Multi View Geometry (Spring '08)

Camera Models

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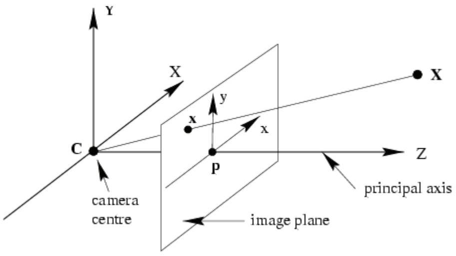
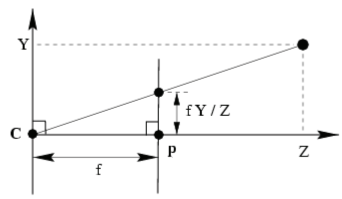
Camera Models – Finite Cameras

- **Pinhole camera model**:





$$\frac{y}{f} = \frac{Y}{Z} \Rightarrow y = f\frac{Y}{Z}$$

$$(X, Y, Z)^\top \mapsto (fX/Z, fY/Z)^\top$$

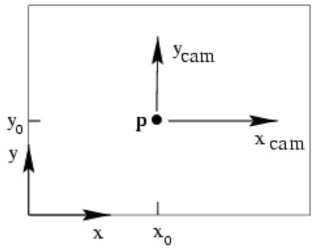
$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

In homogeneous coordinates

$$\mathrm{x} = \mathrm{PX} \qquad \mathrm{P} = \mathrm{diag}(f, f, 1)[\mathrm{I} \mid 0]$$

Finite cameras

- Principal point offset model:
  - ✓ Set the coordinates of the principal point (image center) be $(p_x, p_y)^T$
  - ✓ Then the mapping becomes



$$(X,Y,Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

  - ✓ Or in homogeneous representation

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{x} = \mathrm{K}[\mathrm{I} \mid 0]\mathrm{X}_{cam} \qquad \mathbf{K} = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$ calibration matrix

Finite cameras

- Camera Rotation and Translation:

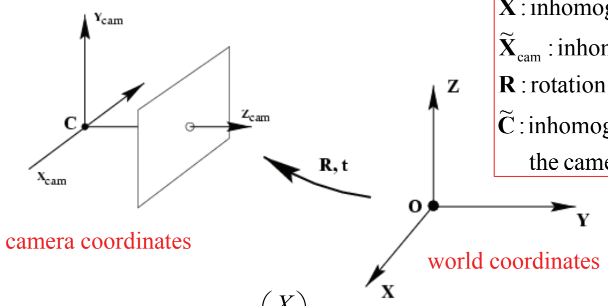
$\tilde{\mathrm{X}}_{cam} = \mathbf{R}(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})$

$\tilde{\mathbf{X}}$: inhomogeneous world coordinates

$\tilde{\mathrm{X}}_{cam}$: inhomogeneous camera coordinates

$\mathbf{R}$: rotation matrix of camera coordiantes

$\tilde{\mathrm{C}}$: inhomogeneous coordinates of the camera center



$$\mathrm{X}_{cam} = \begin{bmatrix} \mathrm{R} & -\mathrm{R}\tilde{\mathrm{C}} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} \mathrm{R} & -\mathrm{RC} \\ 0 & 1 \end{bmatrix} \mathrm{X}$$

$$\Rightarrow \mathrm{x} = \mathrm{K}[\mathrm{I} \mid 0]\mathrm{X}_{cam} \Rightarrow \mathrm{x} = \mathrm{KR}[\mathrm{I} \mid -\tilde{\mathrm{C}}]\mathrm{X} \Rightarrow \boxed{\mathrm{x} = \mathrm{PX}}$$

$\mathrm{P} = \mathrm{K}[\mathrm{R} \mid \mathrm{t}]$

$\mathrm{t} = -\mathrm{R}\tilde{\mathrm{C}}$

Finite cameras

- CCD Cameras:
  - ✓ Non-square pixels

If m_x and m_y pixels per unit distance in x and y directions

$$\mathbf{K} = \begin{bmatrix} \alpha_x & & x_0 \\ & \alpha_x & y_0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} m_x & & \\ & m_x & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix} = \begin{bmatrix} fm_x & & m_x p_x \\ & fm_y & m_y p_y \\ & & 1 \end{bmatrix}$$

  - ✓ 10 DOF

- Finite projective camera: {finite cameras}={P_{4x3} =[M|p_4]|det M≠0}
  - ✓ Including skew parameter s (zero for most normal cameras)

$$\mathbf{K} = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix} \qquad \mathrm{P} = \mathrm{KR}\left[\mathrm{I} \mid -\tilde{\mathrm{C}}\right]$$

M: 3x3 nonsingular

  - ✓ 11 DOF (5+3+3)

$$\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4] = \mathbf{M}\left[\mathbf{I} \mid \mathbf{M}^{-1}\mathbf{p}_4\right]$$

Camera Models - The projective camera

- General projective cameras: $\mathbf{P} = \left[\mathbf{M} \mid \mathbf{p}_4\right]$

Camera centre. The camera centre is the 1-dimensional right null-space $\mathbf{C}$ of P, i.e. $\mathrm{P}\mathbf{C} = \mathbf{0}$.

- ◇ **Finite camera** (M is not singular) $\mathbf{C} = \begin{pmatrix} -\mathrm{M}^{-1}\mathbf{p}_4 \\ 1 \end{pmatrix}$ $\tilde{\mathbf{C}}$
- ◇ **Camera at infinity** (M is singular) $\mathbf{C} = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}$ where $\mathbf{d}$ is the null 3-vector of M, i.e. $\mathrm{M}\mathbf{d} = \mathbf{0}$.

Column points. For $i = 1, \ldots, 3$, the column vectors $\mathbf{p}_i$ are vanishing points in the image corresponding to the X, Y and Z axes respectively. Column $\mathbf{p}_4$ is the image of the coordinate origin.

Principal plane. The principal plane of the camera is $\mathbf{P}^3$, the last row of P.

Axis planes. The planes $\mathbf{P}^1$ and $\mathbf{P}^2$ (the first and second rows of P) represent planes in space through the camera centre, corresponding to points that map to the image lines $x = 0$ and $y = 0$ respectively.

Principal point. The image point $\mathbf{x}_0 = \mathrm{M}\mathbf{m}^3$ is the principal point of the camera, where $\mathbf{m}^{3\top}$ is the third row of M.

Principal ray. The principal ray (axis) of the camera is the ray passing through the camera centre $\mathbf{C}$ with direction vector $\mathbf{m}^{3\top}$. The principal axis vector $\mathbf{v} = \det(\mathrm{M})\mathbf{m}^3$ is directed towards the front of the camera.

Table 5.1. *Summary of the properties of a projective camera* P. *The matrix is represented by the block form* $\mathrm{P} = [\mathrm{M} \mid \mathbf{p}_4]$.

The projective camera

- **Camera center**:
  - ✓ is the null-space of the camera projection matrix

$$\mathbf{PC} = \mathbf{0}$$
$$\mathbf{X} = \lambda\mathbf{A} + (1-\lambda)\mathbf{C}$$
$$\mathbf{x} = \mathbf{PX} = \lambda\mathbf{PA} + (1-\lambda)\mathbf{PC} = \lambda\mathbf{PA}$$

  - ✓ For all **A**, all points on the line **AC** project on image of **A**, therefore **C** is the camera center
  - ✓ Image of camera center is $(0,0,0)^T$, i.e. undefined
  - ✓ Finite Camera case:

$$\mathbf{C} = \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{p}_4 \\ 1 \end{pmatrix} \quad \tilde{C}$$

  - ✓ Infinite Camera case:

$$\mathbf{C} = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}, \mathbf{Md} = 0$$

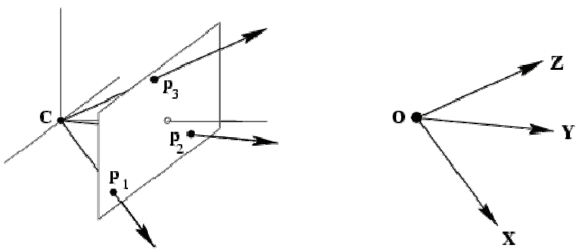
The projective camera

- **Column vectors of P**:

$\mathbf{p}_1 = \mathbf{PD}_X,\ \ \mathbf{D}_X = [1 \quad 0 \quad 0 \quad 0]^T \Rightarrow$ direction of X-axis, point on π_∞

Similarly to $\mathbf{p}_2$, and $\mathbf{p}_3$.

While, $\mathbf{p}_4 = \mathbf{PD}_o,\ \ \mathbf{D}_o = [0 \quad 0 \quad 0 \quad 1]^T \Rightarrow$ world origin



The projective camera

- **Row vectors of P**:

$$\mathbf{P} = \begin{bmatrix} p^{1T} \\ p^{2T} \\ p^{3T} \end{bmatrix}$$

✓ The principal plane:

- the plane passing **C** parallel to the image plane, set of **X**, s.t.,

$$\mathbf{PX} = (x, y, 0)^T \Rightarrow \mathbf{P}^{3T}\mathbf{X} = 0$$

$$\Rightarrow \mathbf{P}^3 \text{ is the vector representing the principal plane}$$

✓ Axis plane:

- Points on $\mathbf{P}^1$,

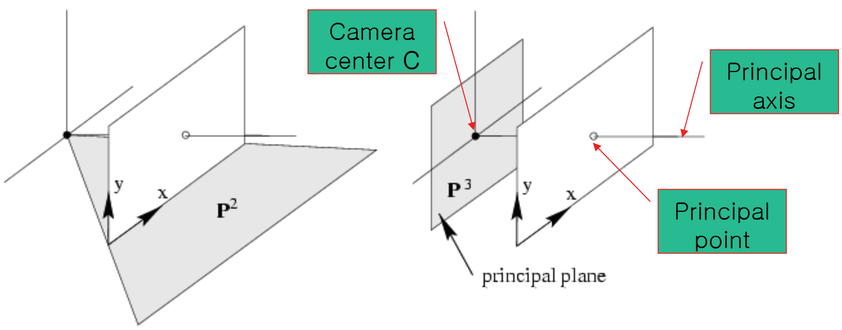
$$\mathbf{P}^{1T}\mathbf{X} = 0 \Rightarrow \mathbf{PX} = (0, y, \omega)^T \Rightarrow \text{points on y-axis}$$

$$\mathbf{P}^{1T}\mathbf{C} = 0 \Rightarrow \mathbf{C} \text{ lies on } \mathbf{P}^1$$

Thus, $\mathbf{P}^1$ is a plane defined by $\mathbf{C}$ and the line $x = 0$

- Similarly, $\mathbf{P}^2$ is a plane defined by **C** and the line $y=0$

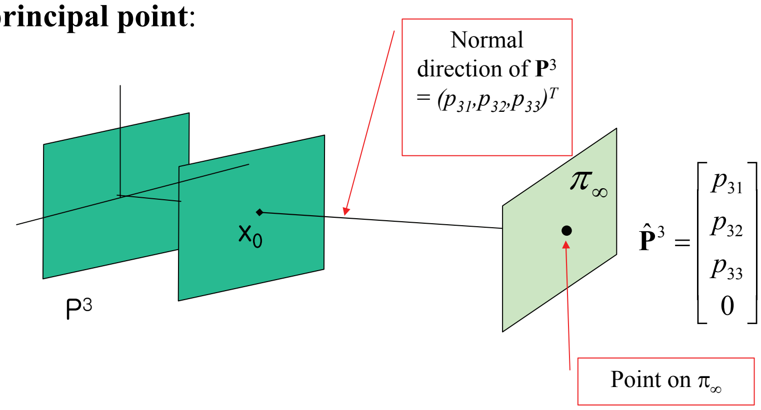
The projective camera



- **C** is the intersection of $\mathbf{P}^1$, $\mathbf{P}^2$ and $\mathbf{P}^3$:

$$\begin{bmatrix} \mathbf{P}^{1T} \\ \mathbf{P}^{2T} \\ \mathbf{P}^{2T} \end{bmatrix} \mathbf{C} = \mathbf{0}$$

- **The principal point**:



$$\mathbf{x}_0 = \mathbf{P}\hat{\mathbf{P}}^3 = [\mathbf{M} \mid \mathbf{P}_4]\hat{\mathbf{P}}^3 = \mathbf{M}\mathbf{m}^3$$

- **Principal axis vector**:
  - ✓ There is an ambiguity between $\mathbf{m}^3$ and $-\mathbf{m}^3$

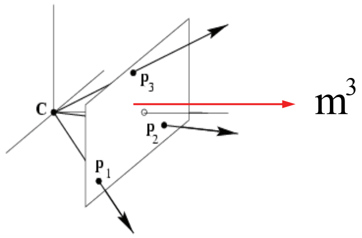
Let $\mathbf{v} = \det(\mathbf{M})\mathbf{m}^3$, then

for $\mathbf{P} = k\mathbf{K}[\mathbf{R} \mid -\mathbf{R}\tilde{\mathbf{C}}] = [\hat{\mathbf{M}} \mid \hat{\mathbf{p}}_4]$

$$\hat{\mathbf{v}} = \det(\hat{\mathbf{M}})\hat{\mathbf{m}}^3 = k^4 \det(\mathbf{K}\mathbf{R})\mathbf{m}^3$$
$$= k^4 \det(\mathbf{M})\mathbf{m}^3 = k^4\mathbf{v}$$

$\Rightarrow$ direction is independent on k

$\Rightarrow$ $\mathbf{v}$ is the direction toward the front of the camera



Action of projective camera on point

- Forward projection:

$$\mathbf{x} = \mathtt{P}\mathbf{X}$$

✓ Projection of vanishing point:

$\mathbf{D}=(\mathbf{d}^T,0)^T$

$$\mathbf{x} = \mathtt{P}\mathbf{D} = [\mathtt{M} \mid \mathbf{p}_4]\mathbf{D} = \mathtt{M}\mathbf{d}$$

- Back-projection:

$\mathbf{P}^+ = \mathbf{P}^T(\mathbf{PP}^T)^{-1}$ $\quad \mathrm{PP}^+ = \mathrm{I}$

$$\mathrm{X} = \mathrm{P}^+\mathrm{x} \qquad \mathbf{X}(\lambda) = \mathtt{P}^+\mathbf{x} + \lambda\mathbf{C}.$$

A ray passing through C and P⁺x

✓ Finite camera case:

$$\tilde{\mathbf{C}} = -\mathbf{M}^{-1}\mathbf{p}_4$$

$$\mathbf{X}(\mu) = \mu\begin{pmatrix}\mathtt{M}^{-1}\mathbf{x}\\ 0\end{pmatrix} + \begin{pmatrix}-\mathtt{M}^{-1}\mathbf{p}_4\\ 1\end{pmatrix} = \begin{pmatrix}\mathtt{M}^{-1}(\mu\mathbf{x} - \mathbf{p}_4)\\ 1\end{pmatrix}.$$

Back projected point on π_∞ C

The projective camera

- Depth of points:

✓ Consider the projection of a space point **X** onto an image point **x**

$$\mathbf{x} = \omega\begin{bmatrix}x\\ y\\ 1\end{bmatrix} = \mathbf{PX} = \begin{bmatrix}\mathbf{P}^{1T}\\ \mathbf{P}^{2T}\\ \mathbf{P}^{3T}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{X}}\\ 1\end{bmatrix}$$

Since PC=0

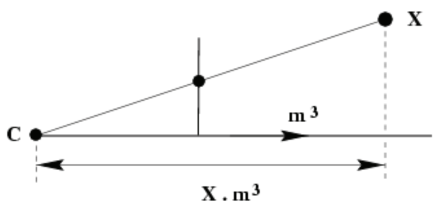
Then,

$$\omega = \mathbf{P}^{3T}\mathbf{X} = \mathbf{P}^{3T}(\mathbf{X}-\mathbf{C}) = \mathbf{m}^{3T}(\tilde{\mathbf{X}}-\tilde{\mathbf{C}})$$

dot product of the ray from C to X, with the principal ray direction

Thus, if $\det \mathbf{M} > 0$ and $\|\mathbf{m}^3\| = 1$, ω represent the depth of $\mathbf{X}$ from $\mathbf{C}$ in the (positive) direction of $\mathbf{m}^3$.

The projective camera



- In general, for $X = (X, Y, Z, T)^T$

$$\text{depth}(\mathbf{X};\mathbf{P}) = \frac{\text{sign}(\det \mathbf{M})\omega}{T\|\mathbf{m}^3\|}$$

The projective cameras - Decomposition of P

- Finding the camera center:

$$\mathbf{PC} = \mathbf{0},\ \mathbf{C} = (X, Y, Z, T)^T \rightarrow \text{null space of } \mathbf{P} \text{ using SVD}$$

$$X = \det([\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4]) \quad Y = -\det([\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4])$$
$$Z = \det([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4]) \quad T = -\det([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3])$$

p. 67

- Finding the camera orientation and internal parameters:

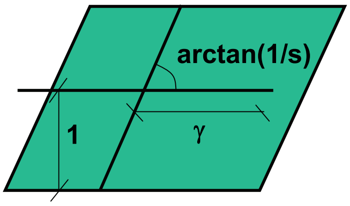
$$P = [M \mid -M\tilde{C}] = K[R \mid -R\tilde{C}].$$

✓ Decompose **M** as $\mathbf{M} = \mathbf{KR}$ using **RQ** factorization method
✓ **K**: Upper triangular matrix
✓ **R**: rotation matrix (orthogonal)
✓ Keep **K** to have positive diagonals to resolve ambiguity

Decomposition of P

- **When is skew non-zero?**

$$\mathbf{K} = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$



✓ for CCD/CMOS, always s=0
✓ Image from image, s≠0 possible (non coinciding principal axis)

Decomposition of P

- **Euclidean vs. projective**
  ✓ general projective interpretation

$$\mathbf{P} = [3\times 3 \text{ homography}] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} [4\times 4 \text{ homography}]$$

✓ Meaningfull decomposition in **K**,**R**,**t** requires Euclidean image and space
✓ Camera center is still valid in projective space
✓ Principal plane requires affine image and space
✓ Principal ray requires affine image and Euclidean space

Cameras at infinity – Affine cameras

- Camera at infinity
  - ✓ Cameras center lying on the plane at infinity
  - ✓ *affine* and *non-affine* cameras
- Affine cameras:
  - ✓ the last row of **P**, $\mathbf{P}^{3T}$ is the form of *(0,0,0,1)*
  - ✓ Thus, *points at infinity* are mapped to *points at infinity*
  - ✓ Consider moving back a finite projection camera while zooming to keep the size of object be the same

$\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4]$
$\mathbf{M}$ is singular
$\mathbf{PC} = 0$

$$\mathtt{P}_0 = \mathtt{KR}[\mathtt{I} \mid -\tilde{\mathbf{C}}] = \mathtt{K}\begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{3\top} & -\mathbf{r}^{3\top}\tilde{\mathbf{C}} \end{bmatrix}$$

$\mathbf{r}^3$: the direction of principal ray

$d_0 = -\mathbf{r}^{3T}\tilde{\mathbf{C}}$: the distance of the world origin from the camera center in the direction of the principal ray

Affine cameras

- Moving back at unit speed for a time ***t***, so that $\tilde{\mathbf{C}} \to \tilde{\mathbf{C}} - t\mathbf{r}^3$

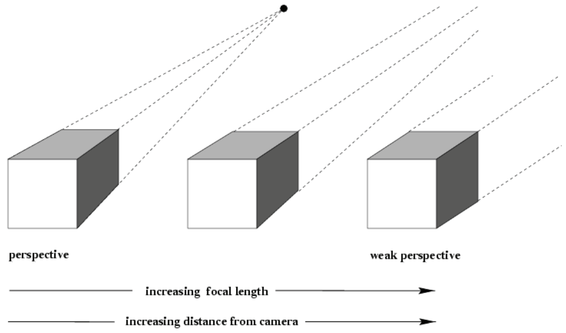
$$\mathtt{P}_t = \mathtt{K}\begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}(\tilde{\mathbf{C}} - t\mathbf{r}^3) \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}(\tilde{\mathbf{C}} - t\mathbf{r}^3) \\ \mathbf{r}^{3\top} & -\mathbf{r}^{3\top}(\tilde{\mathbf{C}} - t\mathbf{r}^3) \end{bmatrix} = \mathtt{K}\begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{3\top} & d_t \end{bmatrix}$$

$d_t = -\mathbf{r}^{3T}\tilde{\mathbf{C}} + t$: the distance of the world origin from the camera center in the direction of the principal ray at time t

- Next, by zooming with a zooming factor $k = d_t / d_0$ so that the image size remains fixed. Then,

$$\mathtt{P}_t = \mathtt{K}\begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix}\begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{3\top} & d_t \end{bmatrix} = \frac{d_t}{d_0}\mathtt{K}\begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{3\top}d_0/d_t & d_0 \end{bmatrix}$$

Affine cameras



Affine cameras

- As d_t tends to infinity,

$$P_\infty = \lim_{t \to \infty} P_t = K \begin{bmatrix} \mathbf{r}^{1\top} & -\mathbf{r}^{1\top}\tilde{\mathbf{C}} \\ \mathbf{r}^{2\top} & -\mathbf{r}^{2\top}\tilde{\mathbf{C}} \\ \mathbf{0}^\top & d_0 \end{bmatrix}$$

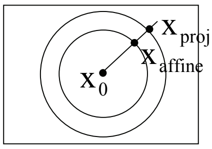
Affine camera

Affine cameras-Error

- For point on plane parallel with principal plane and through the world origin, $\mathbf{X} = \begin{pmatrix} \alpha\mathbf{r}^1 + \beta\mathbf{r}^2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{P}_0\mathbf{X} = \mathbf{P}_t\mathbf{X} = \mathbf{P}_\infty\mathbf{X}$

- For general points $\mathbf{X} = \begin{pmatrix} \alpha\mathbf{r}^1 + \beta\mathbf{r}^2 + \Delta\mathbf{r}^3 \\ 1 \end{pmatrix}$

$$\mathbf{x}_{\text{proj}} = \mathbf{P}_0\mathbf{X} = \mathbf{K}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 + \Delta \end{pmatrix} \qquad \mathbf{x}_{\text{affine}} = \mathbf{P}_\infty\mathbf{X} = \mathbf{K}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ d_0 \end{pmatrix}$$



$$\tilde{\mathbf{x}}_{\text{affine}} - \tilde{\mathbf{x}}_0 = \frac{d_0 + \Delta}{d_0}\left(\tilde{\mathbf{x}}_{\text{proj}} - \tilde{\mathbf{x}}_0\right)$$

Affine cameras-Error

- Thus, the error in employing an affine camera:

$$\tilde{\mathbf{x}}_{\text{affine}} - \tilde{\mathbf{x}}_{\text{proj}} = \frac{\Delta}{d_0}\left(\tilde{\mathbf{x}}_{\text{proj}} - \tilde{\mathbf{x}}_0\right)$$

- Approximation error is small if
  - ✓ Δ is small compared to the average depth d_0
  - ✓ The distance of the point from the principal ray is small (i.e. small field of view)

Affine cameras-Decomposition

- Decomposition of $\mathbf{P}_\infty$

$$P_\infty = \begin{bmatrix} K_{2\times2} & \tilde{x}_0 \\ \hat{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{t} \\ 0^\top & d_0 \end{bmatrix} = \begin{bmatrix} d_0^{-1}K_{2\times2} & \tilde{x}_0 \\ \hat{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{t} \\ 0^\top & 1 \end{bmatrix}$$

or for $\tilde{\mathbf{x}}_0 = \mathbf{0}$

$$P_\infty = \begin{bmatrix} K_{2\times2}R & K_{2\times2}\hat{t} + \tilde{x}_0 \\ \hat{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} K_{2\times2} & \hat{0} \\ \hat{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{t} + K_{2\times2}^{-1}\tilde{x}_0 \\ 0^\top & 1 \end{bmatrix}$$

$$= \begin{bmatrix} K_{2\times2} & K_{2\times2}\hat{t} + \tilde{x}_0 \\ \hat{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{0} \\ 0^\top & 1 \end{bmatrix}.$$

Thus,

$$P_\infty = \begin{bmatrix} K_{2\times2} & \hat{0} \\ \hat{0}^\top & 1 \end{bmatrix} \begin{bmatrix} \hat{R} & \hat{t} \\ 0^\top & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{2\times2} & \hat{\mathbf{0}} \\ \hat{\mathbf{0}}^T & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Affine cameras

- Summary of parallel projection

$$\mathbf{P}_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ canonical representation

$$\mathbf{K} = \begin{bmatrix} K_{2\times2} & 0 \\ 0 & 1 \end{bmatrix}$$ calibration matrix

principal point is not defined

Hierarchy of affine cameras

- Orthographic projection:

$$\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \qquad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

- Incorporating rotation and translation

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1T} & t_1 \\ \mathbf{r}^{2T} & t_2 \\ \mathbf{0} & 1 \end{bmatrix} = [\mathbf{M} \mid \mathbf{t}]$$

unit norm

5 DOF

Hierarchy of affine cameras

- Scaled orthographic projection:

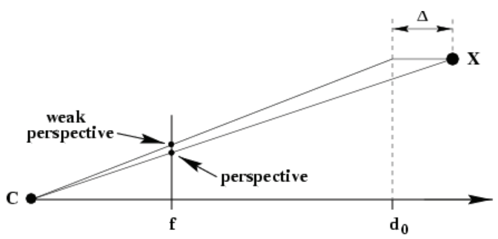
$$\mathtt{P} = \begin{bmatrix} k & & \\ & k & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1\top} & t_1 \\ \mathbf{r}^{2\top} & t_2 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{1\top} & t_1 \\ \mathbf{r}^{2\top} & t_2 \\ \mathbf{0}^\top & 1/k \end{bmatrix}.$$

6 DOF

- Weak perspective projection:

$$\mathtt{P} = \begin{bmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1\top} & t_1 \\ \mathbf{r}^{2\top} & t_2 \\ \mathbf{0}^\top & 1 \end{bmatrix}.$$

7 DOF



Hierarchy of affine cameras

- **Affine camera**:

$$\mathtt{P}_\mathtt{A} = \begin{bmatrix} \alpha_x & s & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}^{1\top} & t_1 \\ \mathbf{r}^{2\top} & t_2 \\ \mathbf{0}^\top & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

8 DOF

3x4 affine

$$= \begin{bmatrix} \mathbf{K}_{2\times 2} & \hat{\mathbf{0}} \\ \hat{\mathbf{0}}^T & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

3x3 affine

4x4 affine

- In inhomogeneous coordinates

$$\tilde{\mathbf{x}} = \mathtt{M}_{2\times 3}\tilde{\mathbf{X}} + \tilde{\mathbf{t}}\ .$$

linear mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} \begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Hierarchy of affine cameras

- Properties of the affine camera:

$$\mathbf{P}_A \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix}$$

plane at infinity = principal plane

point at infinity

✓ Any projective camera with *the principal plane is the plane at infinity* is an affine camera

✓ The camera center is also lies on the plane at infinity

✓ Parallel lines in space are mapped to parallel image lines

✓ The vector **d** satisfying $\mathbf{M}_{2x3}\mathbf{d}=0$ is the direction of parallel projection, and $(\mathbf{d}^T, 0)^T$ is the camera center.

$$\mathbf{P}_A \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = \mathbf{0}$$

General cameras at infinity

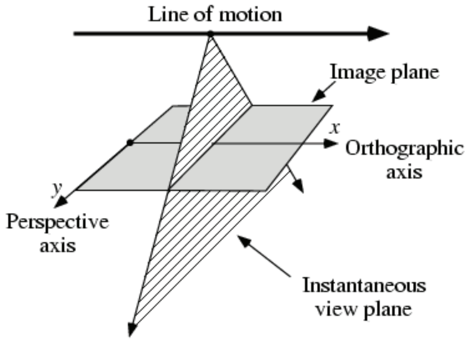
- Non-affine camera at infinity

$$\mathbf{P} = [\mathbf{M} \mid \mathbf{p}_4]$$
$\mathbf{M}$ is singular but the last row is not zero
$$\mathbf{PC} = 0$$

	Affine camera	Non-affine camera
Camera centre on π_∞	yes	yes
Principal plane is π_∞	yes	no
Image of points on π_∞ on $\mathbf{l}_\infty$	yes	no in general

Other camera models

- **Pushbroom cameras**:
- Linear Pushbroom (LP) camera – satellite sensor, SPOT



(11dof)

Other camera models

$$\begin{bmatrix} x\\ y\\ \omega \end{bmatrix} = \mathbf{P}\begin{bmatrix} X\\ Y\\ Z\\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} \tilde{x}\\ \tilde{y} \end{bmatrix} = \begin{bmatrix} x\\ y/\omega \end{bmatrix} \qquad \tilde{x} = x = \mathrm{P}^{1\top}\mathbf{X} \qquad \tilde{y} = y/z = \frac{\mathrm{P}^{2\top}\mathbf{X}}{\mathrm{P}^{3\top}\mathbf{X}}$$

- straight lines are not mapped to straight lines

$$\begin{aligned} \tilde{x} &= \mathbf{P}^{1\top}(\mathbf{X}_0 + t\mathbf{D}) \\ \tilde{y} &= \frac{\mathbf{P}^{2\top}(\mathbf{X}_0 + t\mathbf{D})}{\mathbf{P}^{3\top}(\mathbf{X}_0 + t\mathbf{D})}. \end{aligned} \qquad \Rightarrow \alpha\tilde{x}\tilde{y} + \beta\tilde{x} + \gamma\tilde{y} + \delta = o \Rightarrow \text{hyperbola}$$

Other camera models

- **Line cameras**:
  - ✓ 2D to 1D projection

$$\begin{bmatrix} x\\ y \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13}\\ p_{21} & p_{22} & p_{23} \end{bmatrix}\begin{bmatrix} \mathrm{X}\\ \mathrm{Y}\\ \mathrm{Z} \end{bmatrix} = \mathrm{P}_{2\times 3}\mathbf{x}$$

  - ✓ 5 DOF

camera center

$$\mathbf{PC} = \mathbf{0}$$

$$\mathrm{P}_{2\times 3} = \mathrm{K}_{2\times 2}\mathrm{R}_{2\times 2}[\mathrm{I}_{2\times 2} \mid -\tilde{\mathbf{c}}]$$

$$\mathrm{K}_{2\times 2} = \begin{bmatrix} \alpha_x & x_0\\ & 1 \end{bmatrix}$$