*Multi View Geometry (Spring '08)*

# Projective 3D Geometry

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## Points, Planes, Lines and Quadrics in 3D



- 3D points:

Homogeneous representation

$$\mathbf{X} = (X_1, X_2, X_3, X_4)^T \ in\ \mathbf{P}^3$$
$$\Leftrightarrow (\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1)^T, X_4 \neq 0$$

$\Longleftrightarrow$

Inhomogeneous representation

$$(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4})^T \ in\ \mathbf{R}^3$$

- Projective transform in $\mathbf{P}^3$:

$$\mathbf{X}' = \mathbf{H}_{4\times4}\mathbf{X}$$

✓ Collinear
✓ Lines are mapped to lines
✓ 15 (4x4-1) DOF

## Planes

- 3D planes:

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\downarrow \quad X = \frac{X_1}{X_4}, Y = \frac{X_2}{X_4}, Z = \frac{X_3}{X_4}$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\Rightarrow \boxed{\pi^{\mathsf{T}} \mathrm{X} = 0}$$

$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)^T$ : homogeneous plane

$\mathbf{X} = (X_1, X_2, X_3, X_4)^T$ : homogeneous 3D point

Transformation

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$$

$$\boxed{\pi' = \mathbf{H}^{-\mathsf{T}}\,\pi}$$





- Euclidean representation:

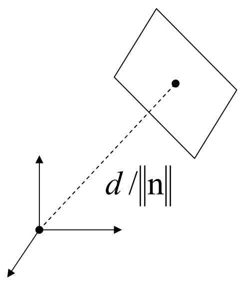
$$\mathbf{n}.\tilde{\mathbf{X}} + d = 0$$

$\mathbf{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}}$ : plane normal

$\tilde{\mathbf{X}} = (X, Y, Z)^{\mathsf{T}}$

$X_4 = 1$

$d = \pi_4$




- Duality: points ↔ planes
  lines ↔ lines

## Determination of a plane by Points



- Three points on a plane satisfies

$$\begin{bmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \mathbf{X}_3^T \end{bmatrix} \boldsymbol{\pi} = 0 \quad \Rightarrow \quad \pi \text{ is the null space of } \begin{bmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \mathbf{X}_3^T \end{bmatrix}$$

- Or, using the coplanarity constraint;

$\forall\, \mathbf{X}$ on $\boldsymbol{\pi}$

$$\det[\mathbf{X}\,\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3] = 0 \quad \Rightarrow \quad \det\begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$\Rightarrow \quad X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^{\mathsf{T}}$$



## Determination of a plane Points



**Example 3.1.** Suppose the three points defining the plane are

$$\mathbf{X}_1 = \begin{pmatrix} \widetilde{\mathbf{X}}_1 \\ 1 \end{pmatrix} \quad \mathbf{X}_2 = \begin{pmatrix} \widetilde{\mathbf{X}}_2 \\ 1 \end{pmatrix} \quad \mathbf{X}_3 = \begin{pmatrix} \widetilde{\mathbf{X}}_3 \\ 1 \end{pmatrix}$$

where $\widetilde{\mathbf{X}} = (\mathrm{X}, \mathrm{Y}, \mathrm{Z})^{\mathsf{T}}$. Then

$$D_{234} = \begin{vmatrix} \mathrm{Y}_1 & \mathrm{Y}_2 & \mathrm{Y}_3 \\ \mathrm{Z}_1 & \mathrm{Z}_2 & \mathrm{Z}_3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \mathrm{Y}_1 - \mathrm{Y}_3 & \mathrm{Y}_2 - \mathrm{Y}_3 & \mathrm{Y}_3 \\ \mathrm{Z}_1 - \mathrm{Z}_3 & \mathrm{Z}_2 - \mathrm{Z}_3 & \mathrm{Z}_3 \\ 0 & 0 & 1 \end{vmatrix} = \left((\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3)\right)_1$$

and similarly for the other components, giving

$$\boldsymbol{\pi} = \begin{pmatrix} (\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3) \\ -\widetilde{\mathbf{X}}_3^{\mathsf{T}}(\widetilde{\mathbf{X}}_1 \times \widetilde{\mathbf{X}}_2) \end{pmatrix}.$$

This is the familiar result from Euclidean vector geometry where, for example, the plane normal is computed as $(\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_3) \times (\widetilde{\mathbf{X}}_2 - \widetilde{\mathbf{X}}_3)$. △

## Points from Planes



- Three planes meets at a point:

$$\begin{bmatrix}\pi_1^\mathsf{T}\\ \pi_2^\mathsf{T}\\ \pi_3^\mathsf{T}\end{bmatrix}\mathbf{X}=0 \quad \Rightarrow \quad \mathbf{X} \text{ is the null space of } \begin{bmatrix}\pi_1^\mathsf{T}\\ \pi_2^\mathsf{T}\\ \pi_3^\mathsf{T}\end{bmatrix}$$

- A plane can be represented by its span:

$$\boxed{\mathbf{X}=\mathbf{M}\mathbf{x}}$$

$$\mathbf{M}_{4x3}=[\mathrm{X}_1\mathrm{X}_2\mathrm{X}_3]$$

**M**: 3dim null-space of $\pi$

$$\pi^\mathsf{T}\mathbf{M}=\mathbf{0}$$

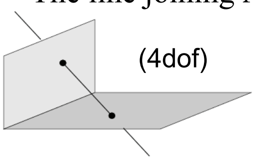
$$\text{if } \pi=(a,b,c,d)^\mathsf{T} \quad \text{then } \mathbf{M}=\begin{bmatrix}\mathbf{p}\\ \mathbf{I}_{3x3}\end{bmatrix} \quad \text{and } \mathbf{p}=\left(-\frac{b}{a},-\frac{c}{a},-\frac{d}{a}\right)$$



## Lines – Null-space and span representation



- The line joining **A** and **B** is the span of the row space of **W**.




$$\mathbf{W}=\begin{bmatrix}\mathbf{A}^\mathsf{T}\\ \mathbf{B}^\mathsf{T}\end{bmatrix} \qquad \lambda\mathbf{A}+\mu\mathbf{B}$$

(i) The span of $\mathtt{W}^\mathsf{T}$ is the pencil of points $\lambda\mathbf{A}+\mu\mathbf{B}$ on the line.

(ii) The span of the 2-dimensional right null-space of $\mathtt{W}$ is the pencil of planes with the line as axis.

- The intersecting line of two planes **P** and **Q** is the span of the row space of $\mathbf{W}^*$.

$$\mathbf{W}^*=\begin{bmatrix}\mathbf{P}^\mathsf{T}\\ \mathbf{Q}^\mathsf{T}\end{bmatrix} \qquad \lambda'\mathbf{P}+\mu'\mathbf{Q}$$

(i) The span of $\mathtt{W}^{*\mathsf{T}}$ is the pencil of planes $\lambda'\mathbf{P}+\mu'\mathbf{Q}$ with the line as axis.

(ii) The span of the 2-dimensional null-space of $\mathtt{W}^*$ is the pencil of points on the line.

## Lines – Null-space and span representation



- $\mathbf{W}^{*}\mathbf{W}^{T} = \mathbf{W}\mathbf{W}^{*T} = \mathbf{0}_{2\times 2}$

**Example 3.2.** The X-axis is represented as

$$\mathtt{W} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathtt{W}^{*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where the points $\mathbf{A}$ and $\mathbf{B}$ are here the origin and ideal point in the X-direction, and the planes $\mathbf{P}$ and $\mathbf{Q}$ are the XY- and XZ-planes respectively. △

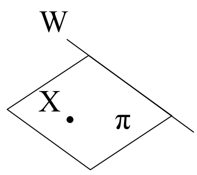


## Points, Lines and Planes



- A plane $\pi$ by a point $\mathbf{X}$ and a line $\mathbf{W}$:

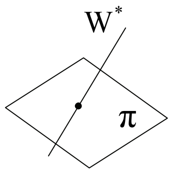
$$\mathbf{M} = \begin{bmatrix} \mathbf{W} \\ \mathbf{X}^{T} \end{bmatrix} \qquad \mathbf{M}\pi = 0$$




If the dim $\mathcal{N}(\mathbf{M}) = 2$, then $\mathbf{X}$ is on $\mathbf{W}$

- A point $\mathbf{X}$ by a line $\mathbf{W}$ and a plane $\pi$ :

$$\mathbf{M} = \begin{bmatrix} \mathbf{W}^{*} \\ \pi^{T} \end{bmatrix} \qquad \mathbf{M}\mathbf{X} = 0$$




If the dim $\mathcal{N}(\mathbf{M}) = 2$, then $\mathbf{W}$ is on $\pi$

## Lines - Plücker matrix representation



- The line joining two points can be represented by the Plücker matrix

$$l_{ij} = A_i B_j - B_i A_j \qquad \mathbf{L} = \mathbf{A}\mathbf{B}^\top - \mathbf{B}\mathbf{A}^\top$$

i) $\mathbf{L}$ is a 4x4 skew-symmetric homogeneous matrix
ii) $\mathbf{L}$ has rank 2 $\quad \mathbf{L}\mathbf{W}^{*\top} = \mathbf{0}_{4\times 2}$
iii) 4 DOF
iv) generalization of $\quad \mathbf{l} = \mathbf{x} \times \mathbf{y}$
v) $\mathbf{L}$ is independent of choice $\mathbf{A}$ and $\mathbf{B}$
vi) Transformation $\quad \mathbf{L}' = \mathbf{H}\mathbf{L}\mathbf{H}^\top$

Ex) X-axis

$$\mathbf{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



## Dual Plücker Matrix



- Dual Plücker matrix: $\mathbf{L}^* = \mathbf{P}\mathbf{Q}^\top - \mathbf{Q}\mathbf{P}^\top$
- Transformation: $\mathbf{L}^{*'} = \mathbf{H}^{-T}\mathbf{L}^*\mathbf{H}^{-1}$
- Correspondence: $l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$
- Join and Incidence

$\boldsymbol{\pi} = \mathbf{L}^*\mathbf{X}$ : plane through point and line

$\mathbf{L}^*\mathbf{X} = 0$ : point on line

$\mathbf{X} = \mathbf{L}\boldsymbol{\pi}$ : intersection point of plane and line

$\mathbf{L}\boldsymbol{\pi} = 0$ : line in plane

$[\mathbf{L}_1, \mathbf{L}_2, \ldots]\boldsymbol{\pi} = 0$ : coplanar lines

## Lines - Plücker line coordinates representation



- Plücker line coordinates: $\mathcal{L} = [l_{12}, l_{13}, l_{14}, l_{23}, l_{42}, l_{34}]^T \in \mathbf{P}^5$
- From det $\mathbf{L}$ =0 $\longrightarrow$ $l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23} = 0 \Leftrightarrow \mathcal{L}^{\mathsf{T}}\mathbf{K}\mathcal{L} = 0$

*Klein quadric constraint*

$$[l_{12} l_{13} l_{14} l_{23} l_{42} l_{34}] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_{12} \\ l_{13} \\ l_{14} \\ l_{23} \\ l_{42} \\ l_{34} \end{bmatrix} = 0$$

$\mathcal{L}, \hat{\mathcal{L}} \leftrightarrow (A,B),(\hat{A},\hat{B})$ two lines are intersect *iff* the 4 points are coplanar

$$\det[\boldsymbol{A}, \boldsymbol{B}, \hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}] = l_{12}\hat{l}_{34} + l_{13}\hat{l}_{42} + l_{14}\hat{l}_{23} + l_{23}\hat{l}_{14} + l_{42}\hat{l}_{13} + l_{34}\hat{l}_{12}$$
$$= \hat{\mathcal{L}}^{\mathsf{T}}\mathbf{K}\mathcal{L} = (\mathcal{L} \mid \hat{\mathcal{L}})$$



## Lines - Plücker line coordinates representation



- Results:

i) Plücker internal constraint:

$$(\mathcal{L}|\mathcal{L}) = 0$$

ii) two lines intersect or coplanar:

$(\mathcal{L}|\hat{\mathcal{L}}) = \det[A, B, \hat{A}, \hat{B}] = 0$ 4 points

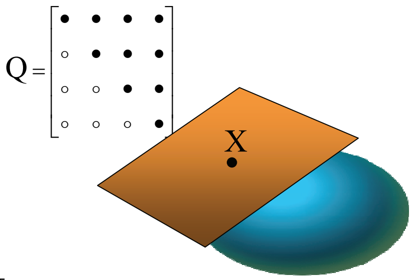
$(\mathcal{L}|\hat{\mathcal{L}}) = \det[P, Q, \hat{P}, \hat{Q}] = 0$ 4 planes

$(\mathcal{L}|\hat{\mathcal{L}}) = (P^T A)(Q^T B) - (Q^T A)(P^T B) = 0$ 2 planes and 2 points

## Quadrics and dual quadrics



- Quadric: $X^T Q X = 0$




  i) Q is a 4x4 symmetric matrix
  ii) 9 DOF
  iii) in general 9 points define quadric
  iv) det $\mathbf{Q}=0 \leftrightarrow$ degenerate quadric
  v) pole – polar $\pi = \mathbf{Q}X$
  vi) (plane ∩ quadric)=conic $\mathbf{C} = \mathbf{M}^T \mathbf{Q} \mathbf{M}$ $\pi : X = \mathbf{M}x$
  vii) transformation $\mathbf{Q}' = \mathbf{H}^{-T} \mathbf{Q} \mathbf{H}^{-1}$

- Dual quadric: $\pi^T \mathbf{Q}^* \pi = 0$
  i) relation to quadric $\mathbf{Q}^* = \mathbf{Q}^{-1}$ (non-degenerate)
  ii) transformation $\mathbf{Q}'^* = \mathbf{H} \mathbf{Q}^* \mathbf{H}^T$



## Classification of quadrics



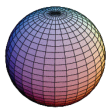
- $\mathbf{Q} = \mathbf{U}^T \tilde{\mathbf{D}} \mathbf{U}$ $\xrightarrow{\text{Scale normalization of D}}$ $\mathbf{Q} = \mathbf{H}^T \mathbf{D} \mathbf{H}$
- **D** represents **Q** up to projective equivalence.
- Signature: $\sigma(\mathbf{Q}) = \sigma(\mathbf{D})$ Independent of H

| Rank | Sign. | Diagonal | Equation | Realization |
|---|---|---|---|---|
| 4 | 4 | (1,1,1,1) | $X^2+Y^2+Z^2+1=0$ | No real points |
| | 2 | (1,1,1,-1) | $X^2+Y^2+Z^2=1$ | Sphere |
| | 0 | (1,1,-1,-1) | $X^2+Y^2=Z^2+1$ | Hyperboloid (1S) |
| 3 | 3 | (1,1,1,0) | $X^2+Y^2+Z^2=0$ | Single point |
| | 1 | (1,1,-1,0) | $X^2+Y^2=Z^2$ | Cone |
| 2 | 2 | (1,1,0,0) | $X^2+Y^2=0$ | Single line |
| | 0 | (1,-1,0,0) | $X^2=Y^2$ | Two planes |
| 1 | 1 | (1,0,0,0) | $X^2=0$ | Single plane |

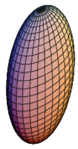
## Classification of quadrics



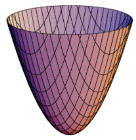
- Nun-ruled quadrics: projectively equivalent to *sphere:*



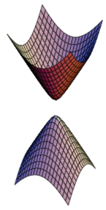
*sphere*



*ellipsoid*

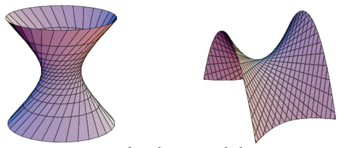


*paraboloid*



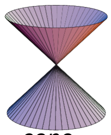
*hyperboloid of two sheets*

- Ruled quadrics: contains straight lines

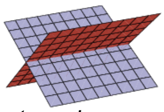


*hyperboloids of one sheet*

- Degenerate ruled quadrics:



*cone*



*two planes*



## Twisted cubics



- A conic in $\Pi^2$:

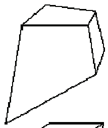
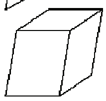
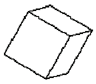
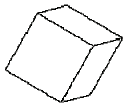
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 \end{pmatrix}$$

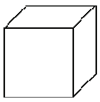
- A twisted cubic in $\Pi^3$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \theta^3 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\theta + a_{13}\theta^2 + a_{14}\theta^3 \\ a_{21} + a_{22}\theta + a_{23}\theta^2 + a_{24}\theta^3 \\ a_{31} + a_{32}\theta + a_{33}\theta^2 + a_{34}\theta^3 \\ a_{41} + a_{42}\theta + a_{43}\theta^2 + a_{44}\theta^3 \end{pmatrix}$$

## The hierarchy of transformations



| Group | Matrix | Distortion | Invariant properties |
|---|---|---|---|
| Projective 15dof | $\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$ |  | Intersection and tangency |
| Affine 12dof | $\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$ |  | Parallellism of planes, Volume ratios, centroids, **The plane at infinity $\pi_\infty$** |
| Similarity 7dof | $\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$ |  | **The absolute conic $\Omega_\infty$** |
| Euclidean 6dof | $\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$ |  | Volume |





## The plane at infinity



- The plane at infinity: $\boldsymbol{\pi}_\infty = (0,0,0,1)^T$
- Contains directions (points) $\mathbf{D} = (X_1, X_2, X_3, 0)^T$
- two planes are parallel ⇔ line of intersection in $\boldsymbol{\pi}_\infty$
- line // line (or plane) ⇔ point of intersection in $\boldsymbol{\pi}_\infty$

- The plane at infinity $\boldsymbol{\pi}_\infty$ is a fixed plane under a projective transformation **H** *iff* **H** is an *affinity*

$$\pi'_\infty = \mathbf{H}_A^{-T}\pi_\infty = \begin{bmatrix} \mathbf{A}^{-T} & 0 \\ -\mathbf{A}\,t & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

## The absolute conic



- the absolute conic $\Omega_\infty$ is defined on $\pi_\infty$ s.t.

$$\left.\begin{matrix} X_1^2+X_2^2+X_3^2 \\ X_4 \end{matrix}\right\}=0 \iff (X_1,X_2,X_3)\mathrm{I}(X_1,X_2,X_3)^{\mathsf{T}}$$

- Thus, a conic with $\mathbf{C} = \mathrm{I}_{\mathbf{3x3}}$
- The absolute conic $\Omega_\infty$ is a fixed conic under the projective transformation **H** *iff* **H** is a *similarity*

  i) $\Omega_\infty$ is only fixed as a set
  ii) Circle intersect $\Omega_\infty$ in two points
  iii) Spheres intersect $\pi_\infty$ in $\Omega_\infty$



## Metric properties



- Once $\Omega_\infty$ is identified in projective 3-space, angles and relative lengths can be measured.

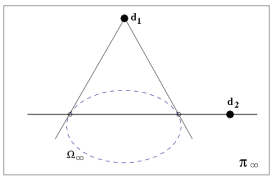
Euclidean: $$\cos\theta = \frac{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\mathbf{d}_1)(\mathbf{d}_2^{\mathsf{T}}\mathbf{d}_2)}}$$

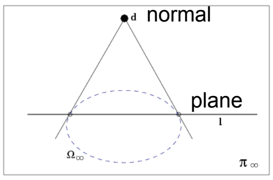
$d_1$, $d_2$ : directions of two lines (Intersection points of lines on $\pi_\infty$)

Projective: $$\cos\theta = \frac{(\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_2)}{\sqrt{(\mathbf{d}_1^{\mathsf{T}}\Omega_\infty\mathbf{d}_1)(\mathbf{d}_2^{\mathsf{T}}\Omega_\infty\mathbf{d}_2)}}$$

Invariant to projective transform

- $\mathbf{d}_1{}^T\Omega_\infty\mathbf{d}_2 = 0 \iff$ orthogonal (conjugacy)



## The absolute dual quadric

- The Absolute dual quadric: $\mathbf{Q}^*_\infty = \begin{bmatrix} I & 0 \\ 0^\mathsf{T} & 0 \end{bmatrix}$

- The absolute dual quadric $\mathbf{Q}^*{}_\infty$ is a fixed quadric under the projective transformation $\mathbf{H}$ *iff* $\mathbf{H}$ is a *similarity*

i) 8 DOF
ii) plane at infinity $\boldsymbol{\pi}_\infty$ is the null-vector of $\mathbf{Q}_\infty$
iii) Angles between $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ :

$$\cos\theta = \frac{\boldsymbol{\pi}_1^\mathsf{T}\mathbf{Q}^*_\infty\boldsymbol{\pi}_2}{\sqrt{\left(\boldsymbol{\pi}_1^\mathsf{T}\mathbf{Q}^*_\infty\boldsymbol{\pi}_1\right)\left(\boldsymbol{\pi}_2^\mathsf{T}\mathbf{Q}^*_\infty\boldsymbol{\pi}_2\right)}}$$