e\frac{dV}{dy} \qquad 1.$$

Electric field (potential) between electrode

$$-\nabla^2 V = \frac{\rho}{\varepsilon_o}$$

$$-\frac{d^2}{dy^2}V = \frac{\rho}{\varepsilon_o} = \frac{1}{\varepsilon_o}\left(-\frac{J}{u}\right) \qquad 2.$$

Eq. 1. becomes

$$m\frac{d}{dt}u = m\frac{dy}{dt}\frac{du}{dy} = mu\frac{du}{dy} = \frac{d}{dy}\left(\frac{1}{2}mu^2\right)$$

Integration of both side results

$$\frac{1}{2}mu^2\Big]_0^y = eV\Big]_0^y$$

$$\frac{1}{2}mu^2 = eV$$ ← conservation of energy

$$KE + PE = \text{const}$$

$$u = \left[\frac{2}{m}eV\right]^{\frac{1}{2}}$$

2． results

$$-\frac{d^2}{dy^2}V=-\frac{1}{\varepsilon_o}\frac{J}{u}=-\frac{J}{\varepsilon_o}\left[\frac{2}{m}eV\right]^{-\frac{1}{2}}$$

$$\left[\frac{1}{2}\left(\frac{dV}{dy}\right)^2\right]_0^y=\frac{J}{\varepsilon_o}\left(\frac{2e}{m}\right)^{-\frac{1}{2}}\left[2V^{\frac{1}{2}}\right]_0^y$$

$$\frac{dV}{dy}=\left[\frac{4J}{\varepsilon_o}\left(\frac{m}{2e}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}V^{\frac{1}{4}}$$

$$\frac{d}{dy}V\,\frac{d^2}{dy^2}V=\frac{d}{dy}\left(\frac{1}{2}\left(\frac{dV}{dy}\right)^2\right) \qquad V^{-\frac{1}{2}}\frac{dV}{dy}=\frac{d}{dy}\left(2V^{\frac{1}{2}}\right)$$

$$\left(\frac{dV}{dy}\right)^2=\frac{4J}{\varepsilon_o}\left(\frac{m}{2e}\right)^{\frac{1}{2}}V^{\frac{1}{2}}$$

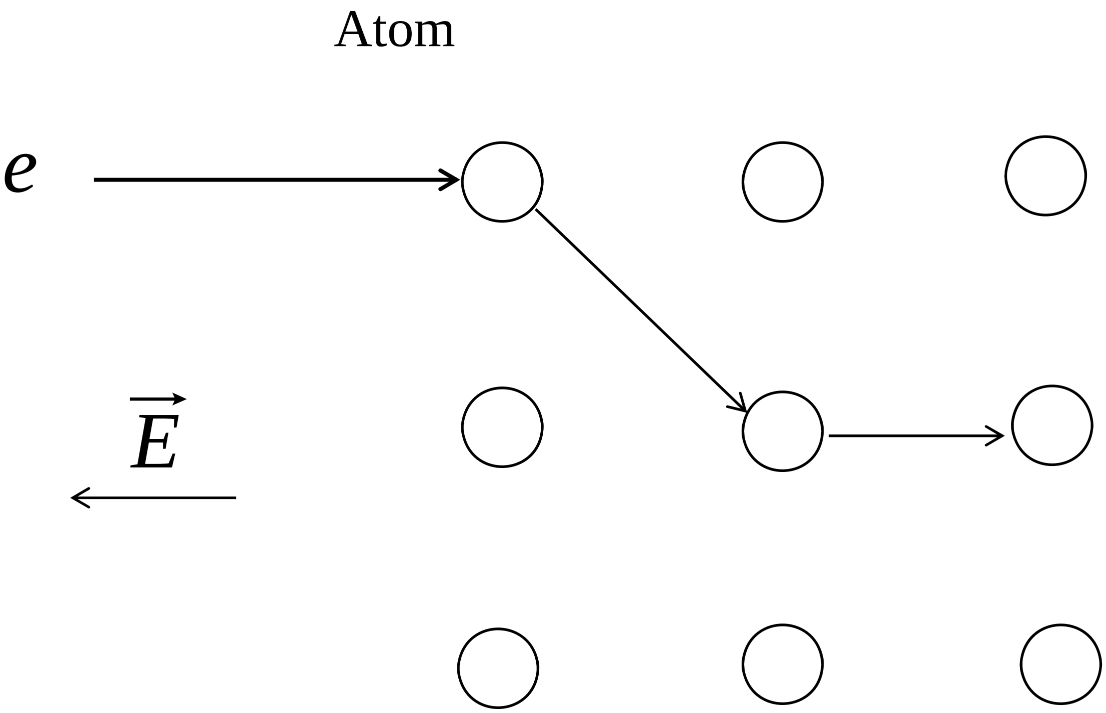
$$\frac{4}{3} V_o{}^{\frac{3}{4}} = \left[\frac{4J}{\varepsilon_o}\left(\frac{m}{2e}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} d$$

$$\boxed{J = \frac{4\varepsilon_o}{9d^2}\sqrt{\frac{2e}{m}}\, V_o{}^{\frac{3}{2}}}$$

Child - Langmuir's
space limited current

$$V^{-1\!/\!4} dV = \left(\frac{4}{3} V^{3\!/\!4}\right)'$$

Movement of electron in a solid



Eq. of motion without collision

$$\overrightarrow{F} = m\frac{d\vec{u}}{dt} = -e\overrightarrow{E}$$

$$\overrightarrow{u} = \overrightarrow{u}_O - \frac{e}{m}\overrightarrow{E}\,t \qquad \longrightarrow \quad \text{acceleration}$$

Eq. of motion with collision

$$\vec{F} = m\frac{d\vec{u}}{dt} = -e\vec{E} - m\vec{u}\nu$$

ν : collision freq

in a steady state

$$0 = -e\vec{E} - m\vec{u}\nu$$

$$\vec{u} = -\frac{e}{m\nu}\vec{E} \equiv -\mu_e\vec{E}$$

where $\mu_e = \dfrac{e}{m\nu}$: mobility

$(m^2/V \cdot S)$

current density

$$\vec{J} = Nq\vec{u} = (-Ne)(\mu_e \vec{E})$$

$$= -\rho_e \mu_e \vec{E} = \sigma \vec{E}$$: Ohm's law

where $\sigma = -\rho_e \mu_e$: conductivity $\begin{pmatrix} A/V \cdot m \\ S/m \end{pmatrix}$

$\rho = 1/\sigma$: resistivity

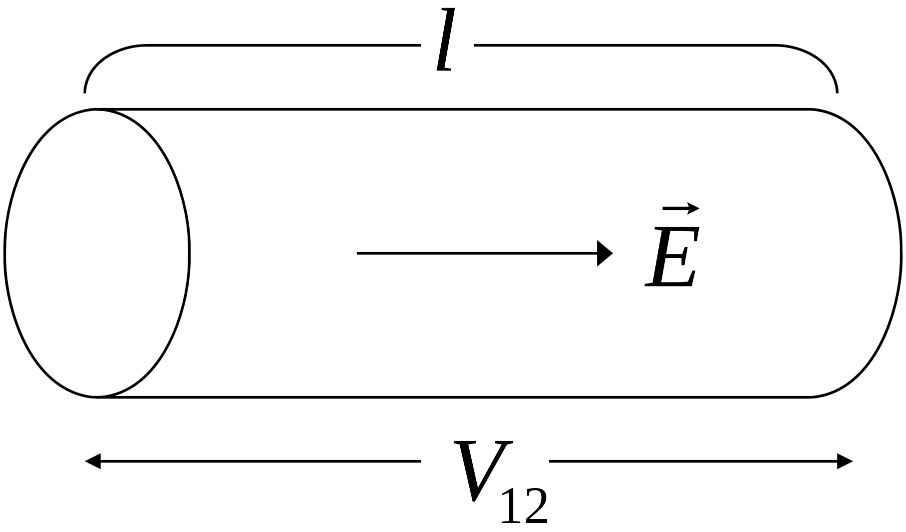
If there are more than one kind of charge carriers (electron , ion, hole)

$$\vec{J} = \Sigma N_i q_i \overline{u_i} \quad (A/m^2)$$

and

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$ for semiconductor

Homogeneous Conductor with a constant cross section



$J = \sigma E$ Holds inside of Conductor

Since total current

$$I = \int \vec{J} \cdot d\vec{S} = JS \ \rightarrow \ J = \frac{I}{S}$$

and

$$V_{12} = El$$

$$J = \frac{I}{S} = \sigma E = \sigma \cdot \frac{V_{12}}{l}$$

$$V_{12} = \frac{l}{\sigma S} I = RI$$

where

$$R = \frac{l}{\sigma S} (\Omega)$$: resistance

5-3 Electromotive force and Kirchhoff's Voltage law [Text p.205]

static electric field ⟶ conservative field

$$\oint_C \vec{E}\cdot d\vec{l} = \oint \frac{1}{\sigma}\vec{J}\cdot d\vec{l} = 0$$

: steady current can't be maintained in the same direction in a closed circuit by an electrostatic field

In order to have a steady current we need a source of nonconservative field

— Battery (chemical energy)
Generator (mechanical energy)
Thermo couple (Thermal energy)
Photovoltaic cell (light energy)
etc.

Electromotive force of source

$$V = \oint (\vec{E} + \vec{E_i}) \cdot d\vec{l}$$

$$= \oint \frac{1}{\sigma} \vec{J} \cdot d\vec{l}$$

$$= RI$$

$\vec{E}$: electrostatic field

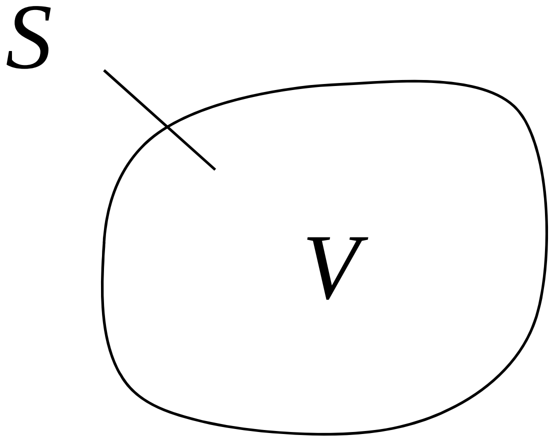
$\vec{E}i$: Internal field by source

or

$$\Sigma V_i = \Sigma R_k I_k$$

: Kirchhoff's voltage law

5-4 Equation of Continuity and Kirchhoff's Current Law [Text p.208]



Conservation of charge

Net current across surface S

$$I = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ}{dt}$$

$$= -\frac{d}{dt}\int_V \rho \, dv = -\int_V \frac{\partial \rho}{\partial t} dv$$

since $$\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J} dv$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0}$$

Equation of continuity

for steady current, $$\frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot \vec{J} = 0$$

$$\oint_S \vec{J} \cdot d\vec{S} = 0$$

or

$$\boxed{\Sigma\, I_j = 0}$$

Kirchhoff's current law

Relaxation time of charges in conductor.

When initial charge density ρ_0 is given at t=0, it relaxes according to the following eq.

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\vec{J}$$

From ohm's law

$$\vec{J} = \sigma\vec{E}$$

Then

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot\vec{J} = -\nabla\cdot\left(\sigma\vec{E}\right)$$

$$= -\sigma\nabla\cdot\vec{E}$$

$$= -\sigma\frac{\rho}{\varepsilon}$$

Sol:

$$\rho(t) = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$$

When $t = \frac{\varepsilon}{\sigma}(relaxation\, time)$,

$$\rho = \rho_0 e^{-1} = \rho_0 \times 0.368$$

With

$$\varepsilon \cong \varepsilon_0 = 8.85 \times 10^{-12}(f/m),$$

$$\sigma = 5.8 \times 10^{7}(s/m)$$

$$t = 1.52 \times 10^{-19}(s)$$

Very short!!

5-5 Power Dissipation and Joule's Law [Text p.210]

Work ΔW done by an electron field $\vec{E}$ in moving a charge q a distance dl

$$\Delta W = q\vec{E}\cdot d\,\vec{l}$$

power

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \lim_{\Delta t \to 0} \frac{q\vec{E}\cdot d\,\vec{l}}{\Delta t} = q\vec{E}\cdot\vec{u}$$

$\vec{u}$: drift velocity

Total power delivered to all charge carriers

$$dp = \Sigma\, p_i$$
$$= \vec{E} \cdot (\Sigma\, N_i q_i \vec{u}_i) dv$$
$$= \vec{E} \cdot \vec{J}\, dv$$

Total electric power converted into heat

$$\boxed{P = \int_V \vec{E} \cdot \vec{J}\, dv}$$

Joule's law

For a conductor with a constant cross section

$$P = \int EJ \, dl dS$$

$$= \int_L E dl \int_S J dS$$

$$= VI$$

$$= RI^2$$

5-6 Boundary Conditions for Current Density [Text p.211]

For a steady current density $\vec{J}$

Without non-conservative energy source

conservation of charge

$$\oint_S \vec{J} \cdot d\vec{S} = 0$$

$$\nabla \cdot \vec{J} = 0$$

$$\boxed{J_{1n} = J_{2n}}$$

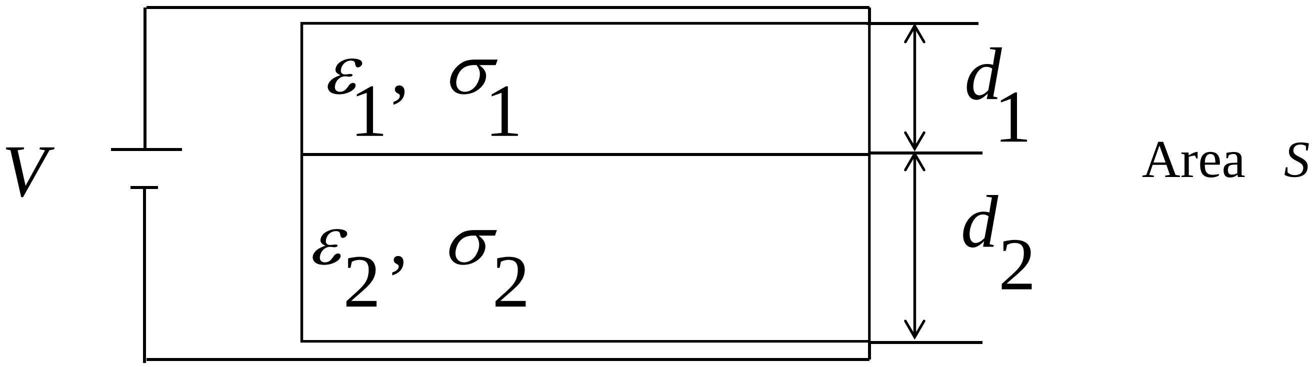
Electrostatic field is conservative field

$$\nabla \times \vec{E} = \nabla \times \left(\frac{\vec{J}}{\sigma} \right) = 0$$

$$\oint_C \frac{1}{\sigma} \vec{J} \cdot d\vec{l} = o$$

$$\boxed{\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}}$$

Example 5-4 [Text p.214]



a) current density between plates

By Kirchhoff's voltage law

$$V = \left(R_1 + R_2\right) I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right) I$$

$$J = \frac{I}{S} = \frac{V}{\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2}} \qquad (A/m^2)$$

b) Electric field intensity in both dielectrics

$$V = E_1 d_1 + E_2 d_2$$

$$J_{1n} = J_{2n} \quad \rightarrow \quad \sigma_1 E_1 = \sigma_2 E_2$$

$$E_1 = \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

$$E_2 = \frac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

c) Surface charge density on plates and interface

$$\sigma_{S1} = \varepsilon_1 E_1 = \frac{\varepsilon_1 \sigma_2}{\sigma_2 d_1 + \sigma_1 d_2} V$$

$$\sigma_{S2} = -\varepsilon_2 E_2 = -\frac{\varepsilon_2 \sigma_1}{\sigma_2 d_1 + \sigma_1 d_2} V$$

for the surface charge density at interface

$$\begin{cases} J_{1n} = J_{2n} \ \rightarrow \ \sigma_1 E_{1n} = \sigma_2 E_{2n} \\ D_{2n} - D_{1n} = \sigma_{si} \ \rightarrow \ \varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \sigma_{si} \end{cases}$$

$$\longrightarrow \sigma_{si} = \left(-\varepsilon_1 + \varepsilon_2 \frac{\sigma_1}{\sigma_2} \right) E_{1n}$$

$$= \left(-\varepsilon_1 \frac{\sigma_2}{\sigma_1} + \varepsilon_2 \right) E_{2n} = \left(-\varepsilon_1 + \varepsilon_2 \frac{\sigma_1}{\sigma_2} \right) \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

$$= \frac{-\sigma_2 \varepsilon_1 + \sigma_1 \varepsilon_2}{\sigma_2 d_1 + \sigma_1 d_2} V$$

5-7 Resistance Calculation [Text p.215]

capacitance between two conductors

$$C = \frac{Q}{V} = \frac{\oint_S \vec{D} \cdot d\vec{S}}{-\int_L \vec{E} \cdot d\vec{l}} = \frac{\oint \varepsilon \vec{E} \cdot d\vec{S}}{-\int_L \vec{E} \cdot d\vec{l}}$$

Lossy dielectric $\longrightarrow$ current flows

Resistance between conductors

$$R = \frac{V}{I} = \frac{-\int_L \vec{E} \cdot d\,\vec{l}}{\oint_S \vec{J} \cdot d\,\vec{S}} = \frac{-\int_L \vec{E} \cdot d\,\vec{l}}{\oint_S \sigma \vec{E} \cdot d\,\vec{S}}$$

Thus

$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$

Chap 6. Static Magnetic Fields

6-2 Fundamental Postulates of Magnetostatics in Free Space [Text p.226]

i) $$\nabla \cdot \vec{B} = 0$$

ii) $$\nabla \times \vec{B} = \mu_o \vec{J}$$

: Ampere's circuital law

where

$$\mu_o = 4\pi \times 10^{-7} \quad (H/m)$$

: permeability of free space

i) Implies

$$\int \nabla \cdot \vec{B}\, dv = \oint_S \vec{B} \cdot d\vec{S} = 0$$

Meaning :

No magnetic charge

No magnetic flow source

Magnetic flux line close on themselves

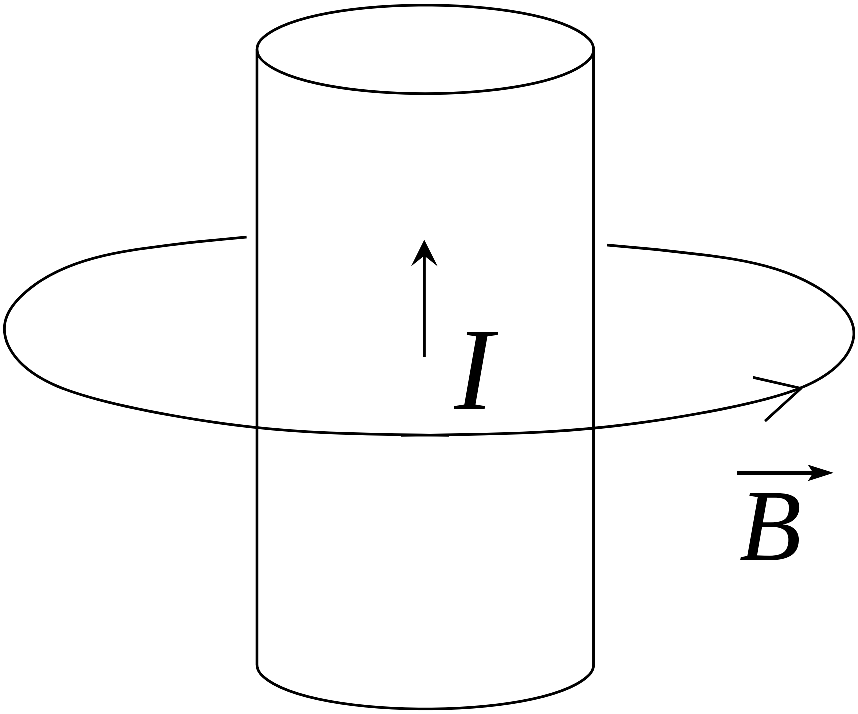
⟶ conservation of magnetic flux

ii) Implies

$$\int \nabla\times\vec{B}\cdot d\vec{S} = \oint \vec{B}\cdot d\vec{l} = \mu_o I$$

Ex 6-1 Infinitely long, straight conductor, radius b, uniform current I

[Text p.228]

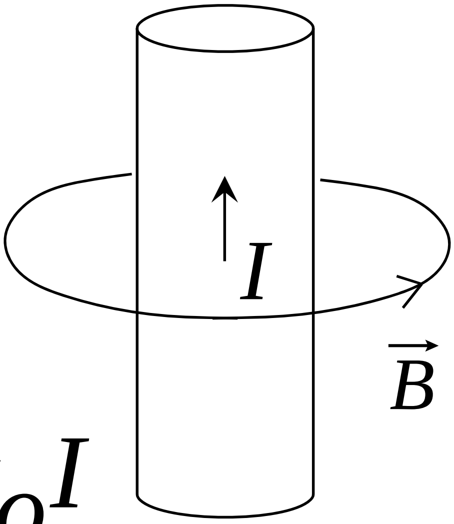


Magnetic flux density : ϕ **- direction , constant**

Inside

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi r = \mu_o \cdot \frac{\pi r^2}{\pi b^2} I$$

$$\vec{B} = \hat{a}_\phi \cdot \frac{\mu_o r}{2\pi b^2} I$$



Outside

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi r = \mu_o I$$

$$\vec{B} = \hat{a}_\phi \cdot \frac{\mu_o I}{2\pi r}$$

Ex 6-2 Toroidal coil with an air core

N turn , current I

Cylindrical Symmetry ⟶ B_ϕ only

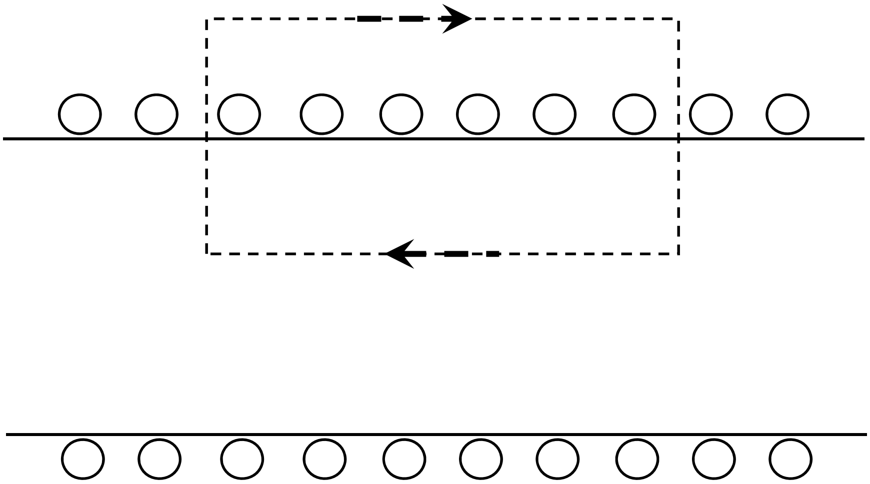
$$\oint \vec{B}\cdot d\vec{l} = 2\pi r\, B_\phi = \mu_o NI$$

$$\vec{B} = \frac{\mu_o NI}{2\pi r}\hat{a}_\phi \quad \text{for} \quad b-a<r<b+a$$

$$\text{and} \quad \vec{B}=0 \qquad \text{for} \quad \begin{array}{l} r<b-a \\ r>b+a \end{array}$$

Ex 6-3 Infinitely long Solenoid

n turn, current I [Text p. 230]



No field outside

$$BL = \mu_o nLI$$

$$B = \mu_o nI$$

6-3 Vector Magnetic Potential [Text p.232]

$$\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B} = \nabla \times \vec{A}$$

$\vec{A}$: Vector Magnetic Potential

Coulomb gauge

$$\nabla \cdot \vec{A} = 0$$

Then since

$$\begin{aligned}\nabla \times \vec{B} &= \nabla \times (\nabla \times \vec{A}) \\ &= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \\ &= \mu_o \vec{J}\end{aligned}$$

and with Coulomb gauge

$$\boxed{-\nabla^2\vec{A}=\mu_o\vec{J}}$$

Vector Poisson's eq

Since the solution of $\nabla^2 V = -\dfrac{\rho}{\varepsilon_o}$ is

$$V = \frac{1}{4\pi\varepsilon_o}\int_{v'} \frac{\rho}{R} dv'$$

The sol of Vector Poisson's eq is

$$\vec{A} = \frac{\mu_o}{4\pi}\int_{v'} \frac{\vec{J}}{R} dv' \qquad (Wb/m)$$

Magnetic flux

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \int_S \nabla\times\vec{A}\cdot d\vec{S}$$

$$= \oint \vec{A}\cdot d\vec{l} \qquad (Wb)$$

6-4 Biot - Savart Law and Applications

[Text p.234]

Magnetic field due to a line current

$$\vec{J}\,dv' \quad \rightarrow \quad I d\vec{l}'$$

and

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}}{R}dv' = \frac{\mu_o I}{4\pi}\oint_{c'} \frac{d\vec{l}'}{R}$$

Then $$\vec{B}=\nabla\times\vec{A}=\nabla\times\left[\frac{\mu_o I}{4\pi}\oint_{C'}\frac{d\vec{l'}}{R}\right]$$

$$=\frac{\mu_o I}{4\pi}\oint_{C'}\nabla\times\left(\frac{d\vec{l'}}{R}\right)$$

$$=\frac{\mu_o I}{4\pi}\oint\left(\nabla\frac{1}{R}\right)\times d\vec{l'}$$

$$=\frac{\mu_o I}{4\pi}\oint\left(\frac{-\hat{a}_R}{R^2}\right)\times d\vec{l'}$$

: Biot-Savart Law

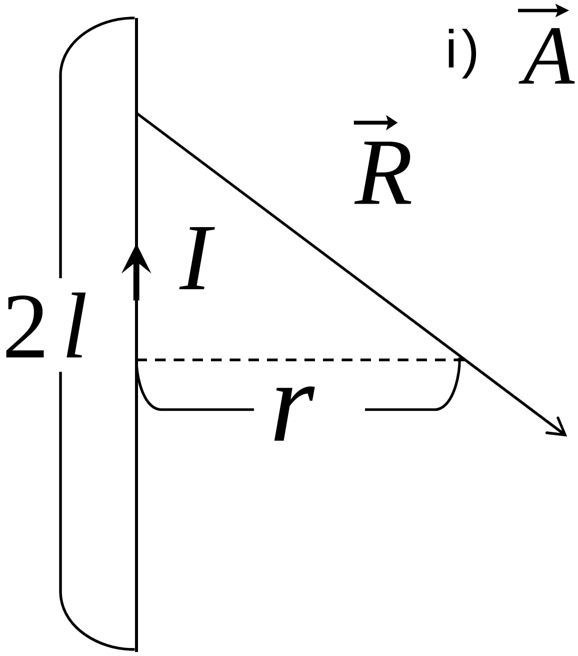
or $$d\vec{B}=\frac{\mu_o I}{4\pi}\frac{d\vec{l'}\times\overrightarrow{a_R}}{R^2}$$

uniform current ⟶ hollow current

Inside $\vec{B}=0$

Outside $\vec{B}=\hat{a}_\phi \cdot \frac{\mu_o I}{2\pi r}$

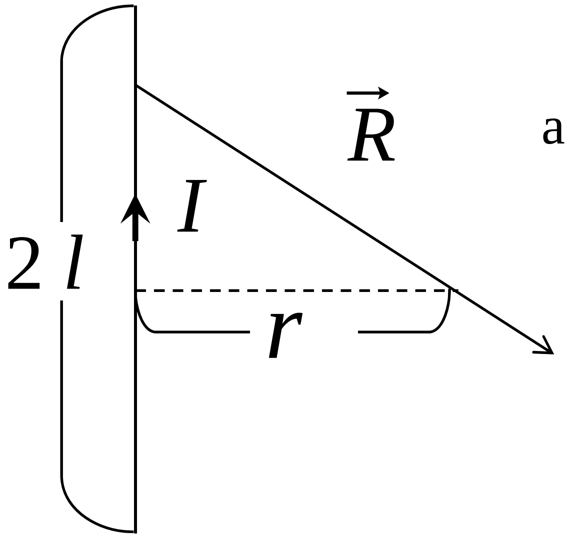
Ex 6-4 [Text p.236]



i) $\vec{A}=\frac{\mu_o I}{4\pi}\int \frac{d\vec{l}}{R}$

$$=\frac{\mu_o I}{4\pi}\int_{-L}^{L} \frac{dz}{\sqrt{z^2+r^2}}\hat{a}_z$$

$$=\frac{\mu_o I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\hat{a}_z$$



and $$\vec{B} = -\hat{a}_\phi \frac{\partial}{\partial r} A_z$$

$$= \hat{a}_\phi \frac{\mu_o IL}{2\pi r\sqrt{L^2 + r^2}}$$

ii) since

$$\vec{R} = r\hat{a}_R - Z\hat{a}_z$$

$$d\vec{l} \times \vec{R} = \hat{a}_z dz \times \left(r\hat{a}_R - z\hat{a}_z \right) = \hat{a}_\phi r dz$$

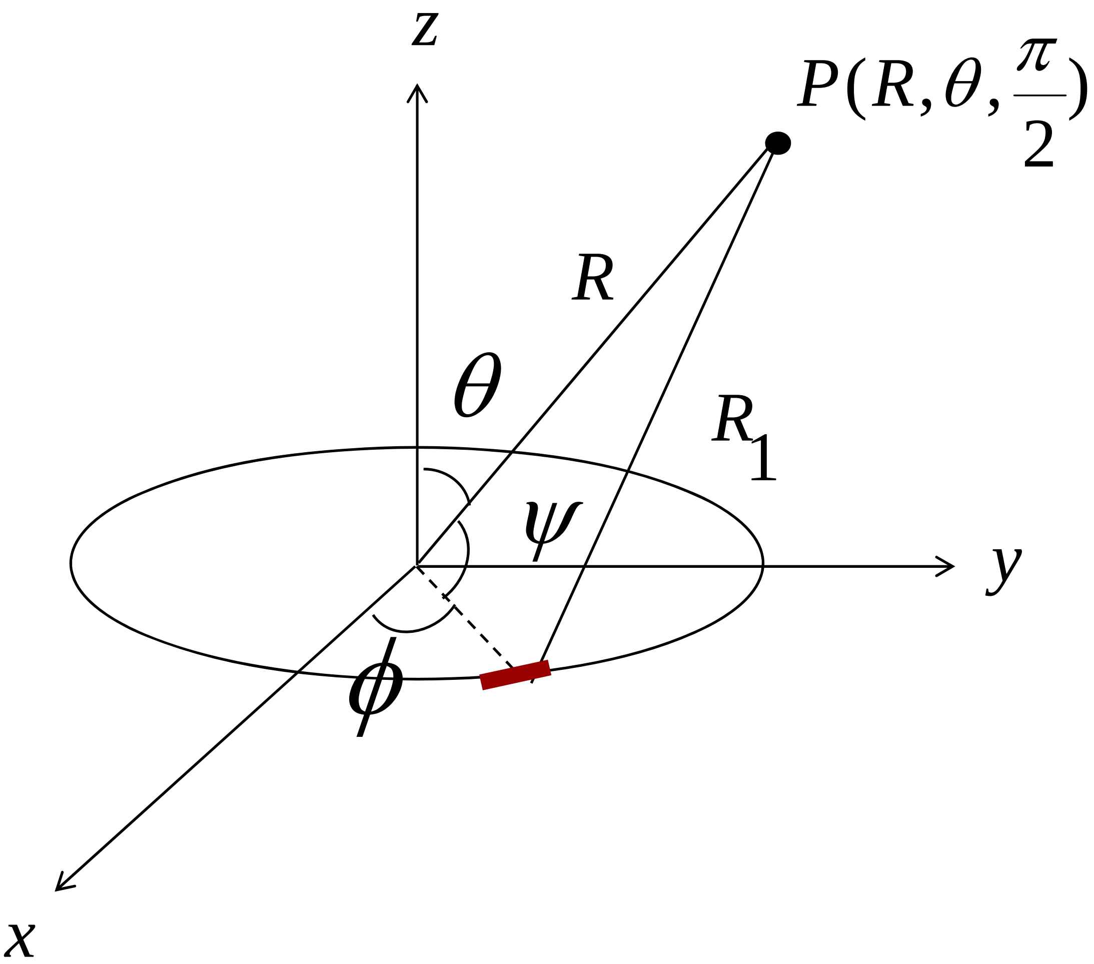
$$\vec{B} = \frac{\mu_o I}{4\pi} \int_{-L}^{L} \frac{r dz}{\left(r^2 + z^2 \right)^{3/2}} \hat{a}_\phi$$

$$= \frac{\mu_o IL}{2\pi r\sqrt{L^2 + r^2}} \hat{a}_\phi$$

6-4 **The Magnetic Dipole** [Text p.239]

Ex 6-7 [Text p.239]

Magnetic flux density due to a circular loop of radius b **carrying current** I



$$\vec{A}=\frac{\mu_o I}{4\pi}\oint\frac{d\vec{l}}{R_1}$$

$$d\vec{l}=\left(-b\hat{a}_x\sin\phi+b\hat{a}_y\cos\phi\right)d\phi$$

$$R_1=\left(R^2+b^2-2Rb\cos\psi\right)^{\frac{1}{2}}$$

$$Rb\cos\psi$$

$$=\left(R\sin\theta\hat{a}_y+R\cos\theta\hat{a}_z\right)\cdot\left(b\cos\phi\hat{a}_x+b\sin\phi\hat{a}_y\right)$$
$$=Rb\sin\theta\sin\phi$$

$$\vec{A}=\frac{\mu_o I}{4\pi}\oint\frac{-b\hat{a}_x\sin\phi\,d\phi}{\left(R^2+b^2-2Rb\sin\theta\sin\phi\right)^{\frac{1}{2}}}$$

$$=\hat{a}_\phi\frac{\mu_o I}{4\pi}\cdot 2b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\phi\,d\phi}{\left(R^2+b^2-2Rb\sin\theta\sin\phi\right)^{\frac{1}{2}}}$$

$$=\hat{a}_\phi\frac{\mu_o Ib}{2\pi}\int_0^{\pi}\frac{-\cos\alpha\,d\alpha}{\left(R^2+b^2+2Rb\sin\theta\cos\alpha\right)^{\frac{1}{2}}}$$

$$= -\hat{a}_\phi \frac{\mu_o I b}{2\pi}\int_0^\pi \frac{\frac{1}{2Rb\sin\theta}\left(R^2+b^2+2Rb\sin\theta\cos\alpha\right) - \frac{R^2+b^2}{2Rb\sin\theta}}{\left(R^2+b^2+2Rb\sin\theta\cos\alpha\right)^{\frac{1}{2}}}\, d\alpha$$

$$= -\hat{a}_\phi \frac{\mu_o I b}{2\pi}\int_0^\pi \left[\frac{1}{2Rb\sin\theta}\left(R^2+b^2+2Rb\sin\theta\cos\alpha\right)^{\frac{1}{2}} - \frac{R^2+b^2}{2Rb\sin\theta(\quad)^{\frac{1}{2}}}\right] d\alpha$$

$$= -\hat{a}_\phi \frac{\mu_o I b}{2\pi}\int_0^\pi \Bigg[\frac{1}{2Rb\sin\theta}\sqrt{\left(R^2+b^2+2Rb\sin\theta\right)\left(1-\frac{4Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}\sin^2\frac{\alpha}{2}\right)}$$

$$- \frac{R^2+b^2}{2Rb\sin\theta\sqrt{\left(R^2+b^2+2Rb\sin\theta\right)\left(1-\frac{4Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}\sin^2\frac{\alpha}{2}\right)}}\Bigg] d\alpha$$

$$\alpha = \phi + \frac{\pi}{2} \qquad \sin\phi = \sin(\alpha - \frac{\pi}{2}) = -\cos\alpha$$

$$\cos\alpha = \cos 2\cdot\frac{\alpha}{2} = 1 - 2\sin^2\frac{\alpha}{2}$$

$$= -\hat{a}_\phi \frac{\mu_o I b}{\pi} \left[\frac{\sqrt{R^2+b^2+2Rb\sin\theta}}{2Rb\sin\theta} \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\beta}\, d\beta \right.$$

$$\left. - \frac{R^2+b^2}{2Rb\sin\theta\sqrt{\left(R^2+b^2+2Rb\sin\theta\right)}} \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} \right]$$

$$\left(\frac{\alpha}{2} = \beta, \quad d\alpha = 2d\beta\right)$$

$$= \hat{a}_\phi \frac{\mu_o I b}{\pi} \frac{1}{2Rb\sin\theta\sqrt{R^2+b^2+2Rb\sin\theta}}$$

$$\cdot \left[\left(R^2+b^2\right) K - \left(R^2+b^2+2Rb\sin\theta\right) E \right]$$

where

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta$$ Complete Elliptic Integral of 2nd kind

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}} d\theta$$ Complete Elliptic Integral of 1st kind

$$=\hat{a}_{\phi}\frac{\mu_o Ib}{\pi}\frac{1}{\sqrt{R^2+b^2+2Rb\sin\theta}}\left[\frac{\left(2-k^2\right)^2-2E}{k^2}\right]$$

$$\frac{R^2+b^2}{2Rb\sin\theta}=\frac{\dfrac{2(R^2+b^2)}{R^2+b^2+2Rb\sin\theta}}{\dfrac{4Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}}$$

$$=\frac{\dfrac{2\left(R^2+b^2+2Rb\sin\theta\right)-4Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}}{k^2}=\frac{(2-k^2)K}{k^2}$$

$$\frac{R^2+b^2+2Rb\sin\theta}{2Rb\sin\theta}=\frac{1}{\dfrac{2Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}}=\frac{2}{k^2}$$

for small $k^2=\dfrac{4Rb\sin\theta}{R^2+b^2+2Rb\sin\theta}$

$$\left(R<<b,\ \ R>>b\ \ \ or\ \ \ \theta<<\pi\right)$$

$$K=\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9}{64}k^4\right)\ \ ,$$

$$E=\frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3}{64}k^4\right)$$

and

$$\frac{(2-k^2)K-2E}{k^2}$$

$$\cong \frac{\left(2-k^2\right)\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9}{64}k^2\right)-2\cdot\frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3}{64}k^4\right)}{k^2}$$

$$=\frac{\pi}{2}\cdot\frac{2-\frac{k^2}{2}-\frac{k^4}{4}-2+\frac{k^2}{2}+\frac{9}{32}k^4+\frac{3}{32}k^4}{k^2}$$

$$=\frac{\pi k^2}{16}$$

Then for $R >> b,$

$$\vec{A} \cong \hat{a}_{\phi} \frac{\mu_o I b}{\pi} \frac{1}{\sqrt{R^2 + b^2 + 2Rb\sin\theta}} \cdot \frac{\pi k^2}{16}$$

$$\doteq \hat{a}_{\phi} \frac{\mu_o I b}{\cancel{\pi}} \frac{\cancel{\pi}}{16} \frac{4Rb\sin\theta}{\left(R^2 + b^2 + 2Rb\sin\theta\right)^{\frac{3}{2}}}$$

$$\cong \hat{a}_{\phi} \frac{\mu_o I b^2 \sin\theta}{4R^2}$$

and

$$\vec{B}=\nabla\times\vec{A}$$

$$\frac{1}{R^{2}\sin\theta}\begin{vmatrix} \hat{a}_{r} & R\hat{a}_{\theta} & r\sin\theta\hat{a}_{\phi} \\ \partial R & \partial\theta & \partial\phi \\ Ar & RA_{\theta} & R\sin\theta A_{\phi} \end{vmatrix}$$

$$\vec{B}=\frac{1}{R^{2}\sin\theta}\left[\hat{a}_{R}\frac{\partial}{\partial\theta}\left(R\sin\theta\, A\phi\right)+R\hat{a}_{\theta}\cdot-\frac{\partial}{\partial R}\left(R\sin\theta\, A_{\phi}\right)\right]$$

$$=\frac{1}{R^{2}\sin\theta}\left[\hat{a}_{R}\frac{\partial}{\partial\theta}\left(\frac{\mu_{o}Ib^{2}}{4R}\sin^{2}\theta\right)-R\hat{a}_{\theta}\frac{\partial}{\partial R}\left(\frac{\mu_{o}Ib^{2}}{4R}\sin^{2}\theta\right)\right]$$

$$=\frac{\mu_{o}Ib^{2}}{4R^{3}}\left(\hat{a}_{R}\cdot 2\cos\theta+\hat{a}_{\theta}\sin\theta\right)$$

with the definition of

$$\vec{m} = I \cdot \pi b^{2} \cdot \hat{a}_{z} = IS\,\hat{a}_{z}$$: magnetic dipole moment

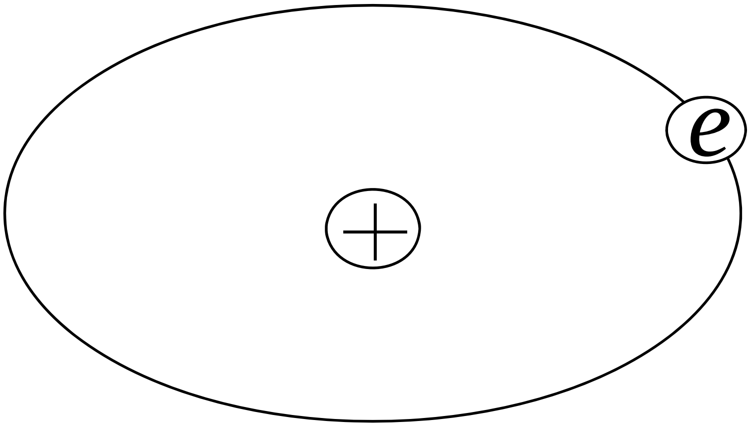
$$\vec{A} = \frac{\mu_{o}}{4\pi R^{2}} \vec{m} \times \hat{a}_{R}$$

and

$$\vec{B} = \frac{\mu_{o} m}{4\pi R^{3}} \left(\hat{a}_{R} \cdot 2\cos\theta + \hat{a}_{\theta} \sin\theta \right)$$

6-6 Magnetization and Equivalent Current Density [Text p.243]

elementary atomic model



spinning electron
spinning nucleus
orbiting electron

→ magnetic moment

Magnetization vector

$$\overrightarrow{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n} \overrightarrow{m}_k}{\Delta v}$$

$\overrightarrow{m}_k$: magnetic dipole moment of an atom

vector magnetic potential due to $d\overrightarrow{M}$ in dv'

$$d\overrightarrow{A} = \frac{\mu_o \overrightarrow{M} \times \hat{a}_R}{4\pi R^2} dv'$$

and

$$\vec{A} = \int d\vec{A}$$

$$= \int \frac{\mu_o}{4\pi} \frac{\vec{M} \times \hat{a}_R}{R^2} dv'$$

$$= \frac{\mu_o}{4\pi} \int \vec{M} \times \nabla' \left(\frac{1}{R} \right) dv'$$

$$= \frac{\mu_o}{4\pi} \int \left[-\nabla' \times \left(\frac{\vec{M}}{R} \right) + \frac{1}{R} \nabla' \times \vec{M} \right] dv'$$

$$= \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \vec{M}}{R} dv' + \frac{\mu_o}{4\pi} \oint_{s'} \frac{\vec{M} \times \hat{a}_n{}'}{R} ds'$$

$$= \frac{\mu_o}{4\pi} \int_{v'} \frac{\vec{J}m}{R} dv' + \frac{\mu_o}{4\pi} \oint_{s'} \frac{\vec{J}ms}{R} ds'$$

$$\left(\because \int_{v'} \nabla' \times \vec{F} dv' = - \oint_{s'} \vec{F} \times d\vec{S}' \right)$$

where

$$\vec{J}m = \nabla \times \overrightarrow{M}$$

: Equivalent volume current density

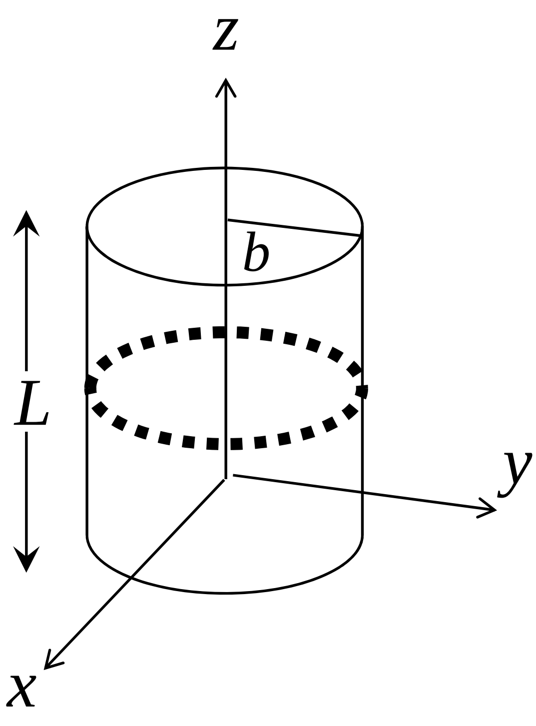
$$\vec{J}ms = \overrightarrow{M} \times \hat{a}_n$$

: Equivalent surface current density

Ex 6-8 Magnetic flux density due to a uniformly magnetized circular cylinder

[Text p.246]

$$\overrightarrow{M} = \hat{a}_z M_o \,,\quad \text{반경}\; b\,,\quad \text{길이}\; L$$



$$\vec{J}ms = \overrightarrow{M} \times \hat{a}_n$$

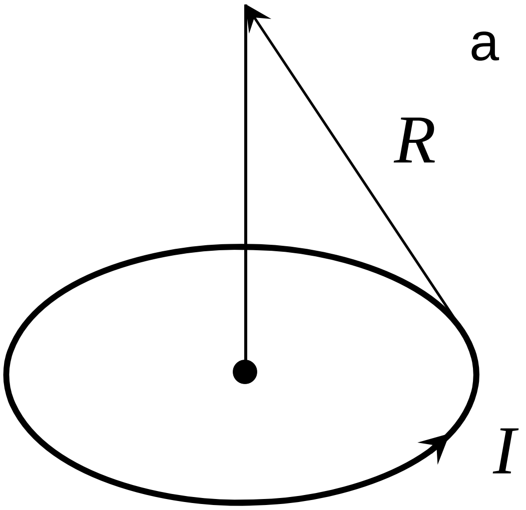
on the side

$$\vec{J}ms = \overrightarrow{M} \times \hat{a}_n$$

$$= \left(\hat{a}_z M_o\right) \times \hat{a}_r$$

$$= M_o \hat{a}_\phi$$

Flux density due to
a circular loop on the axis



$$d\vec{l} = bd\phi'\hat{a}_{\phi}$$

$$\vec{R} = \hat{a}_z z - \hat{a}_r b$$

$$\vec{B}_l = \frac{\mu_o I}{4\pi}\int \frac{d\vec{l}' \times \vec{R}}{\left(z^2 + b^2\right)^{3/2}}$$

$$= \frac{\mu_o I}{4\pi}\int_0^{2\pi} \frac{bz\,d\phi'\hat{a}_r + b^2 d\phi'\,\hat{a}_z}{\left(z^2 + b^2\right)^{3/2}}$$

$$= \hat{a}_z \frac{\mu_o I}{4\pi}\frac{b^2 2\pi}{\left(z^2 + b^2\right)^{3/2}} = \frac{\mu_o I b^2}{2\left(z^2 + b^2\right)^{3/2}}\hat{a}_z$$

Then

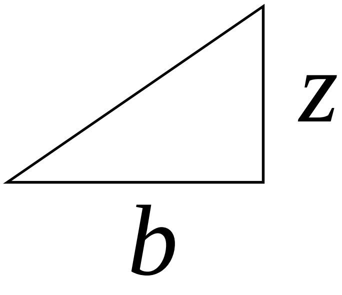
$$\vec{B} = \hat{a}_z \frac{\mu_o M_o b^2}{2} \int_0^L \frac{dz'}{\left[\left(z-z'\right)^2 + b^2\right]^{\frac{3}{2}}}$$

$$= \hat{a}_z \frac{\mu_o M_o b^2}{2} \left[-\frac{1}{b^2} \sin\alpha\right]_{\tan^{-1}\frac{z}{b}}^{\tan^{-1}\frac{z-L}{b}}$$

$$= \hat{a}_z \frac{\mu_o M_o}{2} \left[\frac{z}{\sqrt{b^2+z^2}} - \frac{z-L}{\sqrt{b^2+\left(z-L\right)^2}}\right]$$

$$z - z' = b \tan\alpha$$

$$z' = 0, \quad \alpha = \tan^{-1}\frac{z}{b}$$



$$z' = L, \quad \alpha = \tan^{-1}\frac{z-L}{b}$$

Scalar magnetic potential and equivalent magnetization charge density

In a current - free region $\vec{J} = 0$ and

$$\nabla \times \vec{B} = 0$$

$$\rightarrow \ \vec{B} = -\mu_o \nabla V_m \quad \textbf{(*)}$$

where V_m : Scalar magnetic potential

Then

$$V_{m2} - V_{m1} = -\int_1^2 \frac{\vec{B} \cdot d\vec{l}}{\mu_o}$$

Fictitious magnetic charge

$$V_m = \frac{1}{4\pi}\int \frac{\rho_m}{R} dv' \longrightarrow \quad \textbf{(*)}$$

Field of small bar magnet
= magnetic dipole

$$\vec{m} = q_m \vec{d} = \hat{a}_n IS$$

$$V_m = \frac{\vec{m}\cdot\hat{a}_R}{4\pi R^2}$$

In terms of magnetization vector

$$dV_m = \frac{\overrightarrow{M} \cdot \hat{a}_R}{4\pi R^2}$$

$$V_m = \frac{1}{4\pi} \int \frac{\overrightarrow{M} \cdot \hat{a}_R}{R^2} dv'$$

$$= \frac{1}{4\pi} \oint_{s'} \frac{\overrightarrow{M} \cdot \hat{a}_n{}'}{R} ds' +$$

$$\frac{1}{4\pi} \int_{v'} \frac{-\nabla' \cdot \overrightarrow{M}}{R} dv'$$

Equivalent magnetization surface charge density

$$\rho_{ms} = \overrightarrow{M} \cdot \hat{a}_n$$

Equivalent magnetization volume charge density

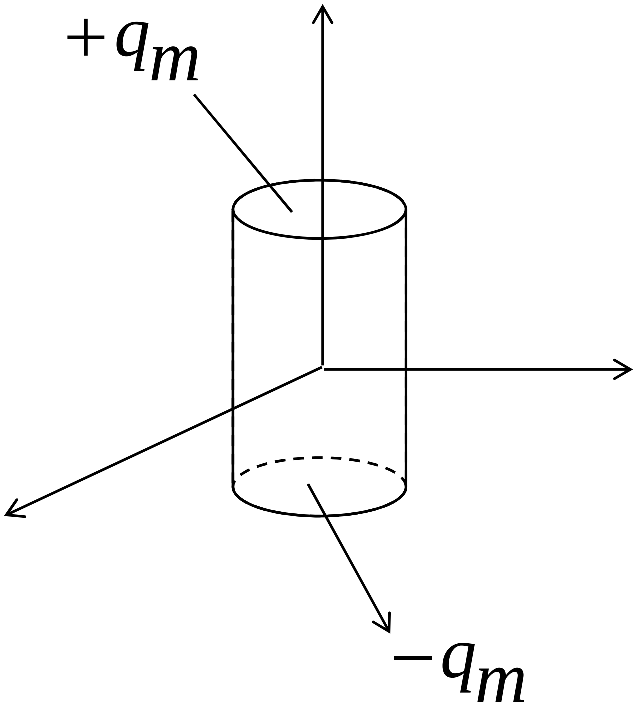
$$\rho_m = -\nabla \cdot \overrightarrow{M}$$

Ex 6. 9 Cylindrical bar magnet [Text p.247]

$$\vec{M} = M_o \hat{a}_z \ ,$$

$\vec{B}$ at a distant point ?

Equivalent magnetization charge density



$$\rho_m = 0$$

$$\rho_{ms} = \vec{M} \cdot \hat{a}_n$$

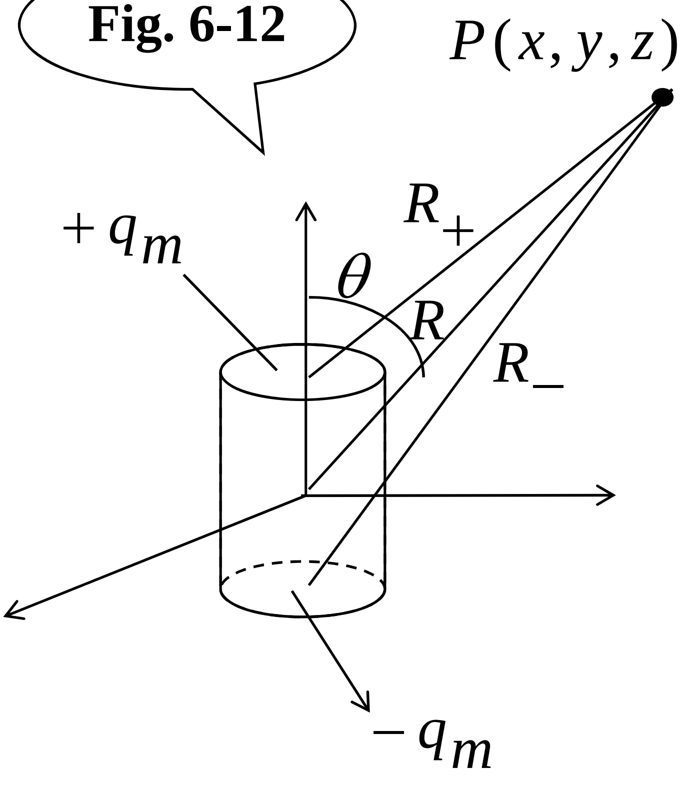
$$= \begin{cases} M_o & top \\ -M_o & bottom \\ 0 & side \end{cases}$$

at a distant point, equivalent charge appears as point charges

$q_m = M_o \cdot \pi b^2$ and

$$V_m = \frac{M_o \pi b^2}{4\pi}\left(\frac{1}{R+} - \frac{1}{R-}\right)$$

Fig. 6-12



$$\simeq \frac{M_o \pi b^2}{4\pi} \cdot \frac{L\cos\theta}{R^2}$$

$$= \frac{M_T \cos\theta}{4\pi R^2}$$

$M_T = M_o \pi b^2 \cdot L$: total dipole moment

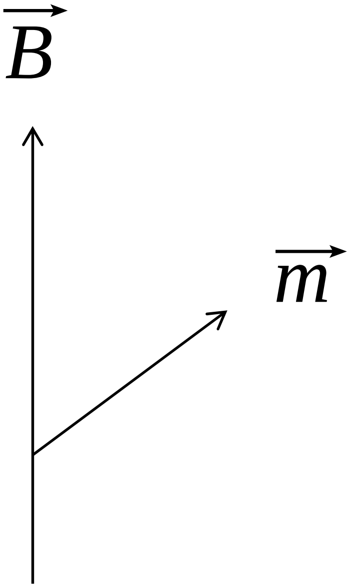
Then

$$\vec{B} = -\mu_o \nabla V_m$$

$$= \frac{\mu_o M_T}{4\pi R^3} \left(\hat{a}_R 2\cos\theta + \hat{a}_Q \sin\theta \right)$$

6-7 Magnetic Field Intensity and Relative Permeability [Text p. 249]

- Application of magnetic field to magnetic material
- Magnetic dipole in an external magnetic field



Torque $\vec{T} = \vec{m} \times \vec{B}$

Potential energy

$$U = -\vec{m} \cdot \vec{B}$$

$\longrightarrow$ alignment of $\vec{m}$ with $\vec{B}$

Then with magnetic material, the Ampere's circuital law becomes

$$\nabla\times\vec{B}=\mu_o\vec{J}total$$

$$=\mu_o\left(\vec{J}+\vec{J}m\right)$$

$$=\mu_o\left(\vec{J}+\nabla\times\overrightarrow{M}\right)$$

$$\nabla\times\left(\frac{\vec{B}}{\mu_o}-\overrightarrow{M}\right)=\vec{J}$$

Then

$$\boxed{\nabla\times\vec{H}=\vec{J}}$$

Where

$$\vec{H}=\frac{\vec{B}}{\mu_o}-\vec{M}\quad\left(A/m\right)$$ magnetic field intensity

$\vec{J}$: free volume current density

For a linear, isotropic medium

$$\vec{M}=\chi_m\vec{H}$$

Where χ_m : magnetic susceptibility

Then $$\vec{B} = \mu_o \left(1 + \chi_m\right) \vec{H}$$

$$= \mu_o \mu_r \vec{H}$$

$$= \mu \vec{H}$$

where $$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_o}$$

: relative permeability

6-8 Magnetic circuits [Text p. 251]

Transformer , Generator , Motor ,

Magnetic recording devices

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\oint \vec{H} \cdot d\vec{l} = NI = V_m$$

V_m : magnetomotive force (mmf)

Ex 6-10 $\vec{B}f\,,\quad \vec{H}f\,,\quad \vec{H}g$

[Text p.252]

$$\nabla\cdot\vec{B}=0 \quad\rightarrow\quad \Phi_f=\Phi_g$$

$$\rightarrow\quad \vec{B}f=\vec{B}g=B_f\,\hat{a}_\phi$$

Neglecting
fringe field

$$\nabla\times\vec{H}=\vec{J} \quad\rightarrow\quad \oint_C\vec{H}\cdot d\vec{l}=NI$$

$$H_f l_f+H_g l_g=NI$$

since $H_f=\dfrac{B_f}{\mu}\ ,\ H_g=\dfrac{B_g}{\mu_o}=\dfrac{B_f}{\mu_o}$

$$\frac{B_f}{\mu}\left(2\pi r_o - l_g\right) + \frac{B_g}{\mu_o} l_g = NI$$

$$\vec{B} f = \hat{a}_\phi \frac{\mu \mu_o NI}{\mu_o\left(2\pi r_o - l_g\right) + \mu l_g}$$

$$\vec{H} f = \hat{a}_\phi \frac{\mu_o NI}{\mu_o\left(2\pi r_o - l_g\right) + \mu l_g}$$

$$\vec{H} g = \hat{a}_\phi \frac{\mu NI}{\mu_o\left(2\pi r_o - l_g\right) + \mu l_g}$$

The flux

$$\Phi = B_f S = \frac{NI}{\left(2\pi r_o - l_g\right)/\mu s + l_g/\mu_o S}$$

$$\equiv \frac{V_m}{R_f + R_g} \quad \left(= \frac{mmf}{\text{Reluctance}} \right)$$

$$\leftrightarrow \qquad I = \frac{V}{R}$$

Electric circuit		Magnetic circuit	
current	I	magnetic flux	Φ
emf	V	mmf	V_m
resistance	R	reluctance	R_m
conductivity	σ	permeability	μ
electric field intensity	$\vec{E}$	magnetic field intensity	$\vec{H}$

$$\Sigma \Phi_j = 0$$

$$\Sigma N_j I_j = \Sigma R_k \Phi_k$$

6-9 Behavior of Magnetic Materials

[Text p.257]

자성체의 분류

Diamagnetic : $\chi_m < 0$

Paramagnetic : $\chi_m > 0$

Ferromagnetic : $\chi_m > 0$ (large)

- Diamagnetism

Orbiting electron

Faraday's induction law + Lenz's law

Universal law

Small, nonpermanent

- Paramagnetism

Spinning electron

Magnetic alignment + thermal derangement

Temp dependent (curie temp)

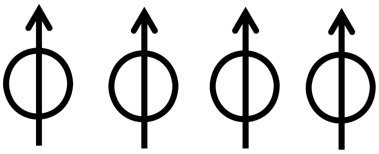
- Ferromagnetism

alignment of magnetized domain

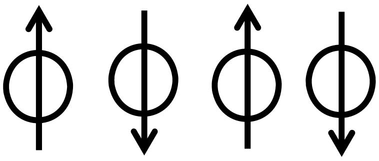
domain : 수 μm ~ 수 mm 크기 ,

10^{15} ~ 10^{16} 원자 포함

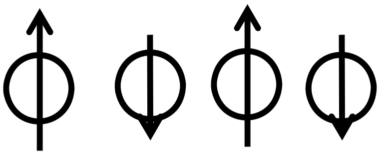
alignment of spinning electron



: Ferromagnetic



: antiferromagnetic



: Ferrimagnetic

❖ Ferrite

1. Magnetic spinel

$$XO\cdot Fe_2O_3 \left(X\ :\ Fe, Co, Ni, Mn, Mg\right)$$

Small eddy current loss due to low conductivity

RF or microwave component

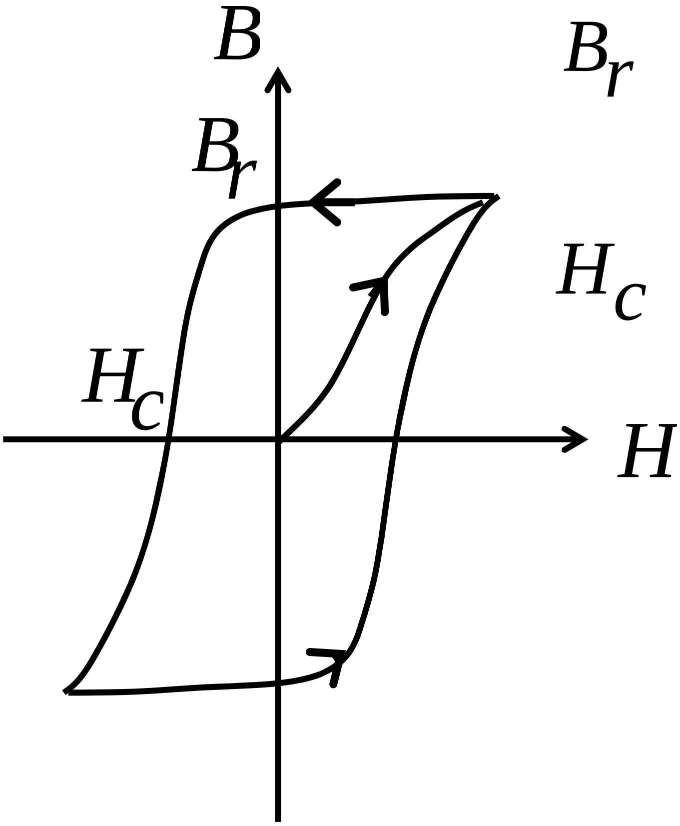
2. Magnetic - oxide garnet

$$\left(YIG, Y_3Fe_5O_{12}\right)$$

Yttrium - iron - garnet

Microwave component

- Hysteresis loop



B_r : residual or remanent flux density

H_c : coercive field intensity

6-10 Boundary conditions for magnetic materials [Text p.262]

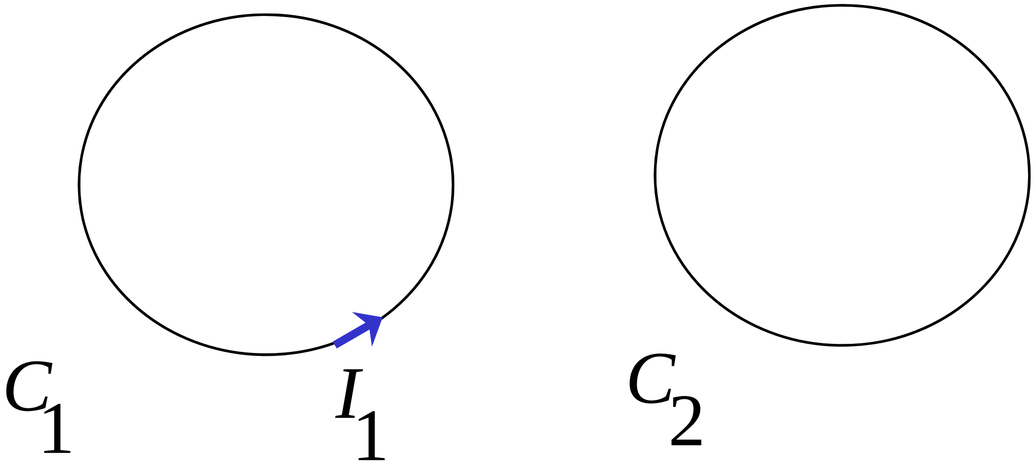
$$\nabla \cdot \vec{B} = 0$$

$$\boxed{B_{1n} = B_{2n}}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\boxed{H_{1t} - H_{2t} = J_{sn}}$$

6-11 Inductances and Inductors [Text p.266]



Magnetic flux through C_2 due to I_1 in C_1

$$\Phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$$

$$= L_{12} I_1$$

L_{12} : Mutual inductance

$\Phi_{11} = L_{11} I_1$ $\qquad$ L_{11} : Self inductance

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2$$

$$= \frac{N_2}{I_1} \int_{S_2} \left(\nabla \times \vec{A}_1 \right) \cdot d\vec{S}_2$$

$$= \frac{N_2}{I_1} \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2$$

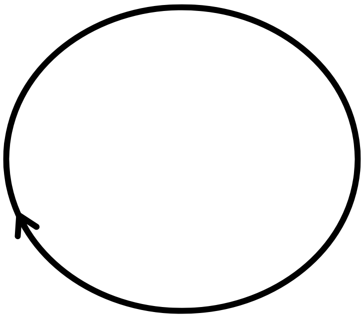
$$= \frac{N_2}{I_1} \oint_{C_2} \left(\frac{\mu_o N_1 I_1}{4\pi} \oint_{C_1} \frac{d\vec{l}_1}{R} \right) \cdot d\vec{l}_2$$

$$= \frac{\mu_o N_1 N_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{R}$$

Neumann formula

$$= L_{21}$$

6-12 Magnetic energy [Text p.277]



L_1 : self - inductance

Current i_1 : $0 \rightarrow I_1$

time varying current ⟶ induced emf

$$v_1 = L_1 \frac{di_1}{dt}$$

Required work

$$W_1 = \int v_1 \cdot i_1 dt = \int L_1 \frac{di_1}{dt} i_1 dt$$

$$= L_1 \int_0^{I_1} i_1 di_1 = \frac{1}{2} L I_1^2$$

$$= \frac{1}{2} I_1 \Phi_1$$: Stored magnetic energy

System of N loops carrying current

$$I_1,\ I_2,\ \cdots\ I_n$$

$$W_m = \frac{1}{2}\sum_{j,k} L_{jk} I_j I_k$$

or $$W_m = \frac{1}{2}\sum_{k} I_k \Phi_k$$

since $$\Phi_k = \sum_{j} L_{jk} I_j$$

for a continuous distribution of current

$$\Phi_k = \int_{S_k} \vec{B} \cdot d\overrightarrow{S'}_k = \oint_{C_k} \vec{A} \cdot d\vec{l}\,'_k$$

$$W_m = \frac{1}{2} \sum_k I_k \Phi_k$$

$$\rightarrow \quad \frac{1}{2} \int J da' \int \vec{A} \cdot d\overrightarrow{l_k}'$$

$$= \frac{1}{2} \int_{v'} \vec{A} \cdot \vec{J} dv'$$

In terms of field quantities

$$\vec{A}\cdot\vec{J}=\vec{A}\cdot\left(\nabla\times\vec{H}\right)$$

$$=\nabla\cdot\left(\vec{H}\times\vec{A}\right)+\vec{H}\cdot\left(\nabla\times\vec{A}\right)$$

$$=\nabla\cdot\left(\vec{H}\times\vec{A}\right)+\vec{H}\cdot\vec{B}$$

and $$W_m=\frac{1}{2}\int_{v'}\vec{A}\cdot\vec{J}dv'$$

$$=\frac{1}{2}\int_{v'}\vec{H}\cdot\vec{B}dv'+\overbrace{\frac{1}{2}\oint\left(\vec{H}\times\vec{A}\right)\cdot d\vec{S}'}^{0}$$

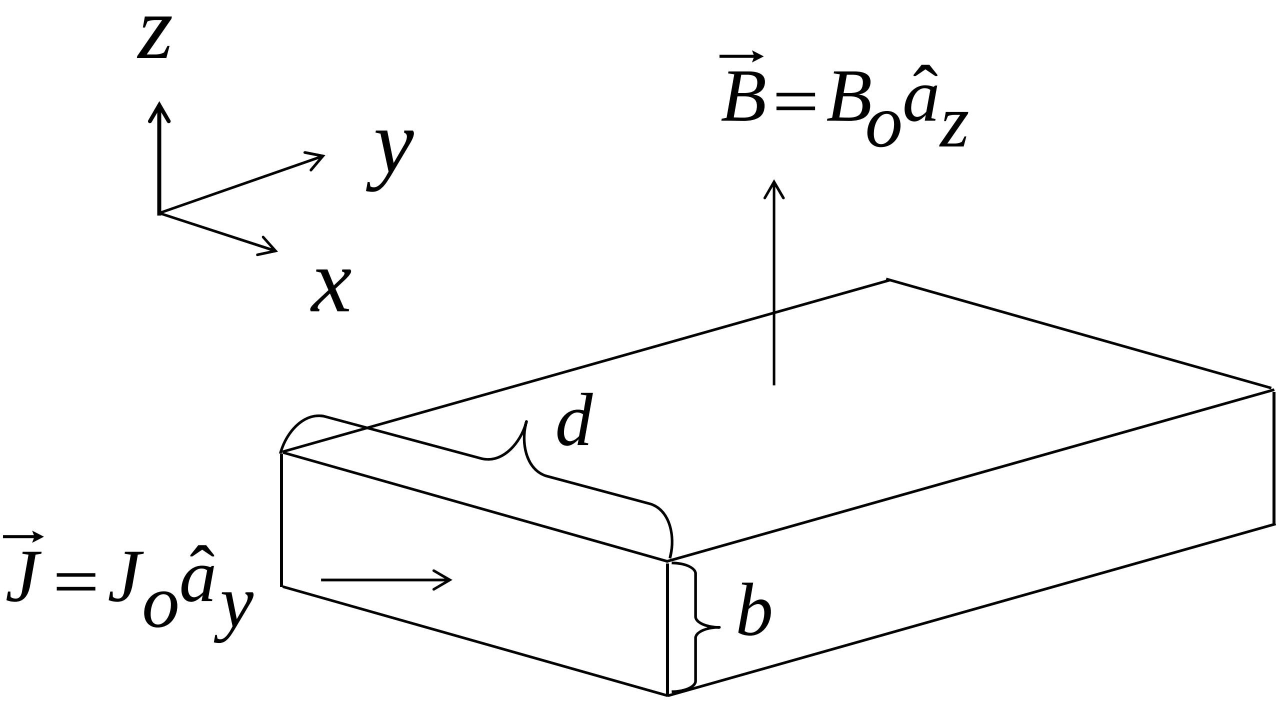
$$=\frac{1}{2}\int_{v'}\vec{H}\cdot\vec{B}dv'$$

and $w_m=\frac{1}{2}\vec{H}\cdot\vec{B}$: magnetic energy density

6-13 Magnetic forces and Torques

[Text p.281]

6-13-1 Hall effect [Text p.282]



Uniform magnetic field $\vec{B} = B_o \hat{a}_z$

Uniform current $\vec{J} = J_o \hat{a}_y = Nq\vec{u}$

Force

$$\vec{F} = q\vec{u} \times \vec{B} \quad \rightarrow \quad x - \text{direction}$$

$$\left(\begin{array}{l} \text{positive charge} \\ \quad \vec{u} \ : \ y - \text{dir} \quad \vec{F} \ : \ x - \text{dir} \\ \\ \text{negative charge} \\ \quad \vec{u} \ : \ -y - \text{dir} \quad \vec{F} \ : \ x - \text{dir} \end{array} \right)$$

accumulation of charge

⟶ electric field : Hall field

$$\vec{E}_h + \vec{u} \times \vec{B} = 0$$

$$\vec{E}_h = -\vec{u} \times \vec{B}$$

$$= -\left(u_o \hat{a}_y\right) \times \left(B_o \hat{a}_z\right)$$

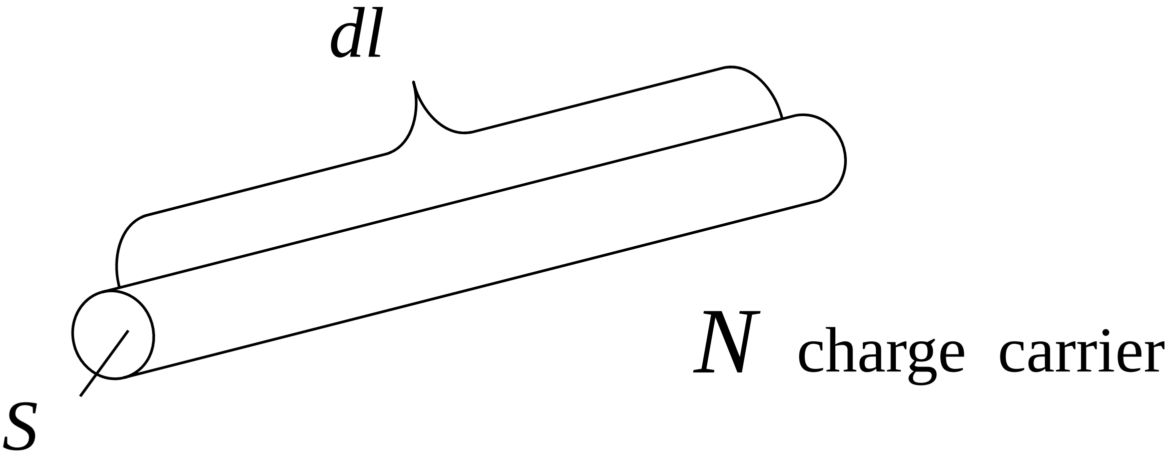
$$= -u_o B_o \hat{a}_x \quad \text{positive charge (hole)}$$

$$V_n = -\int_0^d E_h dx = -u_o B_o d$$: Hall voltage

$$\frac{E_h}{J_y B_z} = \frac{u_o B_o}{NquB_o} = \frac{1}{Nq}$$: Hall coefficient

$$\left.\begin{array}{l}\text{* charge carrier의 종류} \\ \text{charge carrier의 밀도} \\ \text{자장의 세기}\end{array}\right\}\text{측정}$$

6-13-2 Forces and Torques on current - carrying conductors [Text p.283]

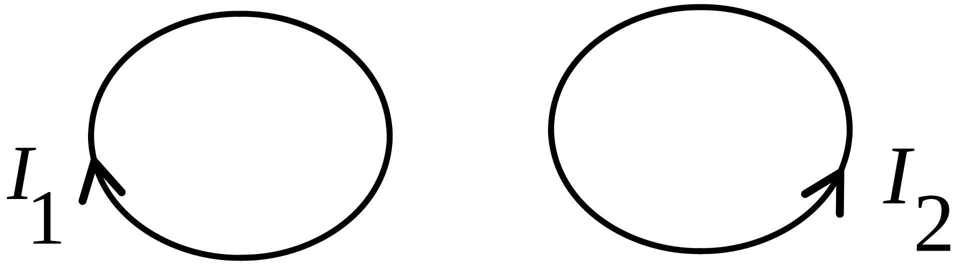


$$d\vec{F}m = \left(-e\vec{u}\times\vec{B}\right)NSdl$$

$$= -NeS\left|\vec{u}\right| d\vec{l}\times\vec{B}$$

$$= I\,d\vec{l}\times\vec{B}$$

$$\vec{F}m = I\oint d\vec{l}\times\vec{B}$$



$$\vec{F}\ 21 = I_1\oint_{c_1} d\vec{l}_1\times\vec{B}\ 21$$

$$= I_1\oint_{c_1} d\vec{l}_1\times\frac{\mu_o I_2}{4\pi}\oint_{c_2}\frac{d\vec{l}_2\times\hat{a}_{R_{21}}}{R_{21}^2}$$

$$= \frac{\mu_o}{4\pi}I_1 I_2\oint_{c_1}\oint_{c_2}\frac{d\vec{l}_1\times\left(d\vec{l}_2\times\hat{a}_{R\ 21}\right)}{R_{21}^2}$$

current carrying loop in a magnetic field

$$\vec{T} = \vec{m}\times\vec{B}$$

6-13-3 Forces and Torques in terms of stored magnetic energy [Text p.289]

System of circuits with constant flux linkage
no emf ⟶ no supplied energy

Mechanical work done by sys
= decrease in stored magnetic energy

$$\vec{F}_{\Phi} \cdot d\vec{l} = -dW_m = -(\nabla W_m) \cdot dl$$

$$\vec{F}_{\Phi} = -\nabla W_m$$

$$\left(T_{\Phi}\right)_z = -\frac{\partial}{\partial \phi} W_m$$

System of circuits with constant current

work done by source
= mechanical work done by sys
+ change in stored magnetic energy

$$dW_s = dW + dW_m$$

since $dW_m = \frac{1}{2} dW_s$, $dW = dW_m$

$$\overrightarrow{F_I} \cdot d\vec{l} = dW_m$$

$$\overrightarrow{F_I} = +\nabla W_m$$

$$\left(T_I\right)_z = \frac{\partial W_m}{\partial \phi}$$

Force and Torque between rigid current – carrying circuits

$$W_m = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + L_{12} I_1 I_2$$

$$\overrightarrow{F_I} = I_1 I_2 \nabla L_{12}$$

$$\left(T_I \right)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi}$$

Ex 6-15 Inductance of Solenoid

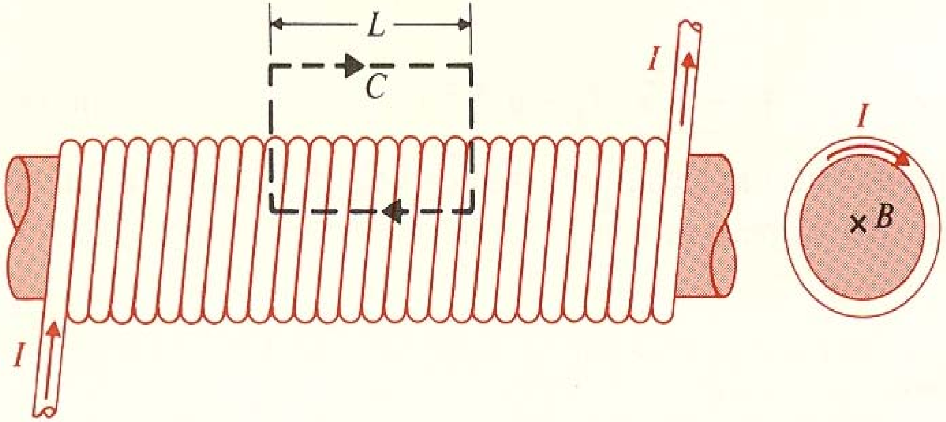


FIGURE 6–4
A current-carrying long solenoid (Example 6–3).

n turn/length, cross - sectional area S

magnetic flux density $B = \mu_0 n I$

$$\Phi = \int \vec{B} \cdot d\vec{s} = \mu_0 n I S$$

flux linkage / unit length is

$$\Lambda' = n\Phi = \mu_0 n^2 I S$$

Inductance / unit length

$$L' = \mu_0 n^2 S (H / m)$$

Ex 6-16 Inductance of coaxial transmission line

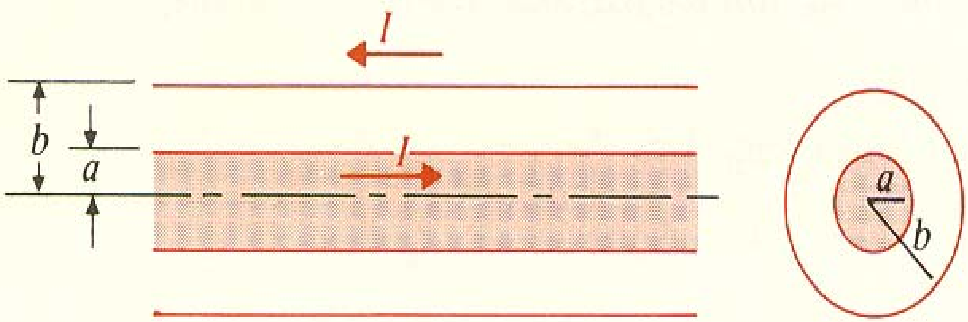


FIGURE 6–24
Two views of a coaxial transmission line (Example 6–16).

$\vec{B}$ for i) $0 \le r \le a$

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

$$2\pi r B = \mu_0 I \frac{r^2}{a^2}$$

$$\vec{B}_1 = \frac{\mu_0 I r}{2\pi a^2} \hat{a}_\phi$$

ii) $a \le r \le b$

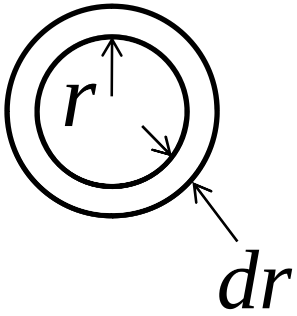
$$2\pi r B = \mu_0 I$$

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi r} \hat{a}_\phi$$

Consider an annular ring in the inner conductor.

between r and r + dr.

The current in the annular ring is linked by the flux



$$d\Phi^{'} = \int_{r}^{a} B_1 dr + \int_{a}^{b} B_2 dr$$

$$= \frac{\mu_0 I}{2\pi a^2}\int_{r}^{a} r dr + \frac{\mu_0 I}{2\pi}\int_{a}^{b} \frac{dr}{r}$$

$$= \frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 I}{2\pi} l\mathrm{n} \frac{b}{a}$$

since the current in the annular ring is

$$I^{'} = \frac{2\pi rdr}{\pi a^{2}} = \frac{2rdr}{a^{2}}$$

the flux linkage for the annular ring is

$$d\Lambda^{'} = \frac{2rdr}{a^{2}} d\Phi^{'}$$

and total flux linkage / unit length

$$\Lambda^{'} = \int_{0}^{a} d\Lambda^{'}$$

$$= \int_{0}^{a} \frac{2r}{a^{2}} \left[\frac{\mu_{0} I}{4\pi a^{2}} (a^{2} - r^{2}) + \frac{\mu_{0} I}{2\pi} \ln \frac{b}{a} \right] dr$$

$$= \frac{\mu_{0} I}{2\pi a^{2}} \left(\frac{1}{4} + ln \frac{b}{a} \right)$$

and inductance / unit length is

$$L^{'} = \frac{\Lambda^{'}}{I} = \frac{\mu_{0}}{2\pi} \left(\frac{1}{4} + ln \frac{b}{a} \right)$$

Chap 7.

Time-varying Fields and Maxwell's Eq.

7.1 Introduction

Electrostatic Phenomena

$\nabla \cdot \overrightarrow{D} = \rho$ $\overrightarrow{D} = \varepsilon \overrightarrow{E}$ ε : permittivity

$\nabla \times \overrightarrow{E} = 0$

Magnetostatic

$\nabla \cdot \overrightarrow{B} = 0$

$\nabla \times \overrightarrow{H} = \overrightarrow{j}$ $\overrightarrow{B} = \mu \overrightarrow{H}$ μ : permeability

Time-varying $\vec{E}$ & $\vec{B}$ ⟶ $\vec{H}$ & $\vec{D}$

Modification of curl eq

⟶ complete eqs for time-varying phenomena

⟶ Maxwell's eq

7.2 Faraday's law of Electrical induction

Postulate :

$$\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}$$

$$\int_s \nabla \times \overrightarrow{E} \cdot d\overrightarrow{S} = \oint \overrightarrow{E} \cdot d\overrightarrow{l} = -\int_s \frac{\partial \overrightarrow{B}}{\partial t} \cdot d\overrightarrow{S}$$

7.2.1 Stationary circuit in a time-varying Magnetic Field

$$V = -\frac{d\Phi}{dt} \qquad \int_s \frac{\partial \overrightarrow{B}}{\partial t} \cdot dS = \frac{d}{dt}\int_S \overrightarrow{B} \cdot d\overrightarrow{S}$$

where

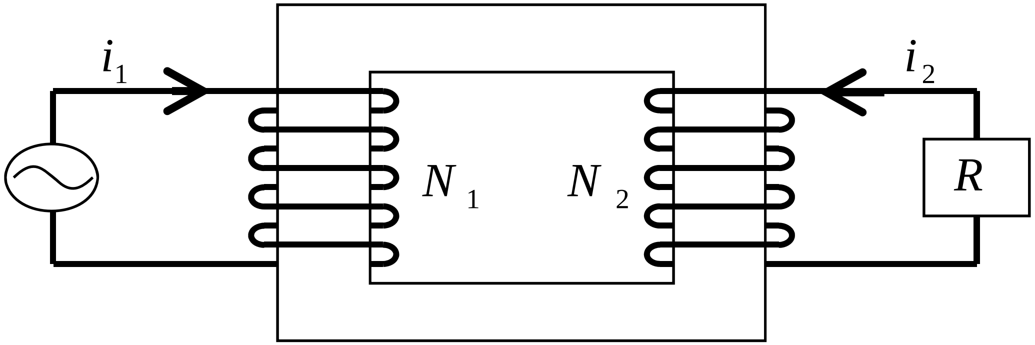
$V = \oint \overrightarrow{E} \cdot d\overrightarrow{l}$: emf induced in circuit (V)

$\Phi = \int_S \overrightarrow{B} \cdot d\overrightarrow{S}$: magnetic flux (Wb)

“ **-** ” sign : Lenz’s law

7.2.2 Transformers

Transformer : AC device which can transform
Voltage, current, impedance



$$\sum N_j I_j = \sum R_k \Phi_k$$

$$N_1 i_1 + \left(- N_2 i_2 \right) = R \Phi$$

Reluctance of magnetic core

$$R = \frac{l}{\mu S}$$

a) Ideal transformer : $\mu \to \infty$

$$N_1 i_1 = N_2 i_2 \quad \to \quad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

Faraday's law

$$v_1 = N_1 \frac{d\Phi}{dt}, \qquad v_2 = N_2 \frac{d\Phi}{dt} \quad \rightarrow \quad \frac{v_1}{v_2} = \frac{N_1}{N_2}$$

Effective load seen by source connected to primary winding

$$(R_1)_{eff} = \frac{v_1}{i_1} = \frac{\frac{N_1}{N_2} v_2}{\frac{N_2}{N_1} i_2} = \left(\frac{N_1}{N_2} \right)^2 R_L$$

같은 방법으로,

$$(Z_1)_{eff} = \left(\frac{N_1}{N_2} \right)^2 Z_L$$

b) Real transformer ⟶ finite μ

⟶ non-vanishing R

$$\Phi = \frac{1}{R}\left[N_1 i_1 - N_2 i_2 \right]$$

using this in Faraday's law

$$v_1 = N_1 \frac{d\Phi}{dt}$$

$$= N_1 \frac{d}{dt}\left[\frac{1}{R}\left(N_1 i_1 - N_2 i_2 \right)\right]$$

$$= \frac{N_1^2}{R}\frac{di_1}{dt} - \frac{N_1 N_2}{R}\frac{d}{dt} i_2$$

$$= L_1 \frac{di_1}{dt} - L_{12}\frac{di_2}{dt}$$

$$v_2 = N_2 \frac{d\Phi}{dt}$$

$$= L_{12} \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

where $L_1 = \frac{N_1^2}{R} = \frac{\mu S}{l} N_1^2$: self-inductance

$L_2 = \frac{\mu S}{l} N_2^2$: self-inductance

$L_{12} = \frac{\mu S}{l} N_1 N_2$: mutual inductance

In genual, $L_{12} = \sqrt{L_1 L_2}$

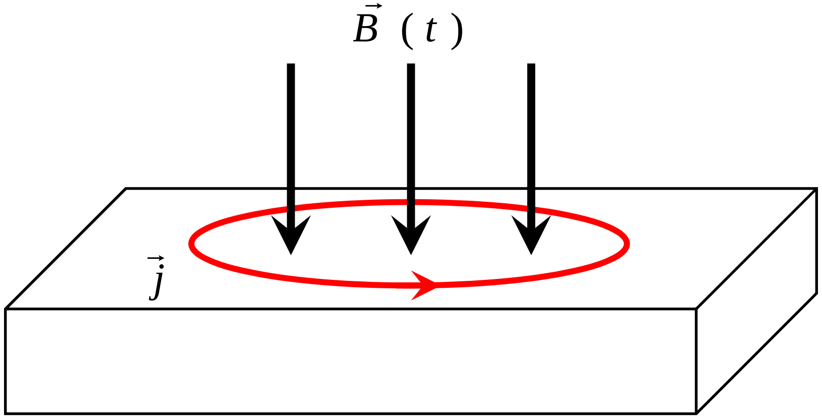
$L_{12} = k\sqrt{L_1 L_2}$,

k(coefficient of coupling) <1

Real transformer :

1. Leakage flux
2. Non-infinite inductance
3. Nonzero winding resistance
4. Hysteresis and eddy-current loss
5. Nonlinear nature of ferromagnetic core

Eddy Current



$$\frac{dB}{dt} \longrightarrow \text{induced emf}$$

$$\longrightarrow \vec{j} = \sigma \vec{E}$$

$$\longrightarrow W_{loss} = \vec{j} \cdot \vec{E} = \sigma E^2$$

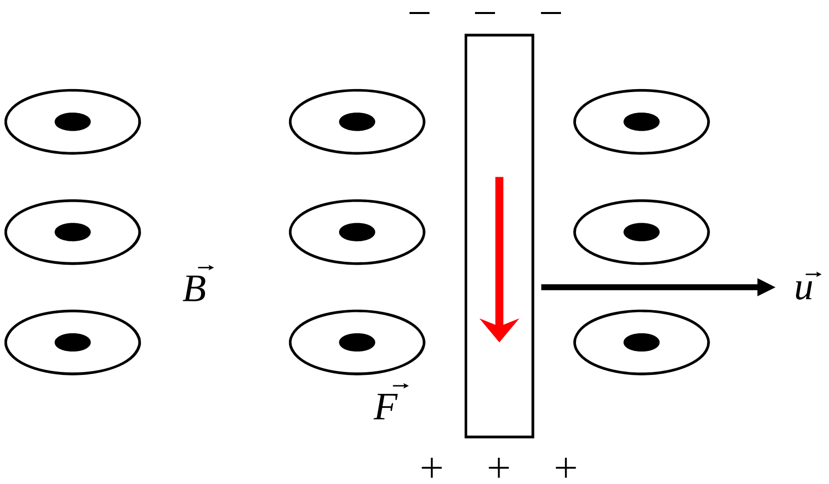
줄이는 방법

1. High μ , low σ material
2. Lamination

변압기에서의 Power loss

1. Hysteresis loss
2. Eddy current loss

7.2.3 Moving conductor in a static magnetic field



Force on the charge in conductor

$$\vec{F} = q\,\vec{u} \times \vec{B}$$

Induced electric field

$$\vec{E}_{ind} = \frac{\vec{F}}{q} = \vec{u} \times \vec{B}$$

Induced emf

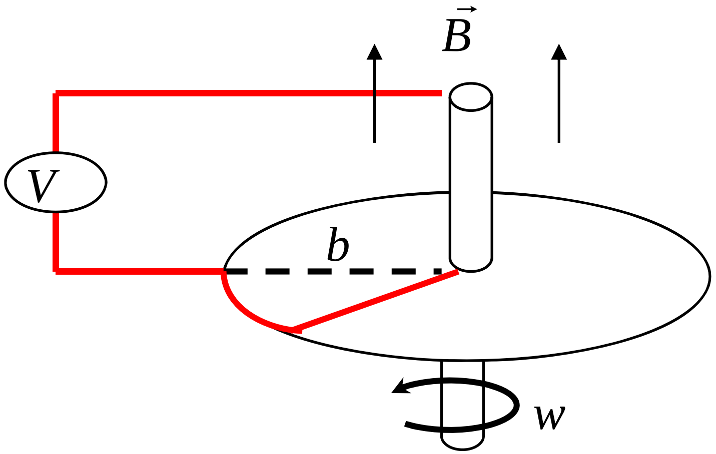
$$V = \oint \vec{u} \times \vec{B} \cdot dl$$

: Motional emf

or flux cutting emf

Ex 7.3 Faraday Disk generator
(Homopolar generator)

Rotating metal disk in a magnetic field



$$V_o = \oint \vec{u} \times \vec{B} \cdot d\vec{l}$$

$$= \oint \left(r\omega\,\hat{a}_\phi \right) \times \left(B_o\,\hat{z} \right) \cdot \hat{a}_r\, dr$$

$$= \omega B_o \int_b^0 r\, dr$$

$$= -\frac{1}{2}\omega B_o b^2$$

7.2.4 Moving circuit in a time-varying magnetic field

Charge q moving with $\vec{v}$ in $\vec{E}$ & $\vec{B}$ field

⟶ force $\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$: Lorentz force

Force measured by an observer in moving frame

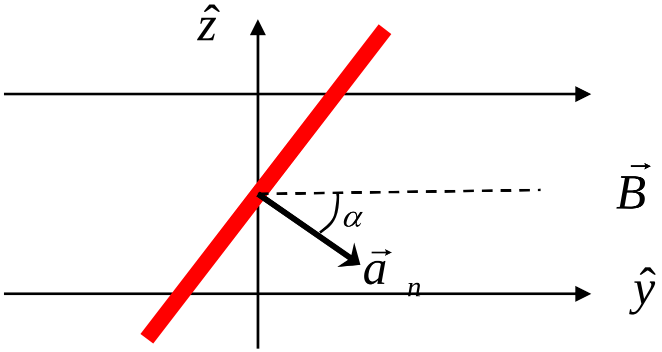
⟶ Electric field $\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$

Electric field seen by an observer

$$\oint_C \vec{E}' \cdot d\vec{l} = \oint_C \vec{E} \cdot d\vec{l} + \oint_C \vec{u} \times \vec{B} \cdot d\vec{l}$$

$$= -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \vec{u} \times \vec{B} \cdot d\vec{l}$$

Ex 7.4



1. $$\Phi = \int \vec{B} \cdot d\vec{S}$$

$$= \left(\hat{a}_y B_o \sin \omega t\right) \cdot \left(hW \hat{a}_n\right)$$

$$= B_o hW \sin \omega t \cos \alpha$$

$$V = -\frac{d\Phi}{dt} = -B_o hW \omega \cos \omega t \cos \alpha$$

2. Motional emf

$$V' = \oint \vec{u} \times \vec{B} \cdot d\vec{l}$$

$$= \left(\hat{a}_n \frac{W}{2}\omega\right) \times \left(\hat{a}_y B_o \sin \omega t\right) \cdot h \times 2$$

$$= hW \omega B_o \sin \omega t \sin \alpha$$

$$V_{total} = V + V'$$

$$= B_o hW \omega (-\cos \omega t \cos \alpha + \sin \omega t \sin \alpha)$$

$$= -B_o S \omega \cos(\omega t + \alpha)$$

if $\alpha = 0$ for $t = 0$, $\alpha = \omega t$ and

$$V_{total} = -B_o S \omega \cos 2\omega t$$

7-3 Maxwell's equation

Static field	Time-varying field
$\nabla \cdot \vec{D} = \rho$ (gauss's law)	$\nabla \cdot \vec{D} = \rho$ ①
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{B} = 0$ ②
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$ (Faraday's law) ③
$\nabla \times \vec{H} = \vec{j}$ (Ampere's circuital law) →	$\nabla \times \vec{H} = \vec{j} \boxed{+\,?}$ ④

Charge conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \qquad ⑤$$

when we use eq ④ , and take divergence

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{j}$$

eq ④ and ⑤ are not consistant

But if we add $\dfrac{\partial \vec{D}}{\partial t}$ to RHS of eq ④ , i.e.

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \qquad ④'$$

then

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{j} + \nabla \cdot \left(\frac{\partial \vec{D}}{\partial t} \right) = \nabla \cdot \vec{j} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \vec{j} + \frac{\partial}{\partial t}\rho$$

$\dfrac{\partial \vec{D}}{\partial t}$: Displacement current density

Maxwell's eq

$\nabla \cdot \vec{D} = \rho$ (Gauss's law)

$\nabla \cdot \vec{B} = 0$

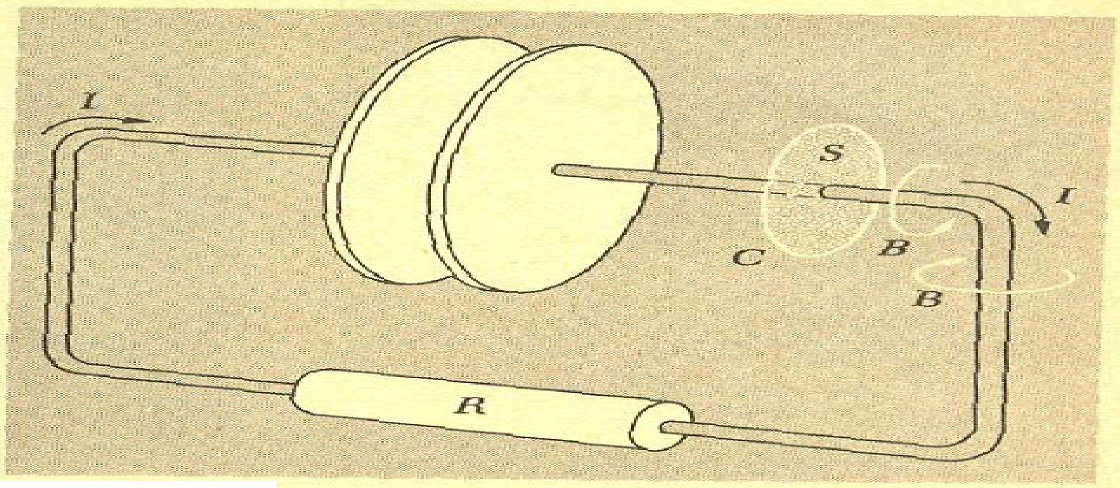
$\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ (Faraday's law)

$\nabla \times \vec{H} = \vec{j} + \dfrac{\partial \vec{D}}{\partial t}$ (Ampere's law)

Something is missing from Ampare's circuital law

$$\nabla \times \vec{H} = \vec{j}$$

consider a capacitor charging as shown in following figure.



Having been charged with the right-hand plate positive, the capacitor is being discharged through the resistor. There is a magnetic field **B** around the wire. The integral of curl **B**, over the surface *S* which passes through the wire, has the value $\mu_0 I$

for surface S.

$$\int_S \nabla \times \vec{H} \cdot d\vec{s} = \oint_C \vec{H} \cdot d\vec{l} = I$$

consider another surface S′ in the following figure. (S′ also spanns contour C)

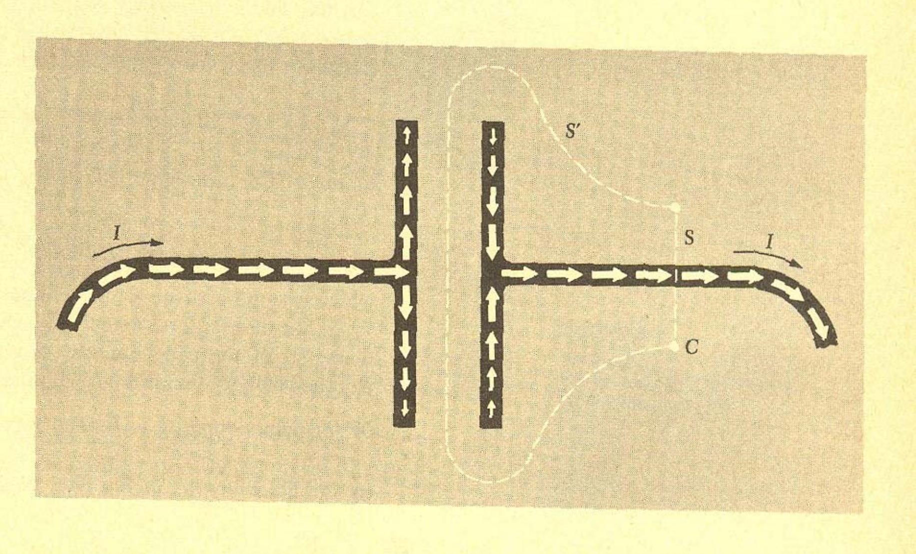


FIGURE 9.2
The white arrows show the current flow in the conductors. The surface *S′*, which like *S* has the curve *C* for its edge, has no current passing through it.

since S′ spanns contour C,

$$\int_{S'} \nabla \times \vec{H} \cdot d\vec{s} = \oint_C \vec{H} \cdot d\vec{l} = 0$$

what is missing ?

Add $\nabla \times \vec{H} = \vec{j} + \boxed{\frac{\partial \vec{D}}{\partial t}}$

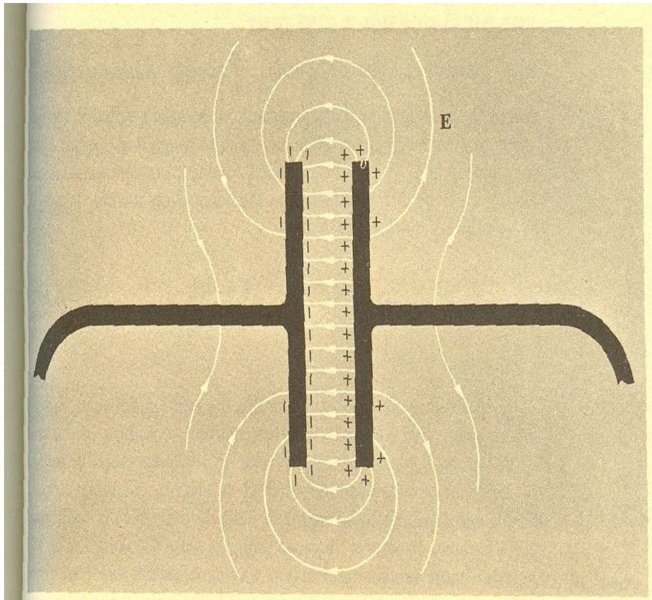


FIGURE 9.3
The electric field at a particular instant. The magnitude of **E** is decreasing everywhere as time goes on.

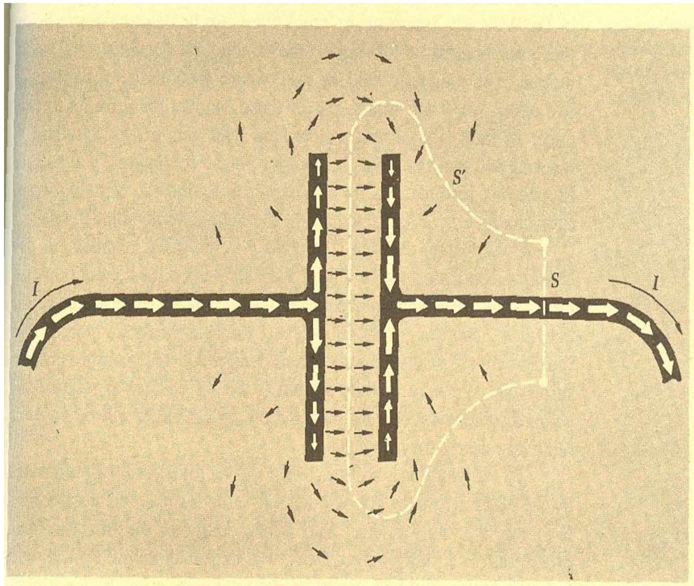


FIGURE 9.4
The conduction current (white arrows) and the displacement current (black arrows).

Electric field

conduction current (white arrow)

displacement current (black arrow)

7-4 Potential functions

since $\nabla \cdot \vec{B} = 0$, $\boxed{\vec{B} = \nabla \times \vec{A} \ \ (*)}$ → eq② (Faraday's law)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

which results in $\boxed{\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad (**)}$

(wave equations)

using (*), (**) in Ampere's eq with $\vec{B} = \mu \vec{H}$, $\vec{D} = \in \vec{E}$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \left(\frac{\vec{B}}{\mu} \right) = \vec{j} + \frac{\partial}{\partial t} \in \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{j} + \mu \in \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{j} - \nabla\left(\mu \in \frac{\partial V}{\partial t} \right) - \mu \in \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 A - \mu \in \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{j} + \nabla\left(\nabla \cdot \vec{A} + \mu \in \frac{\partial V}{\partial t} \right)$$

with $\nabla \cdot \vec{A} + \mu \in \dfrac{\partial V}{\partial t} = 0$ (Lorentz gauge)

$$\boxed{\nabla^2 \vec{A} - \mu \in \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{j}}$$

using (**) in Gauss's eq with $\vec{D} = \in \vec{E}$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \in \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \rho$$

$$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\in}$$

$$\boxed{\nabla^2 V - \mu \in \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\in}}$$

7-5 Electromagnetic Boundary Conditions

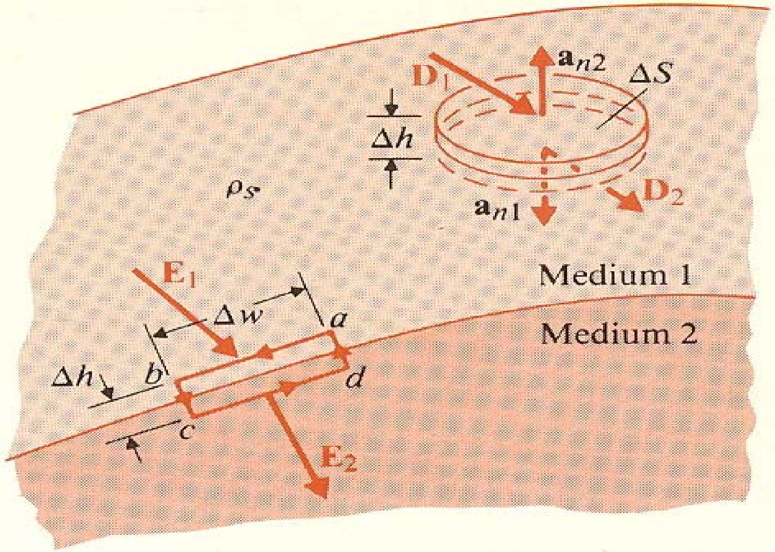


FIGURE 3–23
An interface between two media.

from $\nabla \cdot \vec{D} = \rho,$

$$\int \nabla \cdot \vec{D} dv = \oint \vec{D} \cdot d\vec{s} = (\vec{D}_1 - \vec{D}_2) \cdot \hat{a}_{n2} S = \rho_s S$$

$$\boxed{(\vec{D}_1 - \vec{D}_2) \cdot \hat{a}_{n2} = \rho_s}$$

from $\nabla \cdot \vec{B} = 0$

$$\boxed{B_{1n} = B_{2n}}$$

from $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$$\int \nabla \times \vec{E} \cdot d\vec{s} = \oint \vec{E} \cdot d\vec{l} =$$

$$\int -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{s}$$

with $\Delta h \rightarrow 0$, RHS $\rightarrow 0$

$$\boxed{E_{1t} = E_{2t}}$$

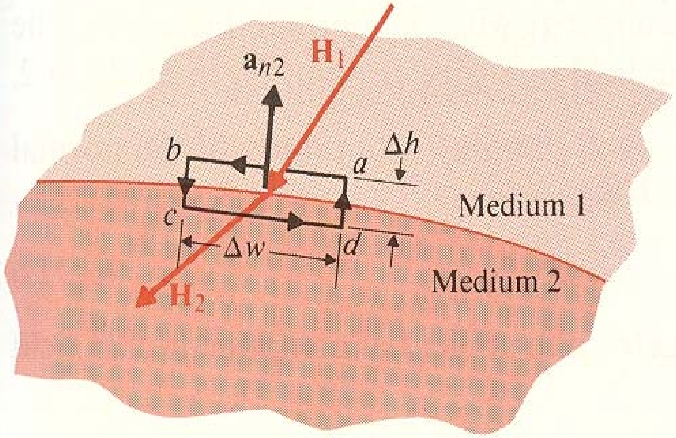


FIGURE 6–19
Closed path about the interface of two media for determining the boundary condition of H_t.

$$\text{from} \quad \nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$$\int \nabla \times \vec{H} \cdot d\vec{s} = \int \vec{j} \cdot d\vec{s} + \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s} \to 0 \quad \text{with} \quad \Delta \mathrm{h} \to 0$$

$$\boxed{H_{1t} - H_{2t} = j_{sn}}$$

$$\text{or} \quad \boxed{\hat{a}_{n2} \times (\vec{H}_1 - \vec{H}_2) = \vec{j}_s}$$

7-6 Wave eq and Thei solutions

7-6-1 Sol of wave eq for potentials

$$\nabla^2 V - \mu \in \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\in}$$

sol for a point charge at the origin at t.

for $\mathrm{r} > 0$, the eq we need to solve is

$$\nabla^2 V - \mu \in \frac{\partial^2 V}{\partial t^2} = 0$$

In spherical coord

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R} V\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin\theta^2}\frac{\partial^2 V}{\partial \phi^2}$$

for a spherically symmetric solution $\left(\frac{\partial}{\partial\theta}V = 0 = \frac{\partial V}{\partial\phi}\right)$

the wave eq becomes

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}V\right) - \mu\in\frac{\partial^2 V}{\partial t^2} = 0$$

with $V(R,\ t) = \frac{1}{R}U(R,\ t)$

$$\frac{\partial^2 U}{\partial R^2} - \mu\in\frac{\partial^2 U}{\partial t^2} = 0$$

$$\text{sol}: \; U(R,\; t) = f\left(t - R\sqrt{\mu \in}\right) + g\left(t + \cancel{R}\sqrt{\mu \in}\right)$$

not physically acceptable solution

and $V(R,\; t) = \dfrac{1}{R}\, f\left(t - R/u\right)$ where $u = \dfrac{1}{\sqrt{\mu \in}}$

since for a static point change $\rho(t)\Delta v'$ at origin gives

$$\Delta V(R) = \frac{\rho(t)\Delta v'}{4\pi \in R}$$

$$V(R,\; t) = \frac{1}{4\pi \in}\int_{V'} \frac{\rho(t - \frac{R}{u})}{R} dv'$$ Retarded scalar potential

similarly

$$\vec{A}(R,\; t) = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{j}(t - \frac{R}{u})}{R} dv'$$ Retarded vector potential

7-6-2 Source -free wave eq

Source free → $\rho, \ \vec{j}$ zero

then Maxwell's eqs reduce to

$$\nabla \cdot \vec{E} = 0 \quad ①$$

$$\nabla \cdot \vec{B} = 0 \quad ②$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad ③$$

$$\nabla \times \vec{H} = \in \frac{\partial \vec{E}}{\partial t} \quad ④$$

(Linear, isotropic, homogeneous, nonconducting medium)

$\nabla \times$③:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H} = -\mu \in \frac{\partial^2}{\partial t^2} \vec{E}$$

$(\nabla \cdot \vec{E} = 0)$

$$\nabla^2 \vec{E} - \frac{1}{u^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0 \qquad u = \frac{1}{\sqrt{\mu \in}}$$

$\nabla \times$④:

$$\nabla \times (\nabla \times \vec{H}) = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \in \frac{\partial}{\partial t} \nabla \times \vec{E} = - \in \mu \frac{\partial^2}{\partial t^2} \vec{H}$$

$(\nabla \cdot \vec{H} = 0)$

$$\nabla^2 \vec{H} - \frac{1}{u^2} \frac{\partial^2}{\partial t^2} \vec{H} = 0$$

7-7 Time-Harmonic Fields

< Review of phasor >

voltage $e(t) = E\cos\omega t = \mathrm{Re}\left[(Ee^{jo})e^{j\omega t}\right] = \mathrm{Re}\left[E_s e^{j\omega t}\right]$

current $i(t) = I\cos(\omega t + \phi) = \mathrm{Re}\left[(Ie^{j\phi})e^{j\omega t}\right] = \mathrm{Re}\left[I_s e^{j\omega t}\right]$

where

$$\left.\begin{aligned} E_s &= Ee^{jo} = E \\ I_s &= Ie^{j\phi} \end{aligned}\right\}\text{phasor}$$

예) A serial RLC circuit with applied voltage e(t)

with $\dfrac{di}{dt} = \mathrm{Re}\left[j\omega I_s e^{j\omega t}\right]$ $\quad\int idt = \mathrm{Re}\left(\dfrac{I_s}{j\omega}e^{j\omega t}\right)$

for $L\dfrac{di}{dt} + Ri + \dfrac{1}{C}\int idt = e(t)$

we have $j\omega LI_s + RI_s + \dfrac{I_s}{Cj\omega} = E_s$

$$I_s = \frac{E_s}{R + j(\omega L - \frac{1}{\omega c})}$$

7-7-2 Time-Harmonic Electromagnetics

Sinusoidal time dependence

$$V(x,y,z,t) = \mathrm{Re}\left[V(x,y,z)e^{j\omega t}\right]$$

$$\vec{A}(x,y,z,t) = \mathrm{Re}\left[\vec{A}(x,y,z)e^{j\omega t}\right]$$

where $V(x,y,z)$, $\vec{A}(x,y,z)$: Potential function phasor

Solution of wave eq for scalar and vector potential V and $\vec{A}$

$$\begin{cases} \nabla^2 V - \mu \in \dfrac{\partial^2}{\partial t^2} V = -\dfrac{\rho}{\in} & ① \\ \nabla^2 \vec{A} - \mu \in \dfrac{\partial^2}{\partial t^2} \vec{A} = -\mu \vec{j} & \end{cases}$$

for phasor $V(x,y,z)$ and $\vec{A}(x,y,z)$

$$\begin{cases} \nabla^2 V + \omega^2 \mu \in V = -\dfrac{\rho}{\in} \\ \nabla^2 \vec{A} + \omega^2 \mu \in \vec{A} = -\mu \vec{j} \end{cases}$$

same notation!

$k \equiv \omega\sqrt{\mu \in} \equiv \omega/u$: wave number

Sol of eq ① has been found as

$$V(R,t)=\frac{1}{4\pi\in}\int\frac{\rho\left(t-\frac{R}{u}\right)}{R}dv'=\frac{1}{4\pi\in}\mathrm{Re}\int\frac{\rho\left(\vec{r}\,'\right)e^{j\omega(t-\frac{R}{u})}}{R}dv'$$

$$=\mathrm{Re}\left[V(R)e^{j\omega t}\right]$$

$$V(R)=\frac{1}{4\pi\in}\int\frac{\rho\left(\vec{r}\,'\right)e^{-j\frac{\omega}{u}R}}{R}dv'$$

similarly

$$\vec{A}(R)=\frac{\mu}{4\pi}\int\frac{\vec{j}e^{-jkR}}{R}dv'$$

when $kR=\frac{\omega}{u}R=\frac{2\pi f}{u}R=2\pi\frac{R}{\lambda}<<1,$

$e^{-j\frac{\omega}{n}R}\sim1$ and

$$V(R)=\frac{1}{4\pi\in}\int\frac{\rho\left(\vec{r}\,'\right)}{R}dv'\rightarrow\text{ static solution}$$

7-7-3 Source-free fields in simple media

$\rho = 0, \quad \vec{j} = 0, \quad \sigma = 0$

from $$\begin{cases} \nabla^2 \vec{E} - \dfrac{1}{u^2} \dfrac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{H} - \dfrac{1}{u^2} \dfrac{\partial^2 \vec{H}}{\partial t^2} = 0 \end{cases}$$

we obtain eq for phasor for $\vec{E}(\vec{r}, t)$ and $\vec{H}(\vec{r}, t)$ as

$$\begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \end{cases}$$

(Homogeneous vector Helmholtz's eq)

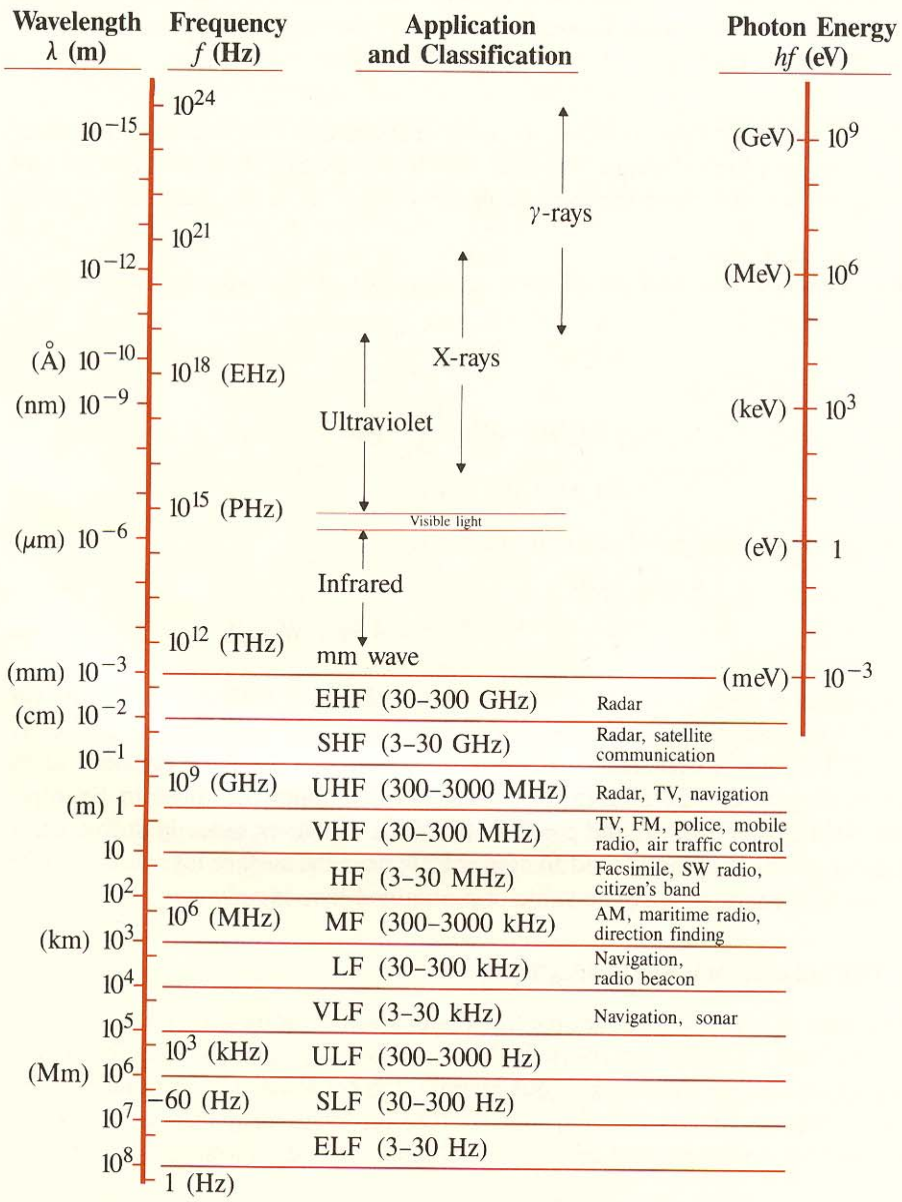


FIGURE 7–9
Spectrum of electromagnetic waves.

전자장 1. (중간시험1)

1. An electric field is given as $\vec{E} = \hat{a}_x y + \hat{a}_y x$ (v/m).
   Find the work needed to move a point charge $Q(C)$ from P_1 (2, 1) to P_2 (8, 2) (unit :m)
   a) Along the parabola $x = 2y^2$ (5)
   b) Along the straight line joining the two points. (5)
   c) Is this $\vec{E}$ a conservative field? (5)

$$\frac{W}{q} = -\int_{P_1}^{P_2} \vec{E} \bullet d\vec{l} = -\int_{P_1}^{P_2} (\hat{a}_x y + \hat{a}_y x) \bullet (\hat{a}_x dx + \hat{a}_y dy) = -\int_{P_1}^{P_2} (ydx + xdy)$$

a) $W = -Q\int_1^2 y \cdot 4ydy + 2y^2 dy = -Q\int_1^2 6y^2 dy = -14Q$

b) 두 점을 연결하는 선의 식은 $y = \frac{1}{6}x + \frac{2}{3} \Rightarrow x = 6y - 4$

$$W = -Q\int_1^2 y \cdot 6dy + (6y-4)dy = -Q\int_1^2 (12y-4)dy = -14Q$$

c) 경로에 상관없이 일정한 값이 가지고, $\nabla \times \vec{E} = \frac{\partial Ey}{\partial x} - \frac{\partial Ex}{\partial y} = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 0$ 을 만족하므로 conservative field이다.

2. Answer to the following questions
   a) Find the electric field on the axis of a circular disk of radius b which carries a uniform surface charge density σ_s lying on x-y plane. (10)
   b) A dielectric cylinder (radius b, height L) has a polarization $\vec{P} = P\hat{z}$ (P is a constant) with its center at the origin and standing along z-axis.
   Find the surface and volume bound-charge density. (10)
   Find the electric field on the z-axis at (0, 0, z) (10)

a) Surface charge distribution $V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\upsilon_s}{R} ds'$ 를 이용하면

$ds' = r' dr' d\phi', R = \sqrt{z^2 + r'^2}$ 이므로 electric potential potential은

$$V = \frac{\rho_s}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^b \frac{r^{'}}{\sqrt{(z^2 + r^{'2})}} dr^{'} d\phi^{'} = \frac{\rho_s}{2\varepsilon_0}\left[\sqrt{z^2 + b^2} - |z|\right]$$

$$\vec{E} = -\nabla V = -\hat{a}_z \frac{\partial V}{\partial z}$$

$$= \begin{cases} \hat{a}_z \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{(z^2 + b^2)}}\right], & z > 0 \\ -\hat{a}_z \frac{\rho_s}{2\varepsilon_0}\left[1 + \frac{z}{\sqrt{(z^2 + b^2)}}\right], & z < 0 \end{cases}$$

b) Surface bound charge density = $\rho_{PS} = \vec{P} \bullet \hat{a}_n$

$$\rho_{PS\_윗면} = P\hat{z} \cdot \hat{a}_z = P$$

$$\rho_{PS\_아랫면} = P\hat{z} \cdot (-\hat{a}_z) = -P$$

$$\rho_{PS\_옆면} = P\hat{z} \cdot (-\hat{a}_r) = 0$$

Volume bound charge density = ρ_P

$$\rho_P = -\nabla \cdot \vec{P} = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (0, 0, P) = 0$$

전기장은 윗면과 아랫면을 고려하면,

i) z > 2/L

$$\vec{E} = \frac{P}{2\varepsilon_0}\left(1 - \frac{z - \frac{1}{2}L}{\sqrt{b^2 + \left(z - \frac{1}{2}L\right)^2}}\right)\hat{a}_z - \frac{P}{2\varepsilon_0}\left(1 - \frac{z + \frac{1}{2}L}{\sqrt{b^2 + \left(z + \frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

$$= \frac{P}{2\varepsilon_0}\left(\frac{z + \frac{1}{2}L}{\sqrt{b^2 + \left(z + \frac{1}{2}L\right)^2}} - \frac{z - \frac{1}{2}L}{\sqrt{b^2 + \left(z - \frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

ii) -2/L < z < 2/L

$$\vec{E} = -\frac{P}{2\varepsilon_0}\left(1+\frac{z-\frac{1}{2}L}{\sqrt{b^2+\left(z-\frac{1}{2}L\right)^2}}\right)\hat{a}_z - \frac{P}{2\varepsilon_0}\left(1-\frac{z+\frac{1}{2}L}{\sqrt{b^2+\left(z+\frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

$$= -\frac{P}{2\varepsilon_0}\left(\frac{z-\frac{1}{2}L}{\sqrt{b^2+\left(z-\frac{1}{2}L\right)^2}} - \frac{z+\frac{1}{2}L}{\sqrt{b^2+\left(z+\frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

iii) z < -2/L

$$\vec{E} = -\frac{P}{2\varepsilon_0}\left(1+\frac{z-\frac{1}{2}L}{\sqrt{b^2+\left(z-\frac{1}{2}L\right)^2}}\right)\hat{a}_z + \frac{P}{2\varepsilon_0}\left(1-\frac{z+\frac{1}{2}L}{\sqrt{b^2+\left(z+\frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

$$= -\frac{P}{2\varepsilon_0}\left(\frac{z+\frac{1}{2}L}{\sqrt{b^2+\left(z+\frac{1}{2}L\right)^2}} + \frac{z-\frac{1}{2}L}{\sqrt{b^2+\left(z-\frac{1}{2}L\right)^2}}\right)\hat{a}_z$$

3. A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a spherical inner wall of radius R_0 . Assume the inner and outer conductor carries charge +Q and –Q respectively.
   a) Find the electric field between the inner and outer conductors. (5)
   b) Find the capacitance. (10)
   c) Find the electric energy stored in the space between the inner and outer conductors. (5)

----

a) Conduction sphere 사이에 Gaussian surface를 잡으면

$$\oint_S \vec{E}\bullet d\vec{s} = \frac{q}{\varepsilon}$$

$\vec{E}$ 이 중심에서 나가는 방향 $\hat{a}_r$ 은 일정한 크기를 가지므로

$$\frac{q}{\varepsilon} = \oint_S \vec{E}\bullet d\vec{s} = E\oint_S \hat{a}_r \bullet \hat{a}_r ds = E\oint_S \ ds = E4\pi r^2$$

$$\therefore \vec{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{a}_r$$

b)두 Conductor간의 전위차 $V = -\int_{R_0}^{R_t}\vec{E}\bullet d\vec{l} = -\int_{R_0}^{R_t}\frac{Q}{4\pi\varepsilon r^2}\bullet dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_i}-\frac{1}{R_0}\right)$

$$C = \frac{Q}{V} = \left(\frac{4\pi\varepsilon}{\left(\frac{1}{R_i} - \frac{1}{R_0} \right)} \right)$$

c)Electric energy

$$W_e = \frac{1}{2} QV = \frac{1}{2} CV^2 = \frac{Q^2}{2C} = \frac{Q^2}{2} \frac{1}{4\pi\varepsilon} \left(\frac{1}{R_i} - \frac{1}{R_0} \right) = \frac{Q^2}{8\pi\varepsilon} \left(\frac{1}{R_i} - \frac{1}{R_0} \right)$$

---

4. A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and outer radius R_0. The dielectric constant of the shell is ϵ_r.
   a) Find the electric field $\vec{E}$(3), potential V(3), displacement $\vec{D}$(2) and polarization vector $\vec{P}$(3) for $R_i < R < R_0$.
   b) Find the surface bound-charge density on the inner(2) and outer(2) surfaces of dielectric shell. Also find the volume bound-charge density inside of dielectric shell. (2)

---

a) $R_i < R < R_0$에서 Gaussian 곡면을 잡으면 $\vec{E}$는 구면에 수직인 방향이고 같은 크기를

갖는다

$$\oint_S \vec{E} \bullet d\vec{s} = E\oint_S \hat{a}_r \bullet \hat{a}_r ds = E\oint_S \ ds = E4\pi R^2 = \frac{Q}{\varepsilon}$$

$$E = \frac{Q}{4\pi R^2 \varepsilon} = \frac{Q}{4\pi R^2 \varepsilon_0 \varepsilon_r}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon R^2}\hat{a}_r$$

$$V = -\int_{\infty}^{R_0} E \cdot dR - \int_{R_0}^{R} E \cdot dR = -\int_{\infty}^{R_0} \frac{Q}{4\pi\varepsilon_0 R^2} \cdot dR - \int_{R_0}^{R} \frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2} \cdot dR = \frac{Q}{4\pi\varepsilon_0}\left[\left(1-\frac{1}{\varepsilon_r}\right)\frac{1}{R_0} + \frac{1}{\varepsilon_r R}\right]$$

$$\vec{D} = \varepsilon\vec{E} = \varepsilon\frac{Q}{4\pi\varepsilon R^2}\hat{a}_r = \frac{Q}{4\pi R^2}\hat{a}_r$$

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = \frac{Q}{4\pi R^2}\hat{a} - \varepsilon_0\frac{Q}{4\pi\varepsilon_0\varepsilon_r R^2}\hat{a}_r = \frac{Q}{4\pi R^2}\left(1-\frac{1}{\varepsilon_r}\right)\hat{a}$$

b) inner surface bound-charge density

$$\rho_{PS}\Big|_{R=R_i} = \vec{P} \bullet (-a_R)\Big|_{R=R_i} = -P\Big|_{R=R_i} = -\frac{Q}{4\pi R_i^{\ 2}}\left(1-\frac{1}{\varepsilon_r}\right)$$

outer surface bound-charge density

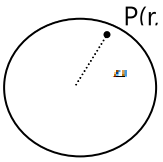
$$\rho_{PS}\Big|_{R=R_0} = \vec{P} \bullet a_R\Big|_{R=R_0} = P\Big|_{R=R_{0i}} = \frac{Q}{4\pi R_0^{\ 2}}\left(1-\frac{1}{\varepsilon_r}\right)$$

Volume bound-charge density

$$\rho_P = -\nabla \bullet \vec{P} = -\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 P_{R2}\right) = 0$$

---

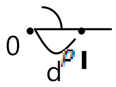
5. Answer to the following questions.
   a) Find the electric potential at a distance r from an infinite, straight line charge of density ρ_l. (5)
   b) An infinite, straight line charge of density ρ_l is placed inside of a parallel, conducting,



circular, hollow cylinder of radius a.

The distance between the line charge and axis of cylinder is d (d<a).

Find the electric potential at P(r, θ) inside of cylinder. (13)



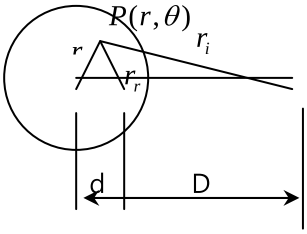
a) 반지름이 r이고 높이가 L인 원기둥을 가정하고 Gauss's law를 적용하면

$$\oint_S \vec{E} \bullet d\vec{s} = \int_0^L \int_0^{2\pi} E_r r d\phi dz = 2\pi r L E_r$$

$$2\pi r L E_r = \frac{\rho_l L}{\varepsilon_0} \qquad E_r = \frac{\rho_l}{2\pi r \varepsilon_0}$$

$$V = -\int_{r_0}^{r} E_r dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

b) M에서의 Electric potential을 구하면



$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$

$\frac{r_i}{r}$ =constant

Real line charge is inside of cylinder

OMPi와 OPM이 같은 지점에 Pi를 두면

$$\frac{\overline{P_iM}}{\overline{\overline{PM}}} = \frac{\overline{OP_i}}{\overline{\overline{OM}}} = \frac{\overline{OM}}{\overline{\overline{OP}}} \qquad or \qquad \frac{r_i}{r} = \frac{d}{a} = \frac{a}{D} = \text{constant}$$

$$\therefore D = \frac{a^2}{d}$$

$$\frac{r_i}{r_r} = \frac{\sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\left(\frac{a^2}{d}\right)\cos\theta}}{\sqrt{r^2 + d^2 - 2rd\cos\theta}}$$

$$V = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{\sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\left(\frac{a^2}{d}\right)\cos\theta}}{\sqrt{r^2 + d^2 - 2rd\cos\theta}}\right)$$

$$= \frac{\rho_l}{4\pi\varepsilon_0}\ln\left(\frac{r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\left(\frac{a^2}{d}\right)\cos\theta}{r^2 + d^2 - 2rd\cos\theta}\right)$$

전자장 1 중간고사 2차 답안지

1.

boundary conditions : $V(0,y) = V_1$, $V(x,0) = 0$, $V(x,b) = 0$, $V(a,y) = V_2$

변수분리법을 이용하면 $V(x,y) = X(x)Y(y)$.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial X^2}{\partial x^2} = -\frac{1}{Y}\frac{\partial Y^2}{\partial y^2} = \lambda = k^2$$

각각의 변수에 대하여 풀면,

$X(x) = Asinh(kx) + Bcosh(kx)$

$Y(y) = C\sin(ky) + Dcos(ky)$

boundary conditions를 superposition으로 나누어서 계산한다.

$V_a(0,y) = V_1$, $V_a(x,0) = 0$, $V_a(x,b) = 0$, $V_a(a,y) = 0$ ------------(1)

$V_b(0,y) = 0$, $V_b(x,0) = 0$, $V_b(x,b) = 0$, $V_b(a,y) = V_2$ ------------(2)

우선 첫 번째 boundary conditions를 대입하여 풀면

$Y(0) = 0$에서 $D = 0$, $Y(b) = 0$에서 $k = \frac{n\pi}{b}$.

$X(a) = 0$에서 $X(a) = Asinh(ka) + Bcosh(ka) = 0$,

$$B = -\frac{Asinh(ka)}{\cosh(ka)} \text{이므로,}$$

$$X(x) = A(\sinh(kx) - \frac{\sinh(ka)}{\cosh(ka)}cosh(kx)) = \frac{A}{\cosh(ka)}(\cosh(ka)\sinh(kx) - \sinh(ka)\cosh(kx))$$

$$= \frac{A}{\cosh(ka)}sinh(k(x-a))$$

$$V_a(x,y) = \sum_{n=1}^{\infty} K_n \sinh(\frac{n\pi}{b}(x-a))\sin(\frac{n\pi}{b}y), \qquad K_n = \frac{AC}{\cosh(ka)}$$

$V_a(0,y) = V_1$ 이므로 $V_a(0,y) = \sum_{n=1}^{\infty} K_n \sinh(\frac{-n\pi a}{b})\sin(\frac{n\pi}{b}y) = V_1$

$$K_n \sinh(\frac{-n\pi a}{b}) = \frac{2}{b}\int_0^b V_1 \sin(\frac{n\pi}{b}y)dy = \begin{cases} 4V_1/n\pi & n\ is\ odd \\ 0 & n\ is\ even \end{cases}$$

$$K_n = -\frac{4V_1}{n\pi \sinh(\frac{n\pi a}{b})}, \quad n\ is\ odd\ .$$

그러므로 $V_a(x,y) = \sum_{n=1}^{\infty} -\frac{4V_1}{n\pi\sinh(\frac{n\pi a}{b})} sinh(\frac{n\pi}{b}(x-a))\sin(\frac{n\pi}{b}y), \quad n\ is\ odd$ -----(3)

마찬가지로 두 번째 boundary conditions의 조건에서 풀면,

$Y(y)$의 조건은 동일하므로 $k=\frac{n\pi}{b}$.

$X(0)=0$ 에서 $B=0$, 그러므로 $X(x)=Asinh(kx)$

$$V_b(x,y)=\sum_{n=1}^{\infty}L_n\sinh(\frac{n\pi}{b}x)\sin(\frac{n\pi}{b}y).$$

boundary condition $V_b(a,y)=\sum_{n=1}^{\infty}L_n\sinh(\frac{n\pi a}{b})\sin(\frac{n\pi}{b}y)=V_2$

$$L_n\sinh(\frac{n\pi a}{b})=\frac{2}{b}\int_0^b V_2\sin(\frac{n\pi}{b}y)dy=\begin{cases}4V_2/n\pi & n\ is\ odd\\0 & n\ is\ even\end{cases}$$

$$L_n=\frac{4V_2}{n\pi\sinh(\frac{n\pi a}{b})}\qquad n\ is\ odd$$

$$V_b(x,y)=\sum_{n=1}^{\infty}\frac{4V_2}{n\pi\sinh(\frac{n\pi a}{b})}sinh(\frac{n\pi x}{b})\sin(\frac{n\pi}{b}y)\qquad n\ is\ odd\ \text{------}(4)$$

그러므로 (3) 과 (4) 를 통해서 $V(x,y)=V_a(x,y)+V_b(x,y)$

$$V(x,y)=\sum_{n=1}^{\infty}\frac{4\sinh(\frac{n\pi}{b}y)}{n\pi\sinh(\frac{n\pi a}{b})}(-V_1\sinh(\frac{n\pi}{b}(x-a))+V_2\sinh(\frac{n\pi}{b}x))\qquad n\ is\ odd$$

2. (a)

$$F=ma=m\frac{du}{dt}=m\frac{du}{dy}\frac{dy}{dt}=mu\frac{du}{dy}\quad\text{------------------}(1)$$

$$F=-eE=-e(-\frac{dV}{dy})=e\frac{dV}{dy}\quad\text{----------------------}(2)$$

(1) 과 (2)에서

$$mu\frac{du}{dy}=e\frac{dV}{dy}\quad\Leftrightarrow\quad mudu=edV\quad\Leftrightarrow\quad\frac{1}{2}mu^2=eV\quad\Leftrightarrow\quad u=\sqrt{\frac{2eV}{m}}$$

$J=-\rho u$, $\nabla^2V=-\frac{\rho}{\epsilon_0}$ 이고, 현재 y 방향만 생각하므로,

$\frac{d^2V}{dy^2}=\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}V^{-\frac{1}{2}}$ 양변에 $2\frac{dV}{dy}$를 곱하면,

우변 : $2\frac{dV}{dy}\frac{d^2V}{dy^2}=\frac{d}{dy}((\frac{dV}{dy})^2)$

좌변 : $\frac{2J}{\epsilon_0}\sqrt{\frac{m}{2e}}V^{-\frac{1}{2}}\frac{dV}{dy}$

dy를 소거 후 적분하면, $(\frac{dV}{dy})^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{\frac{1}{2}}$

(상수항은 initial condition에 의해 0이 된다.)

$$\frac{dV}{dy} = 2\sqrt{\frac{J}{\epsilon_0}}\ \sqrt[\frac{1}{4}]{\frac{m}{2e}}\,V^{\frac{1}{4}} \quad \Leftrightarrow \quad V^{-\frac{1}{4}}dV = 2\sqrt{\frac{J}{\epsilon_0}}\ \sqrt[\frac{1}{4}]{\frac{m}{2e}}\,dy$$

$$\int_0^{V_0} V^{-\frac{1}{4}}dV = \int_0^d 2\sqrt{\frac{J}{\epsilon_0}}\ \sqrt[\frac{1}{4}]{\frac{m}{2e}}\,dy \quad \Leftrightarrow \quad \frac{4}{3}V_0^{\frac{3}{4}} = 2d\sqrt{\frac{J}{\epsilon_0}}\ \sqrt[\frac{1}{4}]{\frac{m}{2e}}$$

그러므로 $J = \frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}\,V_0^{\frac{3}{2}}$

(b) conductivity $\sigma(y) = \sigma_1 + \frac{(\sigma_2 - \sigma_1)}{d}y$ 이다.

I)

$dR = \frac{1}{S\sigma(y)}dy$ 이므로

$R = \int_0^d \frac{1}{S\sigma(y)}dy = \int_0^d \frac{1}{S(\sigma_1 + (\sigma_2 - \sigma_1)y/d)}dy$

$\sigma_1 + (\sigma_2 - \sigma_1)y/d \equiv A$ 라 치환하면

$R = \frac{1}{S}\int_{\sigma_1}^{\sigma_2}\frac{1}{A}\frac{d}{\sigma_2 - \sigma_1}dA = \frac{d}{S(\sigma_2 - \sigma_1)}ln(\frac{\sigma_2}{\sigma_1})$

II)

$$J_0 = \frac{I_0}{S} = \frac{V_0}{S\,R} = \frac{V_0}{S}\frac{S(\sigma_2 - \sigma_1)}{d}\frac{1}{\ln(\sigma_2/\sigma_1)} = \frac{V_0(\sigma_2 - \sigma_1)}{d}\frac{1}{\ln(\sigma_2/\sigma_1)}$$

a. at the top plate.

$\rho_{top} = \epsilon_0 E(d) = \epsilon_0\frac{J_0}{\sigma_2} = \frac{\epsilon_0 V_0(\sigma_2 - \sigma_1)}{\sigma_2 d\ln(\sigma_2/\sigma_1)}$

b. at the bottom plate.

$\rho_{bottom} = -\epsilon_0 E(0) = -\epsilon_0\frac{J_0}{\sigma_1} = -\frac{\epsilon_0 V_0(\sigma_2 - \sigma_1)}{\sigma_1 d\ln(\sigma_2/\sigma_1)}$

III)

II)에서 $J_0 = \frac{V_0(\sigma_2-\sigma_1)}{d}\frac{1}{\ln(\sigma_2/\sigma_1)}$ 이므로

$$\overrightarrow{E(y)} = -\frac{J_0}{\sigma(y)}\overrightarrow{u_y} = -\frac{V_0(\sigma_2-\sigma_1)}{d}\frac{1}{\ln(\sigma_2/\sigma_1)}\frac{1}{\sigma(y)}\overrightarrow{u_y}$$

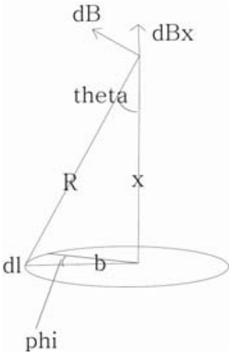
$$\rho_v = \nabla \cdot \overrightarrow{D} = \nabla \cdot \epsilon\overrightarrow{E} = \nabla \cdot \left(-\frac{\epsilon V_0(\sigma_2-\sigma_1)}{d}\frac{1}{\ln(\sigma_2/\sigma_1)}\frac{1}{\sigma(y)}\overrightarrow{u_y}\right) = -\frac{\epsilon V_0(\sigma_2-\sigma_1)}{d\ln(\sigma_2/\sigma_1)}\frac{d}{dy}\left(\frac{1}{(\sigma_2-\sigma_1)y/d+\sigma_1}\right)$$

$$Q_{in} = S\int_0^d \rho_v dy = S\int_0^d -\frac{\epsilon V_0(\sigma_2-\sigma_1)}{d\ln(\sigma_2/\sigma_1)}\frac{d}{dy}\left(\frac{1}{(\sigma_2-\sigma_1)y/d+\sigma_1}\right)dy$$

$$= | -\frac{\epsilon S V_0(\sigma_2-\sigma_1)}{d\ln(\sigma_2/\sigma_1)}\left(\frac{1}{(\sigma_2-\sigma_1)y/d+\sigma_1}\right)|_{y=0}^{y=d} = \frac{\epsilon S V_0(\sigma_2-\sigma_1)}{d\ln(\sigma_2/\sigma_1)}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)$$

그러므로 총 전하 $Q = Q_{in} + \rho_{top}S + \rho_{bottom}S = 0$

3. (a)



미소 구간 $d\ell$ 에 전류 I 가 흐를 때 발생하는 자기장

$$d\overrightarrow{B} = \frac{\mu_0}{4\pi}\frac{I\,d\overrightarrow{\ell}\times\overrightarrow{R}}{R^3}$$

전류가 원형 코일 안을 흐르므로 x 축 방향만 남고 다른 축 성분은 서로 상쇄되어 없어진다.

$$d\overrightarrow{B_x} = \frac{\mu_0}{4\pi}\frac{I\,d\ell}{R^2}sin\theta\overrightarrow{u_x} = \frac{\mu_0 I}{4\pi}\frac{b\,d\phi}{b^2+x^2}\frac{b}{\sqrt{(b^2+x^2)}}\overrightarrow{u_x}$$

$$\overrightarrow{B_x} = \int_0^{2\pi}\frac{\mu_0 I}{4\pi}\frac{b}{b^2+x^2}\frac{b\overrightarrow{u_x}}{\sqrt{(b^2+x^2)}}d\phi = \frac{\mu_0 I b^2}{2(b^2+x^2)^{3/2}}\overrightarrow{u_x}$$

magnetic moment $\overrightarrow{M} = I\overrightarrow{A} = \pi b^2 I\,\overrightarrow{u_x}$

(b)

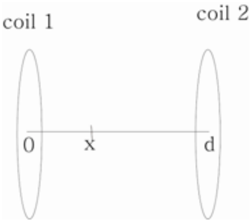
I) 두 코일의 $\overrightarrow{B_x}$ 를 합친다.

midpoint에서 왼쪽 코일은 $x = d/2$, 오른쪽 코일은 $x = -d/2$를 넣고 (a)식에 대입
각각 N turns 돼 있으므로 N배.

$$\overrightarrow{B_{x1}} = \frac{N\mu_0 I b^2}{2(b^2+(d/2)^2)^{3/2}}\overrightarrow{u_x}, \qquad \overrightarrow{B_{x2}} = \frac{N\mu_0 I b^2}{2(b^2+(-d/2)^2)^{3/2}}\overrightarrow{u_x}$$

$$\overrightarrow{B_x} = \overrightarrow{B_{x1}} + \overrightarrow{B_{x2}} = \frac{N\mu_0 I b^2}{(b^2+(d/2)^2)^{3/2}}\overrightarrow{u_x}$$

II)



오른쪽코일의 x변위를 0이라고 하면
두 코일의 자기장의 합

$$\overrightarrow{B_x} = \overrightarrow{B_{x1}} + \overrightarrow{B_{x2}} = \left(\frac{N\mu_0 I\, b^2}{(b^2+(x)^2)^{3/2}} + \frac{N\mu_0 I\, b^2}{(b^2+(d-x)^2)^{3/2}}\right)\overrightarrow{u_x}$$

$$\frac{d\overrightarrow{B_x}}{dx} = N\mu_0 I b^2\ \overrightarrow{u_x}\quad \frac{d}{dx}\left(\frac{1}{(b^2+(x)^2)^{3/2}} + \frac{1}{(b^2+(d-x)^2)^{3/2}}\right)$$

$$\frac{d\overrightarrow{B_x}}{dx} = N\mu_0 I b^2\ \overrightarrow{u_x}\ \left(-\frac{3}{2}\frac{2x}{(b^2+x^2)^{5/2}} + \frac{3}{2}\frac{2(d-x)}{(b^2+(d-x)^2)^{5/2}}\right)$$

그러므로 midpoint에서 값을 구하면 $\frac{d\overrightarrow{B_x}}{dx}|_{x=d/2} = 0$

4.

I)

$\nabla \cdot \vec{J} = \nabla \cdot \sigma\vec{E} = -\frac{d\rho}{dt}$, 이식에 $\nabla \cdot E = \frac{\rho}{\epsilon}$을 대입하면

$$\frac{\sigma\rho}{\epsilon} = -\frac{d\rho}{dt} \quad \Leftrightarrow \quad -\frac{\sigma}{\epsilon}dt = \frac{d\rho}{\rho} \quad \Leftrightarrow \quad \ln\rho - \ln\rho_0 = \frac{\sigma t}{\epsilon}$$

그러므로 $\rho = \rho_0 \exp(-\frac{\sigma}{\epsilon}t)$

II)

$\nabla \cdot \vec{J} = 0$ 에서 양변을 적분하면

$\int_V \nabla \cdot \vec{J} dv = \oint_S \vec{J} \cdot d\vec{s} = J_{1n}S - J_{2n}S = 0$ 그러므로 $J_{1n} = J_{2n}$

$\nabla \times \vec{E} = \nabla \times \frac{\vec{J}}{\sigma} = 0$ 양변을 적분하면

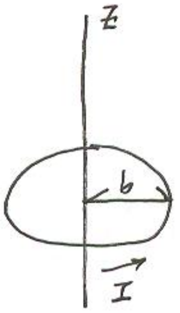
$\int_S \nabla \times \frac{\vec{J}}{\sigma}\ ds = \oint_C \frac{\vec{J}}{\sigma} d\vec{l} = \frac{J_{1t}}{\sigma_1}L - \frac{J_{2t}}{\sigma_2}L = 0$ 그러므로 $\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$

normal: $J_{1n} = J_{2n}$

tangential: $\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$

1.

(1)



$$dB = \frac{u_0}{4\pi}\frac{I\,dl}{R^2} \times a_{\hat{R}}$$

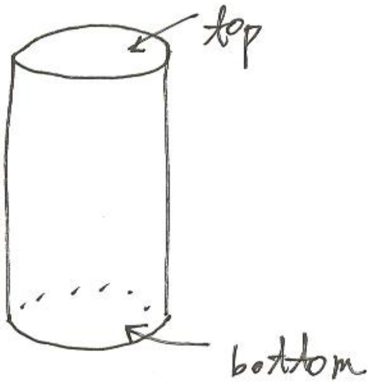
대칭성에 의해서 z 성분만 남음

$$dB_z = \frac{u_0}{4\pi}\cdot\frac{I}{b^2+z^2}\cdot\frac{b}{\sqrt{b^2+z^2}}\,dl$$

$$= \frac{u_0 I}{4\pi} b \frac{b\,d\phi}{(b^2+z^2)^{\frac{3}{2}}} \quad (dl = b\,d\phi)$$

$$B_z = \int_0^{2\pi} \frac{u_0 I b^2}{4\pi} \frac{1}{(b^2+z^2)^{\frac{3}{2}}}\,d\phi = \frac{u_0 I b^2}{2(b^2+z^2)^{\frac{3}{2}}}\hat{a}_z$$

(2)



$$J_{m\theta} = \vec{M} \times \hat{a}_n = \left[\begin{array}{l} 0 \ , (\text{top, bottom}) \\ \vec{M}\times\hat{a}_r = M_0\hat{a}_\phi \\ (\text{side wall}). \end{array}\right.$$

$$dB_z = \hat{a}_z \left(\frac{1}{2}\right) \frac{u_0 M_0 b^2}{2(b^2+(z-z')^2]^{\frac{3}{2}}}\,dz' \frac{1}{z}$$

$$B_z = \frac{u_0 M_0 b^2}{2}\int_0^L \frac{dz'}{(b^2+(z-z')^2)^{\frac{3}{2}}}\,dz'$$

$$z - z' = b\tan\theta$$

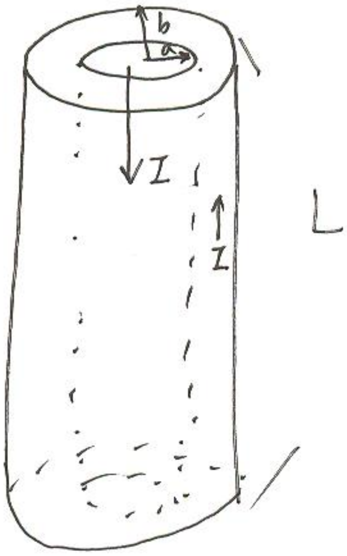
$$dz' = -b\sec^2\theta\, d\theta, \quad z' : 0 \to L, \quad \theta : \theta_1 \to \theta_2$$

$$B_z = \frac{u_0 M_0 b^2}{2} \int_{\theta_1}^{\theta_2} \frac{-b\sec^2\theta}{b^3 \sec^3\theta}\, d\theta$$

$$= -\frac{u_0 M_0}{2} (\sin\theta_2 - \sin\theta_1)$$

$$= \frac{u_0 M_0}{2} \left(\frac{z}{\sqrt{b^2 + z^2}} - \frac{(z - L)}{\sqrt{b^2 + (z - L)^2}} \right)$$

2 .



(1) $r < a$

$$B \cdot 2\pi r = u_0 I \left(\frac{r}{a}\right)^2 , \quad B = \frac{u_0 I}{2\pi a^2} r \,(\hat{\phi})$$

$a < r < b$

$$B = \frac{u_0 I}{2\pi r} \phi$$

$r > b , \quad B = 0$

magnetic energy per unit length in the inner conductor

$$W_1 = \int_0^a \frac{1}{2u_0} B^2 \, dV$$

$$= \int_0^a \frac{1}{2u_0} \cdot \left(\frac{u_0 r I}{2\pi a^2}\right)^2 \cdot L \cdot 2\pi r \, dr .$$

$$W_1 = \frac{1}{2u_0}\left(\frac{u_0^2 I^2}{4\pi^2 a^2}\right)\int_0^a L\, r^2 \cdot 2\pi r\, dr$$

$$= \frac{\pi L}{u_0}\cdot\left(\frac{u_0^2 I^2}{4\pi^2 a^2}\right)\int_0^a r^3\, dr$$

$$= \frac{\pi L}{4 u_0}\,\frac{u_0^2 I^2}{4\pi^2 a^2}\cdot a^4$$

$$W_1 = \frac{u_0 I^2}{16\pi} a^2 L \quad , \quad \frac{W_1}{L} = \frac{u_0 I^2}{16\pi} a^2$$

magnetic E in the inner and out conductor.

$$W_2 = \int_a^b \frac{1}{2u_0}\left(\frac{u_0 I}{2\pi r}\right)^2 2\pi r L\, dr$$

$$= \frac{1}{2u_0}\left(\frac{u_0^2 I^2}{4\pi^2 r^2}\right) 2\pi L \int_a^b r\, dr$$

$$= \frac{\pi L}{u_0}\left(\frac{u_0^2 I^2}{4\pi^2}\right)\int_a^b \frac{1}{r}\, dr$$

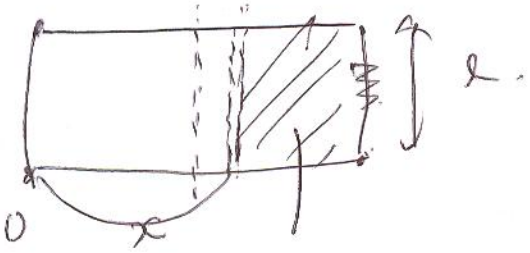
$$= \underline{\frac{u_0 I^2}{4\pi} L \ln\frac{b}{a}} \qquad \frac{W_2}{L} = \frac{u_0 I^2}{4\pi}\ln\frac{b}{a}.$$

(2) inductance per unit Lenght

$$= \frac{2}{I^2} \times \left(\frac{u_0 I^2}{16\pi} + \frac{u_0 I^2}{4\pi} \ln \frac{b}{a} \right)$$

$$= \frac{u_0 I^2}{8\pi} + \frac{u_0}{2\pi} \ln \frac{b}{a}$$

3.



~~ϕ~~ $s = (0.7 - x)\, l$.

$$\boxed{\frac{\sin^2\omega t}{2}}$$

$ds = -l\, dx$

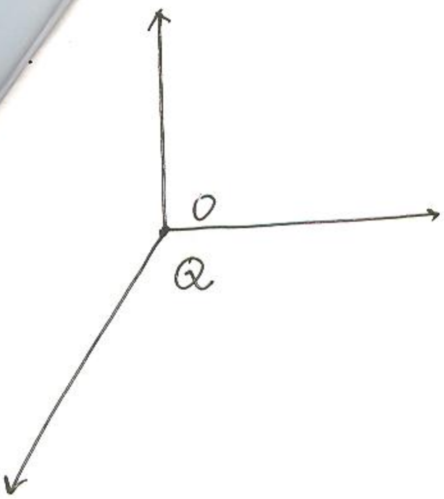
$= -l\,(-0.35)\,\omega \sin\omega t\, dt$

$$V_{emf} = -\frac{\partial}{\partial t}\int^{t} B \cdot ds \quad = -\frac{\partial}{\partial t}\int^{t} (5\cos\omega t) \cdot 0.2 \cdot 0.35\, \omega \sin\omega t\, dt$$

$$= + \frac{\partial}{\partial t}\,(5 \times 0.2 \times 0.35)\, \cancel{\omega}\, \frac{\cos 2\omega t}{4\cancel{\omega}}$$

$$= + \frac{0.35}{\cancel{4}_{2}} \cancel{2}\omega \sin 2\omega t \quad \times 10^{-3}$$

~~[illegible]~~ $i = \frac{V_{emf}}{R} = \frac{0.175}{0.2}\, \omega \sin 2\omega t \times 10^{-3}$ (A)



(1) $\nabla \cdot D = \rho, \quad \nabla \cdot B = 0.$

$$\underbrace{\nabla \times E = -\frac{dB}{dt}}_{①}, \quad \nabla \times H = J + \frac{dD}{dt}, \quad \underbrace{B = \nabla \times A}_{②}$$

$$\nabla \times E = -\frac{d}{dt}(\nabla \times A)$$

$$\nabla \times \left(E + \frac{dA}{dt}\right) = 0. \qquad E + \frac{dA}{dt} = -\nabla V$$

$$\nabla \times B = u_0 J + u_0 \varepsilon_0 \frac{dE}{dt} \leftarrow$$

vaccum 에서.

$$\nabla \times (\nabla \times A) = u_0 J + u_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\nabla V - \frac{dA}{dt}\right)$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = u_0 J + -\nabla u_0 \varepsilon_0 \frac{\partial V}{\partial t} - u_0 \varepsilon_0 \frac{\partial^2 A}{\partial^2 t}$$

$$\nabla \cdot A + u_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$$

$$\nabla^2 A = -u_0 J + u_0 \varepsilon_0 \frac{\partial^2 A}{\partial^2 t} \leftarrow$$

scalar B potential

$$\nabla^2 V - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} V = -\frac{\rho}{\varepsilon_0}$$

유ε

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} = \nabla \cdot \left(- \nabla V - \frac{\partial A}{\partial t} \right)$$

$$- \nabla \cdot \nabla V - \frac{\partial}{\partial t} \left(\nabla \cdot A \right) = \frac{\rho}{\varepsilon_0}$$

$$- \nabla^2 V - \frac{\partial}{\partial t} \left(- u_0 \varepsilon_0 \cdot \frac{\partial V}{\partial t} \right) = \frac{\rho}{\varepsilon_0}$$

$$\underbrace{\nabla^2 V - u_0 \varepsilon_0 \frac{\partial^2 V}{\partial^2 t} = - \frac{\rho}{\varepsilon_0}}$$ (nonhomogenous wave eq. for V)

(2)

~~$V = u(R, t)$~~

~~$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial u}{\partial R} \right) - u_0 \varepsilon_0 \frac{\partial^2 V}{\partial^2 t} = - \frac{\rho}{\varepsilon_0}$, $\rho = 0$~~

~~$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \cdot \frac{\partial V}{\partial R} \right) = u_0 \varepsilon_0 \cdot \frac{\partial^2 V}{\partial^2 t} = k$ (k 상수).~~

~~⇒ 변수분리~~

~~$V =$~~

(2) $\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right)$: 구대칭이므로.

$V = \frac{1}{R}U(R,t)$ 라 두면

$\nabla^2 V = \frac{1}{R}\frac{\partial^2 V}{\partial R^2}$

$\frac{1}{R}\frac{\partial V}{\partial R^2} - \mu_0\epsilon_0\frac{1}{R}\frac{\partial^2 V}{\partial R^2} = -\frac{\rho}{\epsilon_0}$, $\rho = 0$

$\frac{\partial V}{\partial R^2} - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} = 0.$

$U = f(t - R\sqrt{\mu\epsilon})$ 면 윗식 만족.

$\therefore V = \frac{1}{R}f(t - R\sqrt{\mu\epsilon})$

$\therefore V = \boxed{\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{R}dv'} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(t - R\sqrt{\mu\epsilon}}{R}dv'$

$= \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{R}\int\left(\frac{Q_0}{4\pi R^2}\delta(r)\cos\omega(t - R\sqrt{\mu\epsilon})\right)dv'$

$= \frac{Q_0}{4\pi\epsilon_0 R}\cos\omega(t - R\sqrt{\mu\epsilon})$

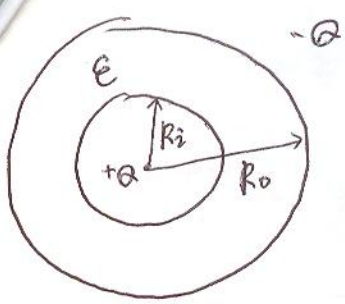
$\therefore (V(R,t)) = \frac{Q_0\cos\omega(t - R\sqrt{\mu\epsilon})}{4\pi\epsilon_0 R}$

교수님께서 틀린줄 알고 다시 적었으나 맞았음.

$\rho = Q_0\cos\omega t\ \delta(r)$

$V = \frac{1}{4\pi\epsilon_0}\int\frac{Q_0\cos\omega(t - R\sqrt{\mu_0\epsilon_0})\delta(r')}{R}dv'$

$= \frac{1}{4\pi\epsilon_0}\ \frac{Q_0\cos\omega(t - R\sqrt{\mu_0\epsilon_0})}{R}$



(1) Potential difference

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a}_r \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon}\cdot\frac{1}{r^2}\hat{a}_r$$

$$\therefore \Delta V = -\int_{R_o}^{R_i} E_r\, dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{R_o}^{R_i} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon}{\left(\frac{1}{R_i} - \frac{1}{R_o}\right)}$$

(2) electric energy

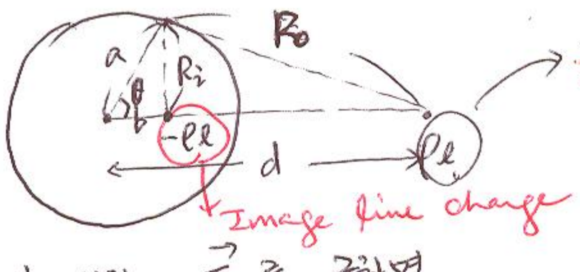
$$D = \frac{Q}{4\pi r^2}\hat{a}_r, \quad w_e = \frac{1}{2}E\cdot D = \frac{D^2}{2\varepsilon}$$

$$\therefore W_e = \int_v w_e\, dv = \frac{1}{2\varepsilon}\int_v D^2 dv = \frac{1}{2\varepsilon}\int_{R_i}^{R_o} \frac{Q^2}{16\pi^2 r^4}\cdot 4\pi r^2 dr$$

$$= \frac{Q^2}{8\pi\varepsilon}\int_{R_i}^{R_o}\frac{1}{r^2}dr = \frac{Q^2}{8\pi\varepsilon}\left[-\frac{1}{r}\right]_{R_i}^{R_o} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right)$$

한편 $W_e = \dfrac{Q^2}{2C}$ 이므로 $C = \dfrac{4\pi\varepsilon}{\left(\frac{1}{R_i} - \frac{1}{R_o}\right)}$

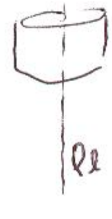
6. (1)



→ ~~Image~~ Real charge density

Image line charge

ρ_l 에 의한 $\vec{E}$ 를 구하면.



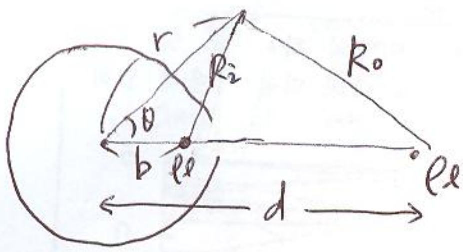
$$\oint \vec{E}\cdot ds = \frac{\rho_l}{\varepsilon_0} \Leftrightarrow E = \frac{\rho_l}{2\pi r\varepsilon_0}\hat{a}_r$$

$$\therefore V = -\int_x^r \frac{\rho_l}{2\pi r\varepsilon_0}dr = -\frac{\rho_l}{2\pi\varepsilon_0}\ln r\Big|_{\textcircled{x}}^{r} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{x}{r}$$

→ V = 0 인 지점.

$$\therefore V = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{x}{R_o} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{x}{R_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{R_i}{R_o}$$

$\frac{R_i}{R_o} = const$ 이므로 $\frac{R_i}{R_o} = \frac{a}{d} = \frac{b}{a}$ $\therefore b = \frac{a^2}{d}$



이제 (r, θ) 에서의 V를 구해보면

$$R_i^2 = r^2 + b^2 - 2rb\cos\theta = r^2 + \frac{a^4}{d^2} - \frac{2ra^2}{d}\cos\theta$$

$$R_o^2 = r^2 + d^2 - 2rd\cos\theta$$ 이므로.

$$V = \frac{\rho_l}{2\pi\varepsilon_0} \ln \frac{\sqrt{r^2 + \frac{a^4}{d^2} - \frac{2ra^2}{d}\cos\theta}}{\sqrt{r^2 + d^2 - 2rd\cos\theta}}$$

(2)

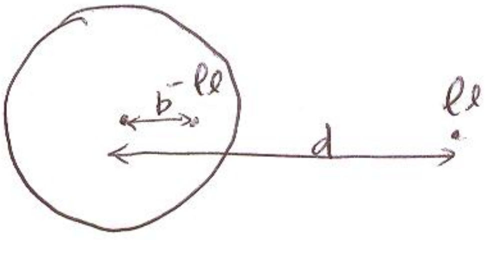


image charge 가 ρ_l 이 있는 지점에 만드는 E를 구하면,

$$\oint E \cdot ds = \frac{-\rho_l}{\varepsilon_0}$$

$$\therefore E \cdot 2\pi(d-b) = -\frac{\rho_l}{\varepsilon_0} \iff E = -\frac{\rho_l}{2\pi\varepsilon_0(d-b)}$$

따라서 line charge에 존재하는 전하 q가 받는 힘 $F = -\frac{q\rho_l}{2\pi\varepsilon_0(d-b)}$

unit length에 n개의 전하가 있다면 $F_{tot} - \frac{nq\rho_l}{2\pi\varepsilon_0(d-b)}$

$n q = \rho_l$ 이고, $b = \frac{a^2}{d}$ 이므로 $F_{tot} = -\frac{\rho_l^2}{2\pi\varepsilon_0\left(d - \frac{a^2}{d}\right)}$

$\boxed{17}$ (1) charge conservation $\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$ …①

ampere's law $\nabla\times\vec{H}=\vec{J}$ …②

②에서 양변에 divergence를 취하면

좌변 $\nabla\cdot\vec{J}=-\frac{\partial\rho}{\partial t}$ ……①′

우변 $\nabla\cdot(\nabla\times\vec{H})=0$ …②′

①′와 ②′가 값이 틀리므로 보정해야 한다

$$\nabla\cdot(\nabla\times\vec{H})=0=\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=\nabla\cdot(\vec{J}+\frac{\partial\vec{D}}{\partial t})$$

$$\therefore\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

(2) $\rho_P=-\nabla\cdot\vec{P}$, $\rho_{PS}=\vec{P}\cdot\hat{a}_n$

(3) Lorentz condition ; $\nabla\cdot\vec{A}+\varepsilon\mu\frac{\partial V}{\partial t}=0$

$\nabla\cdot\vec{B}=0$ 에서 $B=\nabla\times\vec{A}$ …①

$\nabla\times\vec{E}=-\frac{\partial}{\partial t}\vec{B}=-\frac{\partial}{\partial t}(\nabla\times\vec{A})$ $\therefore\nabla\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$

$\therefore\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\nabla V$ …②

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon}=\nabla\cdot(-\nabla V-\frac{\partial\vec{A}}{\partial t})=-\nabla^2V+\frac{\partial}{\partial t}(\mu\varepsilon\frac{\partial V}{\partial t})$$

$$=-\nabla^2V+\varepsilon\mu\frac{\partial^2V}{\partial t}$$

$$\therefore\boxed{\nabla^2V-\varepsilon\mu\frac{\partial^2V}{\partial t}=-\frac{\rho}{\varepsilon}}$$

그리고 $\nabla\times B=\mu J+\varepsilon\mu\frac{\partial E}{\partial t}=\mu J+\varepsilon\mu\frac{\partial}{\partial t}(-\nabla-\frac{\partial A}{\partial t})$

$$=\mu J-\nabla(\varepsilon\mu\frac{\partial V}{\partial t})-\varepsilon\mu\frac{\partial^2A}{\partial t^2}$$

$$=\mu J+\nabla(\nabla\cdot A)-\varepsilon\mu\frac{\partial^2A}{\partial t^2}$$

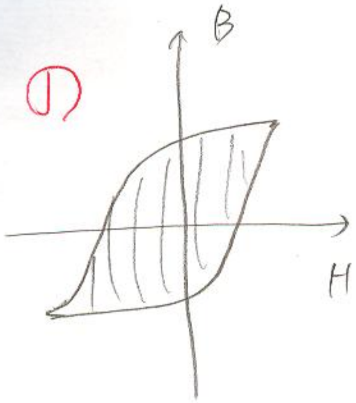
한편 $\nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

$$\therefore \nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J + \nabla(\nabla \cdot A) - \varepsilon\mu \frac{\partial^2 A}{\partial t^2}$$

$$\therefore \boxed{\nabla^2 A - \varepsilon\mu \frac{\partial^2 A}{\partial t} = -\mu J}$$

(4)

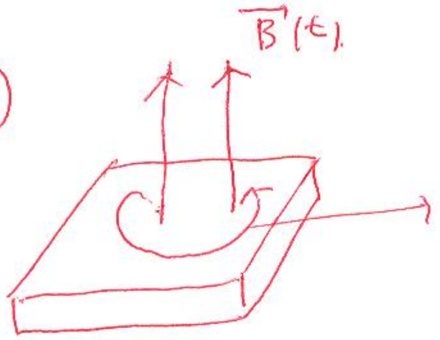
①



변압기를 사용하면 B와 H가 선형적이지 않는다. 1 cycle 돌때 면적에 해당하는 에너지가 손실된다. (열로 변환)

이것을 Hysteresis loss 라한다.

②



Eddy current

↓

Joule 열발생

↓

Eddy current loss.

학번	이름	이론(75)	실험(25)	총점(100)
2007-11638	고휘석	71.31	22.75	94.06
2007-11821	조경훈	71.64	20.88	92.52
2007-11691	박동진	71.02	21.46	92.48
2007-11847	한용수	69.68	21.83	91.51
2007-11830	차영택	68.86	22.50	91.36
2007-11815	정서빈	68.69	22.54	91.23
2007-11784	이재면	69.52	21.42	90.94
2007-11644	길태호	70.91	19.83	90.74
2007-11650	김동수	67.87	22.08	89.95
2007-11837	최윤주	69.11	19.96	89.07
2007-11732	안병용	67.61	21.42	89.03
2007-11648	김동권	65.62	23.25	88.87
2007-11693	박민수	68.67	20.00	88.67
2007-11753	유의상	65.65	22.42	88.07
2007-11721	송기윤	65.69	21.08	86.77
2007-11727	신상민	65.40	21.21	86.61
2006-11747	문영식	64.96	21.29	86.25
2006-12104	장승규	64.49	21.75	86.24
2007-11808	전기연	63.63	22.25	85.88
2007-11660	김영휴	65.51	20.33	85.84
2007-11776	이영문	61.90	22.50	84.40
2005-11891	최형우	63.10	21.21	84.31
2007-11641	권보준	63.01	21.00	84.01
2007-11726	신보경	61.52	22.08	83.60
2007-11800	이태재	63.10	20.50	83.60
2007-11728	신성욱	62.29	21.25	83.54
2007-11662	김우열	62.53	20.92	83.45
2007-11803	이홍민	62.32	21.00	83.32
2007-11689	박경환	63.21	19.54	82.75
2007-11653	김병헌	61.01	21.71	82.72
2007-11853	홍상현	62.41	19.25	81.66
2007-11705	박희천	58.95	22.67	81.62
2007-11678	김한별	62.58	18.50	81.08
2007-11711	변성호	58.11	22.50	80.61
2007-11674	김채빈	61.68	18.63	80.31
2007-11825	조승훈	60.18	19.67	79.85
2007-11824	조성룡	58.84	20.83	79.67
2007-11670	김준석	60.96	18.29	79.25
2007-11765	이동훈	57.46	20.92	78.38
2005-11876	지균철	56.04	22.04	78.08
2007-11746	오형석	56.57	21.33	77.90

2007-11669	김종호	57.76	20.04	77.80
2004-12017	우경재	57.05	20.67	77.72
2005-11695	김경헌	56.78	20.83	77.61
2007-11791	이준행	58.04	19.54	77.58
2005-11718	김윤수	56.07	20.96	77.03
2007-11756	윤용호	54.75	22.25	77.00
2005-11905	황영석	55.20	21.67	76.87
2007-11664	김윤직	55.20	21.67	76.87
2006-11728	김지수	57.83	18.42	76.25
2007-11685	노관우	55.57	20.38	75.95
2004-11943	김판석	56.91	18.96	75.87
2007-11817	정은지	53.81	22.17	75.98
2006-11768	배우람	54.03	21.25	75.28
2002-11851	김태호	52.49	22.25	74.74
2007-11716	선영석	53.78	20.25	74.03
2007-11659	김영찬	53.09	20.92	74.01
2007-11723	송승근	52.94	20.96	73.90
2007-11690	박기태	53.53	20.08	73.61
2005-11874	조창민	53.99	19.54	73.53
2001-12355	윤서영	51.27	22.17	73.44
2007-11849	한정민	52.49	20.83	73.32
2004-12027	윤종순	52.87	20.38	73.25
2006-11804	유승용	51.90	20.92	72.82
2000-14223	류준환	54.97	17.75	72.72
2005-11724	김정기	53.12	19.38	72.50
2003-11954	김의현	50.80	20.96	71.76
2005-11758	박민우	49.85	21.71	71.56
2007-11789	이주형	50.13	21.42	71.55
2005-11715	김승용	48.68	22.17	70.85
2007-11672	김지윤	49.27	21.42	70.69
2004-11934	김진규	48.15	22.21	70.36
2007-11658	김수지	48.41	20.58	68.99
2006-11829	이재국	48.04	20.63	68.67
2007-11725	신다영	49.82	18.75	68.57
2007-11729	신인모	47.75	20.67	68.42
2004-12096	주민수	46.86	20.96	67.82
2007-11677	김태훈	45.05	22.33	67.38
2004-11975	박진홍	48.44	18.88	67.32
2007-11840	최혁준	47.56	19.75	67.31
2003-12036	안정호	50.24	15.88	66.12
2007-11818	정재웅	44.45	21.46	65.91
2005-11819	이광희	46.81	18.71	65.52
2007-11763	이동영	43.96	21.54	65.50

2004-12063	이중현	45.68	19.50	65.18
2004-12125	한승호	43.62	20.38	64.00
2007-11642	권용원	42.40	21.17	63.57
2007-11844	한상범	46.77	16.63	63.40
2007-11735	안준영	42.64	19.50	62.14
2006-11868	조완기	41.26	19.92	61.18
2007-11704	박홍종	43.08	17.67	60.75
2007-11846	한아름	40.17	19.75	59.92
2003-12035	안석기	40.65	18.46	59.11
2007-11734	안보영	39.36	19.50	58.86
2002-11709	김근하	38.30	20.54	58.84
2007-11812	정동석	41.36	16.96	58.32
2004-12055	이우영	36.53	21.21	57.74
2002-11651	강수민	38.71	17.79	56.50
2005-11908	황인환	35.66	19.71	55.37
2007-11810	정다원	34.96	19.21	54.17
2005-11868	정상훈	34.42	19.46	53.88
2003-12042	염장헌	33.91	16.54	50.45
2007-11845	한승주	29.74	20.17	49.91
2003-12038	안혁	32.15	16	48.15
2005-11865	정민철	28.40	17.83	46.23
2002-11749	김상범	22.34	21.25	43.59
2004-11918	김영현	19.17	18.63	37.80
2006-15630	유장현	29.45	4.46	33.91
2005-11878	채경훈	8.26	14.71	22.97
2001-12268	김우준	4.40	9.04	13.44
98420-175	윤동현	Drop		
99420-028	윤여진	Drop		
2000-12342	오재교	Drop		
2004-11992	송인욱	Drop		
2005-11771	박태준	Drop		
2006-11836	이하윤	Drop		
2007-11715	서유민	Drop		
2007-11767	이민재	Drop		
2007-11806	임정우	Drop		