



Do you like math?

Here follows some elegant description of QM...

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Kets and Bras

$$\langle \alpha | \beta \rangle$$

↑ ↑

bracket

$$\langle \alpha | \quad | \beta \rangle$$

P. A. M. Dirac

- Hilbert space



Ket Space

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

$$c|\alpha\rangle = |\alpha\rangle c$$



Operator and Eigenkets

$$A \cdot (|\alpha\rangle) = A|\alpha\rangle$$

- Eigenvalues and eigenkets

$$|a'\rangle, |a''\rangle, |a'''\rangle, \dots$$

$$A|a'\rangle = a'|a'\rangle, \quad A|a''\rangle = a''|a''\rangle$$





Bra Space

- Dual Correspondence (DC)

$$|\alpha\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha|$$

$$|a'\rangle, |a''\rangle, \dots \overset{\text{DC}}{\leftrightarrow} \langle a'|, \langle a''|, \dots$$

$$|\alpha\rangle + |\beta\rangle \overset{\text{DC}}{\leftrightarrow} \langle\alpha| + \langle\beta|$$

$$c_\alpha |\alpha\rangle + c_\beta |\beta\rangle \leftrightarrow c_\alpha^* \langle\alpha| + c_\beta^* \langle\beta|$$



Inner Product

$$\langle \beta | \alpha \rangle = (\langle \beta |) \cdot (| \alpha \rangle)$$

Bra (c) ket

- Postulates

$$\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^*$$

$$\langle \alpha | \alpha \rangle \geq 0$$

where the equality sign holds only if $| \alpha \rangle$ is a null ket.



Orthogonality & Normalization

- Orthogonality

$$\langle \alpha | \beta \rangle = 0$$

- Normalization

$$|\tilde{\alpha}\rangle = \left(\frac{1}{\sqrt{\langle \alpha | \alpha \rangle}} \right) |\alpha\rangle$$

$$\langle \tilde{\alpha} | \tilde{\alpha} \rangle = 1$$



Operators

$$X \cdot (|\alpha\rangle) = X|\alpha\rangle$$

$$X + Y = Y + X$$

$$X + (Y + Z) = (X + Y) + Z$$

$$(\langle\alpha|) \cdot X = \langle\alpha|X$$





Hermitian Adjoint & Hermitian Operator

- X^\dagger is called Hermitian adjoint of X if

$$X|\alpha\rangle \stackrel{\text{DC}}{\leftrightarrow} \langle\alpha|X^\dagger$$

- An operator X is said to be Hermitian if

$$X = X^\dagger$$



Operator Multiplication I

$$XY \neq YX$$

$$X(YZ) = (XY)Z = XYZ$$

$$X(Y|\alpha\rangle) = (XY)|\alpha\rangle = XY|\alpha\rangle, \quad (\langle\beta|X)Y = \langle\beta|(XY) = \langle\beta|XY$$





Operator Multiplication II

$$(XY)^\dagger = Y^\dagger X^\dagger$$

$$XY|\alpha\rangle = X(Y|\alpha\rangle) \leftrightarrow (\langle\alpha|Y^\dagger)X^\dagger = \langle\alpha|Y^\dagger X^\dagger$$

- Outer product

$$(|\beta\rangle) \cdot (\langle\alpha|) = |\beta\rangle\langle\alpha|$$



Associative Axiom

- Important Examples

$$(|\beta\rangle\langle\alpha|) \cdot |\gamma\rangle = |\beta\rangle \cdot (\langle\alpha|\gamma\rangle)$$

☞: If $X = |\beta\rangle\langle\alpha|$, then $X^\dagger = |\alpha\rangle\langle\beta|$

$$\underbrace{(\langle\beta|)}_{\text{bra}} \cdot \underbrace{(X|\alpha\rangle)}_{\text{ket}} = \underbrace{(\langle\beta|X)}_{\text{bra}} \cdot \underbrace{(|\alpha\rangle)}_{\text{ket}} = \langle\alpha|X^\dagger|\beta\rangle^*$$

$$\langle\beta|X|\alpha\rangle = \langle\beta| \cdot (X|\alpha\rangle) = \{(\langle\alpha|X^\dagger) \cdot |\beta\rangle\}^* = \langle\alpha|X^\dagger|\beta\rangle^*$$

- For a Hermitian X, we have

$$\langle\beta|X|\alpha\rangle = \langle\alpha|X|\beta\rangle^*$$



Eigenvalues of Hermitian Operator

Theorem. *The eigenvalues of a Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal.*

Proof.

$$A|a'\rangle = a'|a'\rangle$$

$$\langle a''|A = a''^*\langle a''|$$

$$(a' - a''^*)\langle a''|a'\rangle = 0$$





Eigenkets as base Kets

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$c_{a'} = \langle a' | \alpha \rangle$$

$$|\alpha\rangle = \sum_{a'} |a'\rangle \langle a' | \alpha \rangle$$





Completeness Relation

$$\sum_{a'} |a'\rangle \langle a'| = I$$

$$\langle \alpha | \alpha \rangle = \langle \alpha | \cdot \left(\sum_{a'} |a'\rangle \langle a'| \cdot | \alpha \rangle \right) = \sum_{a'} |\langle a' | \alpha \rangle|^2$$

$$\sum_{a'} |c_{a'}|^2 = \sum_{a'} |\langle a' | \alpha \rangle|^2 = 1$$



Projection Operator

$$A_{a'} \equiv |a'\rangle\langle a'|$$

$$(|a'\rangle\langle a'|) \cdot |\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle = c_{a'}|a'\rangle$$

$$\sum_{a'} \Lambda_{a'} = I$$





Measurement I

- For an observable, there corresponds a Hermitian operator.

If we measure the observable, only eigenvalues of the Hermitian operator can be measured. A measurement always causes the system to jump into an eigenstate of the Hermitian operator.



Measurement II

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle = \sum_{a'} |a'\rangle \langle a' | \alpha \rangle$$

$$|\alpha\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$

$$\text{Probability for } a' = |\langle a' | \alpha \rangle|^2$$

$$|a'\rangle \xrightarrow{A \text{ measurement}} |a'\rangle$$



Expectation value

$$\langle A \rangle = \langle \alpha | A | \alpha \rangle$$

$$\langle A \rangle = \sum_{a'} \sum_{a''} \langle \alpha | a'' \rangle \langle a'' | A | a' \rangle \langle a' | \alpha \rangle = \sum_{a'} \underbrace{\langle a' | \alpha \rangle}_{\text{probability for obtaining } a'}$$

a'
↑
measured value a'

$\underbrace{|\langle a' | \alpha \rangle|}_{\text{probability for obtaining } a'}$



Compatibility & Incompatibility

- Observables A and B are compatible when the corresponding operators commute, i.e.,

$$[A, B] = 0$$

- Observables A and B are incompatible when

$$[A, B] \neq 0$$



Uncertainty Principle

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

where

$$\Delta A = A - \langle A \rangle$$

$$\langle (\Delta A)^2 \rangle = \langle (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \rangle = \langle A^2 \rangle - \langle A \rangle^2$$

- The uncertainty relation can be obtained from the Schwarz inequality in a straight forward manner.



Example of Uncertainty Relation

$$X, \quad P = -i\hbar\nabla$$

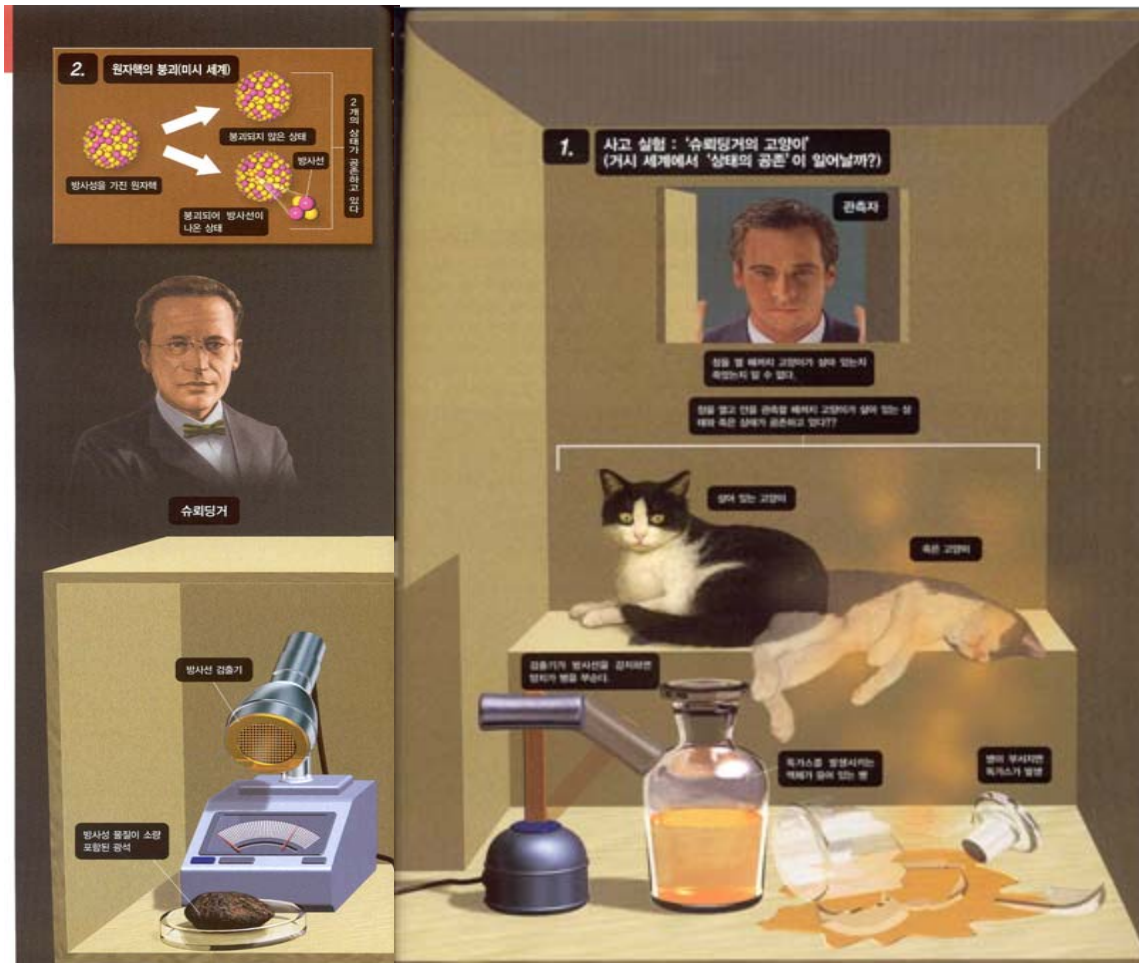
$$[X, P] = i\hbar$$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{1}{4} |i\hbar|^2$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$



Schrödinger's Cat



Newton Highlight – 양자론, 뉴턴코리아, 2006.



Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right)$$
$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger), \quad p = i\sqrt{\frac{m\hbar\omega}{2}} (-a + a^\dagger)$$

a : Annihilation operator

a^\dagger : Creation operator

$$[a, a^\dagger] = \left(\frac{1}{2\hbar} \right) (-i[x, p] + i[p, x]) = 1$$





Number Operator I

- Number Operator

$$N = a^\dagger a$$

$$H = \hbar\omega\left(N + \frac{1}{2}\right)$$

$$N|n\rangle = n|n\rangle$$

$$H|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$



Number Operator II

$$[N, a] = [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a] a = -a$$

$$[N, a^\dagger] = a^\dagger$$

$$Na^\dagger |n\rangle = \left([N, a^\dagger] + a^\dagger N \right) |n\rangle = (n+1) a^\dagger |n\rangle$$

$$Na |n\rangle = \left([N, a] + aN \right) |n\rangle = (n-1) a |n\rangle$$





Annihilation and Creation Operators

$$a|n\rangle = c|n-1\rangle$$

$$\langle n|a^\dagger a|n\rangle = |c|^2$$

$$n = |c|^2$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$n = \langle n|N|n\rangle = (\langle n|a^\dagger) \cdot (a|n\rangle) \geq 0$$

$$E_0 = \frac{1}{2}\hbar\omega$$





Energies of a Harmonic Oscillator

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad (n = 0, 1, 2, 3, \dots)$$





Who Made the Law? Is There God?

Richard Dawkins, *The God Delusion* (2006)

Francis S. Collins, *The Language of God* (2006)

Alister McGrath and Joanna C. McGrath, *The Dawkins Delusion?* (2007)

Antony Flew, *There Is A God – How the world's most notorious atheist changed his mind* (2007)

Ian G. Barbour, *When Science Meets Religion* (2000)