1. Discrete-time signals and systems

- Discrete - time signals: sequences
  - Discrete-time signals are represented as sequences of numbers
  - A sequence is a function whose domain is the set of integers
  - Define the n^{th} number in the sequence by $x[n]$, a set of numbers x can be represented by

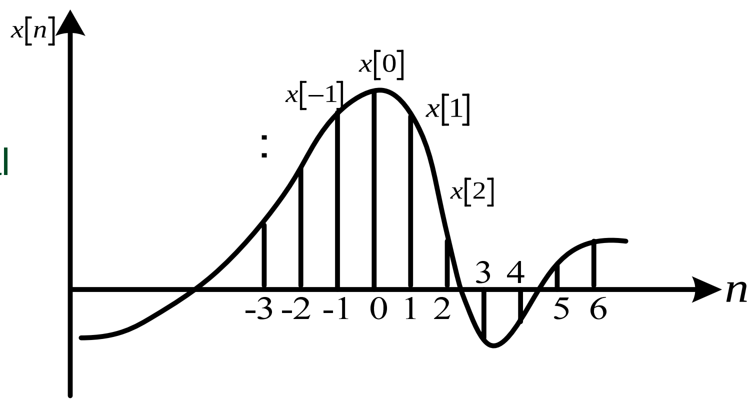
$$x = \{x[n]\}, -\infty < n < \infty$$

  - Note that

$$x[n] = x_a(nT)$$

discrete-time signal ← $x[n]$; $x_a(nT)$ → analog signal

where T denote the sampling period.



  - A delayed or shifted version of $x[n]$:

$$y[n] = x[n - n_0]$$

where n_0 is an integer

– An unit sample sequence is defined by

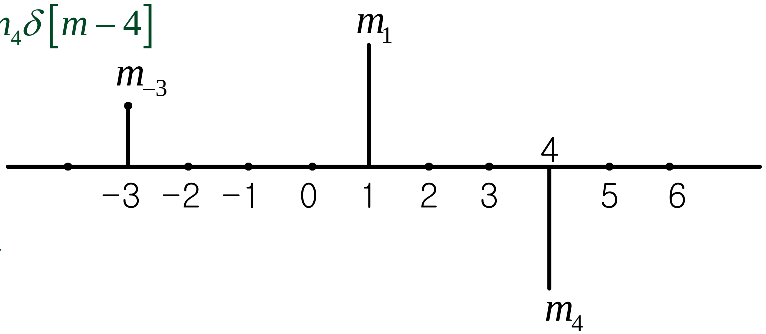
$$\delta[n] = \begin{cases} 0; & n = 0 \\ 1; & n \neq 0 \end{cases}$$

which is also referred to as a *discrete-time impulse* or simply as an *impulse*

– An arbitrary sequence can be represented as a sum of weighted and delayed impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

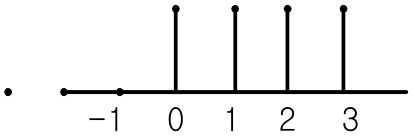
<u>Ex)</u> $x[n] = m_{-3}\delta[n+3] + m_1\delta[n-1] + m_4\delta[m-4]$



– The unit step sequence is defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\Rightarrow u[n] = \delta[n] + \delta[n-1] + \ldots = \sum_{k=0}^{\infty} \delta[n-k]$$



$$\Rightarrow \delta[n] = u[n] - u[n-1]$$

- Let $x[n]$ be an exponential sequence given by

$$x[n] = A\alpha^n u[n]$$

– For complex $\alpha = |\alpha| e^{j\omega_0}$ and $A = |A| e^{j\phi}$,

$$x[n] = |A| e^{j\phi} |\alpha|^n e^{j\omega_0 n}$$
$$= |A||\alpha|^n \left[\cos(\omega_o n + \phi) + j\sin(\omega_0 n + \phi)\right]$$

– Complex exponential sequence $(|\alpha| = 1)$

$$x[n] = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi)$$

where ω_0 is called the *frequency* of the complex sinusoid and ϕ is called the *phase*

– Since $x[n] = Ae^{j(\omega_o + 2\pi)n} = Ae^{j\omega_0 n} e^{j2\pi n} = Ae^{j\omega_0 n}$,

complex exponential sequences with frequencies $(\omega_0 + 2\pi k)$, where *k* is an integer, are indistinguishable from one another

– A sinusoidal sequence can be represented by $x[n] = A\cos(\omega_0 n + \phi), \ \forall n$

– Since $x[n] = A\cos\left[(\omega_0 + 2\pi k)n + \phi\right] = A\cos(\omega_0 n + \phi)$,

we consider ω in a frequency interval of length 2π,

$$-\pi < \omega_0 \le \pi \quad \text{or} \quad 0 \le \omega_0 < 2\pi.$$

- A sequence is called periodic with an integer period of *N*, if

$$x[x] = x[n+N], \quad \forall n$$

- In order for a sinusoidal sequence to be

$$A\ \cos(\omega_0 n + \phi) = A\ \cos[\omega_0 (n+N) + \phi]$$

it requires for $\omega_0 N = 2\pi k$.

- A sinusoidal sequence is not necessarily periodic with a period of $\frac{2\pi}{\omega_0}$ and may not be periodic at all, depending on the value of ω_0

Ex $\omega_0 = \frac{1}{3}\pi \Rightarrow N = \frac{2\pi k}{\pi} \cdot 3 = 6k$

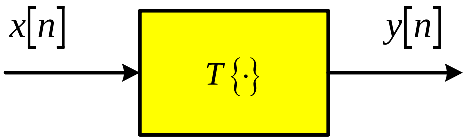
$\omega_0 = 1 \quad \Rightarrow N = 2\pi k \Rightarrow \text{No integer } N$

- High and low frequency concept: A continuous time sinusoidal signal $x[n] = A\cos(\Omega_0 t + \phi)$ oscillates more rapidly as Ω_0 increases. However, $x[n] = A\cos(\omega_0 n + \phi)$ oscillates more rapidly as ω_0 increases from 0 toward π.

  But, as ω_0 increases from π to 2π, the oscillation becomes slower due to symmetry with respect to π.

  Ex: $A\ \cos\left(\frac{\pi}{8} n + \phi\right) = A\ \cos\left(\frac{15}{8}\pi n + \phi\right)$

- Discrete-time systems



$$y[n] = T\{x[n]\}$$

– Ideal delay system

$$y[n] = x[n - n_d], \quad n_d \text{ is a positive integer}$$

– Moving averager

$$y[n] = \frac{1}{N}\{x[n] + x[n-1] + ... + x[n-N+1]\} = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

– Memoryless system

A system is referred to as memoryless if the output $y[n]$ depends only on the input $x[n]$

Ex : $y[n] = x^2[n], \quad \forall n$

– Linear system : principle of superposition

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

Ex: accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Time-invariant system (TIS): A time shift or delay of the input sequence results in a corresponding shift in the output sequence

$$y_1[n] = T\{x[n-n_d]\} \Rightarrow y[n-n_d] = y_1[n]$$

Ex : $y[n] = x^2[n] \rightarrow$ TIS

$y[n] = \sum_{k=-\infty}^{n} x[k] \rightarrow$ TIS

$y[n] = x[Mn]$: compressor

$y_1 = x_1[Mn] = x[Mn - n_o]$

$y[n-n_0] = x[m(n-n_0)] \neq y_1[n] \rightarrow$ not TIS

- Causality: A system is causal if the output sequence at $n = n_o$ depends only on the input sequences for $n \le n_{0,} \quad \forall n_0$

Ex: $y[n] = x[n+1] - x[n] \Rightarrow$ non causal

$y[n] = x[n] - x[n-1] \Rightarrow$ causal

- Stability: A system is stable in the BIBO sense *iff* every bounded input sequence produces a bounded output sequence

$$|x[n]| \le B_x < \infty \Rightarrow |y[n]| \le B_y < \infty, \quad \forall n$$

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0 & n < 0 \\ n+1 & n \ge 0 \end{cases}$$

- Linear time-invariant systems
  - Let $n_k[n]$ be the response of the system to the input $\delta[n-k]$, *i.e.,* $h_k[n] = T\{\delta[n-k]\}$

$$\begin{aligned} y[n] &= T\{x[n]\} \\ &= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \\ &= \sum_{k=-\infty}^{\infty} x[k]h_k[n] \end{aligned}$$

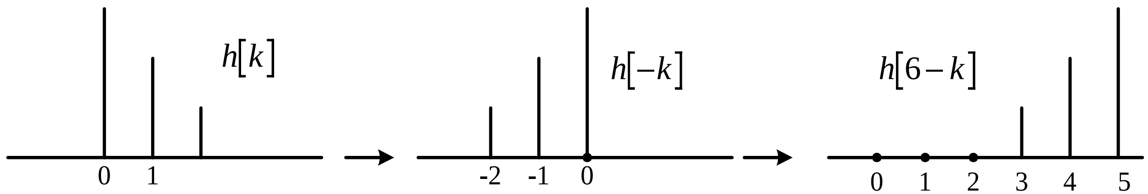
  - Time invariance implies

$$\delta[n] \rightarrow h[n] \;\Rightarrow\; \delta[n-k] \;\Rightarrow\; h[n-k] = h_k[n]$$

  - Discrete-time convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \triangleq x[n] * h[n]$$

  - To compute $h[n-k]$, first reflect $h[k]$ about the origin to obtain $h[-k]$ and then shift the origin to $k = n$;



Ex $h[n]=u[n]-u[n-N]$

$$=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$$

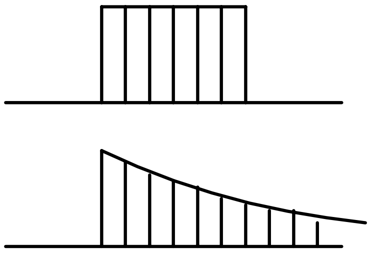
$x[n]=a^n u[n]$

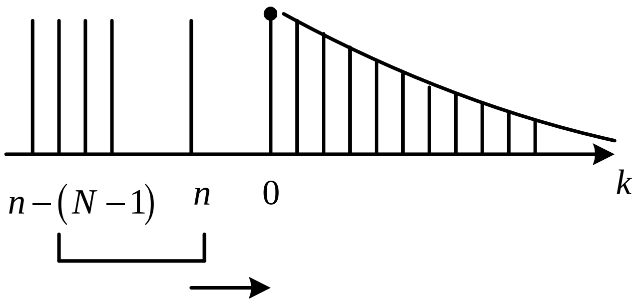
$n<0;\ \ y[n]=0$

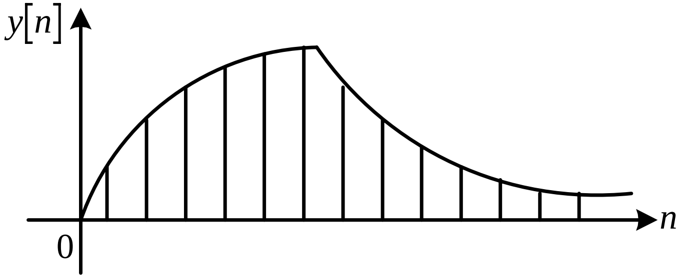
$n\le 0\le N-1;\ \ y[n]=\sum_{k=0}^{n}a^k=\frac{1-a^{n+1}}{1-a}$

$n>N-1;\ \ y[n]=\sum_{k=n-N+1}^{n}a^k=\frac{a^{n-N+1}-a^{n+1}}{1-a}$

$$=a^{n-N+1}\left(\frac{1-a^N}{1-a}\right)$$







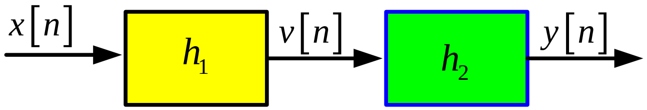
- Properties of LTI system
  - Commutative $x[n]*h[n]=h[n]*x[n]$

$$\Rightarrow y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$x[n]*\left(h_1[n]+h_2[n]\right)=x[n]*h_1[n]+x[n]*h_2[n]$$

  - Cascaded connection



Letting $h[n] \triangleq h_1[n]*h_2[n]$, we have

$$v[n]=x[n]*h_1[n]$$

$$y[n]=v[n]*h_2[n]$$

$$=x[n]*h_1[n]*h_2[n]$$

$$=x[n]*h[n]$$

– A Linear system is stable *iff* the impulse response is absolutely summable, *i,e.,*

$$S \triangleq \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Proof

Sufficient condition:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

If $x[n]$ is bounded, *i.e.*, $|x[n]| \le B_X$

then, $|y[n]| \le B_X \sum_{k=-\infty}^{\infty} |h[k]|$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \quad \Rightarrow \quad |y[n]| < \infty$$

Necessary condition:

Show that if $S = \infty$, a bounded input can cause an unbounded output.

Consider a bounded sequence given by

$$x[n] = \begin{cases} \dfrac{h^*[-n]}{|h[-n]|}, & h[n] \ne 0 \\ 0 \quad , & h[n] = 0 \end{cases} \Rightarrow y[0] = \sum_{k=-\infty}^{\infty} x[-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{|h[k]|^2}{|h[k]|} = S = \infty$$

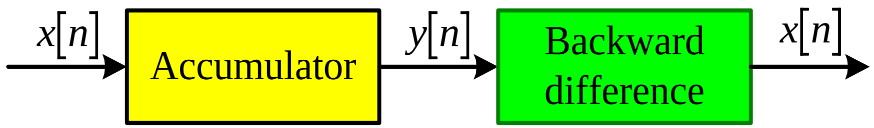
<u>Ex</u>

$$h[n] = a^n u[n]$$

$$\Rightarrow S = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|}$$

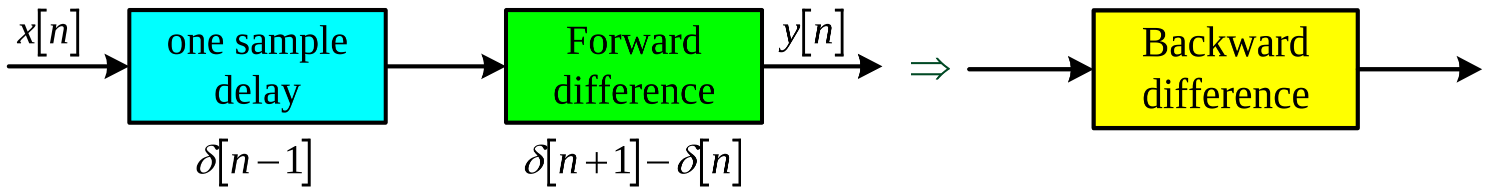
$$|a| < 1 \Rightarrow S < \infty$$

$|a| \geq 1 \Rightarrow S = \infty \Rightarrow$ unstable



$$h[n] = u[n] * (\delta[n] - \delta[n-1])$$
$$= u[n] - u[n-1]$$
$$= \delta[n]$$

Note: The property of convolution can be used to analyze the time-invariant system



$$h[n] = \delta[n-1] * (\delta[n+1] - \delta[n])$$
$$= \delta[n] - \delta[n-1]$$

- Linear constant coefficients difference equations
  - An N^{th} order linear const. difference eq. is represented by

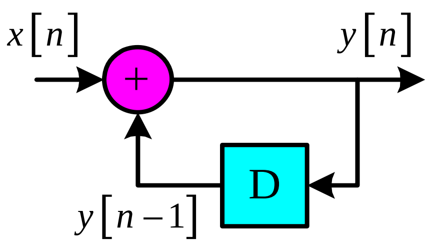
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

When the right side term is equal to zero, it is called the homogeneous difference eqn.

Ex: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k] = x[n] + y[n-1]$$

$$\Rightarrow y[n] - y[n-1] = x[n]$$



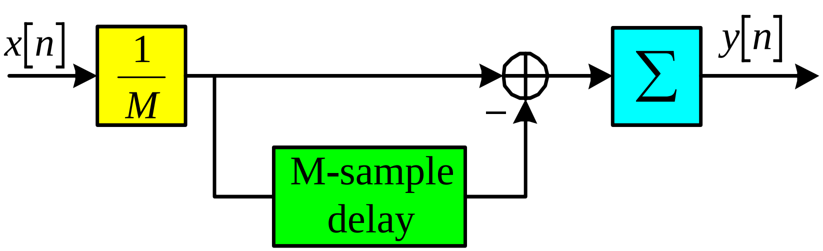
Ex: Moving average

$$h[n] = \frac{1}{M}\left(u[n] - u[n-M]\right)$$

$$\Rightarrow y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

$$h[n] = \frac{1}{M}\left(\delta[n] - \delta[n-M]\right) * u[n]$$

$$y[n] - y[n-1] = \frac{1}{M}\left(x[n] - x[n-M]\right)$$



Ex

$$y[n] = ay[n-1] + x[n]$$

Consider $x[n] = K\delta[n]$ and $y[-1] = c \leftarrow$ auxiliary cond.

For $n \geq 0;$

$$\Rightarrow y[0] = ac + K$$

$$y[1] = ay[0] + 0 = a^2c + aK$$

$$\vdots$$

$$y[n] = a^{n+1} \cdot c + a^n K, n \geq 0 \Rightarrow y[n] = a^{n+1} \cdot c + Ka^n u[n]$$

For $n < -1;$

$$y[-2] = a^{-1}\left[y[-1] - x[-1]\right] = a^{-1} \cdot c$$

$$y[-3] = a^{-1} \cdot a^{-1}c = a^{-2}c$$

$$\vdots$$

$$y[n] = a^{n+1} \cdot c, \quad n \leq -1$$

- Frequency-domain representation of DTS

$$x[n]=e^{j\omega n}$$

$$\Rightarrow y[n]=\sum_{k=-\infty}^{\infty}h[k]e^{j\omega[n-k]}$$

$$=e^{j\omega n}\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}$$

Let $H\left(e^{j\omega}\right)\triangleq\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}$

$$=H_R\left(e^{j\omega}\right)+jH\left(e^{j\omega}\right)$$

$$=\left|H\left(e^{j\omega}\right)\right|e^{j\theta_H\left(e^{j\omega}\right)}$$

$$y[n]=H\left(e^{j\omega}\right)e^{j\omega n}$$

eigen-value ← $H\left(e^{j\omega}\right)$; $e^{j\omega n}$ → eigen-function of the system

Ex: Ideal delay $y[n]=x[n-n_0]$

For $x[n]=e^{j\omega n}$,

$y[n]=e^{j\omega[n-n_0]}=e^{-j\omega n_0}e^{j\omega n}\Rightarrow H(e^{j\omega})=e^{-j\omega n_0}$

Since $h[n]=\delta[n-n_0]$ for an ideal delay system,

$$H(e^{j\omega})=\sum_{n=-\infty}^{\infty}\delta[n-n_0]e^{-j\omega n}=e^{-j\omega n_0}\Rightarrow |H(e^{j\omega})|=1$$

$$Arg\{H(e^{j\omega})\}=-\omega n_0$$

– For $x[n]=\sum_k \alpha_k e^{j\omega_k n}$,

the output of the system is

$$y[n]=\sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n} \quad \leftarrow \text{principle of superposition}$$

Ex: $x[n]=A\cos(\omega_0 n+\phi)=\frac{A}{2}\left(e^{j\phi}e^{j\omega_0 n}+e^{-j\phi}e^{-j\omega_0 n}\right)$

$\Rightarrow y[n]=\frac{A}{2}\left[H(e^{j\omega_0})e^{j\phi}e^{j\omega n}+H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}\right]$

If $h[n]$ is real, $H(e^{-j\omega})=H^*(e^{-j\omega})=|H(e^{j\omega})|e^{-j\theta}$

$\Rightarrow y[n]=A|H(e^{j\omega})|\cos(\omega_0 n+\phi+\theta)$, where $\theta=\arg[H(e^{j\omega})]$

Ex: Delay $\left|H\left(e^{j\omega_0}\right)\right|=1; \qquad \theta=-\omega_0 n_0$

$$\Rightarrow y[n]=A\cos\left(\omega_0\left(n-n_0\right)+\phi\right)$$

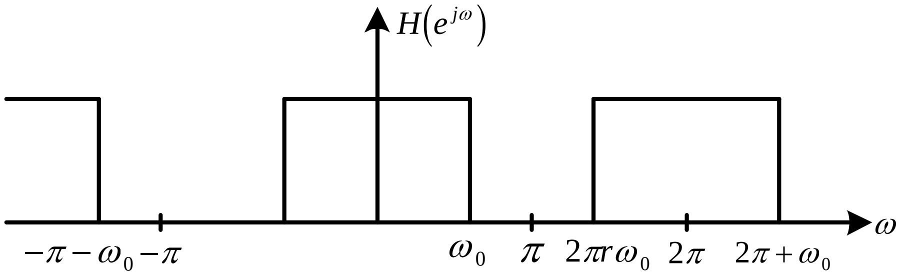
– Discrete-time LTI $\rightarrow$ periodicity of 2π

$$\Rightarrow H\left(e^{j(\omega+2k\pi)}\right)=\sum_{k=-\infty}^{\infty} h[n]e^{-j(\omega+2k\pi)n}$$

$$=\sum_{k=-\infty}^{\infty} h[n]e^{-j\omega n}=H\left[e^{(j\omega)}\right]$$

$\Rightarrow H\left(e^{j\omega}\right)$ is periodic with a period of 2π

Ex: Ideal frequency selective filters (LPF)



Ex:

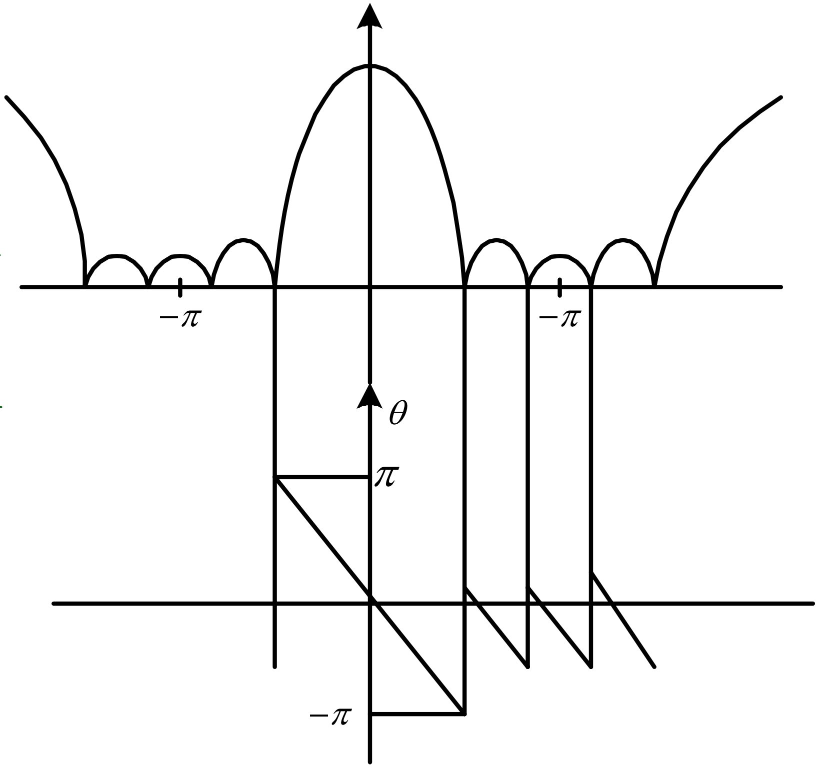
$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1} & -M_1 \le n \le M_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow H\left(e^{j\omega}\right) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}$$

$$= \frac{1}{M_1 + M_2 + 1} \frac{e^{+j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega(M_2+1-M_1)/2}}{M_1 + M_2 + 1} \frac{e^{j\omega(M_1+M_2+1)/2} - e^{-j\omega(M_1+M_2+1)/2}}{1 - e^{-j\omega}}$$

$$= \frac{1}{M_1 + M_2 \_ 1} e^{-j\omega(M_2 - M_1)} \frac{\sin \dfrac{\omega(M_1 + M_2 + 1)}{2}}{\sin \dfrac{\omega}{2}}$$



- Representation of sequences by Fourier transform
  - A stable sequence can be represented by a Fourier integral of the form

$$x[n]=\frac{1}{2\pi}\int_{2\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega \qquad \text{--- (A)}$$

where $X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n} \qquad \text{--- (B)}$

$$=\left|X\left(e^{j\omega}\right)\right|e^{j\theta_X\left(e^{j\omega}\right)} \leftarrow \text{polar cord. form}$$

$$=X_R\left(e^{j\omega}\right)+jX_1\left(e^{j\omega}\right) \leftarrow \text{rectangular cord. form}$$

Since $X\left(e^{j\omega}\right)$ is periodic with period 2π, it is of the form of Fourier series for the continuous-variable periodic function $X\left(e^{j\omega}\right)$.

  - Note that Eq (A) is the inverse of Eq. (B)

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}x[m]e^{-j\omega n}\right)e^{j\omega n}d\omega$$

$$=\sum_{n=-\infty}^{\infty}x[m]\frac{1}{2\pi}\int_{2\pi}^{\pi}e^{j\omega(n-m)}d\omega=x[n]$$

$$\left(\because \ \frac{1}{2\pi}\int_{2\pi}^{\pi}e^{j\omega(n-m)}d\omega=\frac{\sin\pi(n-m)}{\pi(n-m)}=\delta[n-m]\right)$$

– If $x[n]$ is absolutely summable (*i.e.*, stable sequence), $X(e^{j\omega})$ exists.

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

– Furthermore, $x[n]$ can be shown to converge uniformly to a continuous function of ω.

$\Rightarrow$ Any *stable* sequence (or system) have a finite and continuous frequency response:

(Sufficient condition for existence of $H(e^{j\omega})$)

– A finite-length sequence is absolutely summable

Ex: $x[n] = a^n u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}}, \quad \text{if } |ae^{-j\omega}| = |a| < 1$$

– If a sequence is not absolutely summable, but square summable ; *i.e.*,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

it can be represented by a Fourier transform with mean-square convergence, *i.e.*,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - \sum_{n=-M}^{M} x[n] e^{-j\omega n} \right|^2 d\omega = 0$$

Ex: Ideal LPF

$$H_{LP}\left(e^{j\omega}\right)=\begin{cases}1 & |\omega|<\omega_c \\ 0 & \omega_c<|\omega|<\pi\end{cases}$$

$$\Rightarrow \quad h_{LP}(n)=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega=\frac{\sin \pi\omega_c n}{\pi n}, \quad \forall n$$

$\Rightarrow$ non causal and not absolutely summable

$H_{Lp}\left(e^{j\omega}\right)$ is discontinuous at $\omega=\omega_c$

But is mean-square summable

Ex: $x[n]=1, \ \forall n$: neither absolutely nor square summable

$$\Rightarrow X\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega+2k\pi\right)$$

Since $X\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty} 2\pi\delta\left(\omega-\omega_0+2k\pi\right)$

$$\Rightarrow x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta\left(\omega-\omega_0\right)e^{j\omega n}d\omega$$

$$=e^{j\omega_0 n}$$ $\Rightarrow$ When $\omega_0=0, \ x[n]=1$

$\left|X\left(e^{j\omega}\right)\right|$ is not finite for all ϖ

– Theory of generalized functions

$$\sum_{n=-\infty}^{\infty} e^{j(\omega-\omega_0)n} = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2k\pi)$$

$$\Rightarrow x[n] = \sum_m a_m e^{j\omega_m n} \Rightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\sum_m 2\pi a_m \delta(\omega-\omega_m+2k\pi)$$

– Conjugate symmetric sequence: $x_e[n] = x_e^*[-n]$

– Conjugate anti-symmetric sequence: $x_o[n] = -x_o^*[-n]$

$$\Rightarrow x[n] = x_e[n] + x_o[n]$$

$$x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$$

$$x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$$

even sequence : $x_e[n] = x_e[-n]$

odd sequence: $x_o[n] = -x_o[-n]$

$$\Rightarrow X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where

$$X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right)$$

$$X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right)$$

– Note that $X_e(e^{j\omega}) = X^*(e^{-j\omega})$ and $X_o(e^{j\omega}) = -X^*(e^{-j\omega})$

$$\mathrm{Re}\{x[n]\} \leftrightarrow X_e(e^{j\omega})$$
$$j\,\mathrm{Im}\{x[n]\} \leftrightarrow X_o(e^{j\omega})$$
$$x_e[n] \leftrightarrow X_R(e^{j\omega})$$
$$x_o[n] \leftrightarrow jX_I(e^{j\omega})$$

Re $\{x[n]\} \leftrightarrow X(e^{j\omega}) = X^*(e^{j\omega})$

Ex: $X(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}} \quad |a|<1$

$$X^*(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = X(e^{j\omega})$$

$$X_R(e^{j\omega}) = \frac{1}{1+a^2-2a\cos\omega}(1-a\cos\omega) = X_R(e^{j\omega})$$

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

$$\theta_X(e^{j\omega}) = \tan^{-1}\frac{-a\sin\omega}{1-a\cos\omega} = -\theta_X(e^{-j\omega})$$

- Fourier transform theorems

$$F\left\{x[n]\right\} = X\left(e^{j\omega}\right)$$

$$x[n] = F^{-1}\left\{x\left(e^{j\omega}\right)\right\}$$

$$\Rightarrow X[n] \xleftrightarrow{F} X\left(e^{j\omega}\right)$$

  – Linearity: $ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$

  – Delay: $x[n-n_0] \xleftrightarrow{F} X\left(e^{j\omega}\right)e^{-j\omega n_0}$

    $e^{j\omega_0 n} \xleftrightarrow{F} X\left(e^{j(\omega-\omega_0)n}\right)$

  – Time reversal: $x[-n] \xleftrightarrow{F} X\left(e^{-j\omega}\right)$

    If $x[n]$ is real, $x[-n] \xleftrightarrow{F} X^*\left(e^{+j\omega}\right)$

  – Differentiation:

$$nx[n] \xleftrightarrow{F} j\frac{d}{d\omega}X\left(e^{j\omega}\right)$$

  – Parseval's theorem:

$$E \triangleq \sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

– Convolution:

$$x[n] \overset{F}{\longleftrightarrow} X\left(e^{j\omega}\right),\ \ h[n] \overset{F}{\longleftrightarrow} H\left(e^{j\omega}\right)$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$y[n] = x[n]h[n] \overset{F}{\longleftrightarrow} Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\xi}\right)H\left(e^{j(\omega-\xi)}\right)d\xi$$

periodic convolution

$\Rightarrow$ Multiplication of two sequences is equivalent to periodic convolution of corresponding Fourier Transforms.

Ex: $y[n]-\frac{1}{2}y[n-1] = x[n]-\frac{1}{4}x[n-1]$

$$\Rightarrow \delta[n]-\frac{1}{4}\delta[n-1] = h[n]-\frac{1}{2}h[n-1] \ \Rightarrow\ H\left(e^{j\omega}\right) = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}};\ |\alpha|<1 \ \Rightarrow\ h[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- Discrete-time Random Signals
  - Consider an LTI system with impulse response $h[n]$

$$y[n]=\sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

When $x[n]$ is a WSS discrete-time random process,

$$\mu_X[n] \triangleq E\{X[n]\}, \quad \mu_Y[n] \triangleq E\{y[n]\}$$

  - If stationary, $\mu_X[n]=\mu_X$

$$\mu_Y[n]=E\left\{\sum_{k=-\infty}^{\infty} h[n-k]x[k]\right\}$$

$$=\sum_{k=-\infty}^{\infty} h[n-k]E\{x[k]\}=\sum_{k=-\infty}^{\infty} h[k]E\{x[n-k]\}$$

$$=\mu_X \sum_{k=-\infty}^{\infty} h[k]$$

$$=\mu_X H\left(e^{j0}\right)$$

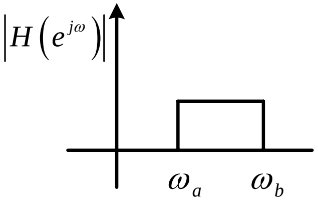
– Autocorrelation of $y[n]$

$$\phi_{yy}[n,n+m]=E\{y[n]y[n+m]\}$$
$$=E\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}h[k]h[l]x[n-k]x[n-l+m]\right\}$$
$$=\sum_{k}\sum_{l}h[k]h[l]E\{x[n-k]x[m+m-l]\}$$
$$=\sum_{k}\sum_{l}h[k]h[l]\phi_{xx}[m+k-l]=\phi_{yy}[m]$$

Let $l-k \triangleq i$. Then,

$$\phi_{yy}[m]=\sum_{i=-\infty}^{\infty}\phi_{xx}[m-i]\sum_{k=-\infty}^{\infty}h[k]h[i+k]$$

Autocorrelation sequence of $h[n]$: $\phi_h[i]\triangleq\sum_{k=-\infty}^{\infty}h[k]h[i+k]$



$$\Rightarrow \phi_{yy}[m]=\sum_{i=-\infty}^{\infty}\phi_h[i]\phi_{xx}[m-i]$$

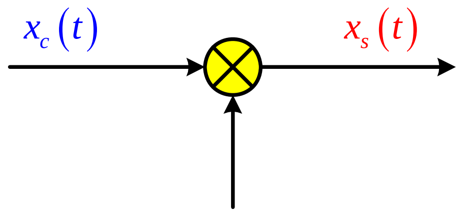
$$\Rightarrow \Phi_{yy}(e^{j\omega})=\Phi_h(e^{j\omega})\Phi_{xx}(e^{j\omega})=|H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega})$$

where $\Phi_h(e^{j\omega})=H(e^{j\omega})H^*(e^{j\omega})=|H(e^{j\omega})|^2$

Note: $E\{y^2[n]\}=\phi_{yy}(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{yy}(e^{j\omega})d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega})d\omega\geq 0$

2. Periodic Sampling of Continuous-time Signals

- Impulse sampling: ideal continuous-to–discrete (C/D) converter



$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

$$s(t) = \sum \delta(t - nT)$$

where $T\left(=\frac{1}{f_s}\right)$ is the sampling period and f_s is called the sampling frequency

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \Leftarrow \text{shifting property of the impulse function}$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

– Fourier transform of $x_s(t)$

$$X_s(f) = F\{x_s(t)\} = \int \sum_{n=\infty}^{\infty} x_c(nT)\delta(t-nT)e^{-j2\pi ft}dt$$

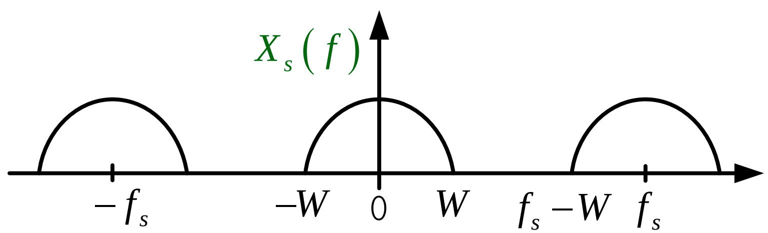
$$= \sum_{n=\infty}^{\infty} x_c(nT)e^{-j2\pi fnT}$$

$$X_s(f) = X_c(f) * F\left\{\sum_{n=-\infty}^{\infty} \delta(t-nT)\right\}$$

$$= \frac{1}{T} X_c(f) * \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

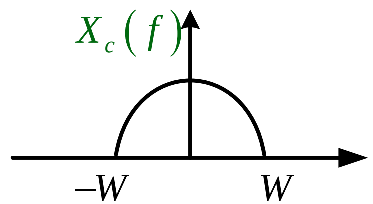
$$= \frac{1}{T} \sum_{n=\infty}^{\infty} X_c\left(f - \frac{n}{T}\right) \Leftarrow \text{periodic function}$$

$$\Omega_s = 2\pi / T = 2\pi f_s$$

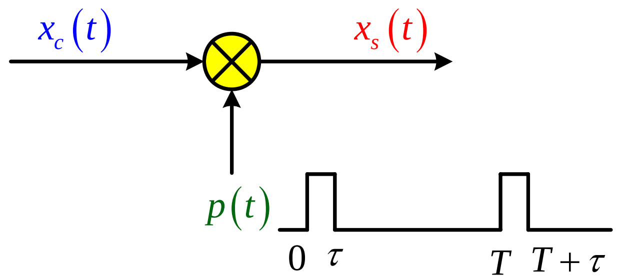


$f_s - W \le W \Leftrightarrow f_s \le 2W$

$\Rightarrow$ Aliasing



- Natural Sampling

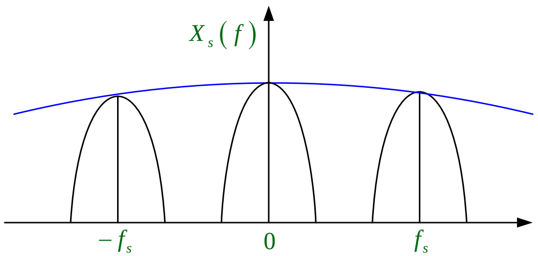


$$p(t) = \sum_{n=-\infty}^{\infty} p_n e^{j2\pi f_s nt}; \quad p_n = \frac{1}{T} \operatorname{sinc} \frac{n\tau}{T}$$

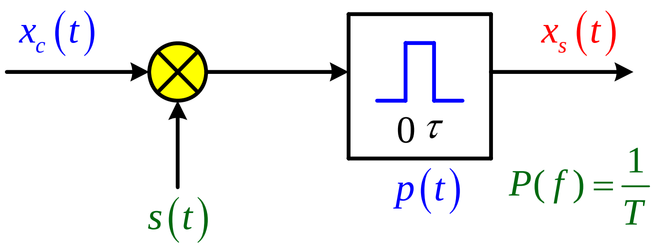
$$x_s(t) = x_c(t)\, p(t)$$

$$\Rightarrow X_s(f) = F\left\{ x_c(t) \sum_{n=-\infty}^{\infty} p_n e^{j2\pi f_s nt} \right\}$$

$$= \sum_{n=-\infty}^{\infty} p_n X_c\left(f - \frac{n}{T} \right)$$



- Sample and hold (S/H)

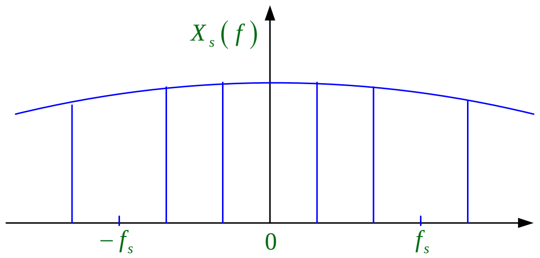


$$P(f)=\frac{1}{T}\mathrm{sinc}\left(\frac{\tau}{T}f\right)$$

$$x_s(t)=\left[x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\right]*p(t)$$

$$\Rightarrow X_s(f)=\left[X_c(f)*\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T}\right)\right]p(f)$$

$$=p(f)\frac{1}{T}\sum_{n=-\infty}^{\infty}X\left(f-\frac{n}{T}\right)$$



- Nyquist Sampling Theorem
  - If $x_c(t)$ is strictly band limited, *i.e.*, $X(f)=0$ for $|f|>W$ and the sampling frequency is chosen such that $f_s=\frac{1}{T}=2W$, then

$$X_s(f)=\sum_{n=-\infty}^{\infty}x_c(nT)e^{-j2\pi fnT}$$

Since

$$X_c(f)=\frac{1}{2W}X_s(f),\quad |f|\le W,$$

if the sample values $x_c(nT)$ are specified for all time, $X_c(f)$ is uniquely determined by using the Fourier series.

$\Rightarrow$ $x_c(t)$ is uniquely determined by $x_c(nT),\ n=0,\pm1,\pm2,\cdots$.

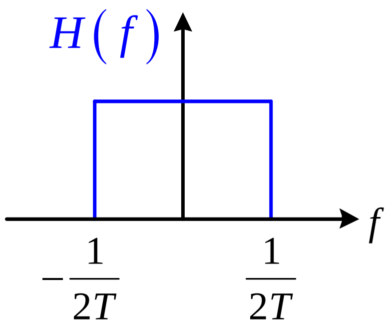
- Reconstruction of a band limited signal from its samples: Interpolation formula

$$
\begin{aligned}
x_c(t) &= \int_{-\infty}^{\infty} X_c(f) e^{j2\pi ft} df \\
&= \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{\pi f}{W}} e^{j2\pi ft} df \\
&= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f(t-nT)} df \\
&= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \pi(t-nT)/T}{\pi(t-nT)/T} \\
&= \sum_{n=-\infty}^{\infty} x[n] h(t-nT); \quad h(t) = \frac{\sin \pi t/T}{\pi t/T}
\end{aligned}
$$

$\Rightarrow$ A band-limited signal of finite energy can be completely recovered by its samples taken at a rate of 2*W*/sec

– Ideal reconstruction filter

$$h(t) = \frac{\sin \pi t/T}{\pi t/T}$$



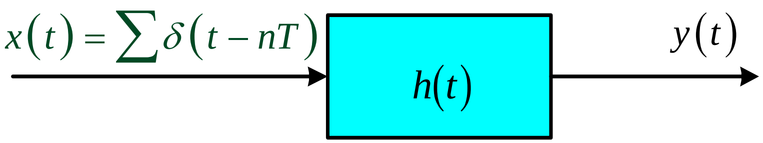
- Poisson sum formula

A periodic signal $x_s(t)$ with period *T* is represented by the Fourier series,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi f_s nt}; \quad f_s = \frac{1}{T}$$

where $x_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_s(t) e^{-j2\pi f_s nt} dt$

Ex: $\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi f_s nt} \quad \left(\because x_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(\tau) e^{-j2\pi f_s nt} dt = \frac{1}{T}\right)$



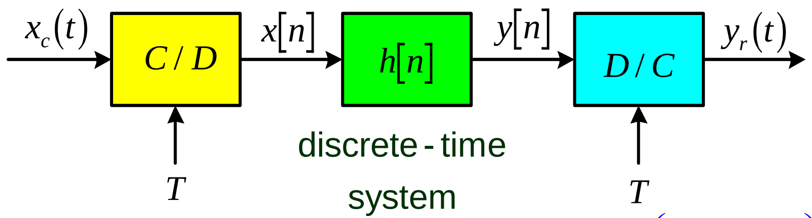
$$y(t) = h(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} h(t-nT)$$

$$y(t) = h(t) * \left[\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi f_s nt}\right]$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} \int h(\tau) e^{j2\pi f_s n(t-\tau)} d\tau = \frac{1}{T}\sum_{n=-\infty}^{\infty} H(nf_s) e^{j2\pi f_s nt}$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty} h(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H(nf_s) e^{j2\pi f_s nt}$$

- Discrete-time processing of continuous-time signals



$$x[n] = x_c(nT) \Rightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right);$$

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin\dfrac{\pi(t-nT)}{T}}{\dfrac{\pi(t-nT)}{T}};$$

$$\Rightarrow Y_r(\Omega) = \underline{H_r(\Omega)}\, Y(e^{j\Omega T})$$

low pass reconstruction filter (← $H_r(\Omega)$)

$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

- The freq-domain representation is easier than the time-domain representation

- LTI discrete-time systems

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

$$\begin{aligned}\Rightarrow Y_r(j\Omega)&=H_r(\Omega)H\left(e^{j\Omega T}\right)X\left(e^{j\Omega T}\right);\quad \omega=\Omega T\\ &=H_r(\Omega)H\left(e^{j\Omega T}\right)\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(\Omega-\frac{2\pi k}{T}\right)\\ &=\begin{cases}H\left(e^{j\Omega T}\right)X_c(\Omega), & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

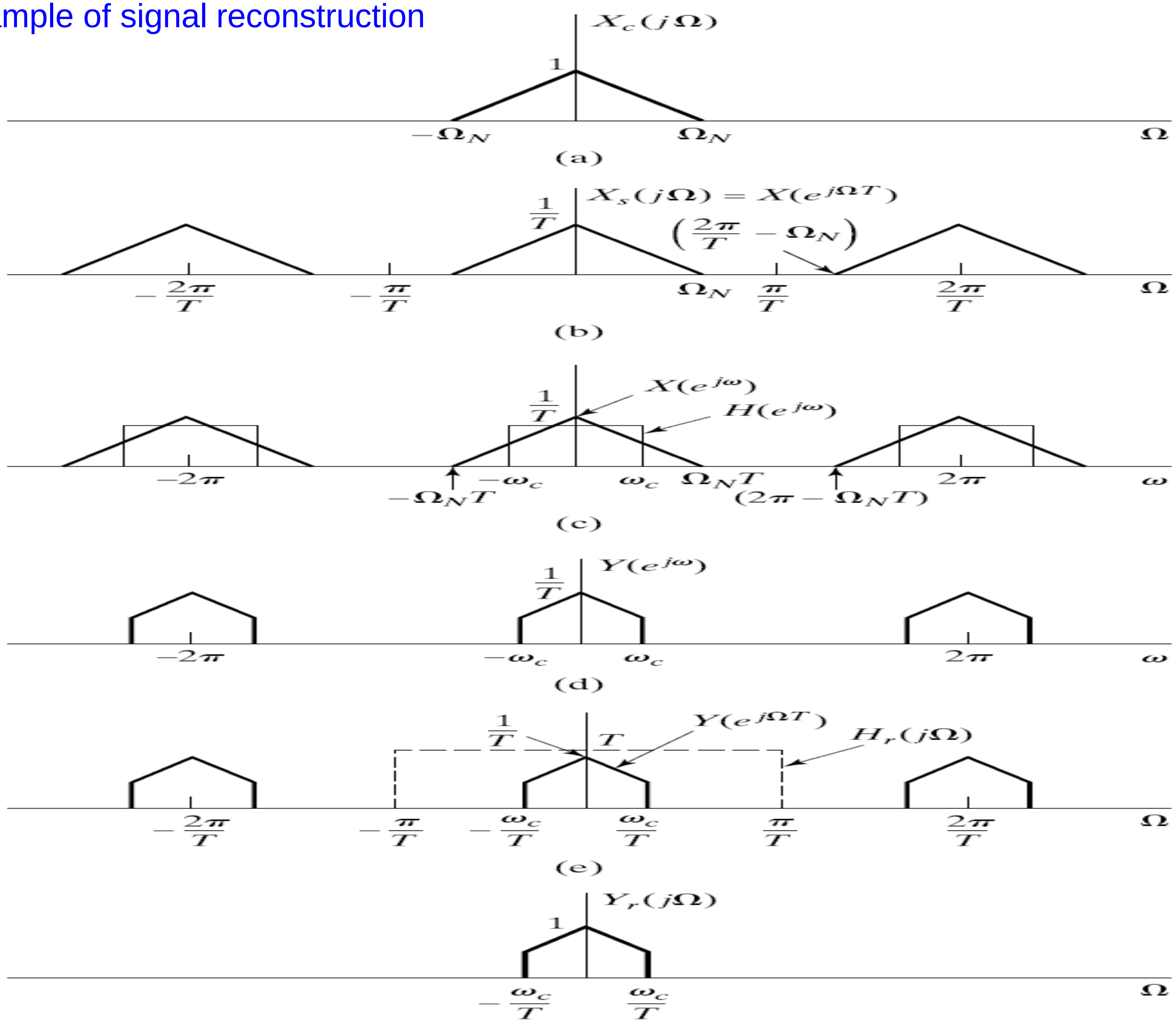
  – If $X_c(\Omega)$ is band limited and the sampling rate is larger than the Nyquist rate,

$$Y_r(\Omega)=H_{eff}(\Omega)X_c(\Omega)$$

the effective frequency response is

$$H_{eff}(\Omega)=\begin{cases}H\left(e^{j\Omega T}\right), & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}$$

- Example of signal reconstruction



Ex: Ideal LPF

$$H_{eff}(\Omega) = \begin{cases} 1, & |\Omega T| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

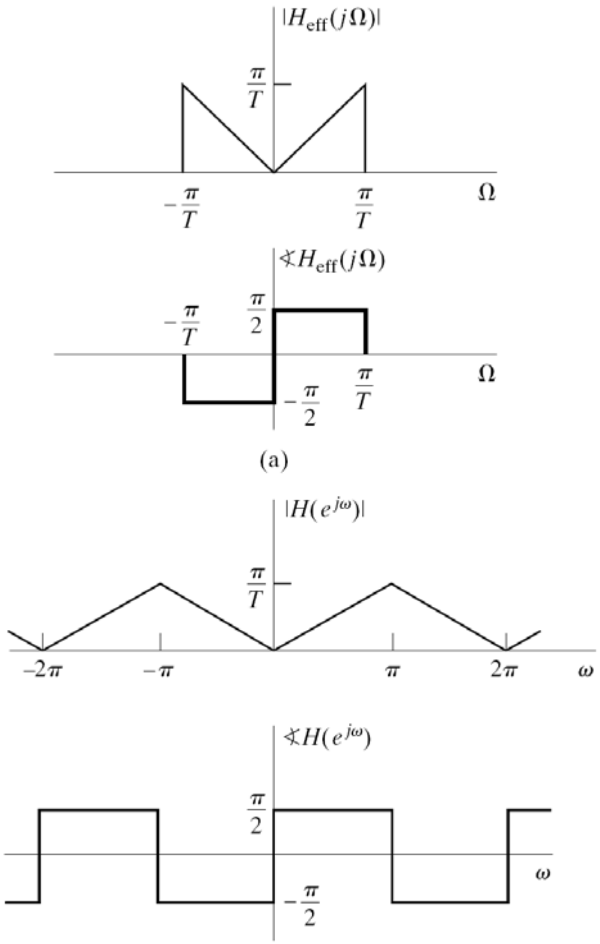
Ex: Ideal band-limited differentiator

$$y_c(t) = \frac{d}{dt} x_c(t) \Rightarrow H_c(\Omega) = j\Omega$$

$$\Rightarrow H_{eff}(\Omega) = \begin{cases} j\Omega, & |\Omega| < \frac{\pi}{T} \\ 0\ , & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = \frac{j\omega}{T}, \quad |\omega| < \pi$$

$$\Rightarrow h[n] = \frac{\pi n \cos \pi n - \sin \pi n}{\pi n^2 T} = \begin{cases} \frac{\cos \pi n}{nT}, & n \neq 0 \\ 0\ , & n = 0 \end{cases}$$



– Impulse invariance

$$H\left(e^{j\omega}\right) = H_c\left(\frac{\omega}{T}\right) = H(\Omega); \quad |\omega| < \pi$$

$\Rightarrow$ When $h[n] = Th_c(nT)$, the discrete-time system is said to be an impulse-invariance version of the continuous-time system.

$$\because H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right) \Leftarrow X_s(\Omega) = X\left(e^{j\omega}\right)\Big|_{\omega=\Omega T} = X\left(e^{j\Omega T}\right)$$

$$\Rightarrow H\left(e^{j\omega}\right) = \frac{1}{T} H_c\left(\frac{\omega}{T}\right), \quad |\omega| < \pi$$

Ex: $h_c(t) = Ae^{s_0 t}u(t) \Rightarrow H_c(s) = \dfrac{A}{s - s_0}$

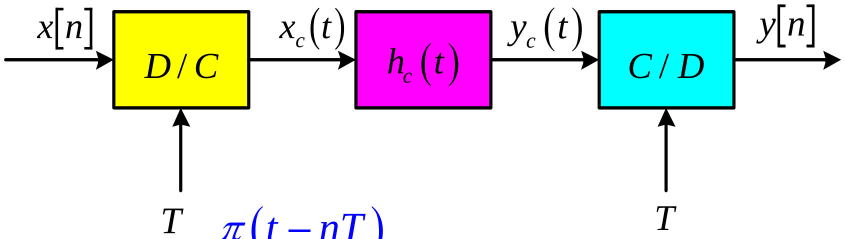
$$h[n] = Th_c(nT) = Ae^{s_0 nT}u[n]$$

$$H(z) = \frac{AT}{1 - e^{s_0 T}z^{-1}} \quad \text{or} \quad H\left(e^{j\omega}\right) = \frac{AT}{1 - e^{s_0 T}e^{-j\omega}} \neq H_c\left(\frac{\omega}{T}\right)$$

$\because h_c(t)$ is not strictly band-limited

The resulting discrete-time frequency response is aliased

- Continuous-time processing of discrete-time signals



$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi(t-nT)}{T}}{\frac{\pi(t-nT)}{T}}; \quad X_c(\Omega) = T\,X\left(e^{j\Omega T}\right), \quad |\Omega| \le \frac{\pi}{T}$$

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin \frac{\pi(t-nT)}{T}}{\frac{\pi(t-nT)}{T}}; \quad Y_c(\Omega) = H_c(\Omega) X_c(\Omega); \quad |\Omega| < \frac{\pi}{T}$$

$$Y\left(e^{j\omega}\right) = \frac{1}{T} Y_c\left(\frac{\omega}{T}\right), \quad |\omega| < \pi$$

$$= H_c(\Omega) X_c(\Omega) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right) \Rightarrow \quad H\left(e^{j\Omega T}\right) = H_c(\Omega) = H_{eff}(\Omega); \quad |\Omega| \le \pi / T$$

Ex: Non-integer delay $H\left(e^{j\omega}\right)=e^{-j\omega\Delta}, \quad |\omega|<\pi$

If Δ is an integer, $y[n]=x[n-\Delta] \quad \Rightarrow \quad h[n]=\delta[n-\Delta]$

If Δ is not an integer, $H_c(f)=e^{-j2\pi f\Delta T}$

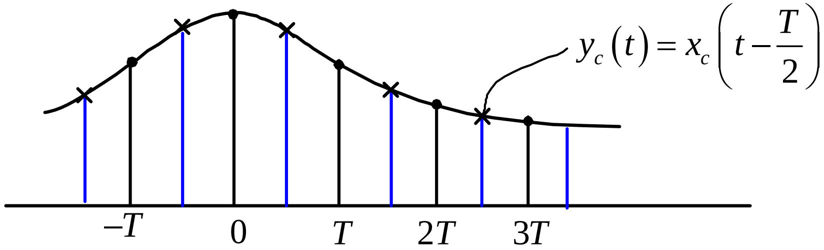
$$y_c(t)=x_c(t-\Delta T)$$

$$\Rightarrow y[n]=x_c(nT-\Delta T)$$

$$=\sum_{k=-\infty}^{\infty} x[k]\left.\frac{\sin\frac{\pi(t-\Delta T-kT)}{T}}{\frac{\pi(t-\Delta T-kT)}{T}}\right|_{t=nT}$$

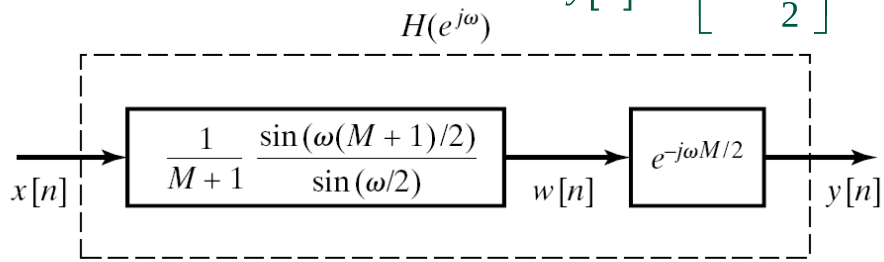
$$=\sum_{k=-\infty}^{\infty} x[k]\frac{\sin\pi(n-k-\Delta)}{\pi(n-k-\Delta)} \quad \Rightarrow \quad h[n]=\frac{\sin\pi(n-\Delta)}{\pi(n-\Delta)}$$

$e.g., \quad \Delta=\frac{1}{2};$



– Ex: Moving average with non-integer delay

$$y[n] = w\left[n - \frac{M}{2}\right]$$



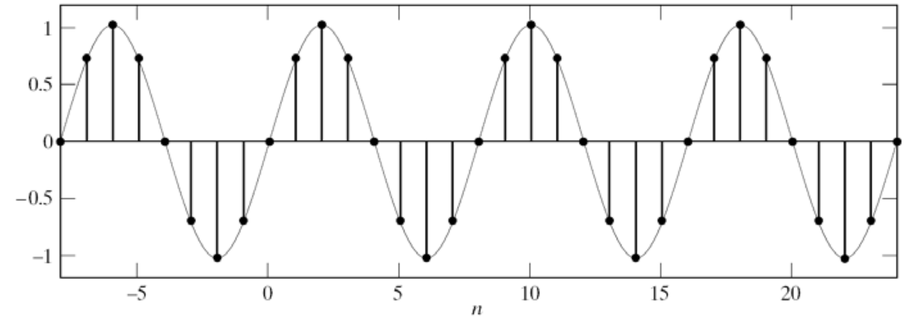
$$h[n] = \begin{cases} \dfrac{1}{M+1}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H\left(e^{j\omega}\right) = \frac{1}{M+1} \frac{\sin \dfrac{\omega(M+1)}{2}}{\sin \dfrac{\omega}{2}} e^{-j\frac{\omega M}{2}}, \quad |\omega| < \pi$$

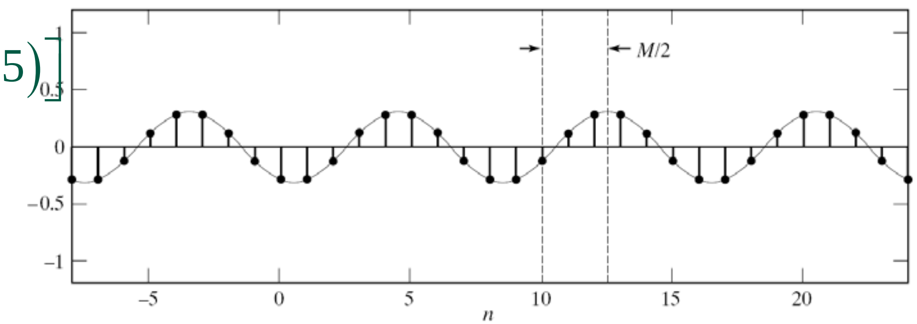
- When M=5,

  for $x[n] = \cos\left(0.25\pi n\right)$

$$\Rightarrow y[n] = 0.308 \cos\left[0.25\pi\left(n - 2.5\right)\right]$$



(a)



- Changing the sampling rate
  - Decimation by *M* (sampling rate compressor)

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT) \;\Rightarrow\; x_d[n] \triangleq x[nM] = x_c(nMT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

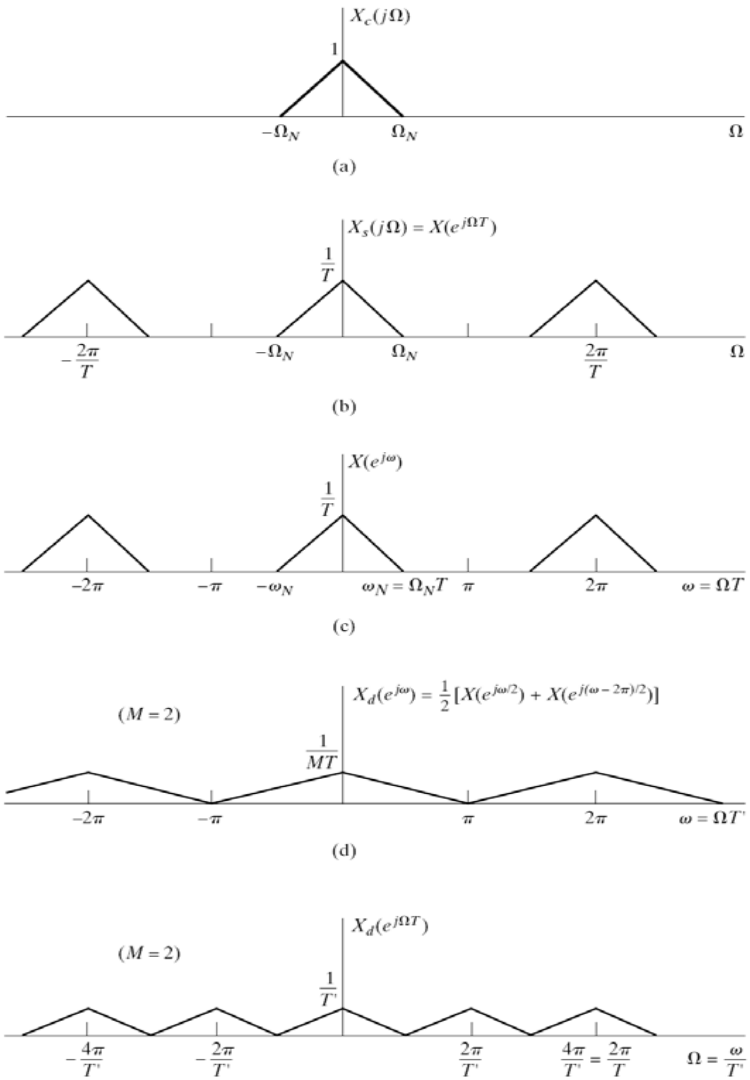
Let $T' = MT$. Then,

$$X_d(e^{j\omega}) = \frac{1}{T'}\sum_{\ell=-\infty}^{\infty} X_c\left(\frac{\omega}{T'} - \frac{2\pi k}{T'}\right)$$

$$= \frac{1}{MT}\sum_{\ell=-\infty}^{\infty} X_c\left(\frac{\omega}{MT} - \frac{2\pi \ell}{MT}\right) \quad \Leftarrow \ell = i + kM$$

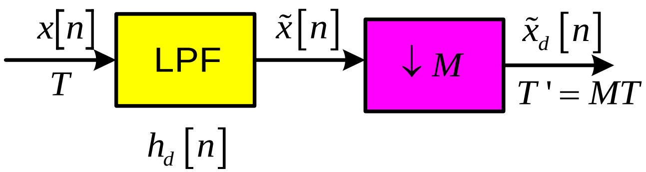
$$= \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right]$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega - 2\pi i}{M}}\right)$$

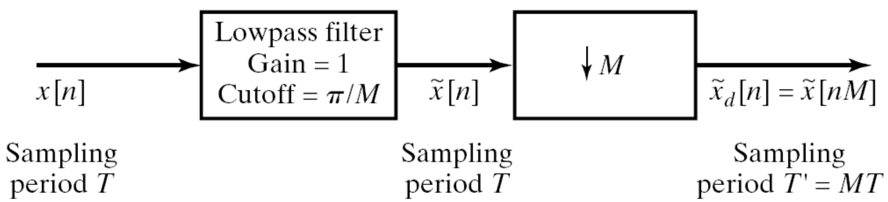


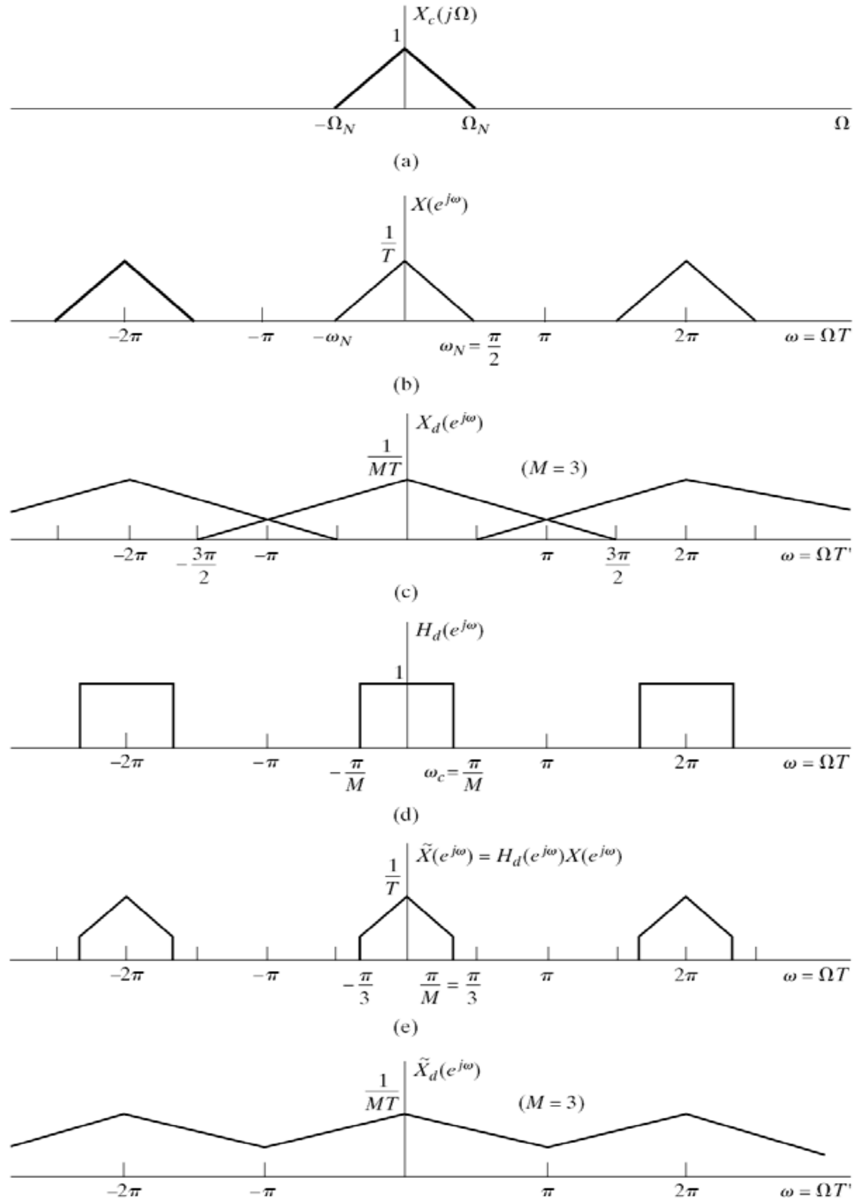
– Decimation by *M* (cont.)

- To avoid aliasing, a LPF is required before down-sampling

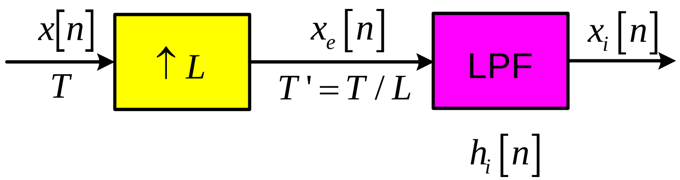


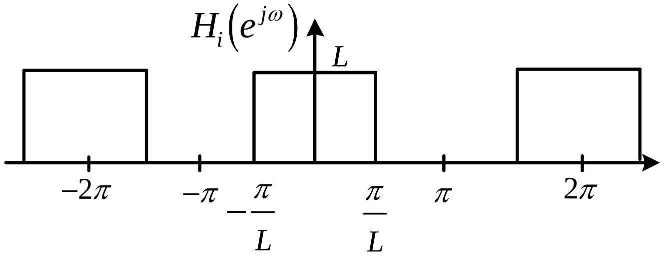
Example: When *M*=3:





– Interpolation by *L* (sampling rate expander)





$$x_i[n] = x_c(nT'); \quad T' = \frac{T}{L}$$

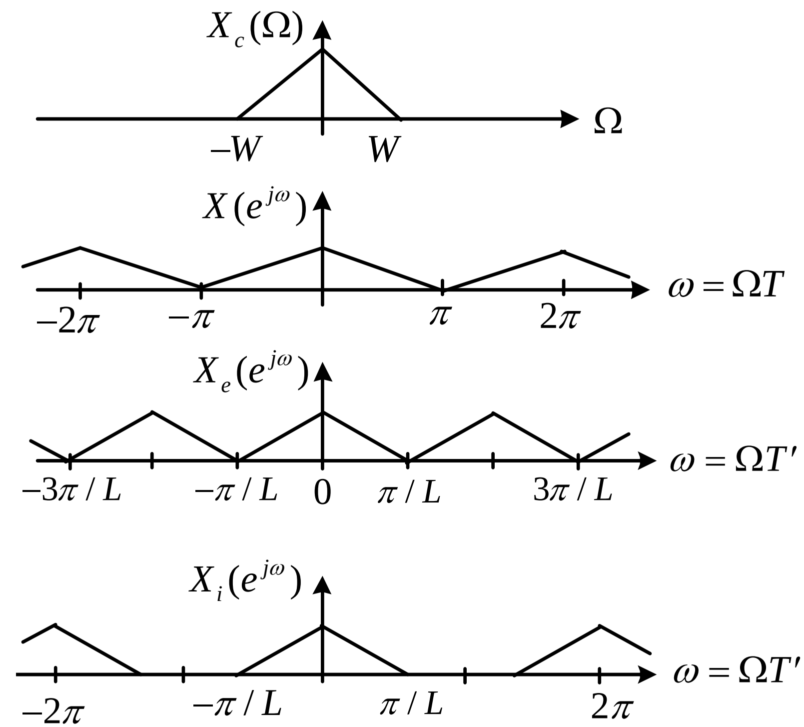
$$= x\left[\frac{n}{L}\right]$$

$$x_e[n] = \begin{cases} x\left[\frac{n}{L}\right], & n = 0, \pm L, \pm 2L, \ldots \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$\Rightarrow X_e\left(e^{j\omega}\right) = \sum_{k=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega n}$$

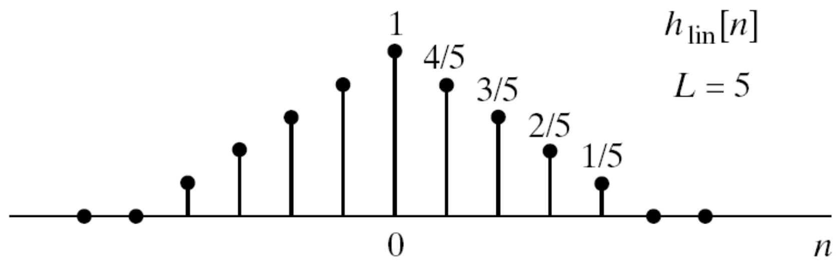
$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X\left(e^{j\omega L}\right)$$



– D/C conversion

$$h_i[n] = \frac{\sin\frac{\pi n}{L}}{\frac{\pi n}{L}} \Rightarrow \begin{cases} 1 & n = 0 \\ 0 & n = \pm L, \end{cases}$$

$$x_i[n] = \sum_k x[k] h_i[n-k] = x\left[\frac{n}{L}\right]$$



In practice, we use an approximate LPF.

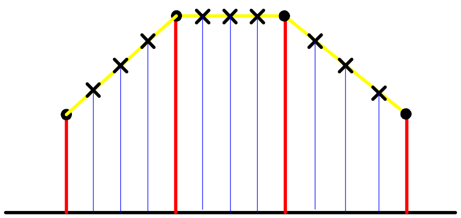
– Linear interpolation

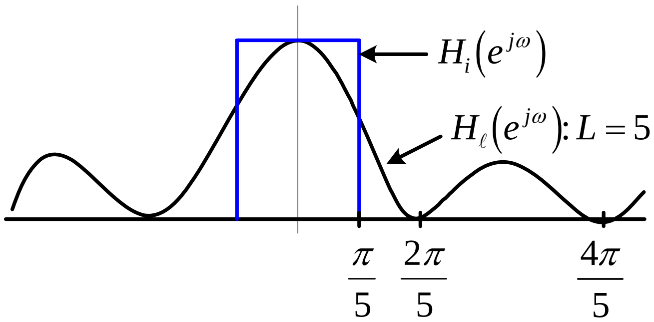
$$h_\ell[n] = \begin{cases} 1 - \frac{|n|}{L}, & |n| < L \\ 0, & \text{otherwise} \end{cases}$$

$$x_\ell[n] = \sum_k x_e[k] h_\ell[n-k]$$

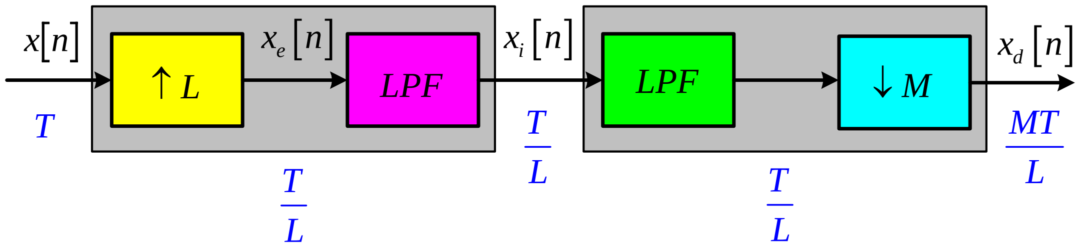
$$= \sum_m x[m] h_\ell[n - mL]$$

$$H_\ell(e^{j\omega}) = \frac{1}{L}\left[\frac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}}\right]^2$$





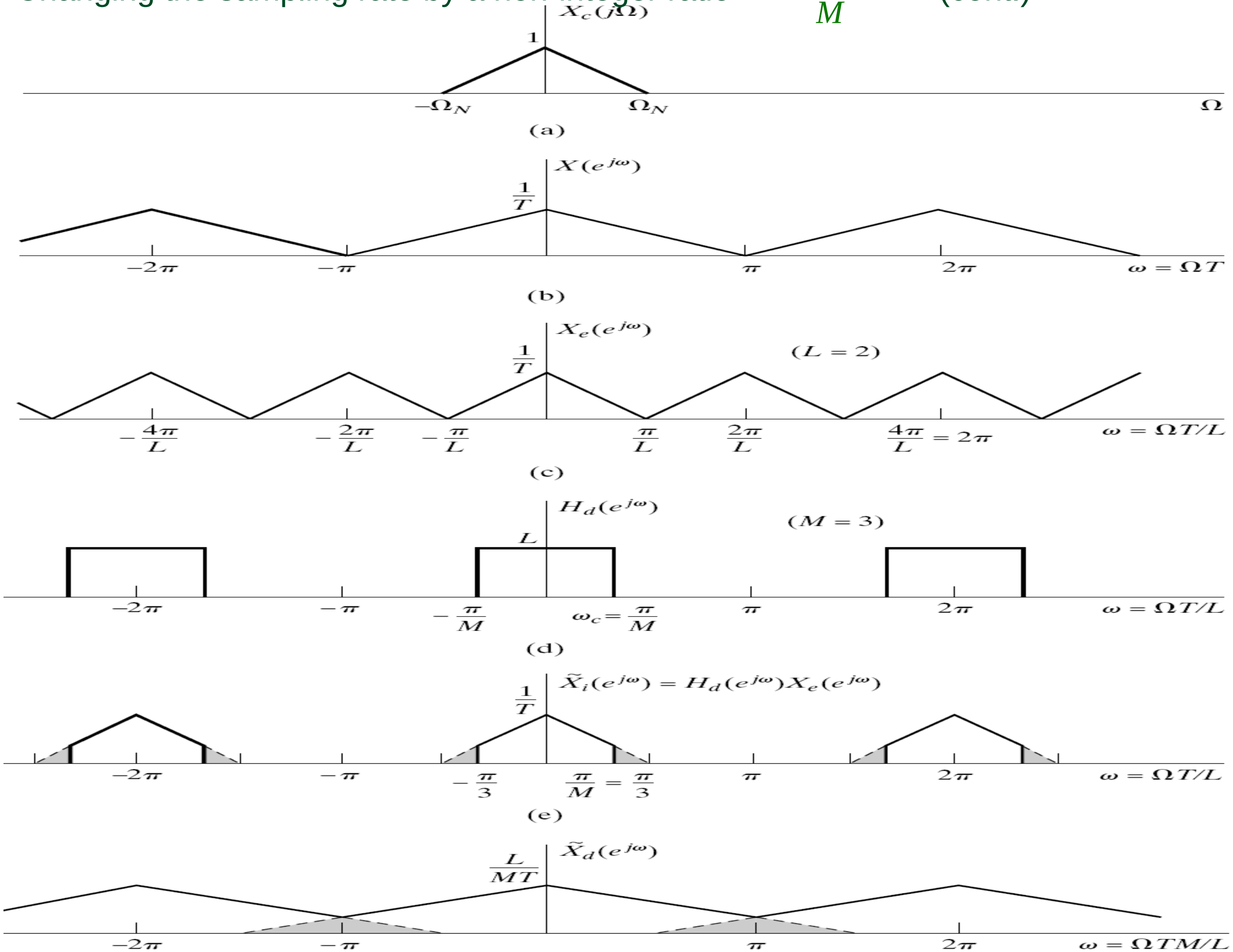
– Changing the sampling rate by a non-integer ratio $R = \frac{L}{M}$



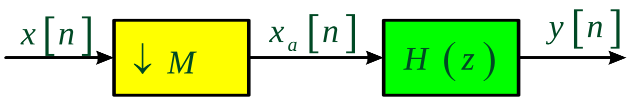
$$G = L$$

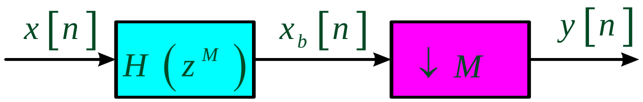
$$\omega_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$$

– Changing the sampling rate by a non-integer ratio $R = \frac{L}{M}$ (cont.)



- Multirate signal processing
  - Interchange the filtering and the down sampling process



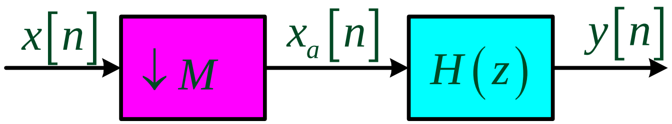


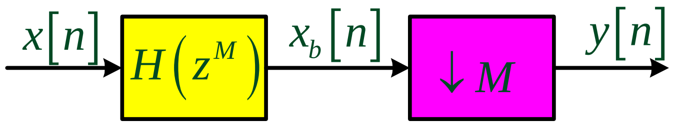
$$X_a\left(e^{j\omega}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right)$$

$$X_b\left(e^{j\omega}\right) = H\left(e^{j\omega M}\right)X\left(e^{j\omega}\right)$$

$$\begin{aligned}
Y\left(e^{j\omega}\right) &= \frac{1}{M}\sum_{i=0}^{M-1} X_b\left(e^{j\frac{\omega-2\pi i}{M}}\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right) H\left(e^{j(\omega-2\pi i)}\right) \\
&= H\left(e^{j\omega}\right)\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right) \\
&= H\left(e^{j\omega}\right)X_a\left(e^{j\omega}\right)
\end{aligned}$$

- Multirate signal processing
  - Interchange the filtering and the down sampling process



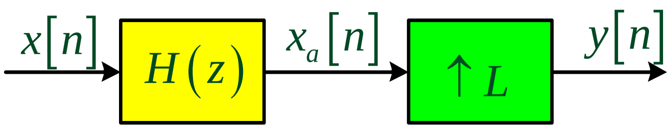


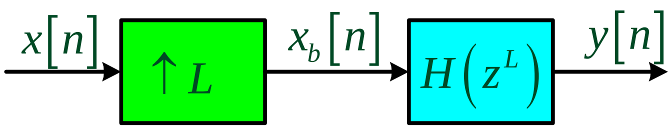
$$X_a(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j\frac{\omega-2\pi i}{M}}) \qquad X_b(e^{j\omega}) = H(e^{j\omega M})X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X_b\left(e^{j\frac{\omega-2\pi i}{M}}\right) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right) H\left(e^{j(\omega-2\pi i)}\right)$$

$$= H(e^{j\omega})\frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right) = H(e^{j\omega})X_a(e^{j\omega})$$

  - Interchange the filtering and the interpolation process

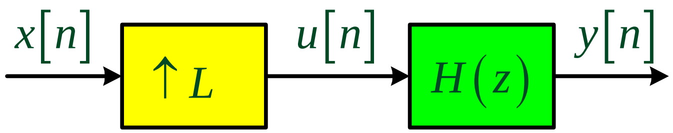




$$Y(e^{j\omega}) = X_a(e^{j\omega L}) = X(e^{j\omega L})H(e^{j\omega L})$$

$$X_b(e^{j\omega}) = X(e^{j\omega L}) \quad \Rightarrow \quad Y(e^{j\omega}) = H(e^{j\omega L})X_b(e^{j\omega}) = X(e^{j\omega L})H(e^{j\omega L})$$

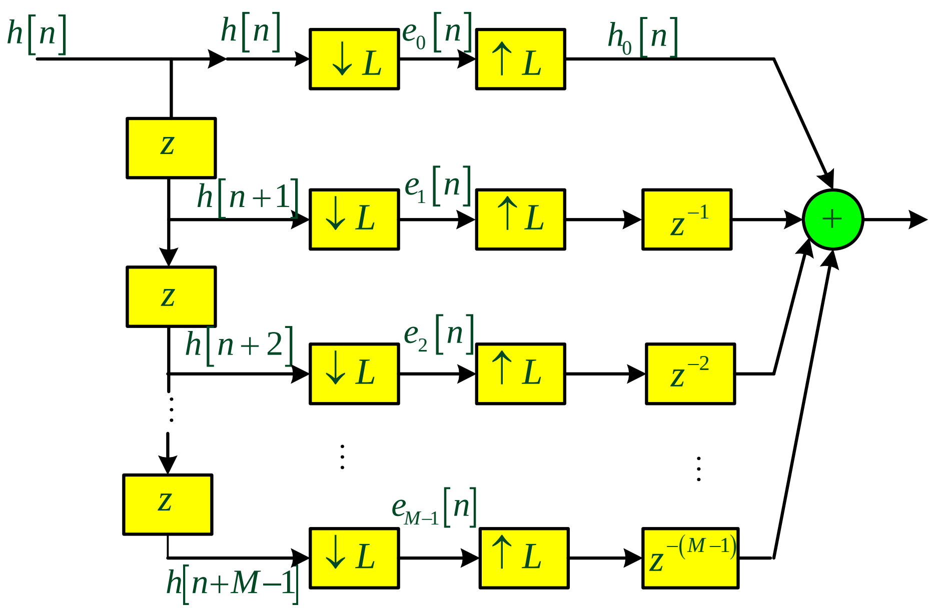
– Poly phase implementation of interpolation filters



- Note that only every L[th] sample of $u[n]$ is nonzero
- Consider decomposition of an impulse response $h[n]$ by

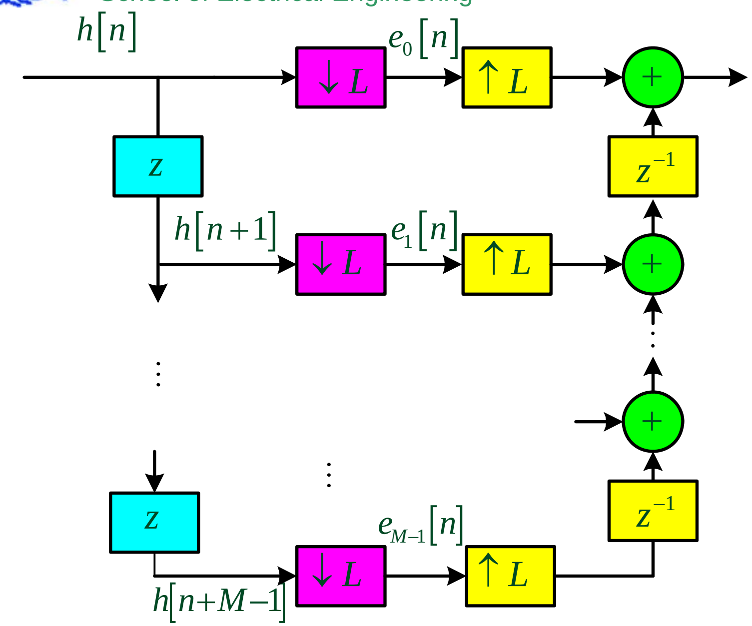
$$h_k[n] = \begin{cases} h[n+k], & n = mL, \quad m = 0, \pm 1, \pm 2, \cdots \\ 0 \quad , & \text{otherwise} \end{cases}$$

- By successively delaying these subsequences, we can reconstruct $h[n]$ by

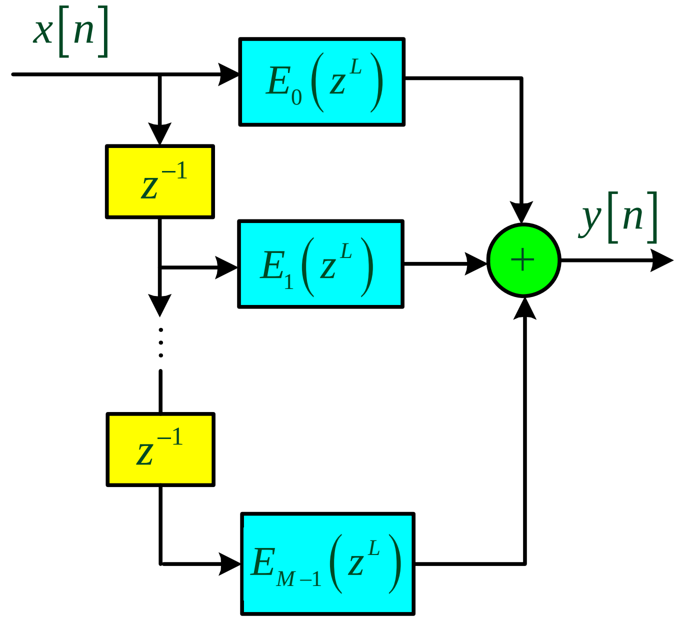


$$h[n] = \sum_{k=0}^{L-1} h_k[n-k]$$

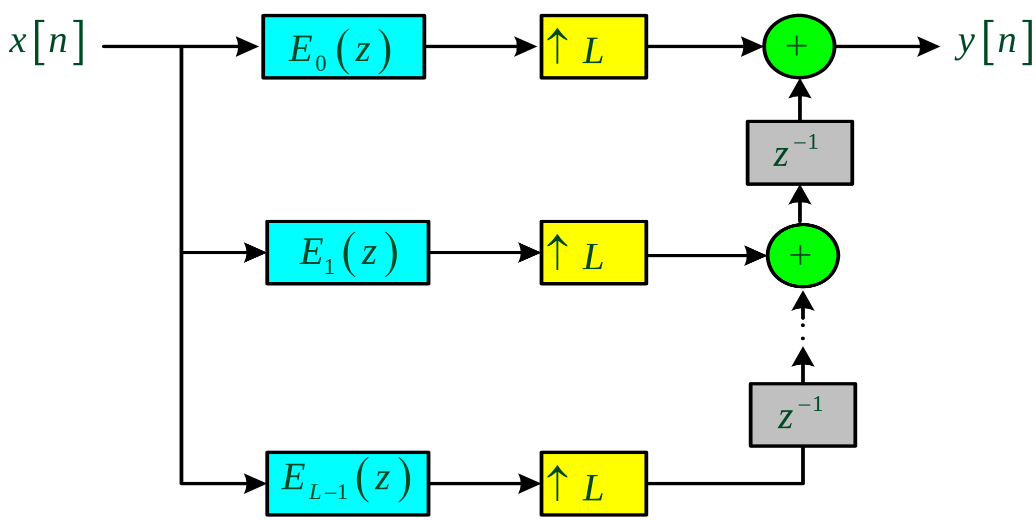
$$e_k[n] = h[nL+k] = h_k[nL]$$


h[n]
e0[n]
↓L
↑L
+
z
z^-1
h[n+1]
e1[n]
h[n+M−1]
eM−1[n]

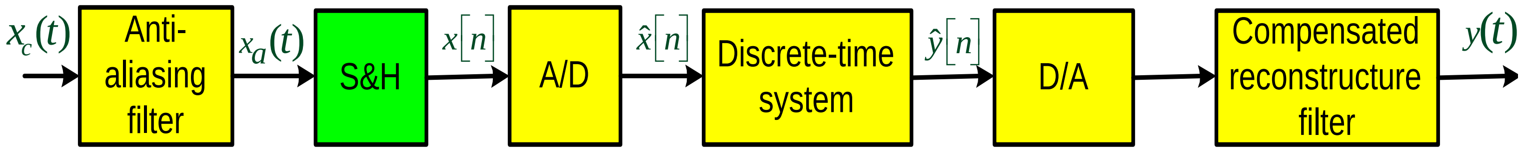
⇒


x[n]
E0(z^L)
z^-1
E1(z^L)
y[n]
+
EM−1(z^L)

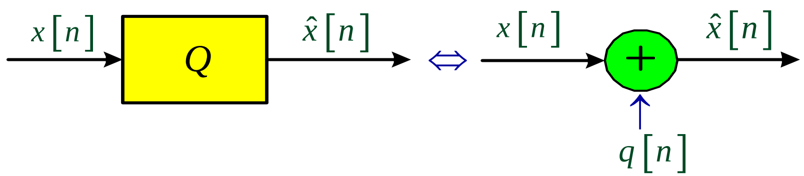
⇒


x[n]
E0(z)
↑L
+
y[n]
z^-1
E1(z)
EL−1(z)

- Digital processing of analog signals



– Quantization error: $-\frac{\Delta}{2} < q[n] = x[n] - \hat{x}[n] \le \frac{\Delta}{2}$



– Assuming that the error sequence $q[n]$ is uncorrelated with $x[n]$ and it is uniformly distributed over $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$,

$$\sigma_q^2 = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 \cdot \frac{1}{\Delta}\, dq = \frac{\Delta^2}{12}$$

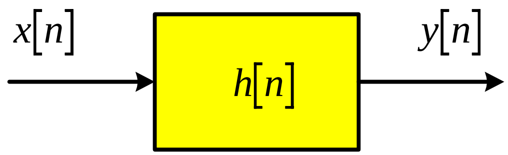
– For a $(K+1)$- bit quantizer with full-scale value X_m, $\sigma_q^2 = \frac{2^{-2k} X_m^2}{12}$

$$\text{SQNR} = 10 \log \frac{\sigma_X^2}{\sigma_q^2} = 6.02K + 10.8 - 20 \log \frac{X_m}{\sigma_X}$$

$\Rightarrow$ SQNR is increases approximately 6dB for each bit added to the quantizer.

Transform analysis of L.T.I. systems

- Frequency response of LTI systems



$$y[n]=\sum_k x[k]h[n-k] \Leftrightarrow Y(z)=H(z)X(z)$$

$$\Rightarrow Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})=\left|H(e^{j\omega})X(e^{j\omega})\right|e^{j\left(ArgH(e^{j\omega})+ArgX(e^{j\omega})\right)}$$

↑ complex gain (or eigenvalue)

← eigenfunction $x[n]=e^{j\omega n}$

– Ideal frequency-selective filters

$$H_{lp}(e^{j\omega})=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \text{otherwise} \quad \omega_c<|\omega|\le\pi\end{cases}$$

$$\Rightarrow H_{hp}(e^{j\omega})=1-H_{lp}(e^{j\omega}):\ \text{ideal high-pass filter}$$

$$\Rightarrow h_{hp}[n]=\delta[n]-h_{lp}[n]$$

$$=\delta[n]-\frac{\sin\omega_c n}{\pi n} \Leftarrow \text{not computationally realizable}$$

– Ideal delay $h_{id}[n] = \delta[n - n_d]$

$$\Rightarrow H_{id}(e^{j\omega}) = e^{-j\omega n_d} = |1| e^{-j\omega n_d}, |\omega| < \pi$$

– Low-pass filter with linear phase (or delay)

$$H_{\ell p}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, |\omega| \le \omega_c \\ 0 \quad, \omega_c \le |\omega| \le \pi \end{cases}$$

$$\Rightarrow h_{hp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}$$ ← Regardless of n_d, it is still noncausal

– Group delay: A measure of the nonlinearity of the phase

$$I(\omega) = -\frac{d}{d\omega} \text{Arg}(H(e^{j\omega}))$$

Example $\text{Arg}(H(e^{j\omega})) \approx -\phi_0 - \omega n_d$

For an input $x[n] = s[n] \cos \omega_0 n$,

$$y[n] \approx s[n - n_d] \cos(\omega_0 (n - n_d) - \phi_0)$$

$\Rightarrow$ The time delay of the envelope $s[n]$ of narrowband signal centered at ω_0 is given by the negative of the slope of the phase at ω_0.

- Linear constant coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Rightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \Leftarrow$$

It does not uniquely specify the impulse response of an LTI system

$$= \frac{b_0 \prod_{k=1}^{M} \left(1 - c_k z^{-1}\right)}{a_0 \prod_{k=1}^{N} \left(1 - d_k z^{-1}\right)} \Rightarrow \text{zeros: } z = c_k; \quad \text{poles: } z = d_k$$

Example

$$H(z) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)} = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)} \Rightarrow \text{zeros: } z=-1,-1; \quad \text{poles: } z = \frac{1}{2}, -\frac{3}{4}$$

$$\Rightarrow y[n] + \frac{1}{4} y[n-1] - \frac{3}{8} y[n-2] = x[n] + 2x[n-1] + x[n-2]$$

- Causal system

  $h[n]$ is a right-sided sequence

  $\Rightarrow$ The region of convergence of $H_c(z)$ should be outside the outermost pole

- Stable system: absolutely summable

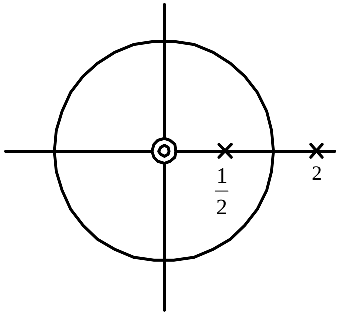
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow \sum_{n} |h[n] z^{-n}| < \infty \quad \text{for} \quad |z| = 1$$

  $\Rightarrow$ The ROC of H(z) includes the unit circle

- Example

$$y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$$

$$\Rightarrow H(z) = \frac{1}{1 - \frac{5}{2} z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2z^{-1}\right)}$$



  To be causal, ROC $\Rightarrow$ $|z| > 2$

  To be stable, ROC $\Rightarrow$ $\frac{1}{2} < |z| < 2$

- In order for an LTI system to be both stable and causal, the ROC must be outside the outermost pole and include the unit circle, i.e., all poles inside the unit circle
- Depending upon the choice of ROC, the same difference equation results in a different impulse response

– Inverse system

- Let $H_i(z)$ be $H_i(z) = \frac{1}{H(z)}$

$$G(z) \triangleq H(z)H_i(z) = 1 \Rightarrow g[n] = h(n) * h_i[n] = \delta[n]$$

- To hold this equation, the ROC of *Hi(z)* and *H(z)* must overlap
- If *H(z)* is causal, the ROC is $|z| > \max_k |d_k|$

$\Rightarrow$ The ROC of $H_i(z)$ should overlap with $|z| > \max_k |d_k|$

Example 1:

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}} \quad \text{with ROC:} \quad |z| > 0.9$$

$$\Rightarrow H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} \quad \Rightarrow \quad \text{ROC:} \quad |z| > 0.5 \text{ (to be overlapped with } |z| > 0.9)$$

$$\Rightarrow \quad h_i[n] = 0.5^n u[n] - 0.9 \cdot 0.5^{n-1} u[n-1]$$

Example 2:

$$H(z) = \frac{z^{-1} - 0.5}{1 - 0.9z^{-1}} \quad \text{with ROC:} \quad |z| > 0.9$$

$$\Rightarrow H_i(z) = \frac{-2 + 1.82z^{-1}}{1 - 2z^{-1}}$$

For an ROC $|z| < 2;\quad h_{i1}[n] = 2 \cdot 2^n u[-n-1] - (1 \cdot 8) \cdot 2^{n-1} u[-n] \Rightarrow$ stable & noncausal

For an ROC $|z| > 2;\quad h_{i2}[n] = -2 \cdot 2^n u[n] + 1 \cdot 8 \cdot 2^{n-1} u[n-1] \Rightarrow$ unstable & causal

- If *H(z)* is causal with zeros at c_k , $H_i(z)$ will be causal *iff* the Roc of $H_i(z)$ is $|z| > \max_k |c_k|$
- If $H_i(z)$ is stable, the RoC of $H_i(z)$ must include the unit circle, $\max_k |c_k| < 1$
- *H(z)* and its inverse are stable and causal *iff* both poles and zeros of *H(z)* are inside the limit circle. $\Rightarrow$ *minimum-phase* system

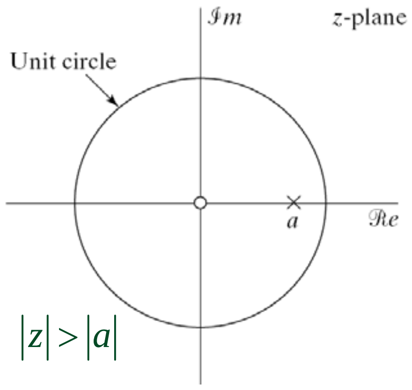
– Impulse response of for rational system functions

$$H(z) = \underbrace{\sum_{\ell=0}^{M-N} B_\ell z^{-\ell}}_{\text{FIR term}} + \underbrace{\sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}}_{\text{IIR term}}, \; M \geq N$$

- If causal,

$$h[n] = \sum_{\ell=0}^{M-N} B_\ell \delta[n-\ell] + \sum_{k=1}^{N} A_k d_k^n u[n], \; M \geq N$$

$$\Rightarrow y[n] = \sum_{\ell=0}^{M-N} B_\ell x[n-\ell] + \sum_{k=1}^{N} A_k d_k^n u[n] * x[n]$$



- Example: A first-order IIR filter

$$y[n] - ay[n-1] = x[n] \;\Rightarrow\; H(z) = \frac{1}{1-az^{-1}}; \quad \text{ROC}: |z| > |a|$$

For stability, $|a| < 1 \Rightarrow h[n] = a^n u[n]$

- FIR system

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

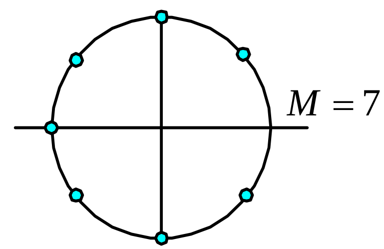
$$\Rightarrow h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Example

$$h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0\ , & \text{otherwise} \end{cases}$$

$$\Rightarrow H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-(M+1)}}{1 - az^{-1}}$$



Zeros: $z_k = ae^{j\frac{2\pi k}{M+1}}, k = 0,1,2,\ldots,M$

$$\Rightarrow y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

or $y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$

- Frequency response for rational systems

$$H\left(e^{j\omega}\right)=H(z)\Big|_{z=e^{j\omega}}=\frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)}$$

$$\Rightarrow \left|H\left(e^{j\omega}\right)\right|=\left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}\left|1-c_k e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-d_k e^{-j\omega}\right|}$$

$$\left|H\left(e^{j\omega}\right)\right|^2=\left(\frac{b_0}{a_0}\right)^2\frac{\prod_{k=1}^{M}\left(1-c_k e^{-j\omega}\right)\left(1-c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k e^{-j\omega}\right)\left(1-d_k^* e^{j\omega}\right)}$$

  - Gain (or attenuation)

$$\text{Gain}=20\log_{10}\left|H\left(e^{j\omega}\right)\right|$$

$$=20\log_{10}\left|\frac{b_0}{a_0}\right|+20\sum_{k=1}^{M}\log_{10}\left|1-c_k e^{-j\omega}\right|-20\sum_{k=1}^{M}\log_{10}\left|1-d_k e^{-j\omega}\right|(dB)$$

  - Phase response

$$\text{Arg}\left(H(e^{j\omega})\right)=\text{Arg}\left(\frac{b_0}{a_0}\right)+\sum_{k=1}^{M}\text{Arg}\left(1-c_k e^{-j\omega}\right)-\sum_{k=1}^{M}\text{Arg}\left(1-d_k e^{-j\omega}\right);\quad -\pi<\text{Arg}(\text{x})\le\pi$$

  - Group delay

$$\text{grd}\left(H(e^{j\omega})\right)=-\sum_{k=1}^{M}\frac{d}{d\omega}\text{Arg}\left(1-c_k e^{-j\omega}\right)+\sum_{k=1}^{M}\frac{d}{d\omega}\text{Arg}\left(1-d_k e^{-j\omega}\right)$$

- Frequency response of a single pole or zero
  - Single zero

$$H(z)=1-az^{-1} \;\Rightarrow\; H(e^{j\omega})=(1-ae^{-j\omega});\;\; a=re^{j\theta}$$

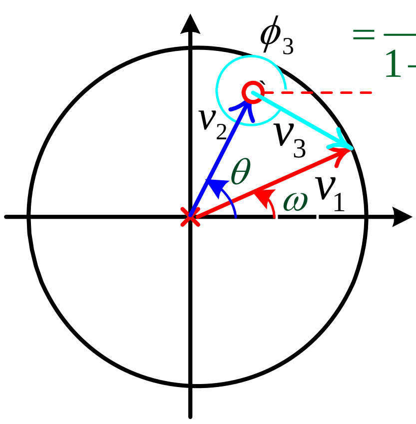
$$\Rightarrow |H(e^{j\omega})|=|1-re^{j\theta}e^{-j\omega}|^2=(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{j\omega})$$

$$=1+r^2-2r\cos(\omega-\theta)$$

$$\Rightarrow Arg(H(e^{j\omega}))=\tan^{-1}\frac{r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}$$

$$\Rightarrow \mathrm{grd}(H(e^{j\omega}))=\frac{d}{d\omega}\left[Arg(H(e^{j\omega}))\right]$$

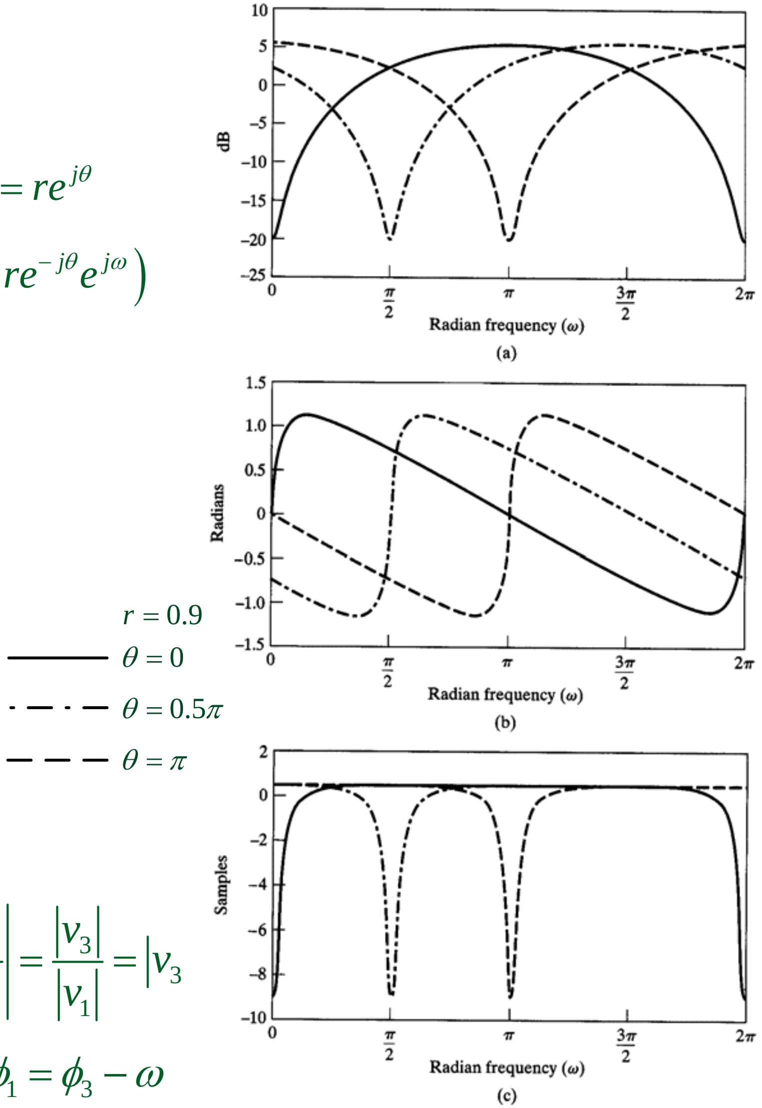
$$=\frac{r^2-r\cos(\theta-\omega)}{1+r^2-2r\cos(\theta-\omega)}$$



$$H(z)=\frac{z-a}{z}$$

$$\Rightarrow |H(e^{j\omega})|=\left|\frac{e^{j\omega}-re^{j\theta}}{e^{j\omega}}\right|=\frac{|v_3|}{|v_1|}=|v_3|$$

$$Arg(H(e^{j\omega}))=\phi_3-\phi_1=\phi_3-\omega$$



– Example: Real single zero

$$H(z) = 1 - az^{-1}$$

$$\Rightarrow \quad H(e^{j\omega}) = (1 - ae^{-j\omega}); \quad a = re^{j\theta}$$

If $\theta = \pi, \; a = -r$

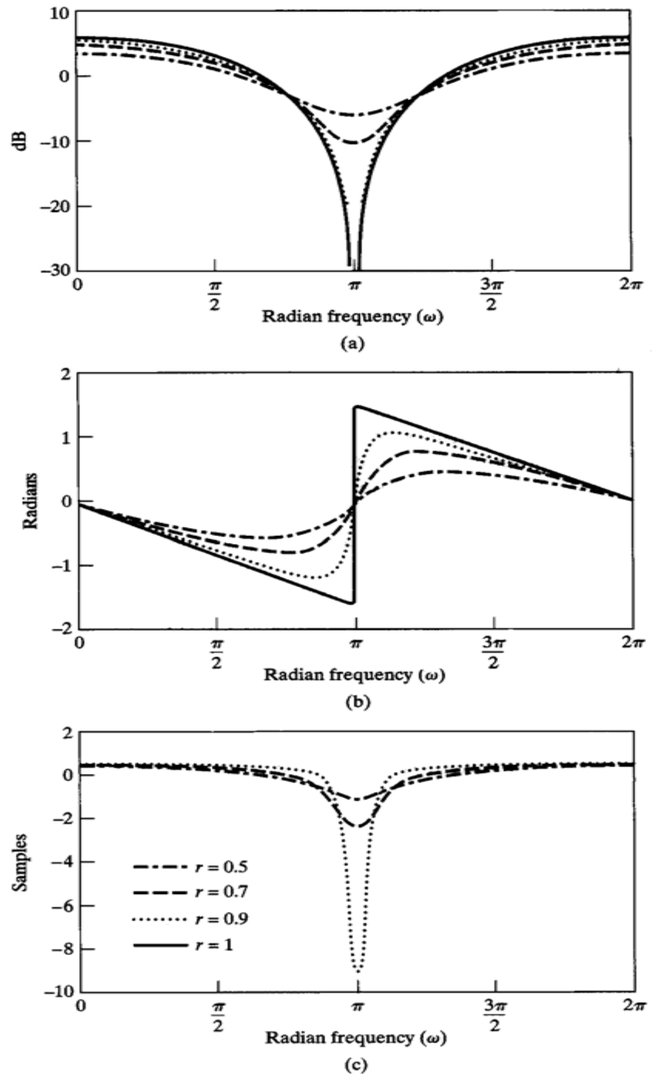
$$\Rightarrow \left|H(e^{j\omega})\right| = \left|1 + re^{-j\omega}\right|^2$$

$$= 1 + r^2 - 2r\cos\omega$$

$$\Rightarrow Arg\left(H(e^{j\omega})\right) = \tan^{-1}\frac{r\sin\omega}{1 + r\cos\omega}$$

$$\Rightarrow \text{grd}\left(H(e^{j\omega})\right) = \frac{d}{d\omega}\left[Arg\left(H(e^{j\omega})\right)\right]$$

$$= \frac{r^2 + r\cos\omega}{1 + r^2 + 2r\cos\omega}$$

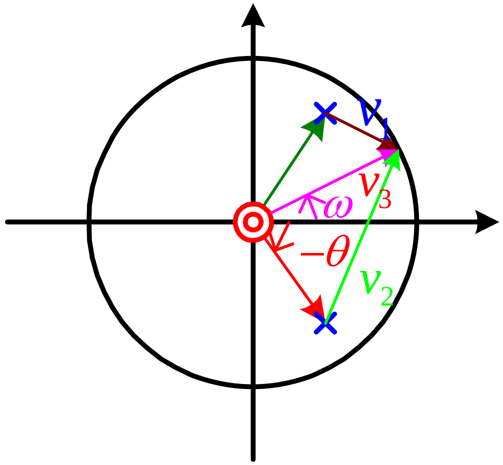


– Second-order IIR system

$$H(z)=\frac{1}{(1-re^{j\theta}z^{-1})(1-re^{j\theta}z^{-1})}=\frac{1}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

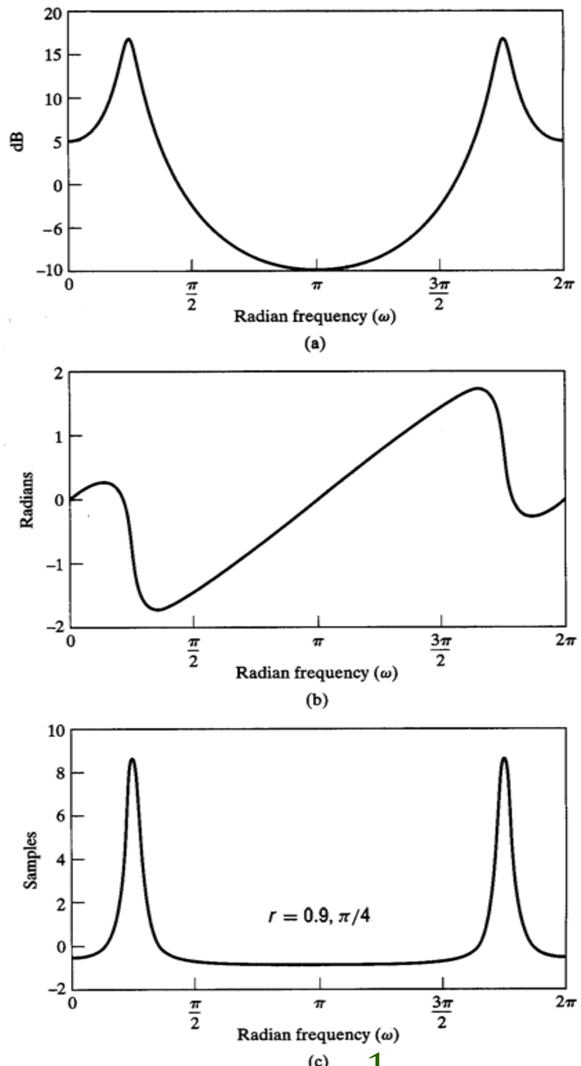
$$\Leftrightarrow y[n]-2r\cos\theta y[n-1]+r^2y[n-2]=x[n]$$

$$\Rightarrow h[n]=\frac{r^n\sin(n+1)\theta}{\sin\theta}u[n]$$



$$\left|H(e^{j\omega})\right|=\frac{|v_3|^2}{|v_1|\cdot|v_2|}=\frac{1}{|v_1||v_2|}$$

$$\mathrm{Arg}(H(e^{j\omega}))=-\tan^{-1}\frac{r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}-\tan^{-1}\frac{r\sin(\theta+\omega)}{1-r\cos(\theta+\omega}$$



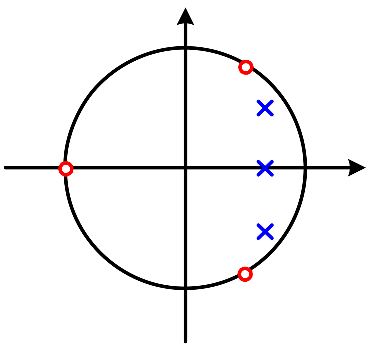
$$H(z)=\frac{1}{1-0.9\sqrt{2}z^{-1}+0.81z^{-2}}$$

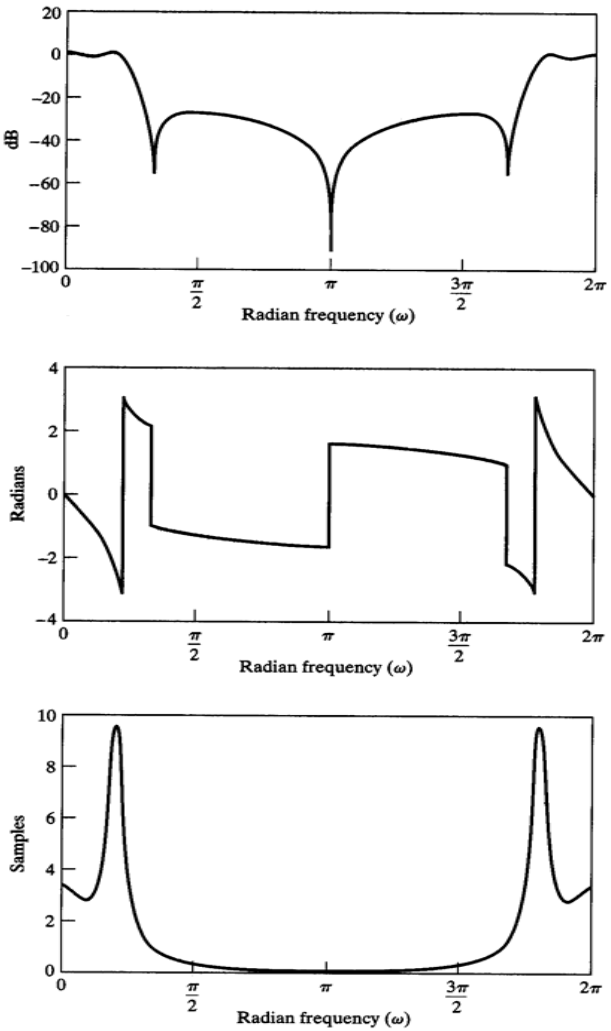
– Third-order IIR system

$$H(z) = \frac{0.05634(1+z^{-1})(1-1.0166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$$

$$\Rightarrow z_z = -1,\ e^{\pm j1.0376}\ (59.45^\circ)$$

$$\Rightarrow z_p = 0.683,\ 0.892e^{\pm j0.6257}\ (35.85^\circ)$$





- Relationship between magnitude and phase
  - In general, magnitude information $\underset{No}{\overset{No}{\rightleftarrows}}$ phase information
  - In cases of rational system functions, there is some constraint between magnitude and phase

$$\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = H(z)H^*\left(\frac{1}{z^*}\right)\Bigg|_{z=e^{j\omega}}$$

$$H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-d_k z^{-1}\right)}; \qquad H^*\left(\frac{1}{z^*}\right) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1-c_k^{*} z\right)}{\prod_{k=1}^{N}\left(1-d_k^{*} z\right)}$$

  - The square of the magnitude frequency response is the evaluation of *z*-transform on the unit circle

$$C(z) \triangleq H(z)H^*\left(\frac{1}{z^*}\right)$$

  - Question: Can we know of *H(z)* from C(z) ?

$$\left.\begin{array}{l} H(z):\ \text{pole } d_k;\ \ \text{zero } c_k \\ H^*\left(\frac{1}{z^*}\right):\ \text{pole } \left(d_k^*\right)^{-1};\ \text{zero } \left(c_k^*\right)^{-1} \end{array}\right\rangle \text{conjugate reciprocal pairs}$$

– If $H(z)$ is assumed to be causal & stable, all its poles are in the unit circle

$\Rightarrow$ Poles of $H(z)$ can be identified uniquely

Example System with the same $C(z)$

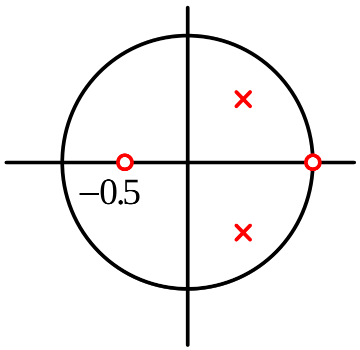
$$H_1(z)=\frac{2\left(1-z^{-1}\right)\left(1+0.5z^{-1}\right)}{\left(1-0.8e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-0.8e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

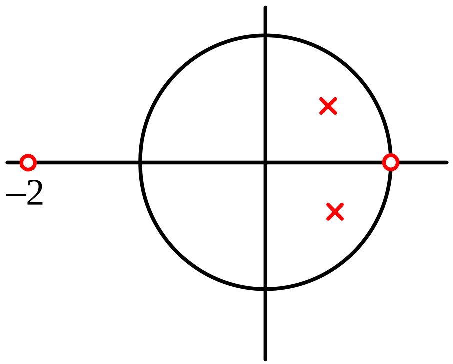
$$H_2(z)=\frac{\left(1-z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-0.8e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-0.8e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

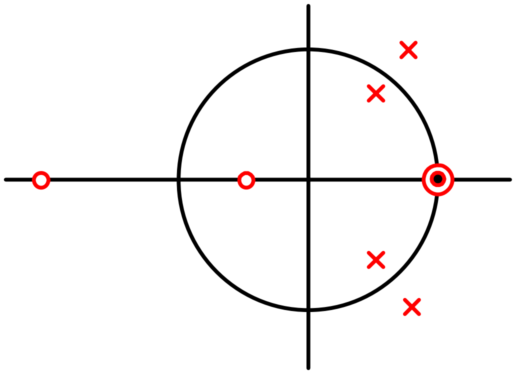
$$\Rightarrow \left|H_1\left(e^{j\omega}\right)\right|=\left|H_2\left(e^{j\omega}\right)\right|$$

$$C_1(z)=H_1(z)H_1^*\left(\frac{1}{z^*}\right)$$

$$C_2(z)=H_2(z)H_2^*\left(\frac{1}{z^*}\right) \quad \Rightarrow \quad C_1(z)=C_2(z)$$







- All pass system

  – A stable system function of the form

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^1}$$

has a frequency-response magnitude independent of ω.

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} e^{-j\omega} \quad \Rightarrow \quad \left| H_{ap}(e^{j\omega}) \right| = 1$$

$$\mathrm{Arg}\left(H_{ap}(e^{j\omega}) \right) = -\omega - 2\tan^{-1} \frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}$$

  – In general, the transfer function of an all-pass system is given by

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}; \quad A > 0, \quad |d_k| < 1, \quad |e_k| < 1$$

  – All-pass system can be used for compensating group delay distortion.

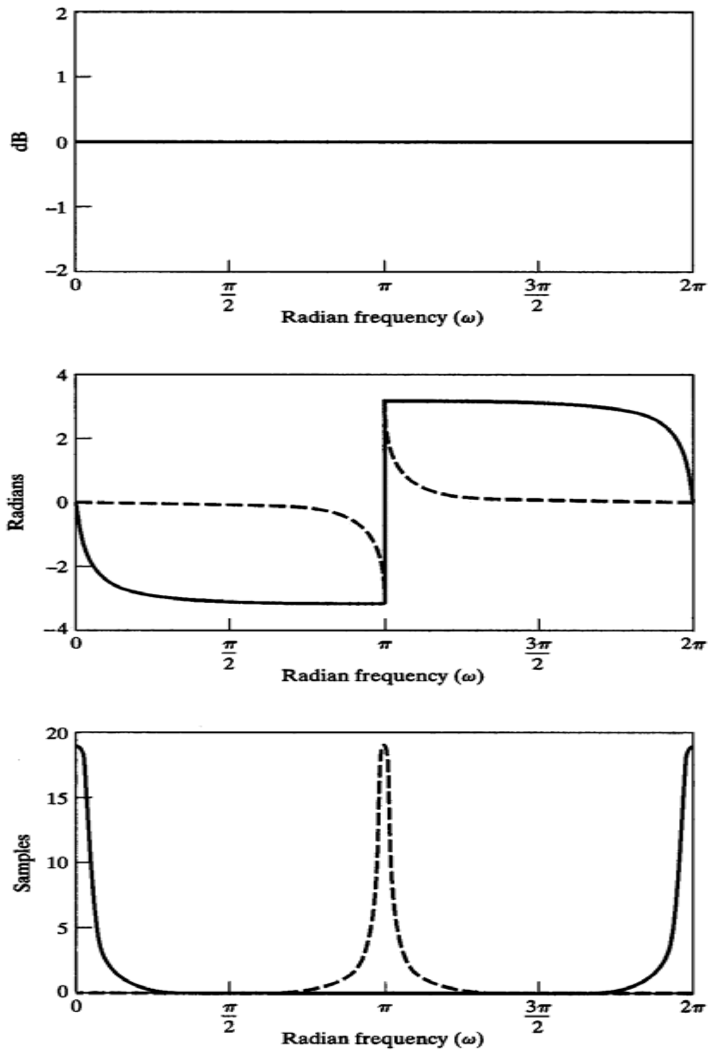
  – Causal all-pass system has positive group delay (but not continuous phase) property

– Example

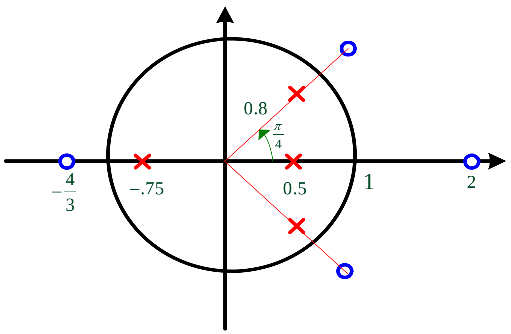
$$H(z)=\frac{z^{-1}\mp 0.9}{1\mp 0.9z^{1}}$$

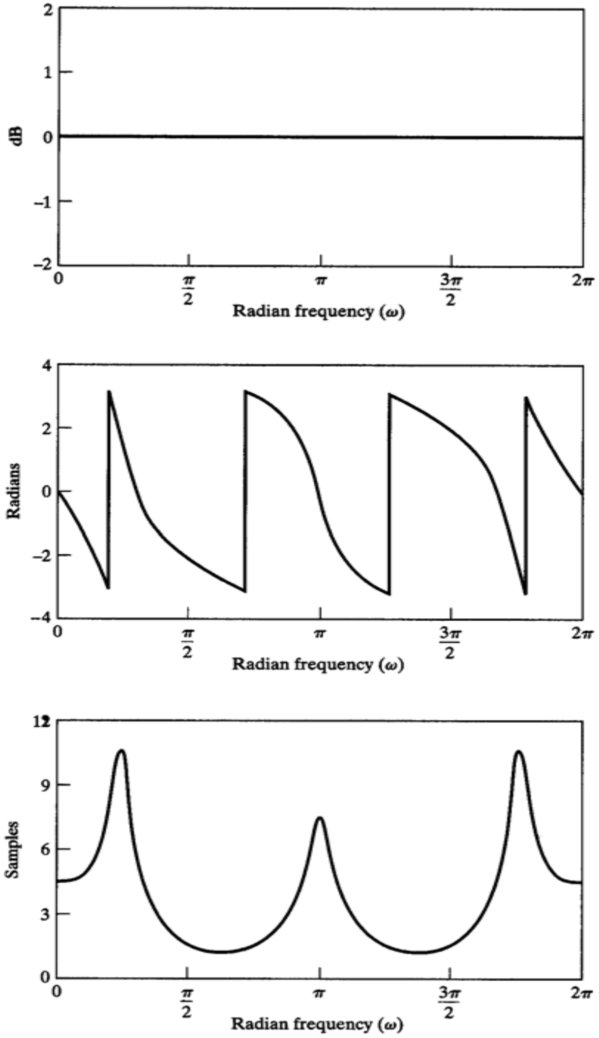
——— $z_p = 0.9$

– – – $z_p = -0.9$



– Example


0.8
π/4
−4/3
−.75
0.5
1
2


dB
Radian frequency (ω)
Radians
Radian frequency (ω)
Samples
Radian frequency (ω)

- Minimum-phase system
  - It has its poles and zeros all inside the unit circle.

    $\Rightarrow$ Any *minimum-phase* system $H(z)$ is causal and stable.

    $\Rightarrow$ $H^{-1}(z)$ is a causal, stable, and min-phase system
  - Cascade of both systems will not introduce any delay

$$H(z)H^{-1}(z)=1\cdot e^{jo}$$

  - Energy concentration
    - For an arbitrary stable and causal system $F(e^{j\omega})$, let $G(e^{j\omega})$ be the min-phase function *s.t.*, $\left|G(e^{j\omega})\right|=\left|F(e^{j\omega})\right|,\ \forall\omega.$
    - Then, for any sequence $x[n]$ and *K*,

$$\sum_{k=0}^{K}\left(x[n]*g[n]\right)_k^2\geq\sum_{k=0}^{K}\left(x[n]*f[n]\right)_k^2$$

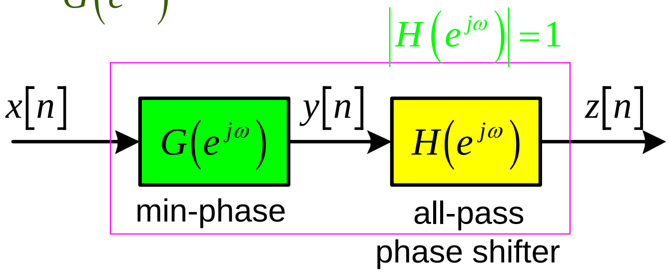
$\Rightarrow$ The signal energy is transported with the smallest possible delay using a minimum-phase system

Example If $x[n]=\delta[n]$, $\sum_{k=0}^{K}g^2[k]\geq\sum_{k=0}^{K}f^2[k]$

– The min-phase function has its energy maximally concentrated near the time origin.

proof

Let $H(e^{j\omega}) \triangleq \dfrac{F(e^{j\omega})}{G(e^{j\omega})}, \ \forall\omega$ be stable, causal and all pass



The energy of $y[n]$

$$E_y \triangleq E\{y^2[n]\} = \frac{1}{\omega_0}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left|Y(e^{j\omega})\right|^2 d\omega, \quad \omega_0 = \frac{2\pi}{T}$$

$$E_z \triangleq E\{z^2[n]\} = \frac{1}{\omega_0}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left|H(e^{j\omega})Y(e^{j\omega})\right|^2 d\omega = E_y$$

Assume $y[n]$ vanishes for $n < 0$ and $n < K$, *i.e.*, $E_y = \sum_{n=0}^{K} y^2[n]$

But $z[n]$ is not necessarily zero for $n > K$

$$E_z = E_y \geq \sum_{n=0}^{K} z^2[n] \quad \square$$

– $H(z)$ cannot be uniquely determined from $G(z)$, because cascading all-pass filters does not affect the magnitude of the frequency response.

– Any rational rational transfer function can be expressed as

$$H(z) = H_{\min}(z) H_{ap}(z)$$

- Assume that $H(z)$ has all zeros and poles inside the unit circle except one zero outside the unit circle at $z = \frac{1}{c^*}, \quad |c| < 1$

$$H(z) = H_1(z)(z^{-1} - c^*)$$

$$= \underbrace{H_1(z) \cdot (1 - cz^{-1})}_{\text{minimum phase}} \cdot \underbrace{\frac{z^{-1} - c^*}{1 - cz^{-1}}}_{\text{all pass}}$$

where $H_1(z)$ is minimum phase

- $H_{\min}(z)$ has poles and zeros of $H(z)$ inside the unit circle plus zeros that are conjugate reciprocals of the zeros of $H(z)$ outside the unit circle
- $H_{ap}(z)$ contains all the zeros of $H(z)$ outside the unit circle
- Example

$$H(z) = \frac{1 + 3z^{-1}}{1 + \frac{1}{2}z^{-1}} = 3 \cdot \underbrace{\frac{1 + \frac{1}{3}z^{-1}}{1 + \frac{1}{2}z^{-1}}}_{H_{\min}(z)} \cdot \underbrace{\frac{z^{-1} + \frac{1}{3}}{1 + \frac{1}{3}z^{-1}}}_{H_{ap}(z)}$$

– The amplitude characteristics of a minimum-phase filter is fully determined by the phase characteristics and vice versa $\Rightarrow$ We can compute $G\left(e^{j\omega}\right)$ from $\left|G\left(e^{j\omega}\right)\right|$

Proof

Since $G(z)$ has its poles and zeros inside the unit circle, $\ln G\left(e^{j\omega}\right)$ is analytic for $|z|>1$

$\Rightarrow \ln G\left(e^{j\omega}\right)$ has a causal and well behaved inverse Fourier transform, *i.e.*,

$$\ln G\left(e^{j\omega}\right)=\sum_{n=0}^{\infty} g[n]e^{-j\omega n}=X\left(e^{j\omega}\right)+j\phi_G\left(e^{j\omega}\right)$$

where $X\left(e^{j\omega}\right)=\ln\left|G\left(e^{j\omega}\right)\right|=\sum_{n=0}^{\infty} g[n]\cos n\omega;\ \ \phi_G\left(e^{j\omega}\right)=-\sum_{n=0}^{\infty} g[n]\sin n\omega$

We can see that $X\left(e^{j\omega}\right)$ and $\phi_G\left(e^{j\omega}\right)$ form a Hilbert transform pair.

Since $X\left(e^{j\omega}\right)$ is the logarithm of a real-valued symmetric function of frequency, it is real and symmetric $\Rightarrow x[n]=x[-n],\ \forall n$

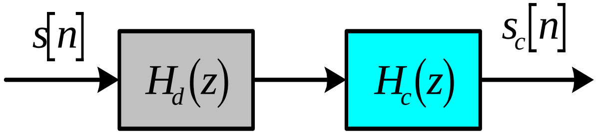
$$\Rightarrow X\left(e^{j\omega}\right)=x[0]+2\sum_{n=0}^{\infty} x[n]\cos n\omega$$

$$\Rightarrow g[0]=x[0] \text{ and } g[n]=2x[n],\ n\geq 1$$

Thus we can compute $\phi_G\left(e^{j\omega}\right)$ □

– Frequency-response compensation

- If $H_d(z)$ is a min-phase system, perfect compensation is generally possible



$$H_d(z) = H_{\min}(z) H_{ap}(z)$$

Let $H_c(z) = \dfrac{1}{H_{\min}(z)} \Rightarrow G(z) = H_d(z) H_c(z) = H_{ap}(z)$

- The frequency response magnitude is exactly compensated with some delay

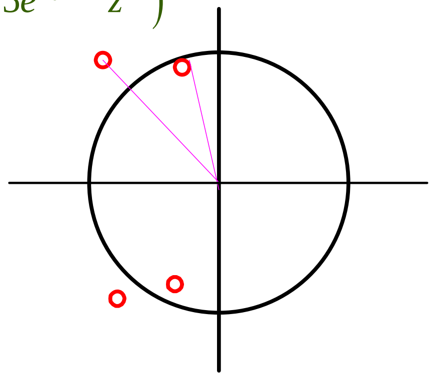
Example

$$H(z) = \left(1 - 0.9e^{j0.6\pi} z^{-1}\right)\left(1 - 0.9e^{-j0.6\pi} z^{-1}\right)\left(1 - 1.25e^{j0.8\pi} z^{-1}\right)\left(1 - 1.25e^{-j0.8\pi} z^{-1}\right)$$

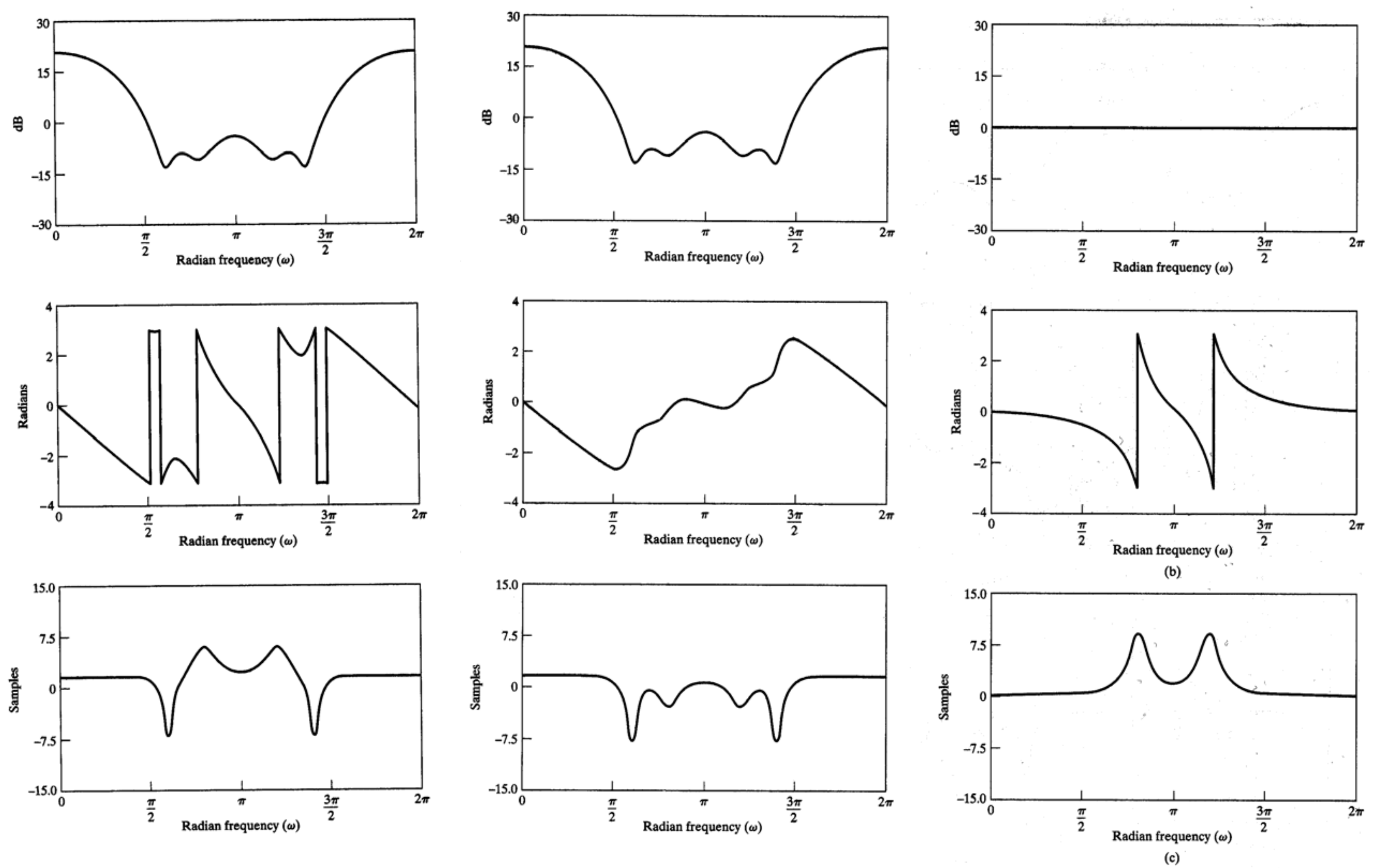
$$\Rightarrow H_{\min}(z) = 1.25^2\left(1 - 0.9e^{j0.6\pi} z^{-1}\right)\left(1 - 0.9e^{-j0.6\pi} z^{-1}\right)$$

$$\cdot\left(1 - 0.8e^{j0.8\pi} z^{-1}\right)\left(1 - 0.8e^{-j0.8\pi} z^{-1}\right)$$

$$\Rightarrow H_{ap}(z) = \frac{\left(1 - 1.25e^{j0.8\pi} z^{-1}\right)\left(1 - 1.25e^{-j0.8\pi} z^{-1}\right)}{\left(1 - 0.8e^{j0.8\pi} z^{-1}\right)\left(1 - 0.8e^{-j0.8\pi} z^{-1}\right)}$$



$H(z)$ $\qquad$ $H_{\min}(z)$ $\qquad$ $H_{ap}(z)$



- In general, min-energy delay property $|h[0]| \le |h_{\min}[0]|$

$$H(z) = H_{\min}(z)\frac{z^{-1} - a}{1 - az^{-1}}, \quad |a| < 1$$

$$\lim_{z\to\infty} H_{\min}(z) = \lim_{z\to\infty} \frac{1 - az^{-1}}{z^{-1} - a} H(z)$$

$$\Rightarrow h_{\min}[0] = -\frac{1}{a} h[0] \quad \Rightarrow \quad h[0] = -a h_{\min}[0] \qquad |a| < 1$$

$$\Rightarrow |h[0]| \le |h_{\min}[0]|$$

– Maximum phase system is a stable system whose poles and zeros are all outside the unit circle

$$H(z) = \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{N}(z - d_k)} \qquad |c_k| > 1, |d_k| > 1$$

stable $\Rightarrow$ Roc : $|z| <$ smallest d_k

$\Rightarrow$ all poles contribute to $h[n]$ with terms of the form $-(d_k)^n u[-n-1]$

$\Rightarrow$ anti causal!

- Note that FIR max-phase sequence can be made causal by introducing a finite delay

$$H_{max}(z) = h_{min}[0]\prod\left(1 - c_k^*\right)\prod \frac{z^{-1} - c_k}{1 - c_k^* z}$$
$$= H_{min}(z) H_{ap}(z)$$

– Property of min-phase systems

- minimum phase-lag

$$\arg\left[H\left(e^{j\omega}\right)\right] = \arg\left[H_{\min}\left(e^{j\omega}\right)\right] + \arg\left[H_{ap}\left(e^{j\omega}\right)\right]$$

negative for $0 \le \omega \le \pi$

- minimum group-delay
- minimum energy delay property

$$\left|h[0]\right| \le \left|h_{min}[0]\right|$$

Structure for discrete-time systems

- Characterization of an LTI system can be represented by difference equation, impulse response or system function

  Example: $H(z)=\dfrac{b_0+b_1z^{-1}}{1-az^{-1}},\quad |z|>|a|$

  $\Rightarrow$ Impulse response: $h[n]=b_0a^nu[n]+b_1a^{n-1}u[n-1]$

  $\Rightarrow$ Infinite duration impulse response

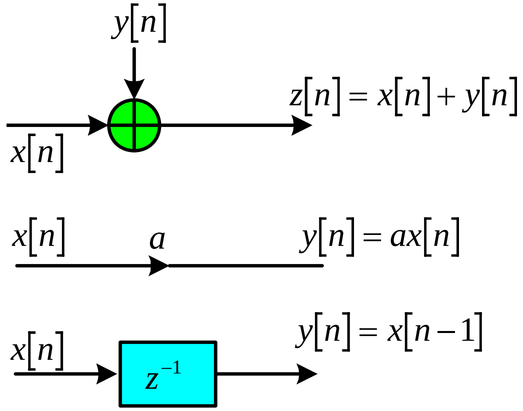
  $\Rightarrow$ Not possible to implement the system by discrete time convolution

  $\Rightarrow$ The output can be calculated by a recursive computation algorithm

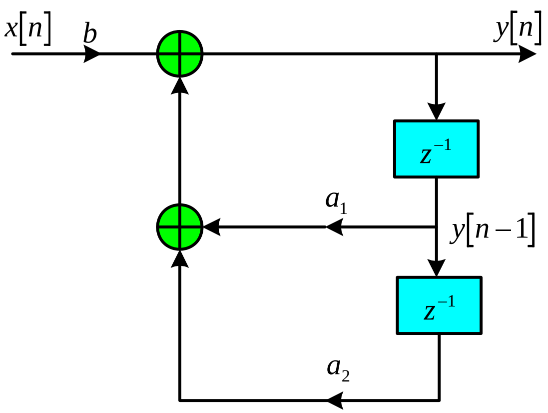
  $$y[n]=ay[n-1]+b_0x[n]+b_1x[n-1]$$

- For description of systems for implementation, use the block diagram and/or signal flow graph method

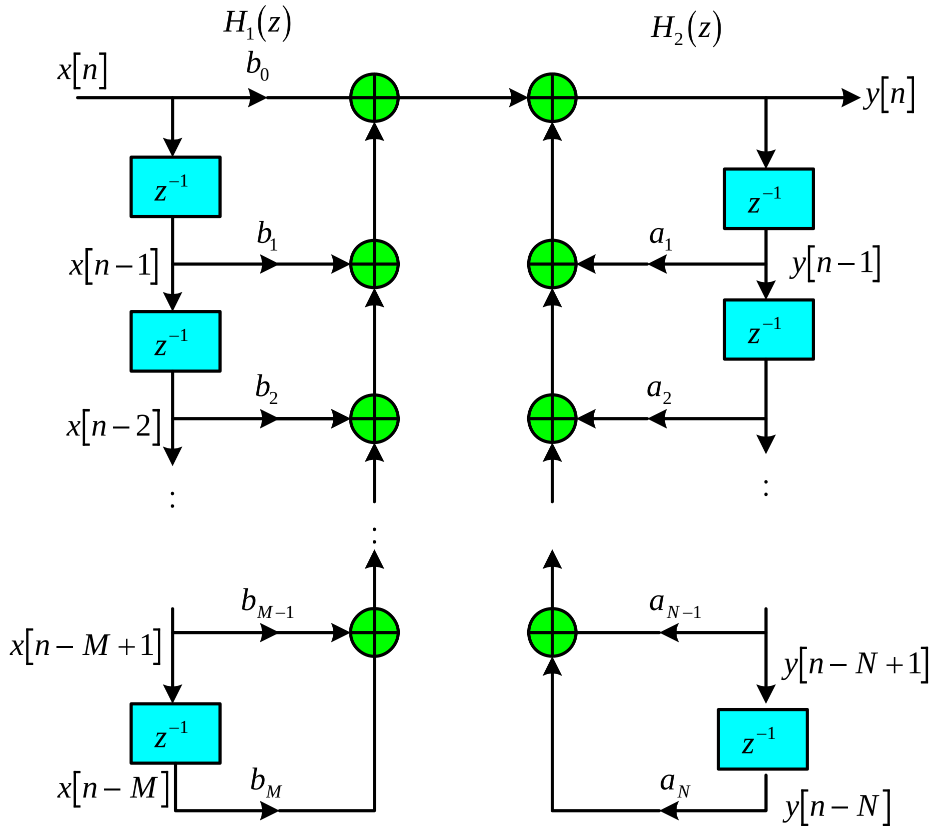
- Basic block diagram symbols



Example $y[n] = a_1 y[n-1] + a_2 y[n-2] + bx[n]$



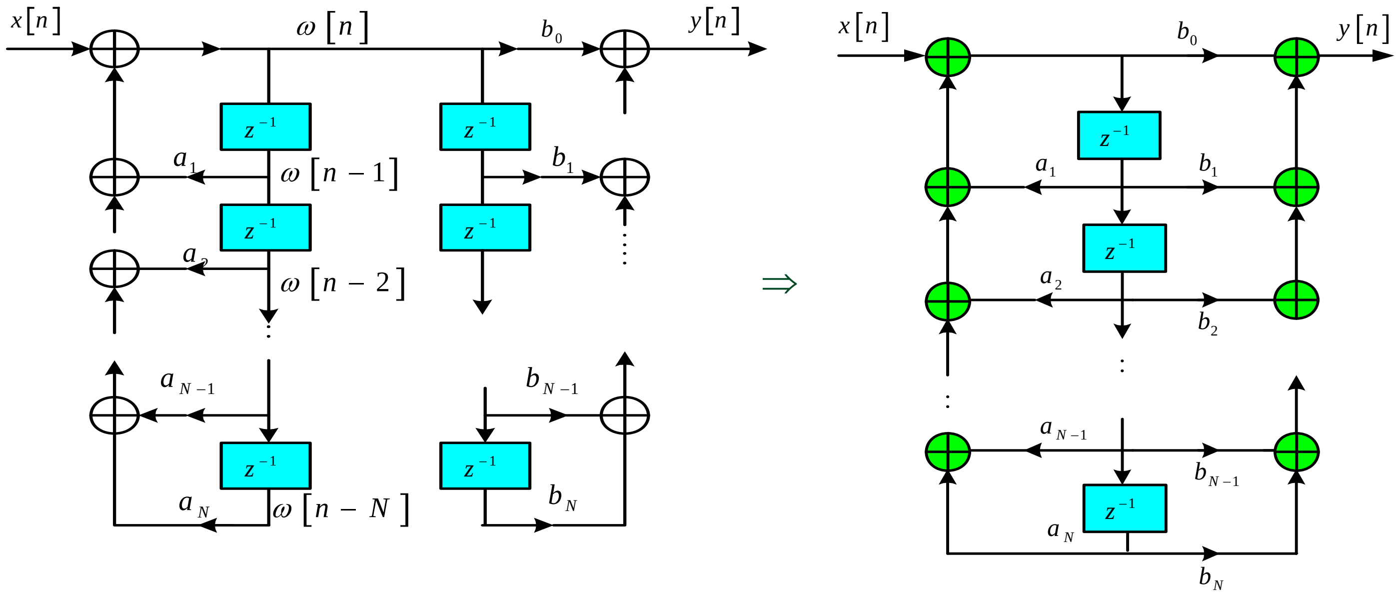
– $y[n]=\sum_{k=1}^{N}a_k y[n-k]+\sum_{k=0}^{M}b_k x[n-k]$



< direct form I>

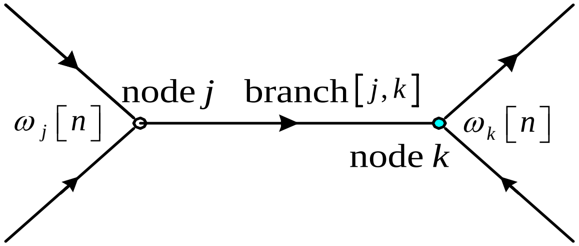
– By interchanging the cascaded blocks,

$$\frac{Y(z)}{X(z)} = H(z)$$
$$= H_1(z)H_2(z)$$
$$= H_2(z)H_1(z)$$

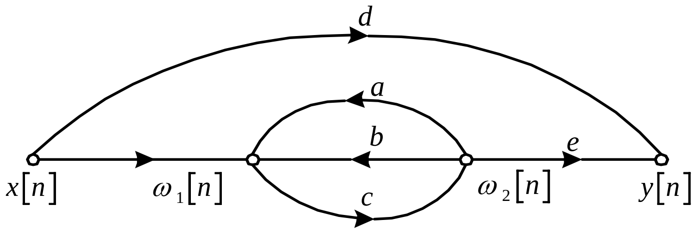


< direct form II or canonic direct form >

- Signal flow graph representation



Example

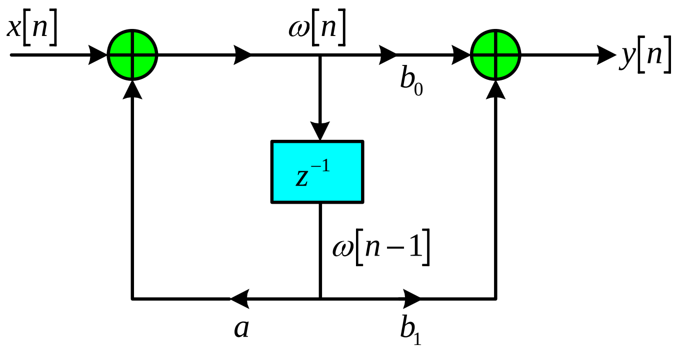


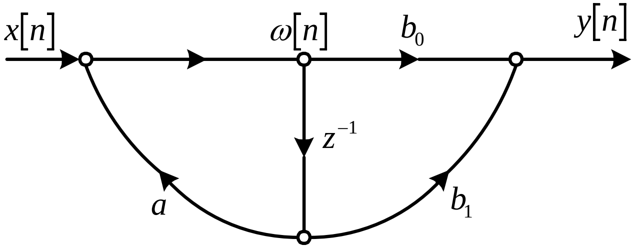
$$\omega_1[n] = x[n] + a\omega_2[n] + b\omega_2[n]$$

$$\omega_2[n] = c\omega_1[n]$$

$$y[n] = dx[n] + e\omega_2[n]$$

Example





$$\omega[n] = x[n] + a\omega[n-1]$$

$$y[n] = b_0\omega[n] + b_1\omega[n-1]$$

$$\Rightarrow W(z) = X(z) + aW(z)z^{-1} \Rightarrow X(z) = \left(1 + az^{-1}\right)W(z)$$

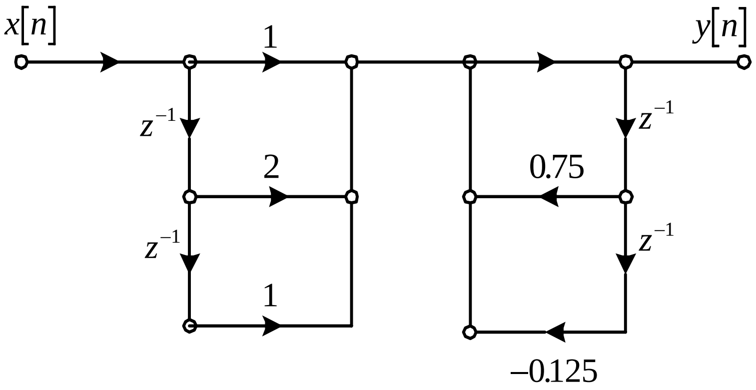
$$Y(z) = b_0W(z) + b_1W(z)z^{-1} \Rightarrow Y(z) = \left(b_0 + b_1z^{-1}\right)W(z)$$

$$\Rightarrow W(z) = \frac{Y(z)}{b_0 + b_1z^{-1}} = \frac{X(z)}{1 + az^{-1}}$$

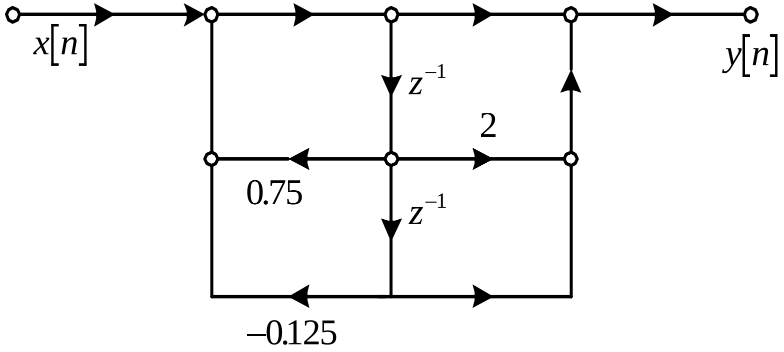
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1}}{1 + az^{-1}}$$

- Basic structures for IIR systems
  - Direct form I

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$$



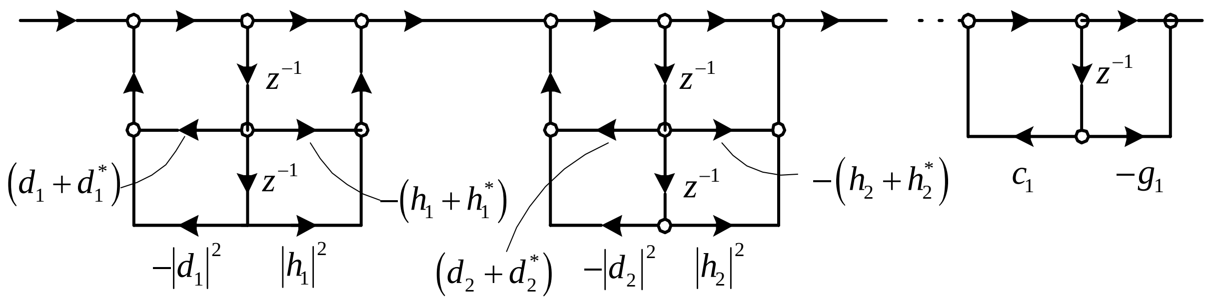
  - Direct form II



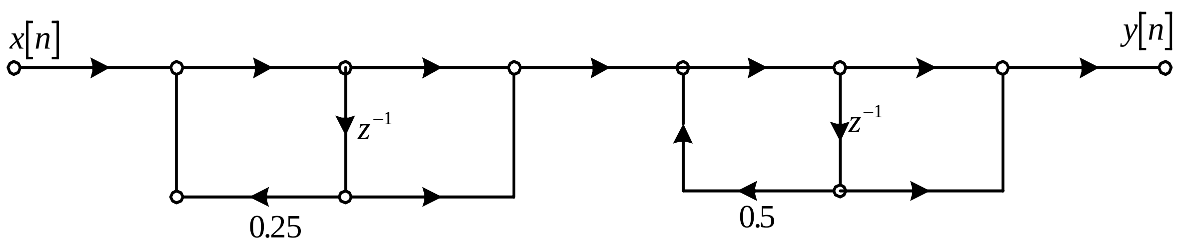
– Cascade form

$$H(z) = A\frac{\prod_{k=1}^{M_1}\left(1-g_k z^{-1}\right)\prod_{k=1}^{M_2}\left(1-h_k z^{-1}\right)\left(1-h_k^* z^{-1}\right)}{\prod_{k=1}^{N_1}\left(1-c_k z^{-1}\right)\prod_{k=1}^{N_2}\left(1-d_k z^{-1}\right)\left(1-d_k^* z^{-1}\right)}$$

$$\left(1-d_k z^{-1}\right)\left(1-d_k^* z^{-1}\right) \rightarrow 1-\left(d_k+d_k^*\right)z^{-1}+\left|d_k\right|^2 z^{-2}$$



Example $H(z) = \dfrac{\left(1+z^{-1}\right)\left(1+z^{-1}\right)}{\left(1-0.5z^{-1}\right)\left(1-0.25z^{-1}\right)}$



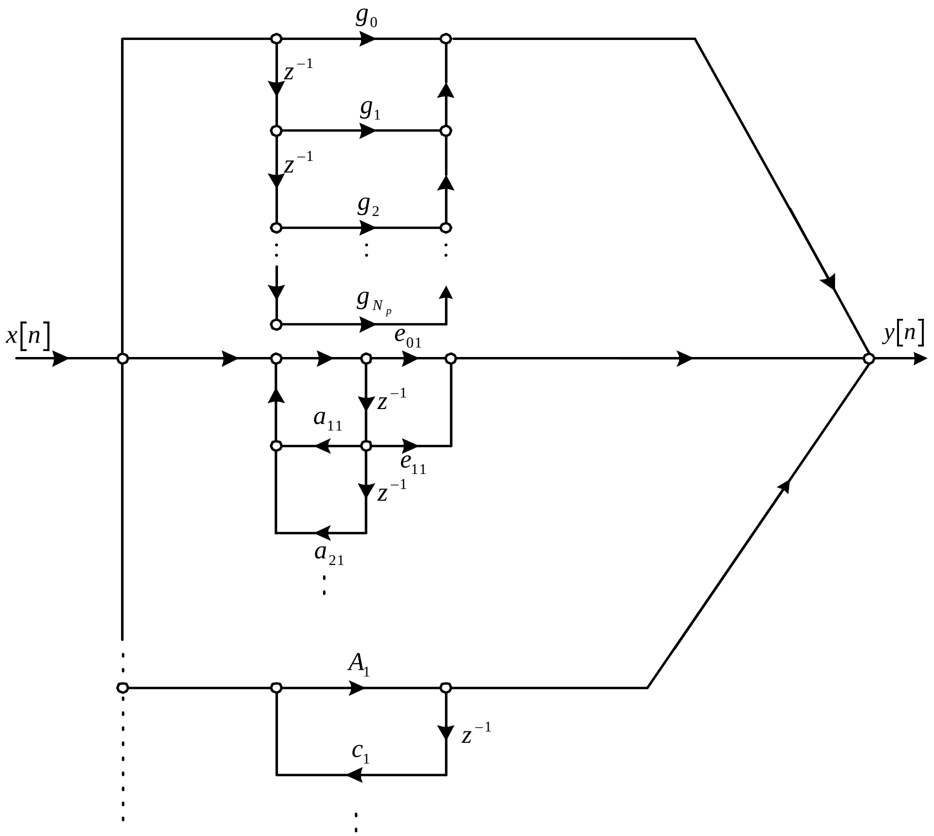
– Parallel form

$$H(z)=\sum_{k=0}^{N_p} g_k z^{-k}+\sum_{k=1}^{N_1}\frac{A_k}{1-c_k z^{-1}}+\sum_{k=1}^{N_2}\frac{B_k\left(1-e_k z^{-1}\right)}{\left(1-d_k z^{-1}\right)\left(1-d_k^* z^{-1}\right)}$$

$$\sum_{k=1}^{N_2}\frac{e_{0k}+e_{1k}z^{-1}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$

$$N=N_1+2N_2$$

$$M\ge N,\quad N_p=M-N$$

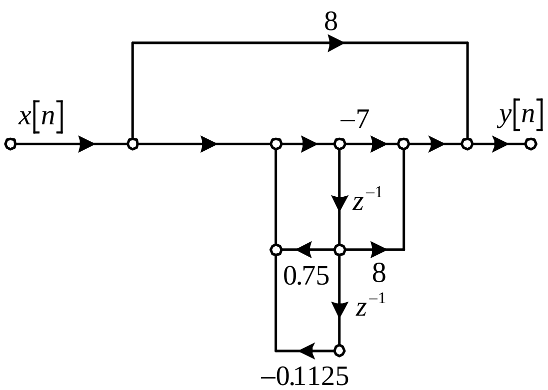


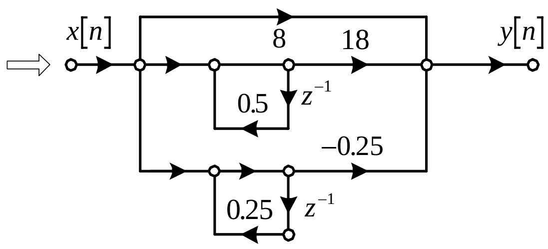
Example

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-1}}$$

$$=8+\frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-1}}$$

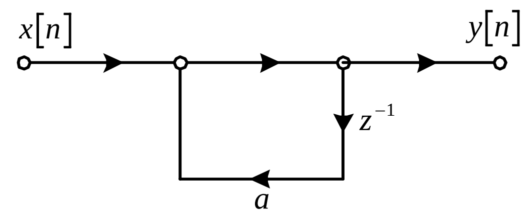
$$=8+\frac{18}{1-0.5z^{-1}}-\frac{25}{1-0.25z^{-1}}$$



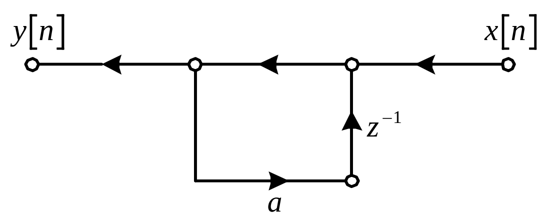


- Transposed forms
  - Signal flow graph methods provide procedures for transforming graphs into different forms
  - Transposition of a flow graph (flow graph reversal) is obtained by reversing the directions of all branches, while keeping the branch transnittances and reversing the roles of the input and output

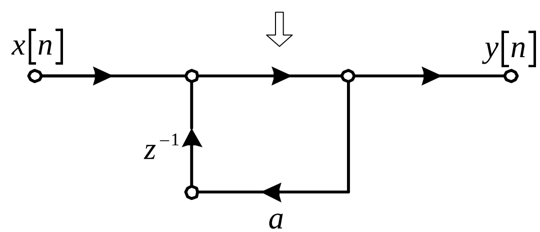
Example $H(z)=\dfrac{1}{1-az^{-1}}$



Transposed form ⇩

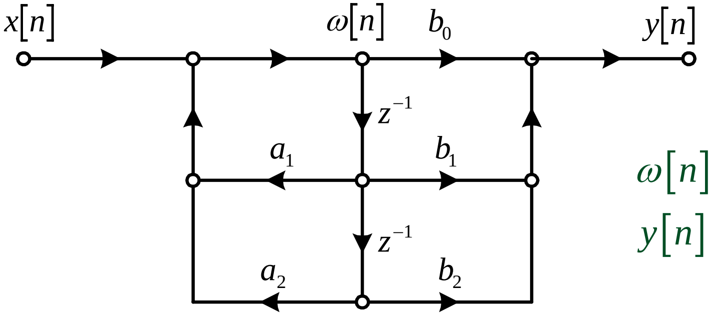


⇩



The difference is the change in ordering

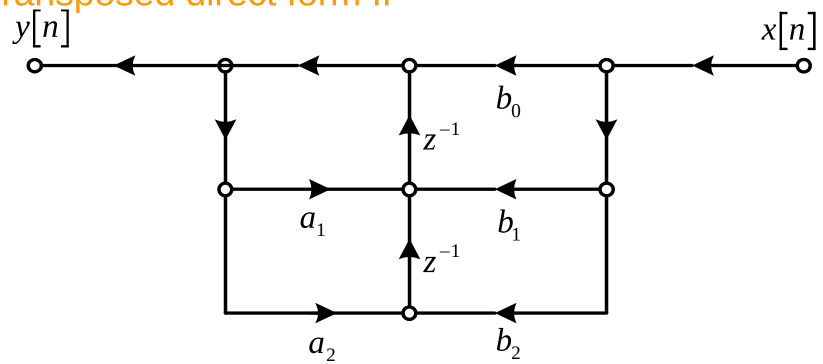
Example Transposed form of a second-order structure

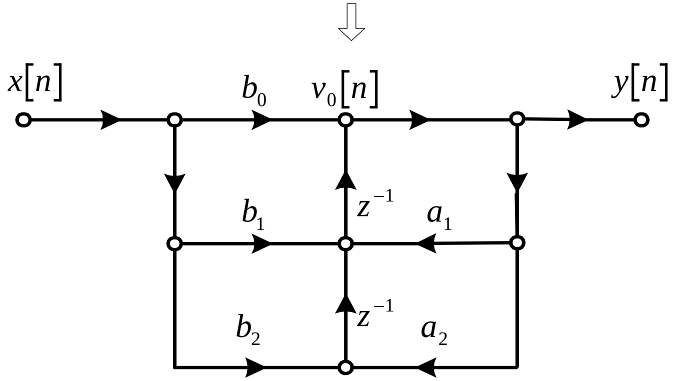


$$\omega[n] = a_1\omega[n-1] + a_2\omega[n-2] + x[n]$$

$$y[n] = b_0\omega[n] + b_1\omega[n-1] + b_2\omega[n-2]$$

Transposed direct form II

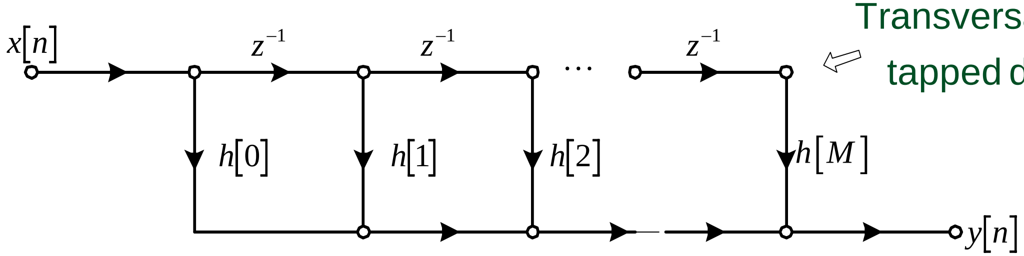




- Basic network structure for FIR systems

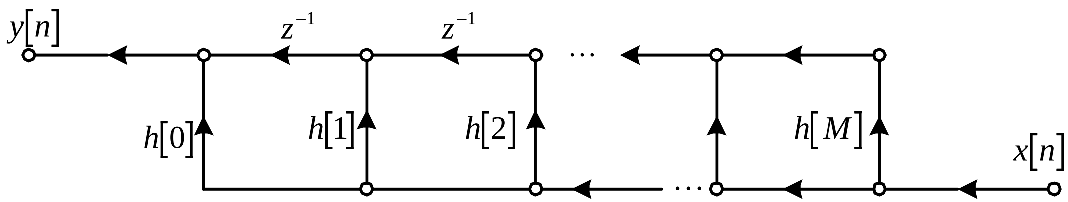
  – Direct form

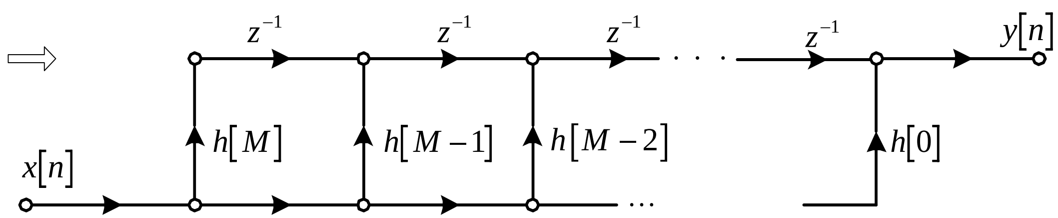
Causal FIR systems $y[n]=\sum_{k=0}^{M} b_k x[n-k] \Rightarrow h[n]=\begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$



Transversal filter or tapped delay line

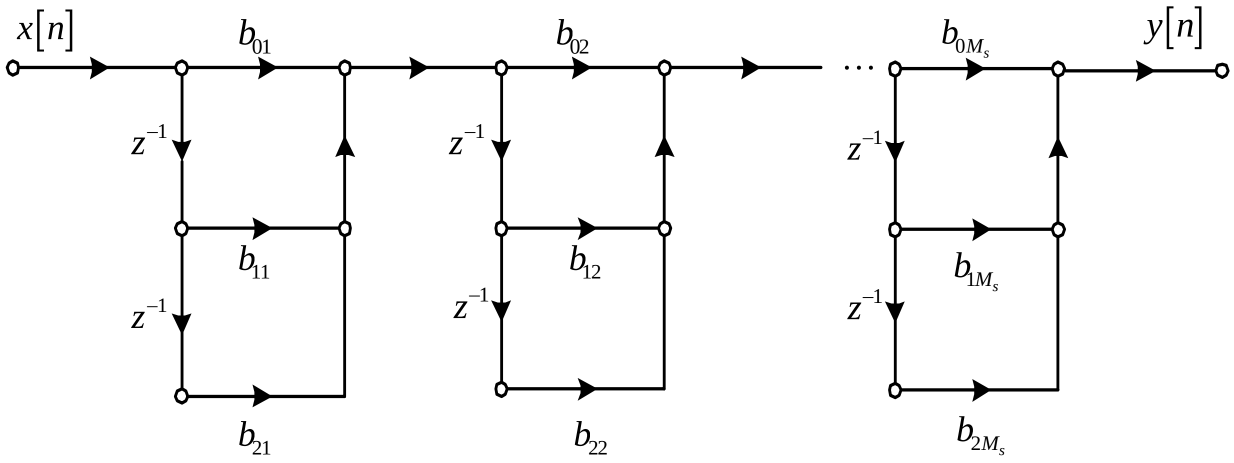
In the transposed form,





– Cascade form

$$H(z)=\sum_{k=0}^{M} h[n]z^{-n}=\prod_{k=1}^{M_s}\left(b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}\right);\quad M_s=\lfloor (M+1)/2 \rfloor$$



– Structure for linear phase FIR systems

- Causal linear phase FIR systems

$$h[M-n]=h[n] \quad \text{or} \quad h[M-n]=-h[n] \quad \text{for} \quad 0\le n\le M$$

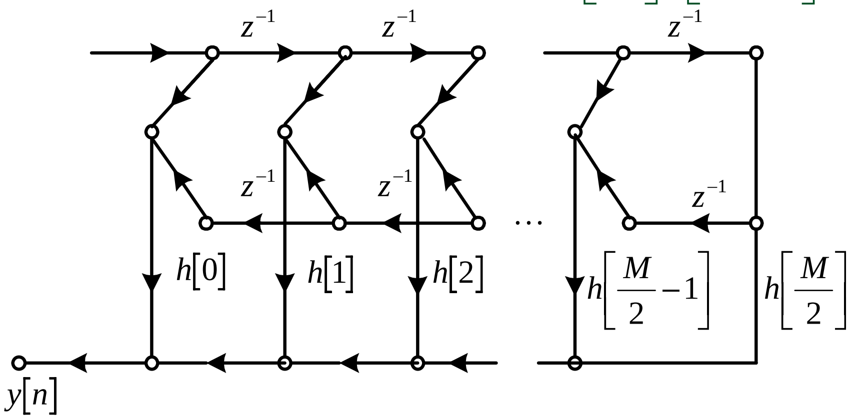
$$\Rightarrow y[n]=\sum_{k=0}^{M} h[k]x[n-k]$$

$$=\sum_{k=0}^{\frac{M}{2}-1} h[k]x[n-k]+h\left[\frac{M}{2}\right]x\left[n-\frac{M}{2}\right]+\sum_{k=M/2+1}^{M} h[k]x[n-k]$$

$$=\sum_{k=0}^{\frac{M}{2}-1} h[k]x[n-k]+h\left[\frac{M}{2}\right]x\left[n-\frac{M}{2}\right]+\sum_{k=0}^{\frac{M}{2}-1} h[M-k]x[n-M+k]$$

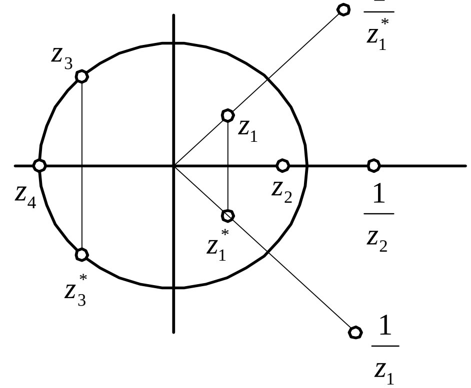
When $h[M-n]=h[n]$ (type - I system),

$$y[n]=\sum_{k=0}^{\frac{M}{2}-1} h[k]\left(x[n-k]+x[n-M+k]\right)+h\left[\frac{M}{2}\right]x\left[n-\frac{M}{2}\right]$$



- Symmetry condition

$\Rightarrow$ zeros of $H(z)$ occur mirror image pairs



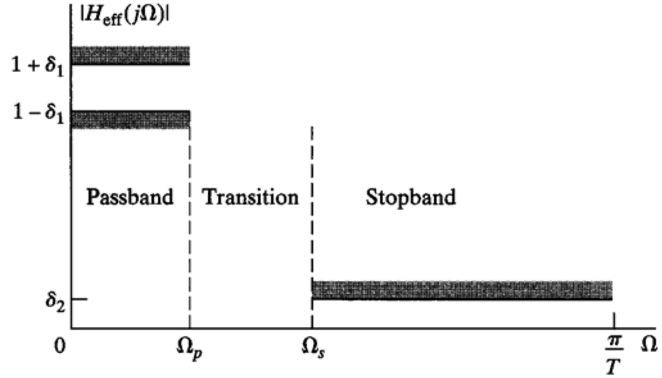
If $h[n]$ is real, zeros of $H(z)$ occur in complex conjugate pairs.

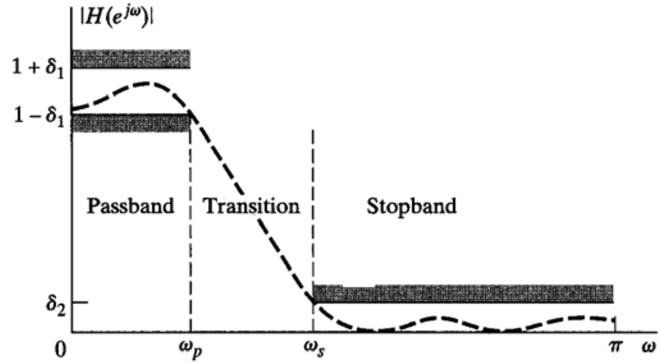
$$H(z)=h[0](1+z^{-1})(1+az^{-1}+z^{-2})(1+bz^{-1}+z^{-2})(1+cz^{-1}+dz^{-2}+cz^{-3}+z^{-4})$$

where $a=z_2+\frac{1}{z_2},\; b=2\,\mathrm{Re}\{z_3\},\; c=-2\,\mathrm{Re}\left\{z_1+\frac{1}{z_1}\right\},\; d=2+\left|z_1+\frac{1}{z_1}\right|^2$

Filter Design Techniques

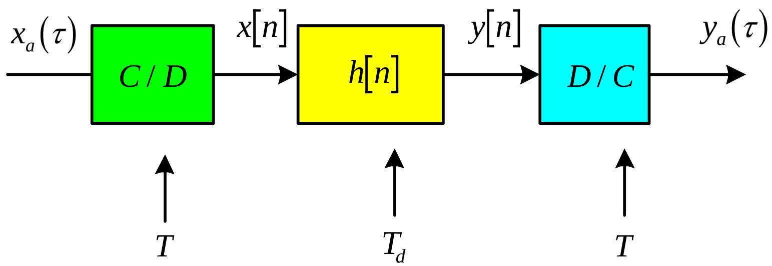
- Design procedure
  - Specify the desired frequency Response
  - Approximate the specifications
  - Realize the filter $H\left(e^{j\omega}\right)=H_{eff}\left(\frac{\omega}{T}\right), \quad |\omega|<\pi$





- Design of discrete-time IIR filters from continuous-time filters
  - Filter design by impulse invariance

$$h[n] = T_d h_c(nT_d)$$



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right) \qquad \frac{\omega}{T_d} \Leftrightarrow 2\pi f\,(=\Omega)$$

- The discrete-time filter specifications are transformed to continuous-time filter specifications by $H(e^{j\omega}) = H_c\left(\frac{\omega}{T_d}\right), \quad |\omega| \le \pi,$ if $H_c(f) = 0, \quad |f| \ge \frac{1}{2T_d}$
- Design a suitable continuous-time filter $H_c(s)$
- Transform to the desired discrete-time filter $H(z)$

- For $H_c(s)=\sum_{k=1}^{N}\frac{A_k}{s-s_k}$, $h_c(t)=\begin{cases}\sum_{k=1}^{N}A_k e^{s_k t}, & t\geq 0\\ 0, & t>0\end{cases}$

By sampling $h_c(t)$,

$$h[n]=T_d h_c[nT_d]=\sum_{k=1}^{N}T_d A_k e^{s_k n T_d}u[n]=\sum_{k=1}^{N}T_d A_k\left(e^{s_k T_d}\right)^n u[n]$$

$$\Rightarrow H(z)=\sum_{k=1}^{N}\frac{T_d A_k}{1-e^{s_k T_d}z^{-1}}$$

- A pole at $s=s_k$ in the s-plane is transformed to a pole at $z=e^{s_k T_d}$ in the z-plane.
- If $H_c(s)$ is stable, the real part of $s_k<0$

$\Rightarrow \left|e^{s_k T_d}\right|<1$

$\Rightarrow$ pole is inside the unit circle

$\Rightarrow$ $H(z)$ is also stable

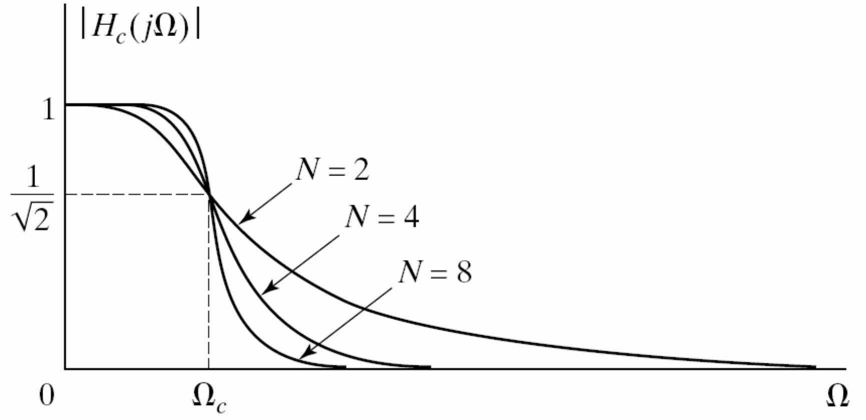
Although the poles in the s-plane are mapped in the z-plane in a simple manner, the zeros are not.

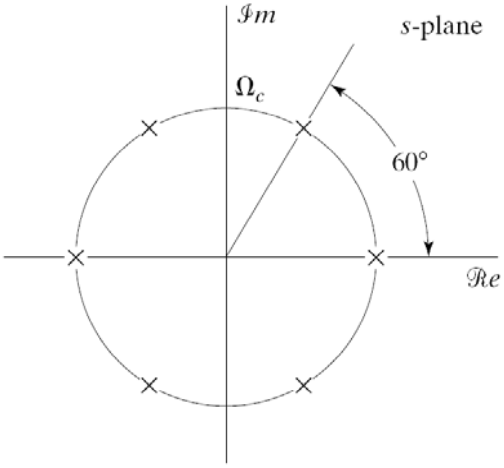
– Butterworth (BW) filter

- The magnitude response of a BW filter is

$$\left|H_c(f)\right|^2 = \frac{1}{1+\left(\frac{f}{f_c}\right)^{2N}} \Leftarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{2\pi f_c}\right)^{2N}}$$

where f_c is the 3dB frequency





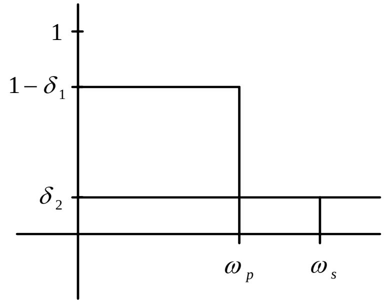
$\Rightarrow$ poles are $\left\{ s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N-1)}; k = 0,1,2,\cdots,2N-1 \right\}$

Example Design a continuous-time Butterworth (BW) filter *s.t.*

$$0.89125 \le |H_c(f)| \le 1, \quad 0 \le f \le 0.1$$

$$|H_c(f)| \le 0.17783 \qquad 0.15 \le f \le \frac{1}{2}$$

Let $T_d = 1$ so that $\omega = \Omega$



- The magnitude response of a BW filter is

$$|H_c(f)|^2 = \frac{1}{1+\left(\frac{f}{f_c}\right)^{2N}} \Leftarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{2\pi f_c}\right)^{2N}}$$

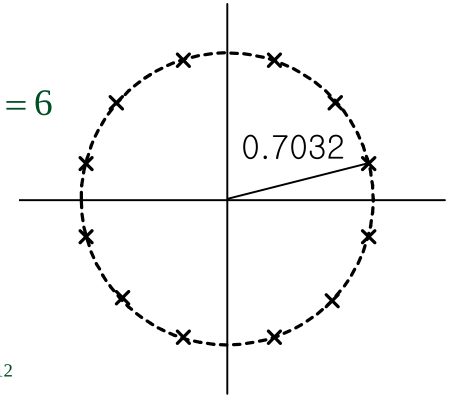
$$\delta_1 = 0.10875$$
$$\delta_2 = 0.17783$$
$$\omega_p = 0.2\pi$$
$$\omega_s = 0.3\pi$$

where f_c is the $3dB$ frequency

By solving

$$\begin{cases} 1+\left(\frac{0.1}{f_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \\ 1+\left(\frac{0.15}{f_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 \end{cases} \Rightarrow N = 5.8858 \Rightarrow N = 6$$

$$f_c = 0.11216$$



$$s_k = 2\pi f_c e^{j\frac{\pi}{2N}(2k+N-1)} \Rightarrow \text{poles are} \begin{pmatrix} -0.182 \pm j0.679 \\ -0.497 \pm j0.497 \\ -0.679 \pm j0.182 \end{pmatrix}$$

$$1+\left(\frac{0.1}{f_c}\right)^{12} = 1.258928$$

$$\Rightarrow f_c = 0.111918$$

$$H_c(s)=\frac{0.12093}{\left(s^2+0.3645+0.4945\right)\left(s^2+0.99455+0.4945\right)\cdot\left(s^2+1.35855+0.4945\right)}$$

$$=\frac{A_1}{\left(s+0.182+j0.679\right)}+\frac{A_2}{\left(s+0.182-j0.679\right)}$$

$$+\frac{A_3}{\left(s+0.497+j0.497\right)}+\frac{A_4}{\left(s+0.497-j0.497\right)}$$

$$+\frac{A_5}{\left(s+0.679+j0.182\right)}+\frac{A_6}{\left(s+0.679-j0.182\right)}$$

$$\Rightarrow H(z)=\frac{A_1}{\left(1-e^{-0.182}e^{-j0.679}z^{-1}\right)}+\frac{A_2}{\left(1-e^{0.182}e^{+j0.679}z^{-1}\right)}+\cdots$$

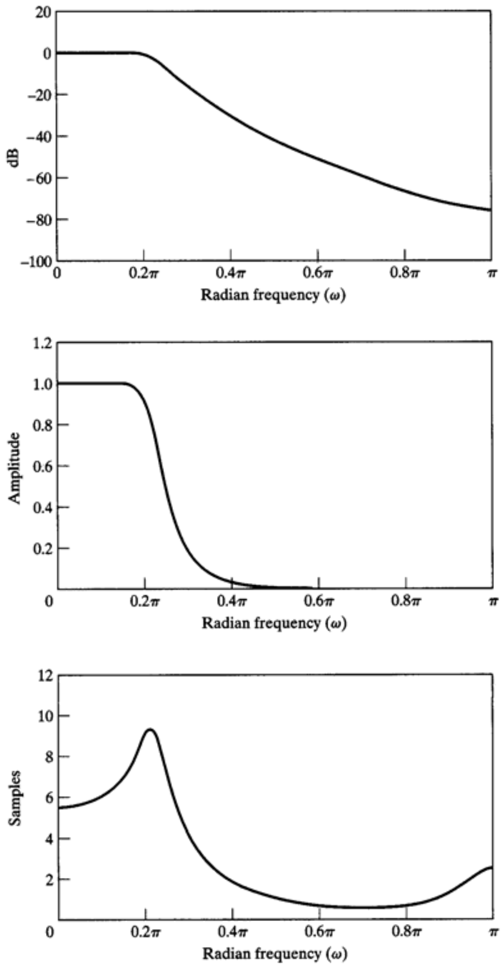
$$z_1=e^{-0.182-j0.679}=0.8336\left(0.7782-j0.6280\right)$$

$$=0.6487-j0.5235\Rightarrow\left|z_1\right|^2=0.69489$$

$$H(z)=\frac{0.2871-0.4466z^{-1}}{1-1.2974z^{-1}+0.6949z^{-2}}$$

$$+\frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}}$$

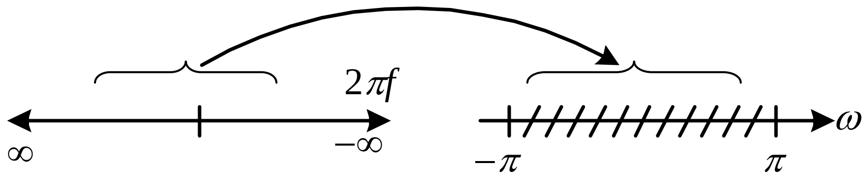
$$+\frac{1.8557-0.63032z^{-1}}{1-0.9972z^{-1}+0.2570z^{-2}}$$



- This method is useful only for band-limited filters

– Bilinear transformation

- To avoid aliasing, we employ the bilinear transform $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$



$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow z = \frac{1+\frac{\pi}{2}s}{1-\frac{T_d}{2}s} \qquad \Leftarrow s = \sigma + j2\pi f$$

$$= \frac{1+\frac{\sigma T_d}{2} + j\pi f T_d}{1-\frac{\sigma T_d}{2} - j\pi f T_d}$$

$\Rightarrow |z| < 1,$ if $\sigma < 0$

- The $j2\pi f$ axis of the s-plane maps onto the unit circle of the z-plane

For $s = j2\pi f$,

$$z = \frac{1 + j\pi f T_d}{1 - j\pi f T_d} \Rightarrow |z| = 1 \quad \Rightarrow e^{j\omega} = \frac{1 + j\pi f T_d}{1 - j\pi f T_d}$$

or

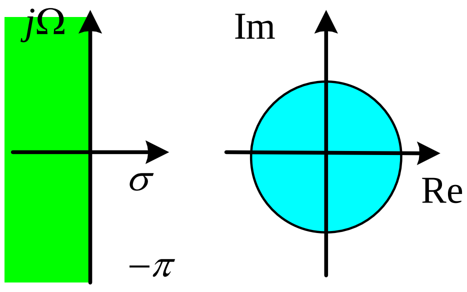
$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right) (= \sigma + j2\pi f)$$

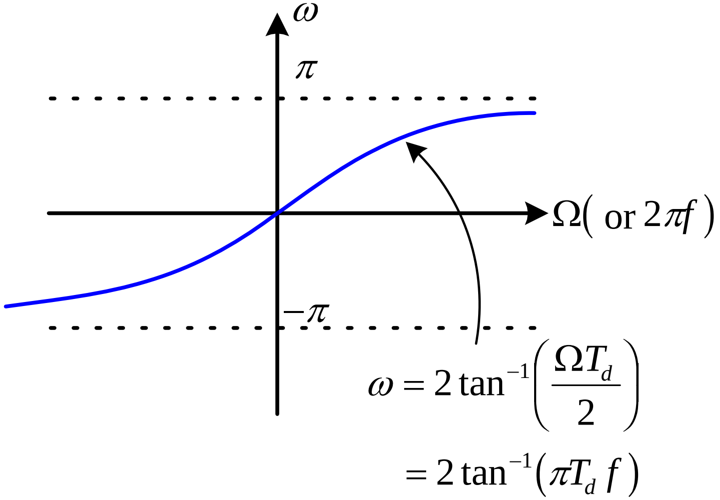
$$= \frac{2}{T_d} \frac{2e^{-j\omega/2} j \sin\left(\frac{\omega}{2}\right)}{2e^{-j2/\omega} \cos\left(\frac{\omega}{2}\right)}$$

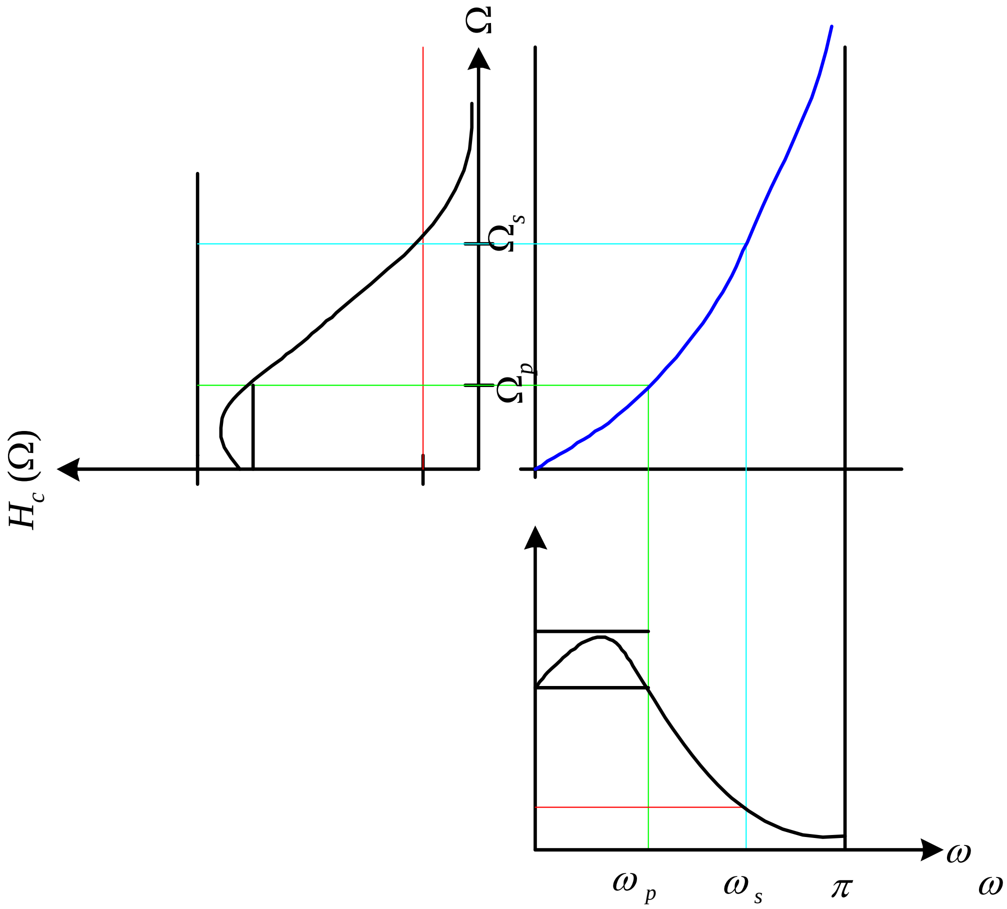
$$= j\frac{2}{T_d}\tan\frac{\omega}{2}$$

$$\Rightarrow \sigma = 0 \quad \text{and} \quad f = \frac{1}{\pi T_d}\tan\frac{\omega}{2} \left(\text{or} \quad \Omega = \frac{2}{T_d}\tan\frac{\omega}{2}\right)$$

$$\text{or} \quad \omega = 2\tan^{-1}\left(\frac{T_d \Omega}{2}\right)$$






Ω
Ω_s
Ω_p
H_c (Ω)
ω
ω_p
ω_s
π
ω

Example Bilinear transformation of a BW filter

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1, \quad 0 \le \omega \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783, \qquad 0.3\pi \le \omega \le \pi$$

$$0 \le \omega \le 0.2\pi \Rightarrow \quad 0 \le \Omega \le \frac{2}{T_d}\tan\frac{0.2\pi}{2} = 0.6498; \quad T_d = 1$$

$$0.3\pi \le \omega \Rightarrow \quad \frac{2}{T_d}\tan\frac{0.3\pi}{2} \le \Omega \Rightarrow 1.01905 \le \Omega$$

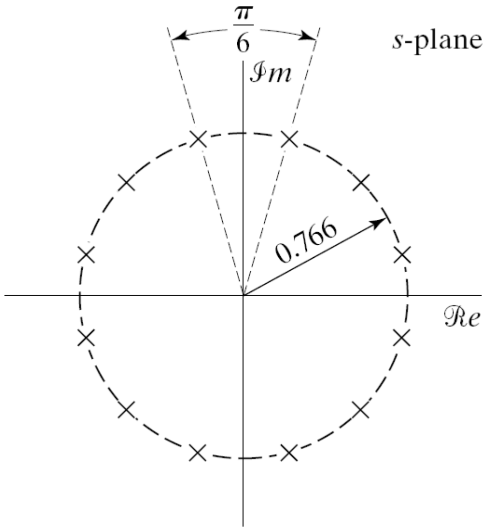
$$\left|H_c(\Omega)\right|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \Rightarrow 1+\left(\frac{0.6498}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 = 1.2589$$

$$1+\left(\frac{1.01905}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2 = 31.6220$$

$$\Rightarrow N = \frac{\left\{\left[\left(\frac{1}{0.17783}\right)^2 - 1\right] \Big/ \left[\left(\frac{1}{1.89125}\right)^2 - 1\right]\right\}}{2\log\left(\frac{1.01905}{0.6498}\right)} \cong \frac{2.0729}{0.3909} \cong 5.30267$$

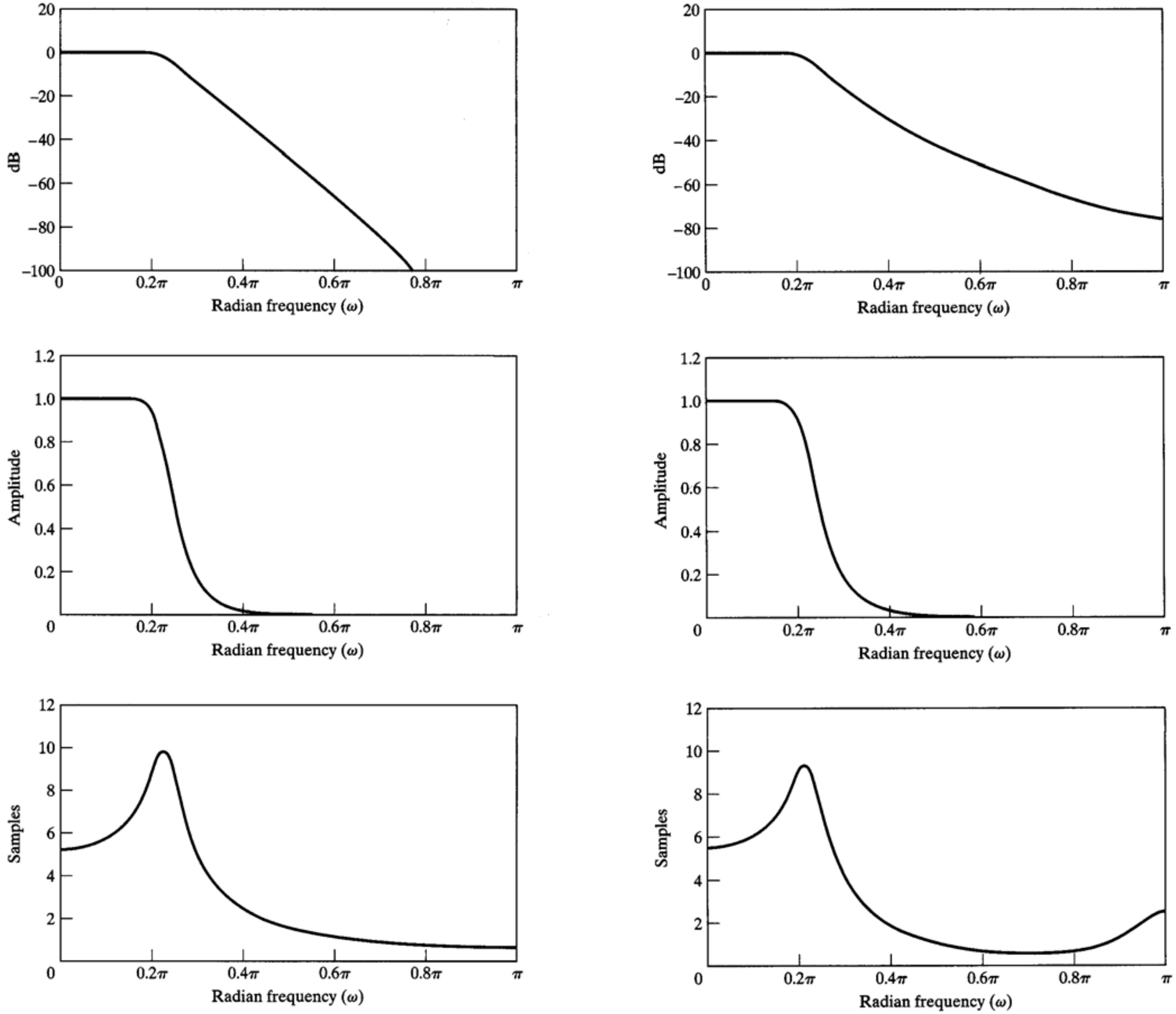
$$\Rightarrow N = 6 \Rightarrow \Omega_c = 0.76622$$

$\Rightarrow$ Poles are
$$\begin{cases} 0.76622 \cdot e^{\pm j\frac{\pi}{12}}, = 0.7401 \quad \pm j0.1983 \\ 0.76622 \cdot e^{\pm j\frac{3}{12}\pi} = 0.5418 \quad \pm j0.5418 \\ 0.76622 \cdot e^{\pm j\frac{5}{12}\pi} = 0.1983 \quad \pm j0.7401 \end{cases}$$



$$\Rightarrow H_c(s) = \frac{0.20238}{\left(s^2 + 0.39965 + 0.5871\right)\left(s^2 + 1.08365 + 0.5871\right)} \times \frac{1}{\left(s^2 + 1.48025 + 0.5871\right)}$$

$$\Rightarrow H(z) = \frac{0.0007378\left(1 + z^{-1}\right)^6}{\left(1 - 1.2686z^{-1} + 0.7051z^{-2}\right)\left(1 - 1.0106z^{-1} + 0.3583z^{-2}\right)} \times \frac{1}{\left(1 - 0.9044z^{-1} + 0.2155z^{-2}\right)}$$


dB
20
0
−20
−40
−60
−80
−100
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)
Amplitude
1.2
1.0
0.8
0.6
0.4
0.2
Samples
12
10
8
6
4
2

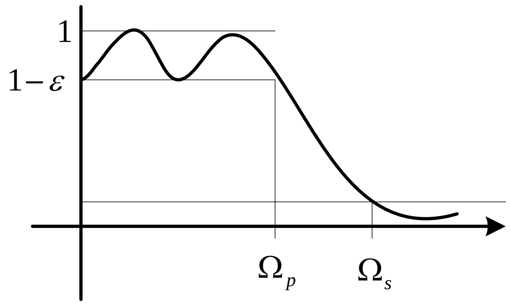
- The magnitude of *N-th* order Butterworth filter can be represented by

$$\left|H\left(e^{j\omega}\right)\right|^2 = \frac{1}{1+\left(\dfrac{\tan\frac{\omega}{2}}{\tan\frac{\omega_c}{2}}\right)^{2N}}; \qquad \tan\frac{\omega_c}{2} = \frac{\Omega_c T_D}{2} \Leftarrow \Omega = \frac{2}{T_d}\tan\frac{\omega}{2}$$

- Note that this BW filter is maximally flat, $\left|H\left(e^{j\omega_c}\right)\right| = \frac{1}{2}$, and periodic with a period of 2π.
- Maximally flat ⇔ The first (2*N*-1) derivatives of are zero at $\omega = 0$
- It is not easy to directly find the poles and zeros in the *z*-plane from $\left|H\left(e^{j\omega}\right)\right|^2$ since the magnitude-squared function should be factorized into $H(z)H\left(z^{-1}\right)$ to determine $H(z)$.

- Design of type-I Chebyshev filters

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 V_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$



– $V_N(x)$ is the *N-th* order Chebyshev polynomial defined by

$$V_N(x) = \cos\left(N\cos^{-1}x\right)$$

$$V_0(x) = \cos 0 = 1$$

$$V_1(x) = \cos\left(\cos^{-1}x\right) = x$$

$$V_2(x) = \cos\left(2\cos^{-1}x\right) = 2x^2 - 1$$

$$V_3(x) = 2x\left(x^2-1\right) - x = 4x^3 - 3x$$

$$\vdots$$

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

$$\Rightarrow 0 \le x \le 1 \Rightarrow 0 \le \left|V_N^2(x)\right| \le 1$$

$x > 1 \Rightarrow \cos^{-1}x$ is imaginary $\Rightarrow V_N(x)$ ~ hyperbolic cosine

$\Rightarrow V_N(x)$ increases monotonically

– Property of $V_N(x)$

- $\{V_N(x)\}$ forms an orthogonal set over $-1 \le x \le 1$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} V_N(x)\, V_M(x)\, dx = \begin{cases} 0, & M \ne N \\ \pi/2, & M = N \ne 0 \\ \pi, & M = N = 0 \end{cases}$$

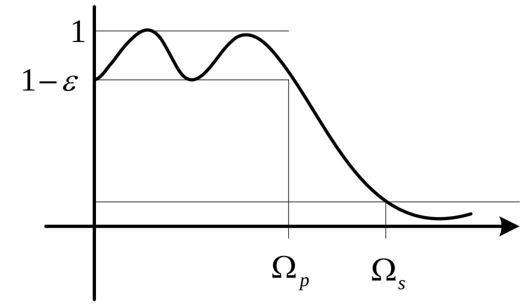
- For each *N*, $V_N(1) = 1$
- For even *N*, $V_N(-1) = 1; \quad V_N(0) = (-1)^{N/2}$

  $V_N(x) = V_N(-x)$
- For odd *N*, $V_N(-1) = -1; \quad V_N(0) = 0$

  $V_N(x) = -V_N(-x)$
- The zeros of $\{V_N(x)\}$ lie in $-1 \le x \le 1$
- The range of $\{V_N(x)\}$ is between –1 and 1 for $-1 \le x \le 1$

$\Rightarrow$ For $0 \le \frac{\Omega}{\Omega_c} \le 1, \;\; 1 \le |H(\Omega)|^2 \le \frac{1}{1+\varepsilon^2}$

$\frac{\Omega}{\Omega_c} > 1, \;\; |H(\Omega)|$ decreases monotonically



- Ripple factor: $\varepsilon^2 \;\; \Rightarrow$ ripple amplitude $\delta = 1 - \frac{1}{\sqrt{1+\varepsilon^2}}$
- Since $V_N(1) = 1$

$$\left|H\left(e^{j\omega_p T}\right)\right| = \frac{1}{1+\varepsilon^2} \;\; \Leftarrow \left|H\left(e^{j\omega T}\right)\right| = \frac{1}{1+\varepsilon^2 V_N\left[k\tan\left(\omega T/2\right)\right]}$$

- Rule: choose ε, N and Ω_c

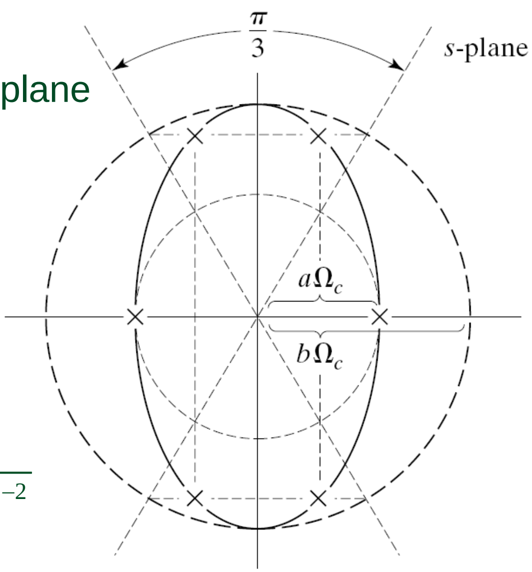
– Consider the first zeros of $1+\varepsilon^2 V_N^2(x)=0$

- Letting $\cos r \equiv \cos(u+jv) = x \Rightarrow \cos^{-1} x = r$

$$V_N(x) = \cos\left(N\cos^{-1}x\right) = \cos Nr$$
$$= \cos Nu \cosh Nv - j\sin Nu \sinh Nv$$
$$= \pm j/\varepsilon$$
$$\Rightarrow \cos Nu \cosh Nv = 0 \;\; \& \;\; \sin Nu \sinh Nv = \pm\frac{1}{\varepsilon}$$
$$\Rightarrow u = (2M+1)\pi/2N, M = 0,1,2,\cdots,2N-1$$
$$v = \frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}$$

- The poles of the Chebyshev filter lie on an ellipse in the *s*-plane

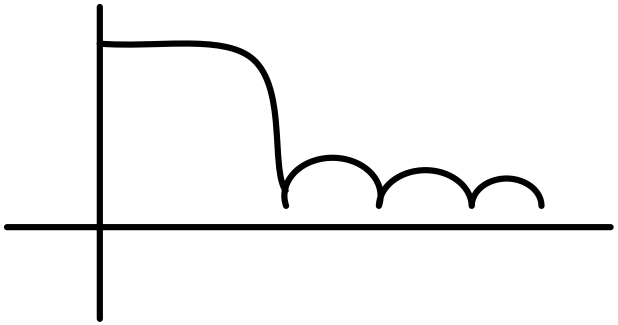
$$\frac{\sigma^2}{\sinh^2 v} + \frac{\Omega^2}{\cosh^2 v} = 1; \quad s = \sigma + j\Omega$$
$$\sigma_p = (\sinh v)\cos\theta = a\Omega_c \cos\theta$$
$$\Omega_p = (\cosh v)\sin\theta = b\Omega_c \sin\theta$$
$$a = \frac{1}{2}\left(\alpha^{\frac{1}{N}} - \alpha^{-\frac{1}{N}}\right), \;\; b = \frac{1}{2}\left(\alpha^{\frac{1}{N}} + \alpha^{-\frac{1}{N}}\right), \;\; \alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$



$N=3$

- Design of Type II Chebyshev Filters

$$\left|H_c(\Omega)\right|^2 = \frac{1}{1+\left[\varepsilon^2 V_N\left(\frac{\Omega_c}{\Omega}\right)\right]^{-1}}$$

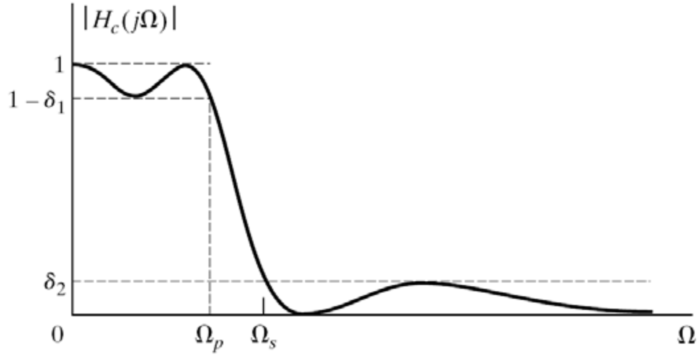


- Elliptic filter
  - The error is distributed over the entire frequency band $\Rightarrow$ Equal ripple in all band

$$\left|H_c(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

$\Rightarrow$ min order of the filter

$U_N(\Omega)$: Chebyshev rational function of degree *N*



- Example
  - Filter specifications
    - Passband

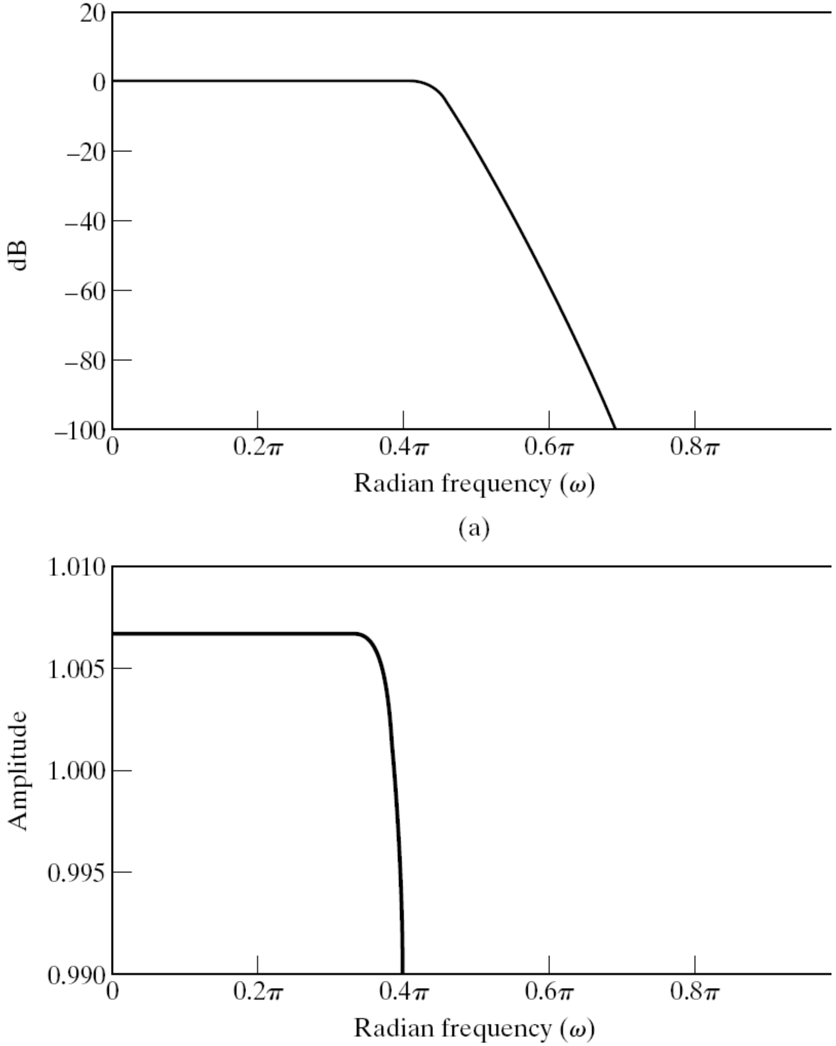
$$0.99 \le \left|H\left(e^{j\omega}\right)\right| \le 1.01; \qquad |\omega| \le 0.4\pi$$

    - Stopband

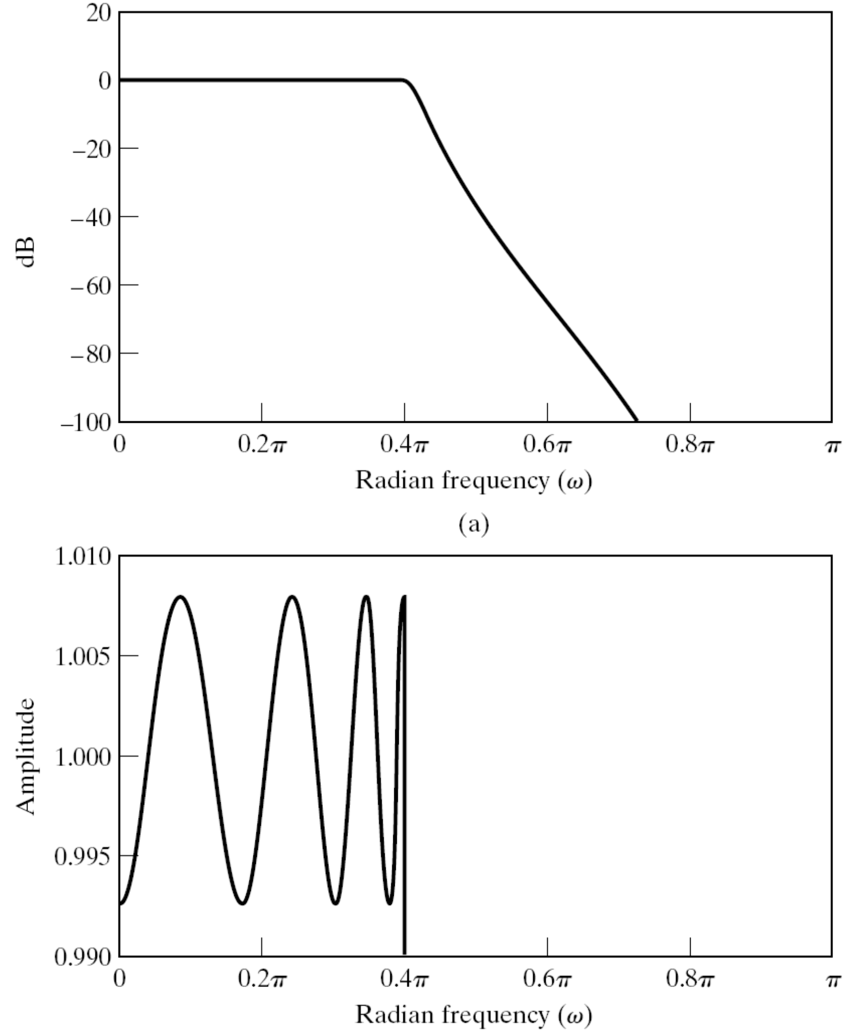
$$\left|H\left(e^{j\omega}\right)\right| \le 0.001; \qquad 0.6\pi \le |\omega| \le \pi$$

    - This implies that $\delta_1 = 0.01,\ \delta_2 = 0.001,\ \omega_p = 0.4\pi$ and $\omega_s = 0.6\pi$

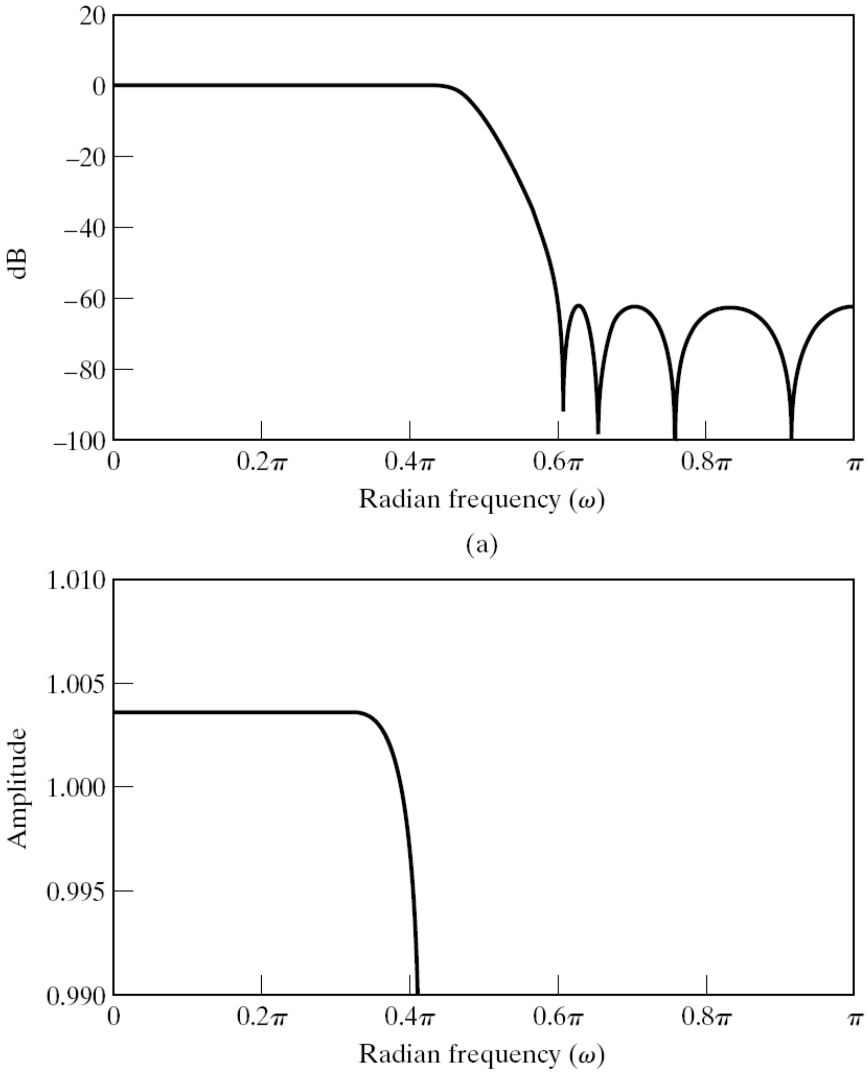
Butterworth filter with *N*=14



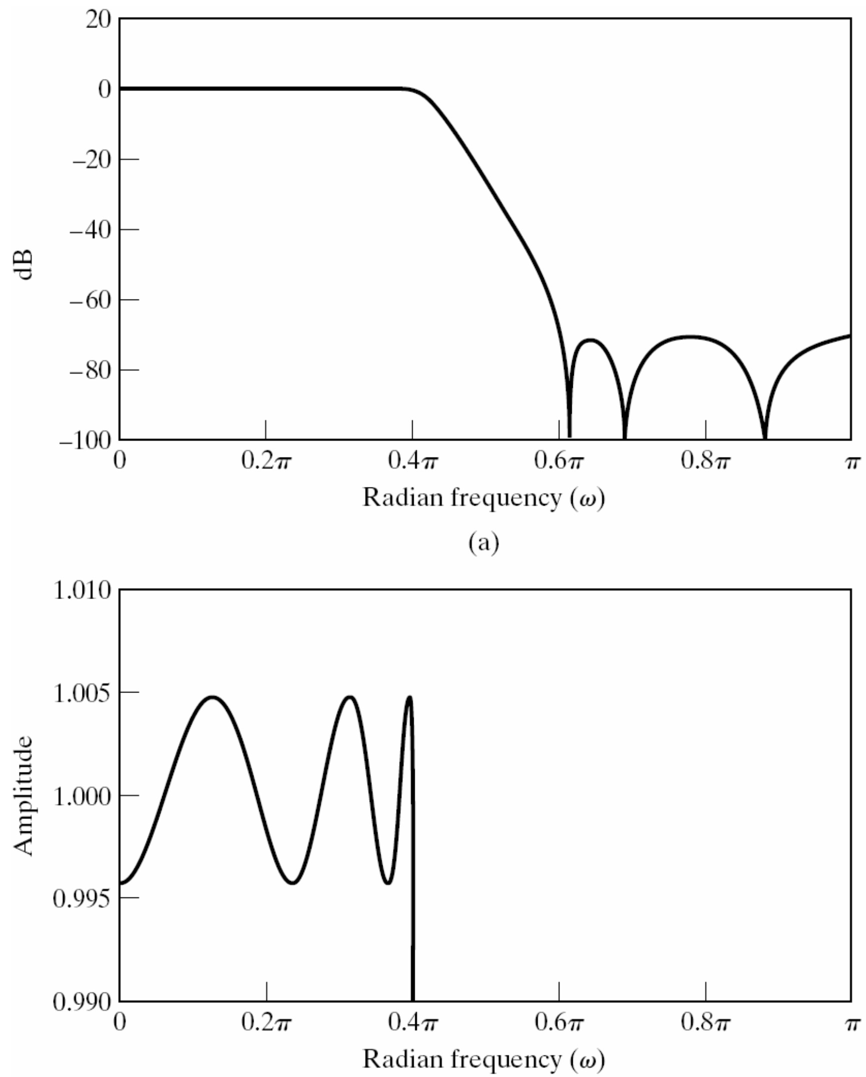
Chev-type I filter with *N*=8



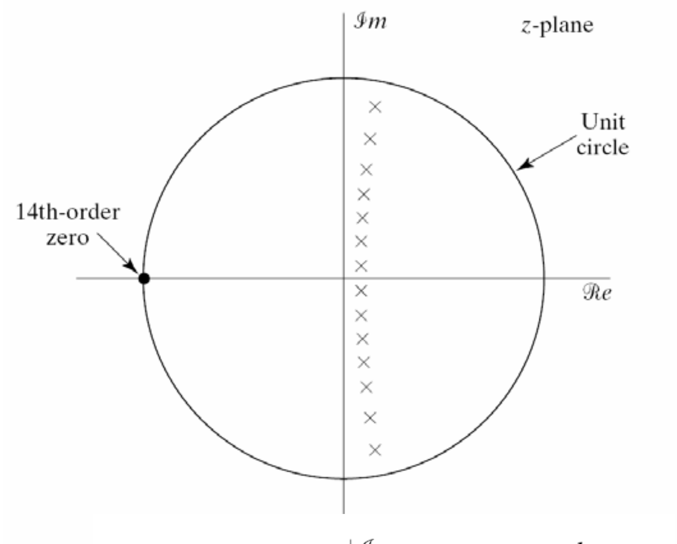
Chev-type II filter with N=8

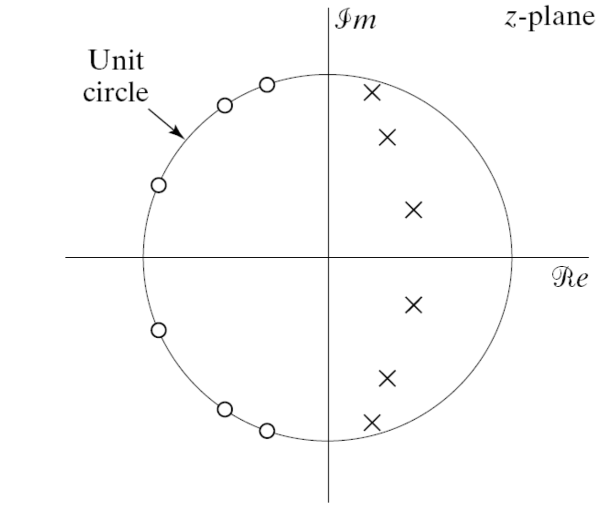


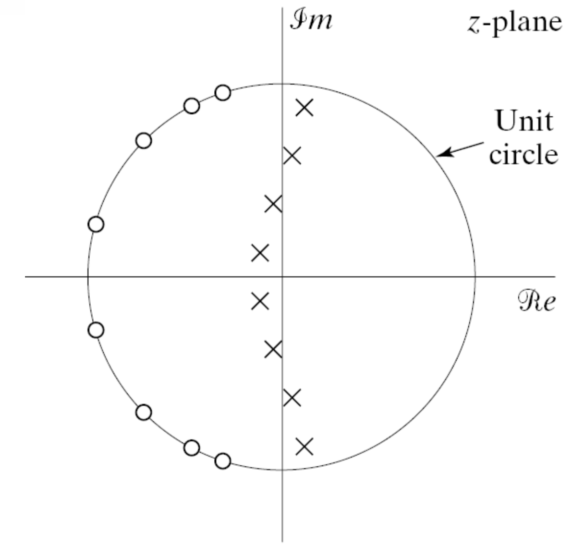
Elliptic filter with N=6

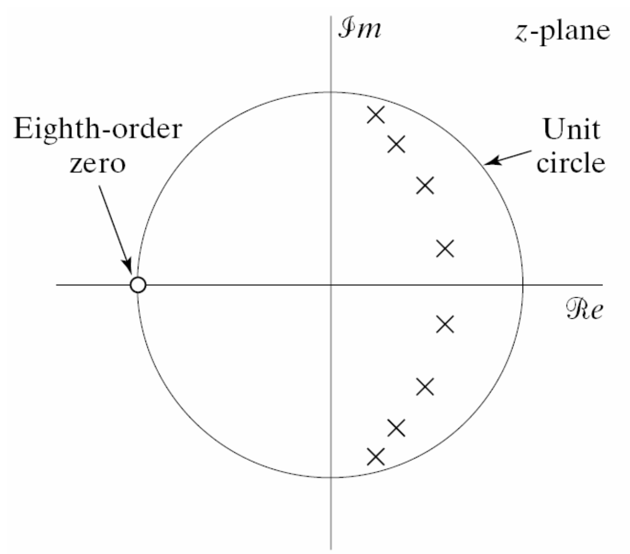


– Poles and zeros of the designed filters









- Design of FIR filters by windowing
  - FIR filter design directly approximates the desired frequency response of the discrete-time system
  - When the desired frequency response is

$$H_d\left(e^{j\omega}\right)=\sum_{h=-\infty}^{\infty} h_d[n]e^{-j\omega n};\quad h_d[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)d\omega$$

consider the design of a causal FIR filter that approximates the ideal response

  - The truncation of the desired impulse response results in the Gibbs phenomenon
  - Rectangular Windowing

$$h[n]=\begin{cases} h_d[n], & 0\le n\le M \\ 0\ , & \text{otherwise} \end{cases} \Rightarrow h[n]=h_d[n]\omega_r[n]$$

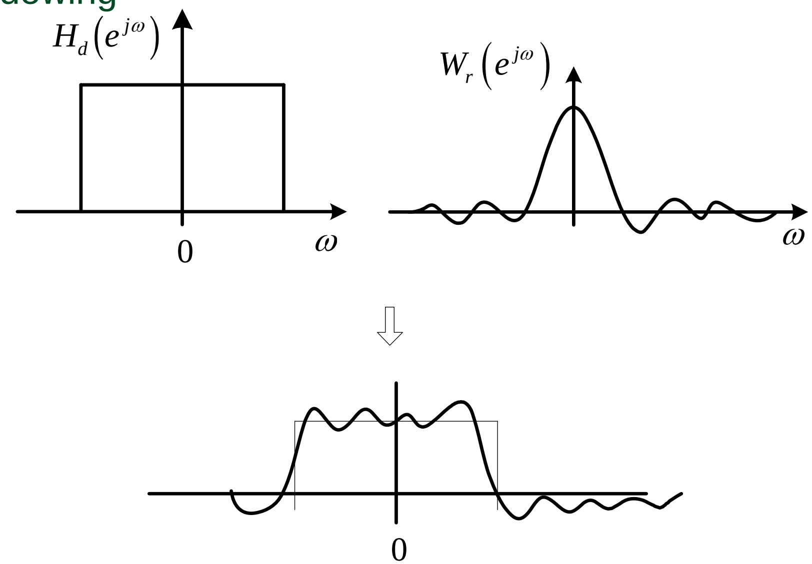
where $\omega_r[n]$ is the windowing function given by

$$\omega_r[n]=\begin{cases} 1, & 0\le n\le M \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow H\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\theta}\right)\omega_r\left(e^{j(\omega-\theta)}\right)d\theta$$

Note that $H\left(e^{j\omega}\right)$ is a smeared version of $H_d\left(e^{j\omega}\right)$

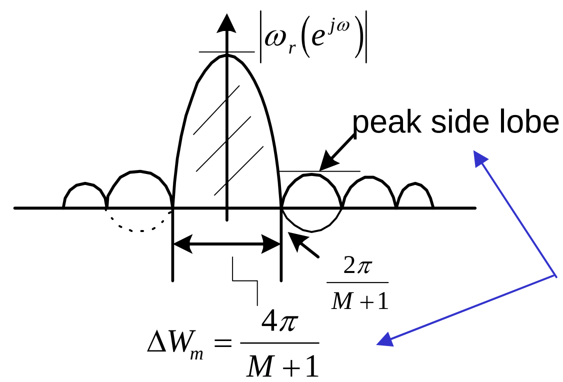
– Effect of windowing



$$W_r(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n}$$

$$= \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

$$= \frac{\sin\frac{\omega(M+1)}{2}}{\sin\frac{\omega}{2}} e^{-j\frac{\omega M}{2}}$$



design paramters

- Common windowing functions
  - Bartlett (triangular)

$$w_B[n]=\begin{cases} \dfrac{2n}{M}, & 0\le n\le \dfrac{M}{2} \\ 2-\dfrac{2n}{M}, & \dfrac{M}{2}\le n\le M \\ 0, & \text{otherwise} \end{cases}$$

  - Hanning

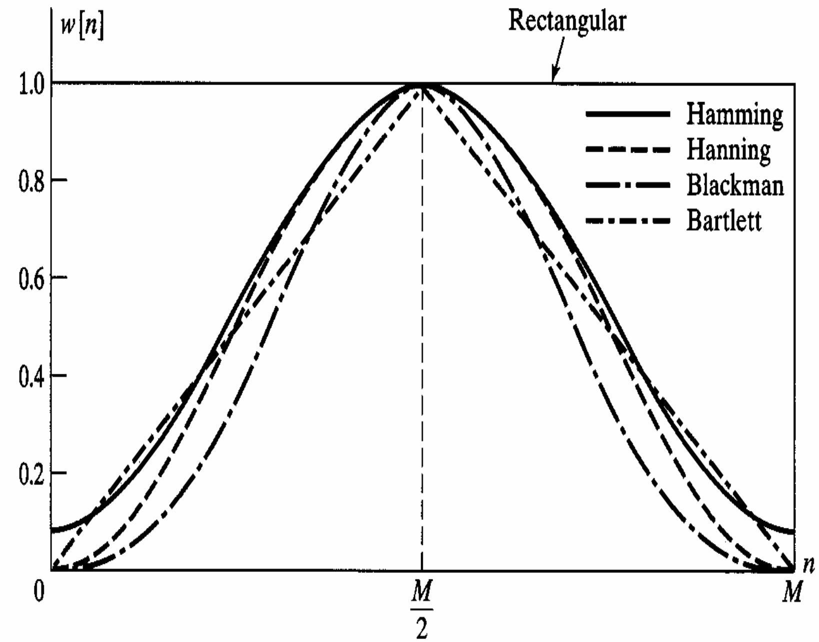
$$w_{han}[n]=\begin{cases} 0.5\left(1-\cos\dfrac{2\pi n}{M}\right), & 0\le n\le M \\ 0 & \text{, otherwise} \end{cases}$$

  - Hamming

$$w_{ham}[n]=\begin{cases} 0.54-0.46\cos\dfrac{2\pi n}{M}, & 0\le n\le M \\ 0 & \text{, otherwise} \end{cases}$$

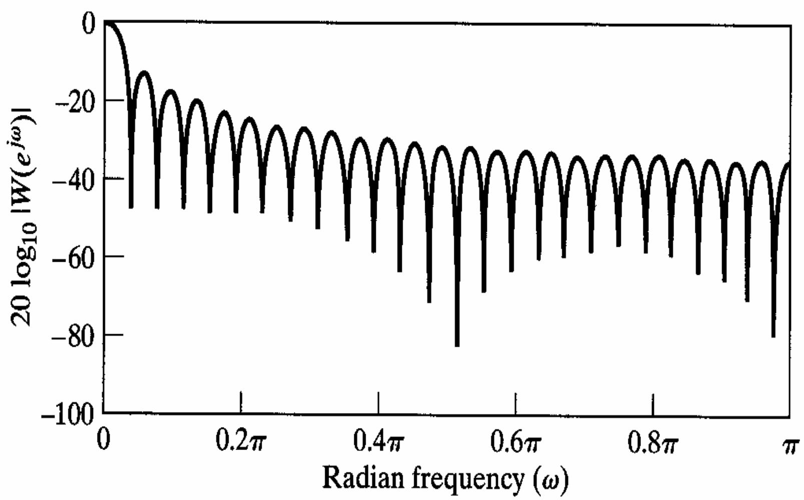
  - Blackman

$$w_{BL}[n]=\begin{cases} 0.42-0.5\cos\left(\dfrac{2\pi n}{M}\right)+0.08\cos\dfrac{4\pi n}{M}, & 0\le n\le M \\ 0 \end{cases}$$

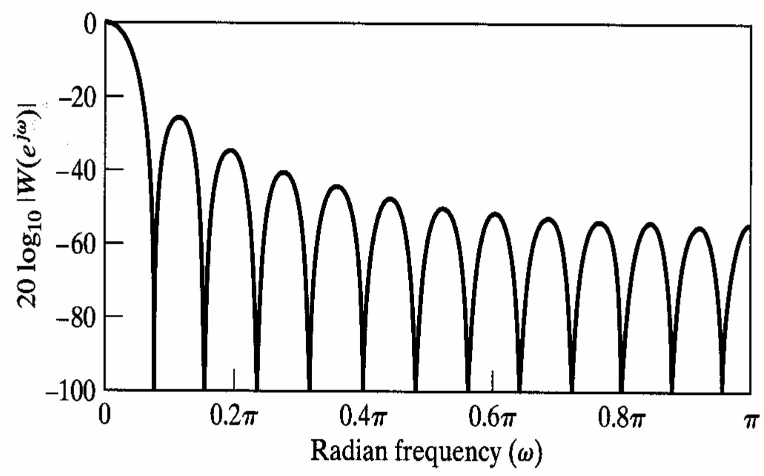


Trade off between the main lobe width and the peak of the side lobe

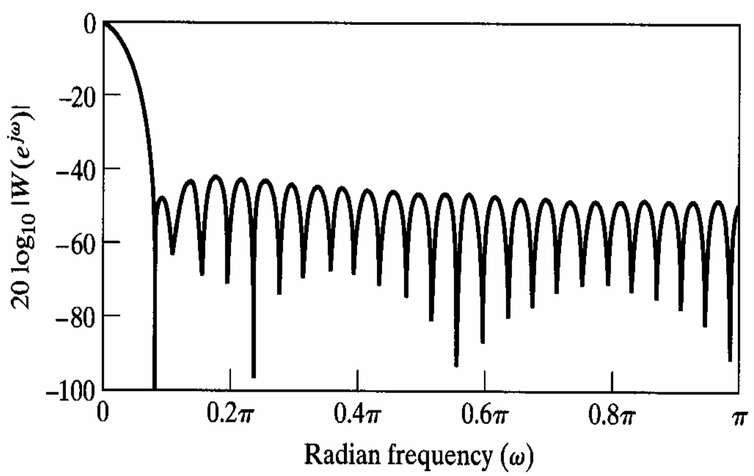
	Peak side lobe	Main-lobe width	Peak approx. error
Rectangular	-13	$4\pi/(M+1)$	-21
Bartlett	-25	$8\pi/M$	-25
Hanning	-31	$8\pi/M$	-44
Hamming	-41	$8\pi/M$	-53
Blackman	-57	$12\pi/M$	-74



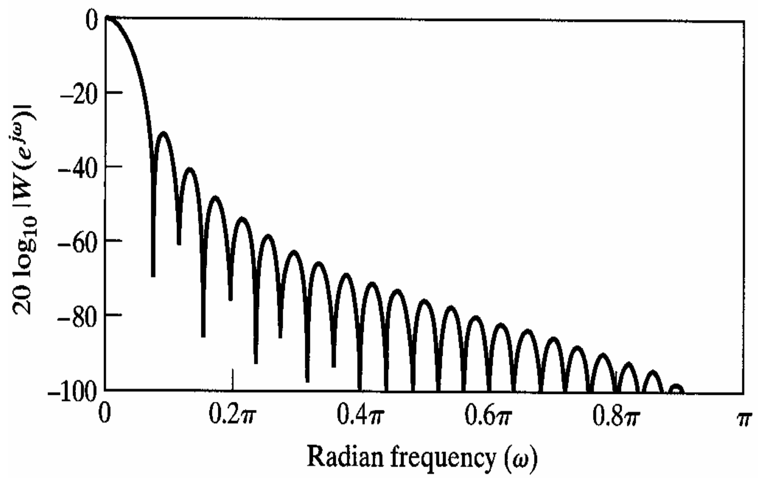
Rectangular window



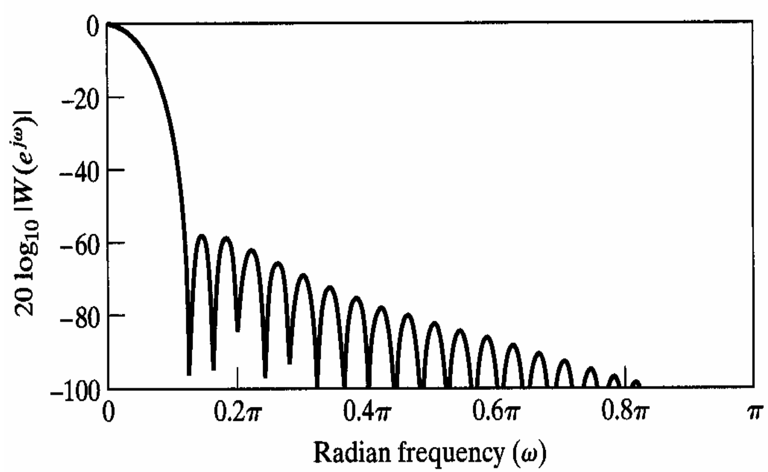
Bartlett window



Hamming window



Hanning window



Blackman window

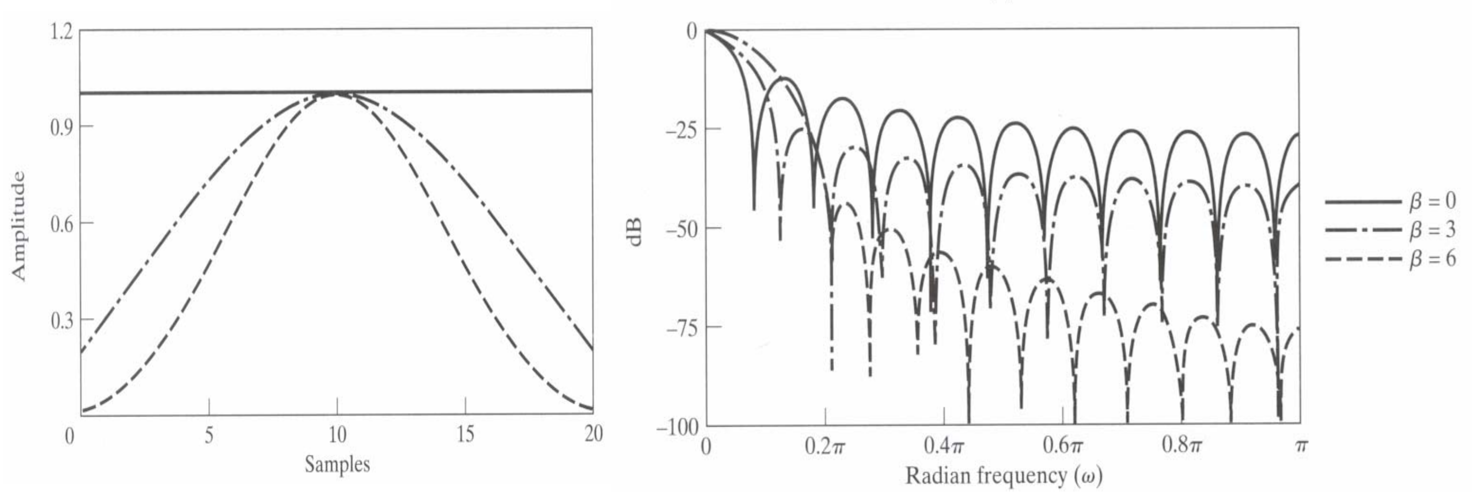
– Kaiser window

$$w[n]=\begin{cases}\dfrac{I_0\left[\beta\sqrt{1-\left[\dfrac{n-\alpha}{\alpha}\right]^2}\right]}{I_0(\beta)}, & 0\le n\le M,\ \alpha=\dfrac{M}{2}\\ 0 & ,\text{otherwise}\end{cases}$$

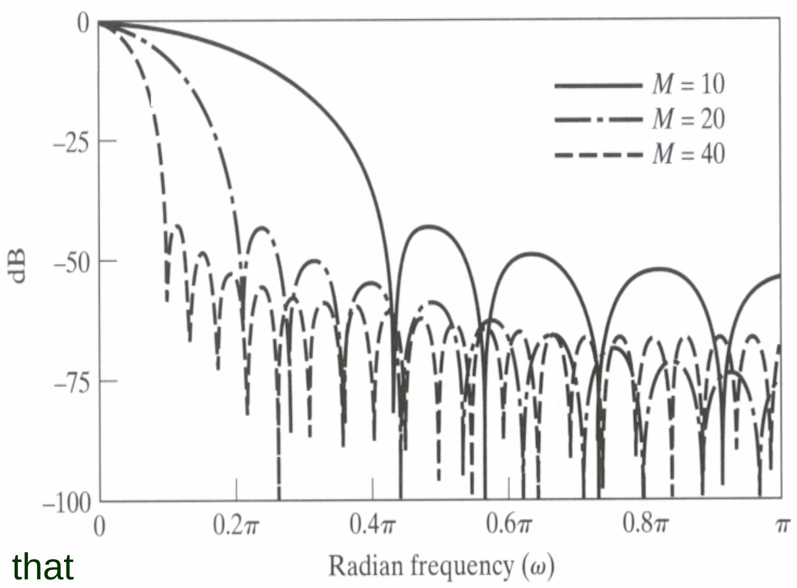
where $I_0(\cdot)$ is the zero[th]-order modified Bessel function of the first kind defined by

$$I_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{jx\sin\theta}d\theta\approx\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right)\ (\text{for large } x)$$

- As $\beta = 0 \rightarrow$ rectangular window case
- If the window is tapered more (*i.e.,* β is increased), the main lobe width becomes wider and the peak of the side lobe becomes lower



- Increasing *M* while holding β constant results in the main lobe width decreased but does not affect the peak of the side lobe.



- Given δ is fixed, ω_p is defined to be the highest frequency such that

$$\left|H\left(e^{j\omega}\right)\right| \geq 1-\delta$$

and ω_s to be the lowest frequency such that

$$\left|H\left(e^{j\omega}\right)\right| \leq \delta$$

- Then β can be empirically determined by

$$\beta = \begin{cases} 0.1102(A-8.7), & A>50 \\ 0.5842(A-21)^{0.4}+0.07886(A-21) & 21\leq A\leq 50 \\ 0 &, \quad A<21 \end{cases}$$

$$M = \frac{A-8}{2.285\Delta\omega};$$

where $\Delta\omega = \omega_s - \omega_p, \ A = -20\log\delta$

– Comparison of commonly used windows

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Design example

$$\omega_p = 0.4\pi; \quad \omega_s = 0.6\pi \Rightarrow \omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi$$

$$\Delta\omega = \omega_s - \omega_p = 0.2\pi$$

$$\delta_1 = 0.01; \quad \delta_2 = 0.001 \Rightarrow \delta = 0.001$$

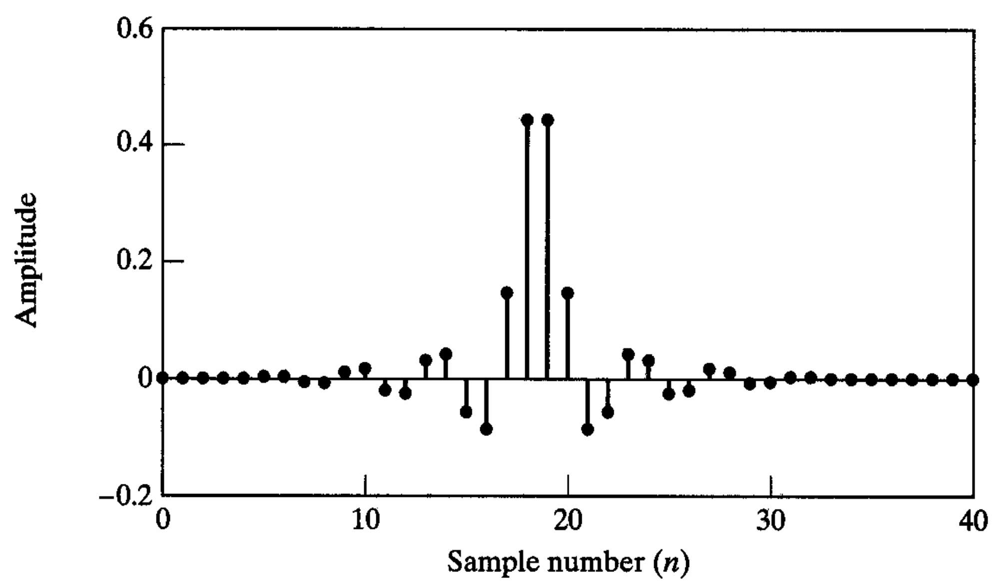
$$\Rightarrow A = -20\log\delta = 60dB$$

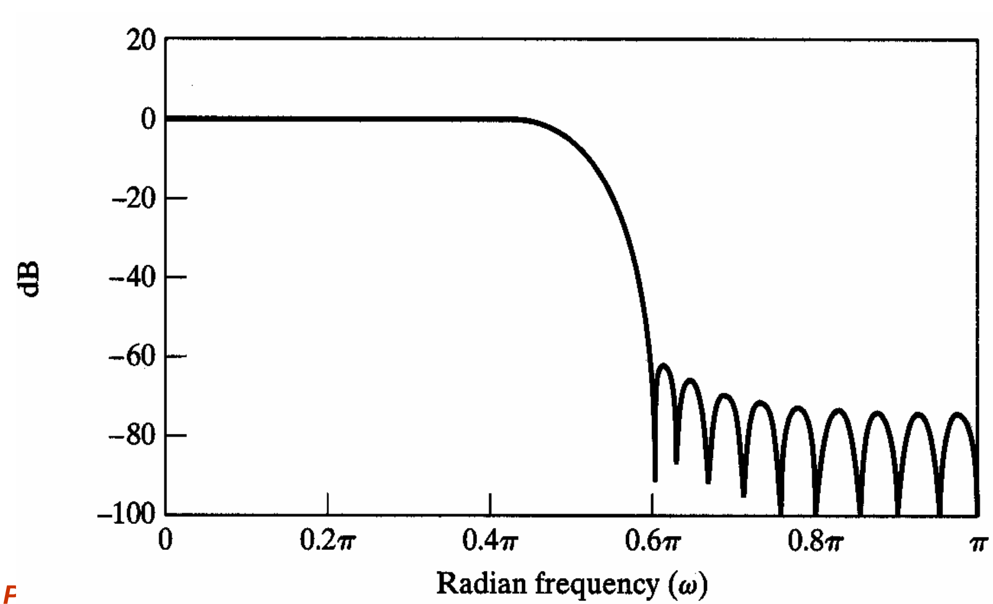
$$\Rightarrow \beta = 0.1102(A - 8.7) = 5.65326$$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{60-8}{2.285 \times 0.2\pi} = 36.219 \Rightarrow M = 37; \text{ Type-II FIR}$$

$$\Rightarrow h[n] = h_d[n] \cdot \omega[n]$$

$$= \begin{cases} \dfrac{\sin 0.5\pi(n-18.5)}{\pi(n-18.5)} \cdot \dfrac{I_0\left[5.65326\left(1-\left(\dfrac{n-18.5}{18.5}\right)^2\right)^{\frac{1}{2}}\right]}{I_0(5.65326)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$


Amplitude
0.6
0.4
0.2
0
−0.2
0
10
20
30
40
Sample number (n)


dB
20
0
−20
−40
−60
−80
−100
0
0.2π
0.4π
0.6π
0.8π
π
Radian frequency (ω)

- Linear system with generalized linear phase
  - For casual system, zero phase is not achievable
    - Phase distortion exists
    - Linear phase results in a simple time shift
  - System with linear phase
    - Ideal delay

$$H_{id}\left(e^{j\omega}\right)=e^{-j\omega\alpha},\quad |\omega|<\pi$$

$$\Rightarrow \left|H_{id}\left(e^{j\omega}\right)\right|=1$$

$$\mathrm{Arg}\left[H_{id}\left(e^{j\omega}\right)\right]=-\alpha\omega$$

$$\mathrm{grd}\left[H_{id}\left(e^{j\omega}\right)\right]=-\frac{\partial}{\partial\omega}\alpha\omega=\alpha$$

From the inverse Fourier transform, $h_{id}[n]=\dfrac{\sin\pi(n-\alpha)}{\pi(n-\alpha)},\quad -\infty<n<\infty$

For an input $x[n]$,

$$y[n]=x[n]*\frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}$$

$$=\sum_{k=-\infty}^{\infty}x[k]\frac{\sin\pi(n-k-\alpha)}{\pi(n-k-\alpha)}$$

In particular, if $\alpha = n_d$, $h_{id}[n] = \delta[n - n_d]$

$$\Rightarrow y[n] = x[n] * h_{id}[n] = x[n - n_d]$$

- Ideal LPF with linear phase

$$H_{\ell p}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \Rightarrow h_{\ell p}[n] = \frac{\sin \omega_c (n - \alpha)}{\pi (n - \alpha)}$$

- If $\alpha = n_d$, the impulse response is symmetric about $n = \alpha$, *i.e.*,

$$h_{\ell p}[2\alpha - n] = h_{\ell p}[n]$$

– Generalized linear phase

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j(\alpha\omega-\beta)}$$

$$= A(e^{j\omega})\cos(\beta-\omega\alpha) + jA(e^{j\omega})\sin(\beta-\omega\alpha) \Rightarrow \quad \tau(\omega) = \text{grd}\left[H(e^{j\omega})\right]$$

$$= -\frac{d}{d\omega}\left\{\text{Arg}\left[H(e^{j\omega})\right]\right\}$$

$$= \alpha$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h[n]\cos\omega n - j\sum_{n=-\infty}^{\infty} h[n]\sin\omega n$$

$$\tan(\beta-\omega\alpha) = \frac{\sin(\beta-\omega\alpha)}{\cos(\beta-\omega\alpha)} = -\frac{\sum h[n]\sin\omega n}{\sum h[n]\cos\omega n} \quad \Rightarrow \sum_n h[n]\sin\left[\omega(n-\alpha)+\beta\right] = 0, \quad \forall\omega$$

This is satisfied when $\beta = 0 \text{ or } \pi,\ 2\alpha$ is an integer, and $h[2\alpha-n] = h[n]$

Alternatively, if $\beta = \pi/2$ or $3\pi/2,\ \sum h[n]\cos\left[\omega(n-\alpha)\right] = 0$

$\Rightarrow 2\alpha = M =$ an integer and $h[2\alpha-n] = -h[n]$

- This is a necessary condition for $h[n]$ to have a constant group delay
- There are other systems having linear phase characteristics without this symmetry condition

- Causal generalized linear phase systems
  - If causal, $\sum_{n=0}^{\infty} h[n]\sin\left[\omega(n-\alpha)+\beta\right]=0,\ \forall\omega$
  - Causal FIR systems of length *(M+1)* have generalized linear phase, if

$$h[n]=h[2\alpha-n] \quad \text{or} \quad h[n]=-h[2\alpha-n]$$

  - If

$$h[n]=\begin{cases} h[M-n], & 0\le n\le M \\ 0 & , \text{otherwise} \end{cases}$$

then $H(e^{j\omega})=A_e(e^{j\omega})e^{-\frac{j\omega M}{2}}$

where $A_e(e^{j\omega})$ is a real, even periodic function of ω

  - If

$$h[n]=\begin{cases} -h[M-n], & 0\le n\le M \\ 0 & , \text{otherwise} \end{cases}$$

then $H(e^{j\omega})=jA_o(e^{j\omega})e^{-\frac{j\omega M}{2}}=A_o(e^{j\omega})e^{-\frac{j\omega M+\pi}{2}}$

where $A_o(e^{j\omega})$ is a real, odd periodic function of ω

- Type I FIR linear phase systems

$$h[n] = h[M-n] \quad 0 \le n \le M$$

where M is an ever integer

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-j\omega n} + h\left[\frac{M}{2}\right]e^{-j\frac{\omega M}{2}} + \sum_{n=\frac{M}{2}+1}^{M} h[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]e^{-j\omega n} + h\left[\frac{M}{2}\right]e^{-j\frac{\omega M}{2}} + \sum_{n=0}^{\frac{M}{2}-1} h[M-n]e^{-j\omega(M-n)}$$

$$= \sum_{n=0}^{\frac{M}{2}-1} h[n]\left(e^{-j\omega n} + e^{-j\omega(M-n)}\right) + h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}}$$

$$= 2\sum_{n=0}^{\frac{M}{2}-1} h[n]\cos\left[n-\frac{M}{2}\right]\omega + h\left[\frac{M}{2}\right]e^{-j\omega\frac{M}{2}} = \sum_{k=0}^{\frac{M}{2}} a[k]\cos\omega k \cdot e^{-j\frac{\omega M}{2}}$$

where

$$a[0] = h\left[\frac{M}{2}\right], \quad a[k] = 2h\left[\frac{M}{2}-k\right], \quad k = 1,2,....\frac{M}{2}.$$

$$\Rightarrow H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\frac{\omega M}{2}}$$

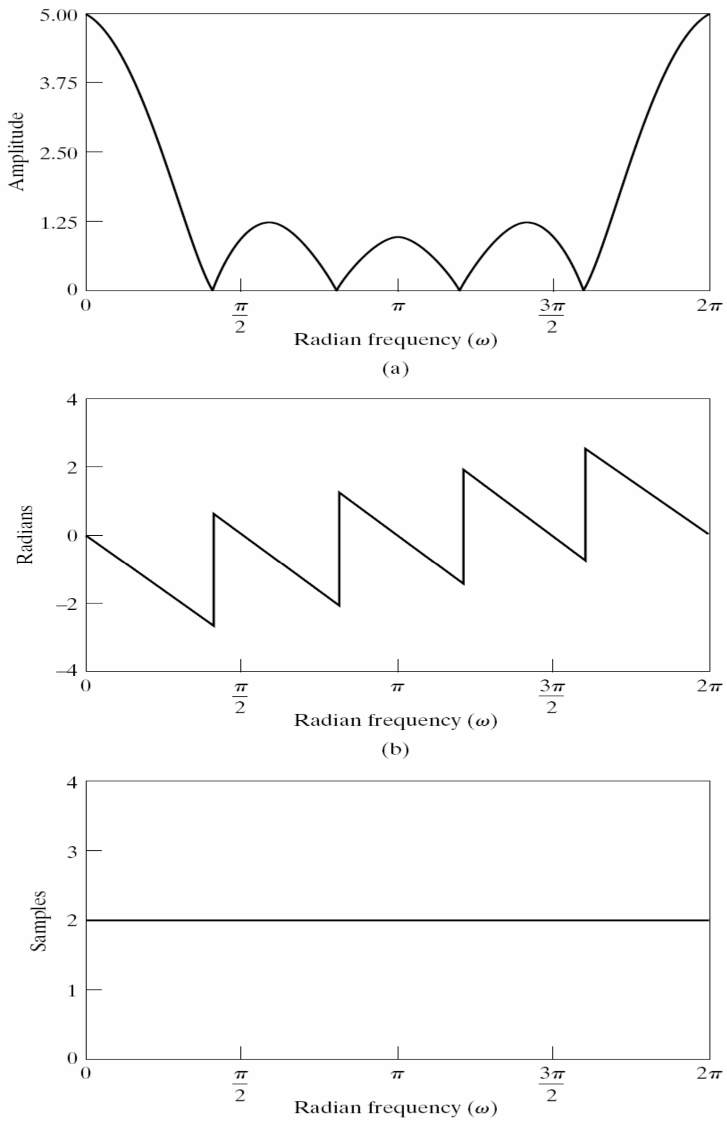
where

$$A_e(e^{j\omega}) = \sum_{k=0}^{\frac{M}{2}} a[k]\cos\omega k$$

Example

$$h[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & \text{otherwise}\end{cases}$$

$$\Rightarrow H\left(e^{j\omega}\right)=\sum_{n=0}^{4}e^{-j\omega n}=\frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}e^{-j2\omega}$$



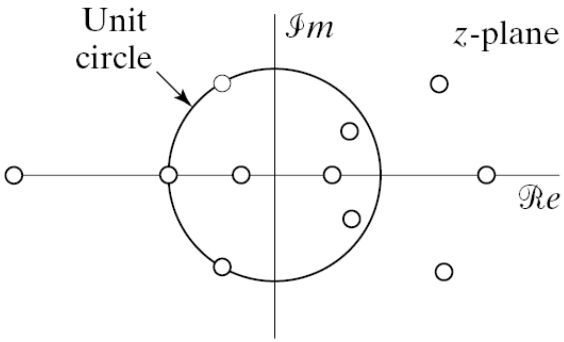
- Type II linear phase systems

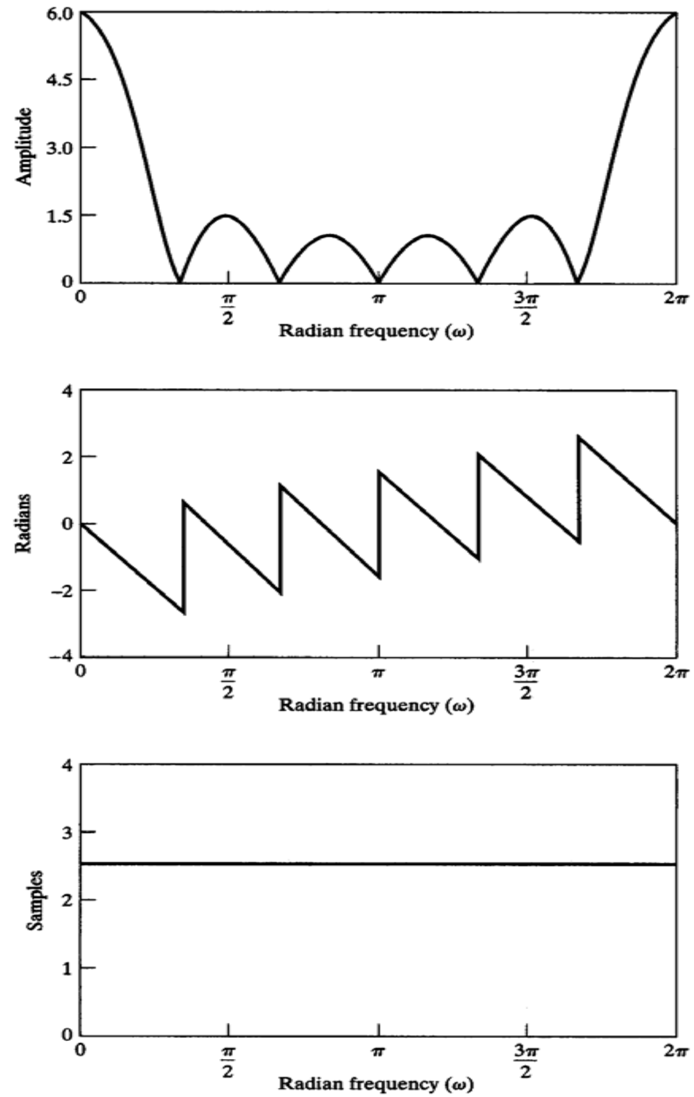
$$h[n] = h[M-n], \quad 0 \le n \le M$$

where M is an odd integer

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

$$= \sum_{k=1}^{\frac{M+1}{2}} 2h\left[\frac{M+1}{2} - k\right] \cos\left[\omega\left(k - \frac{1}{2}\right)\right] \cdot e^{-j\frac{\omega M}{2}}$$





Example

$$h[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

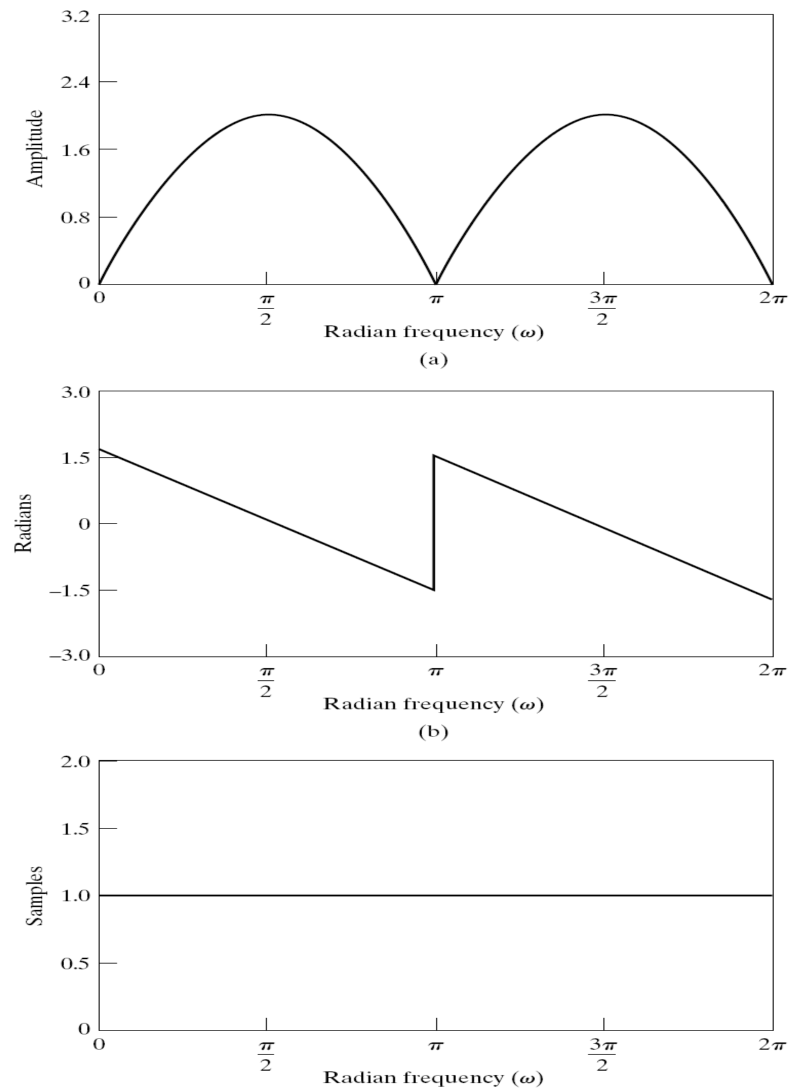
$$\Rightarrow H(e^{j\omega}) = \frac{\sin 3\omega}{\sin \frac{\omega}{2}} e^{-j\frac{5}{2}\omega}$$

- Type III FIR linear- phase systems

$$h[n] = -h[M-n], \quad 0 \le n \le M$$

where M is an even integer

$$\Rightarrow H(e^{j\omega}) = j2\sum_{k=1}^{\frac{M}{2}} h\left[\frac{M}{2}-k\right]\cdot \sin\omega k \cdot e^{-j\frac{\omega M}{2}}$$



Example

$$h[n] = \delta[n] - \delta[n-2]$$
$$H(e^{j\omega}) = 1 - e^{-j2\omega} = j2\sin\omega \cdot e^{-j\omega}$$

- Type IV FIR linear- phase systems

$$h[n] = -h[M-n], \quad 0 \le n \le M$$

where is an odd integer

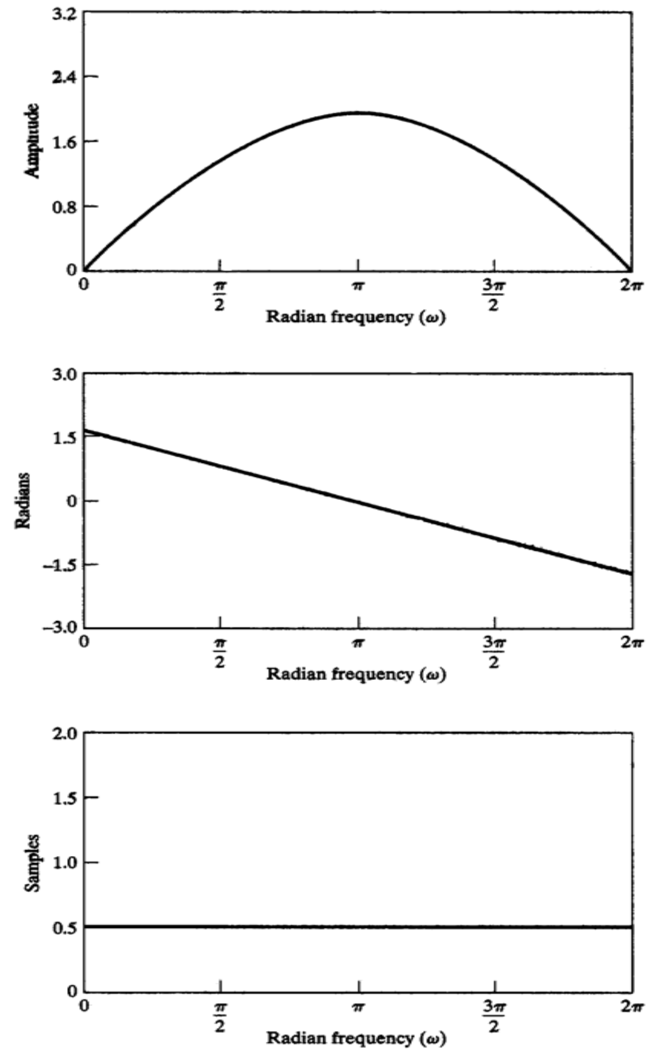
$$\Rightarrow H\left(e^{j\omega}\right) = j2\sum_{k=1}^{\frac{M+1}{2}} h\left[\frac{M+1}{2} - k\right]\cdot \sin\left[\omega\left(k-\frac{1}{2}\right)\right]\cdot e^{-j\frac{\omega M}{2}}$$

Example

$$h[n] = \delta[n] - \delta[n-1]$$

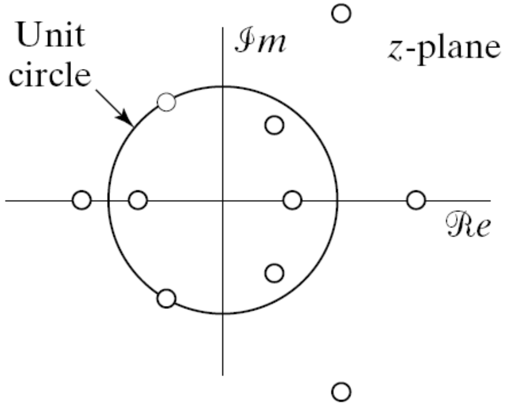
$$\Rightarrow H\left(e^{j\omega}\right) = 1 + \left(e^{j\omega}\right) = 1 - e^{-j\omega}$$

$$= j2\sin\frac{\omega}{2}\cdot e^{-j\frac{\omega}{2}}$$



– Location of zeros for FIR linear-phase systems

$$H(z)=\sum_{n=0}^{M} h[n]z^{-n}$$



- If $\{h[n]\}$ are symmetric,

$$H(z)=\sum_{n=0}^{M} h[M-n]z^{-n}=\sum_{k=M}^{0} h[k]z^{k-M}=z^{-M}H(z^{-1})$$

- If $z_0=re^{j\theta}$ is a zero of $H(z)$, then

$$H(z_0)=z_0^{-M}H(z_0^{-1})=0$$

  That is, $z_0^{-1}=r^{-1}e^{-j\theta}$ is also a zero of $H(z)$.

- When $h[n]$ is real and z_0 is a zero of $H(z)$, $z_0^{*}=re^{-j\theta}$ will also be a zero of $H(z)$, and so will $(z_0^{*})^{-1}$.

  Therefore, when $h[n]$ is real, there will be four conjugate reciprocal zeros of the form

$$(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})(1-r^{-1}e^{j\theta}z^{-1})(1-r^{-1}z^{-j\theta}z^{-1})$$

  unless $r=1$.

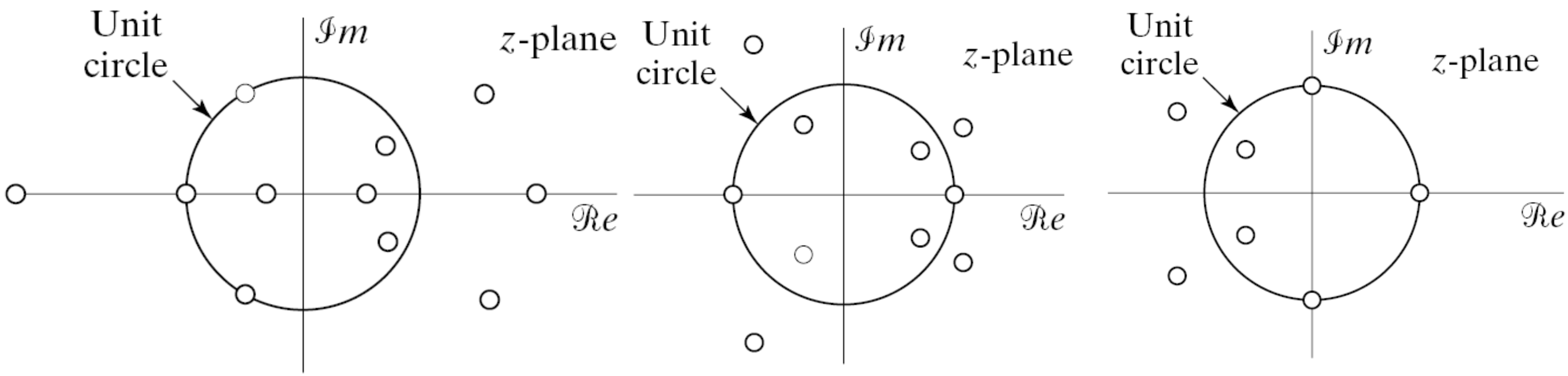
- If $\{h[n]\}$ are anti-symmetric, we have

$$H(z) = -\sum_{n=0}^{M} h[M-n]z^{-n} = -\sum_{k=M}^{o} h[k]z^{k-M} = -z^{-M}H(z^{-1})$$

In particular, when $z=1,\ H(1) = -H(1)$ and if $z=-1,\ H(-1) = (-1)^{M+1}H(-1).$

Thus, $H(z)$ must have a zero at $z=1.$

It also must have a zero at $z=-1 \Leftrightarrow \omega = \pi$ when M is even.



type - II   type - III   type - IV

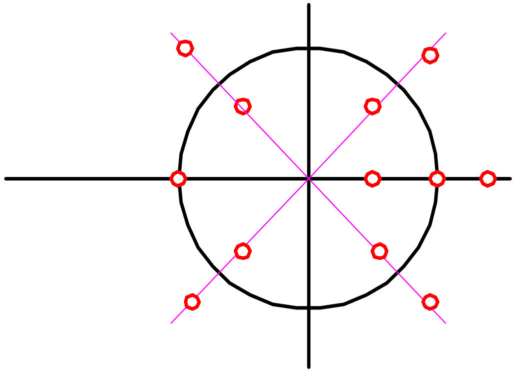
- Relation of FIR linear-phase systems to minimum-phase systems
  - Any FIR linear-phase system can be represented by

$$H(z) = H_{\min}(z) H_{\max}(z) H_{uc}(z)$$

where $H_{\max}(z) = H_{\min}(z^{-1}) z^{-M_i}$, $M_i =$ no. of zeros of $H_{\min}(z)$

and $H_{uc}(z)$ has only M_0 zeros on the unit circle.

$\Rightarrow$ The order of M is $2M_i + M_0$

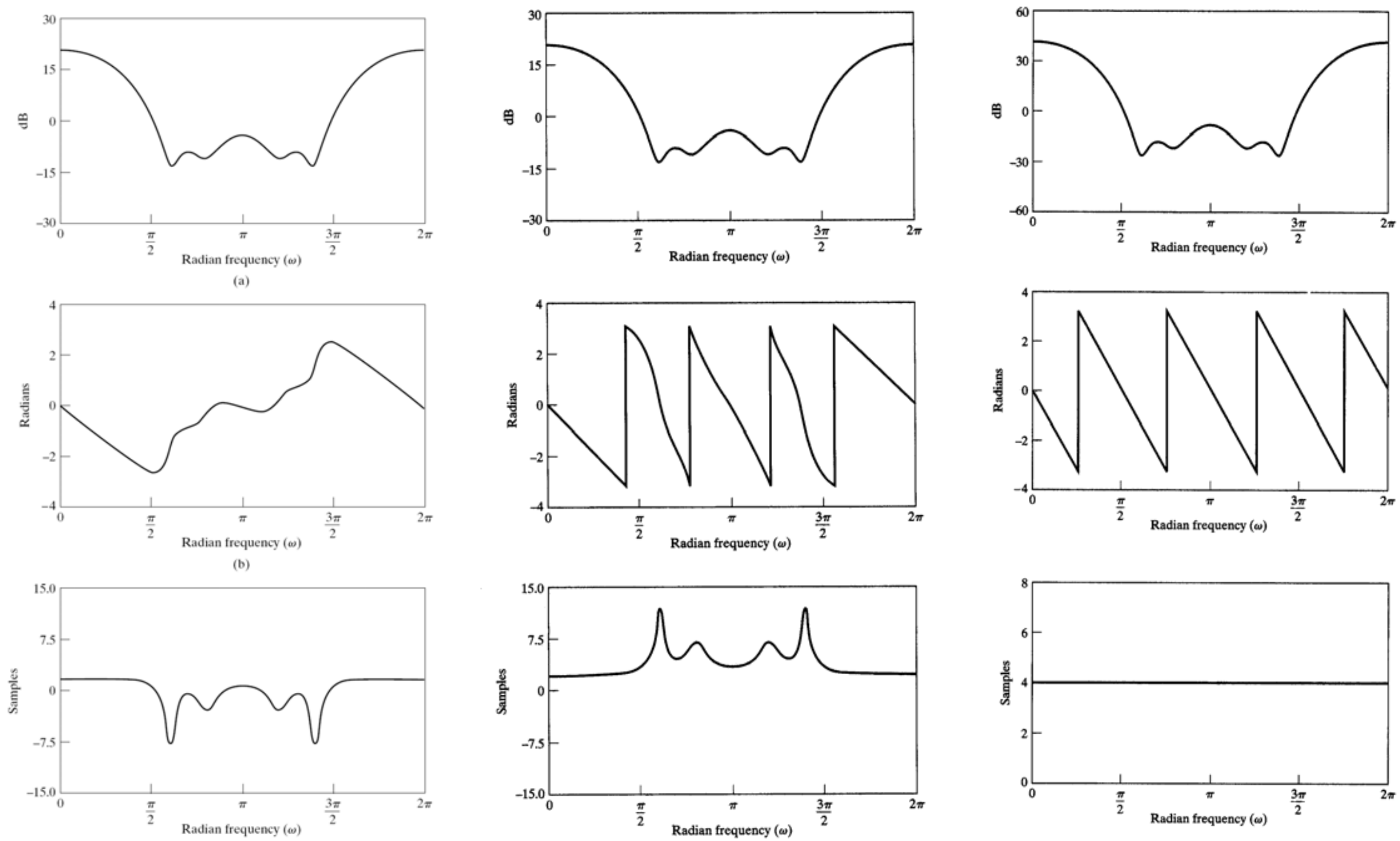


Example: Decomposition of a linear-phase system

$$H_{\min}(z) = 1.25^2 \left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)\left(1-0.8e^{j0.8\pi}z^{-1}\right)\left(1-0.8e^{-j0.8\pi}z^{-1}\right)$$

$$\Rightarrow H_{\max}(z) = (0.9)^2 \left(1-1.1111e^{j0.6\pi}z^{-1}\right)\left(1-1.1111e^{-j0.6\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{-j0.8\pi}z^{-1}\right)$$

- The overall system $H(z) = H_{\min}(z)H_{\max}(z)$ has linear phase



– Causal and linear phase FIR filter

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \end{cases} \Rightarrow W\left(e^{j\omega}\right) = W_e\left(e^{j\omega}\right) e^{-j\omega \frac{M}{2}}$$

even function of ω

- The impulse response of a desired filter is also symmetric; $h_d[n] = h_d[M-n]$

$$H_d\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right) e^{-j\frac{\omega M}{2}}$$

- The overall response

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{1}{2\pi}\int H_d\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta \\ &= \frac{1}{2\pi}\int H_e\left(e^{j\theta}\right) e^{-j\frac{\theta M}{2}} W_e\left(e^{j(\omega-\theta)}\right) e^{-j\frac{(\omega-\theta)M}{2}} d\theta \\ &= \frac{1}{2\pi}\int H_e\left(e^{j\theta}\right) W_e\left(e^{j(\omega-\theta)}\right) e^{-j\frac{\omega M}{2}} d\theta \\ &= A_e\left(e^{j\omega}\right) e^{-j\frac{\omega M}{2}} \end{aligned}$$

where $$A_e\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e\left(e^{j\theta}\right) W_e\left(e^{j(\omega-\theta)}\right) d\theta$$

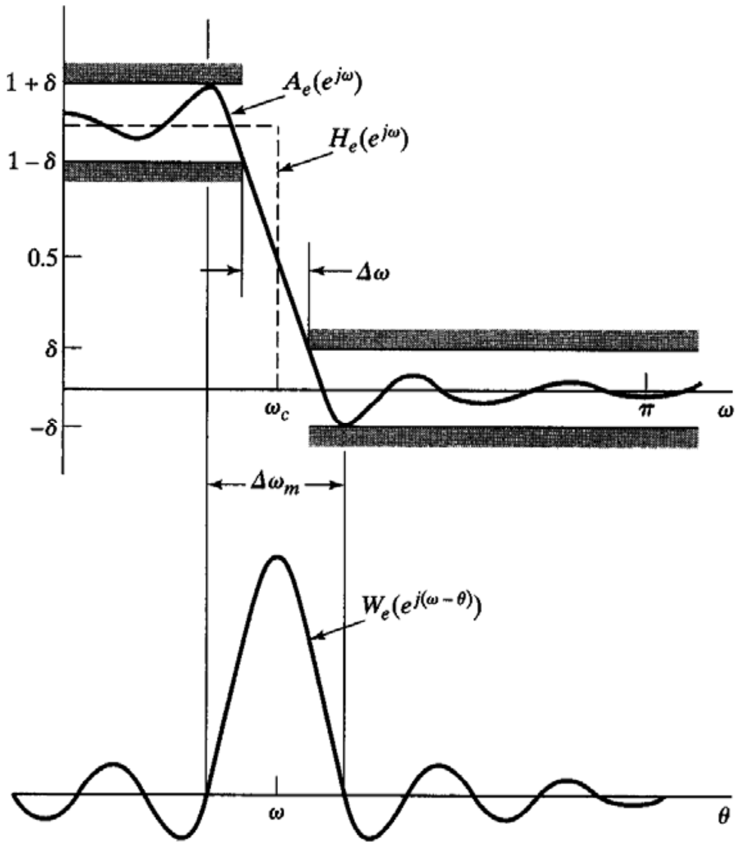
- The resulting response has generalized linear phase and its magnitude is also an even real function.

Example Linear-phase lowpass filter

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\frac{\omega M}{2}}, & |\omega|<\omega_c \\ 0, & \omega_c<|\omega|\le\pi \end{cases}$$

$$\Rightarrow h_d[n]=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\frac{\omega M}{2}} e^{j\omega n}\,d\omega$$

$$=\frac{\sin\left[\omega_c\left(n-\frac{M}{2}\right)\right]}{\pi\left(n-\frac{M}{2}\right)}$$



$$h[n]=h_d[n]w[n]$$

– Filter design by Kaiser method

- High-pass filter

$$H_{hp}\left(e^{j\omega}\right)=\begin{cases}0 & 0\le|\omega|\le\omega_c\\ e^{-j\frac{\omega M}{2}} & \omega_c\le|\omega|\le\pi\end{cases}$$

$$H_{hp}\left(e^{j\omega}\right)=e^{-j\frac{\omega M}{2}}-H_{ep}\left(e^{j\omega}\right)$$

$$\Rightarrow h_{hp}[n]=\frac{\sin\pi\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}-\frac{\sin\omega_c\left(n-\frac{M}{2}\right)}{\pi\left(n-\frac{M}{2}\right)}$$

- Design example:

If $\omega_s=0.35\pi,\quad \omega_p=0.5\pi$ and $\delta_1=\delta_2=\delta=0.021$

$$\Rightarrow A=33.56$$

$$\Rightarrow \beta=0.5842(A-21)^{0.4}+0.07886(A-21)$$

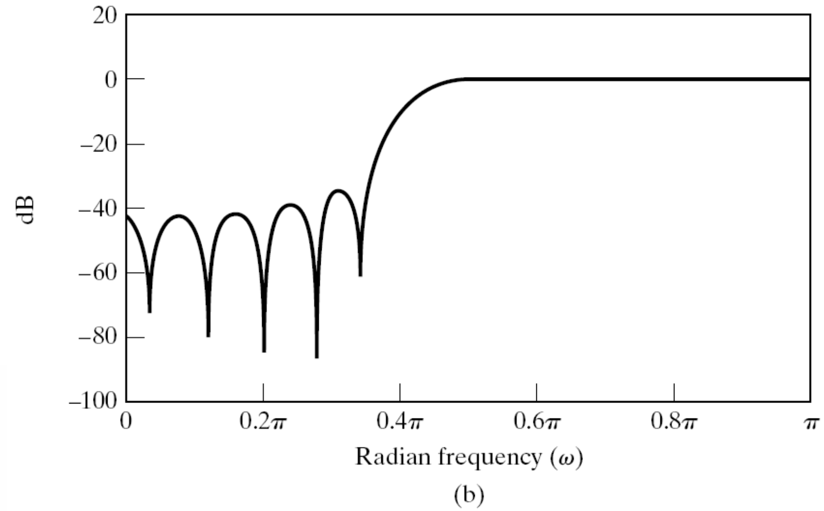
$$=2.5974$$

$$M=\frac{33.56-8}{2.285\times0.15\pi}=23.73\Rightarrow M=24$$

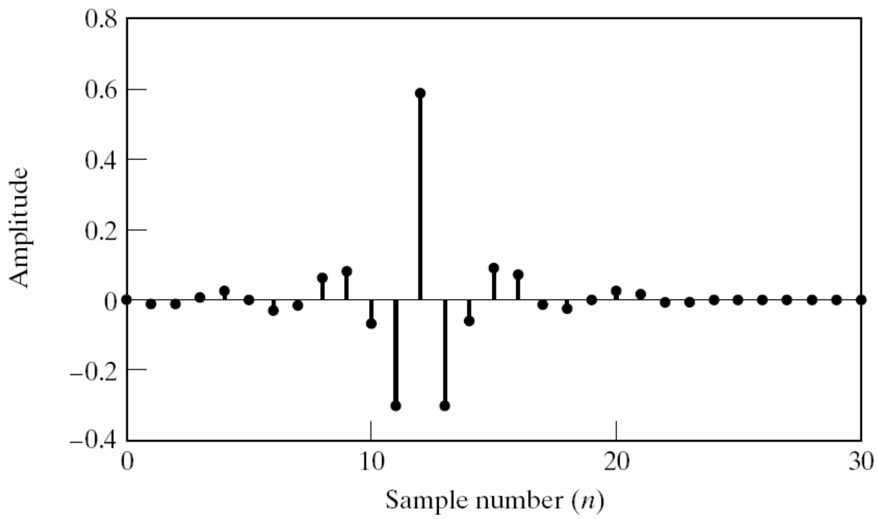
- High-pass filter example

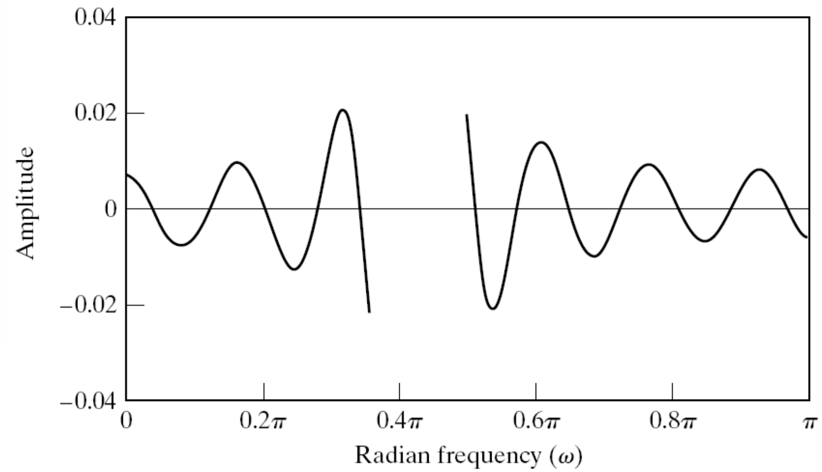
$$A = 33.56$$
$$\beta = 2.5974$$
$$M = 24 \quad \Rightarrow \text{type I FIR}$$

Actual error=0.0213>0.021



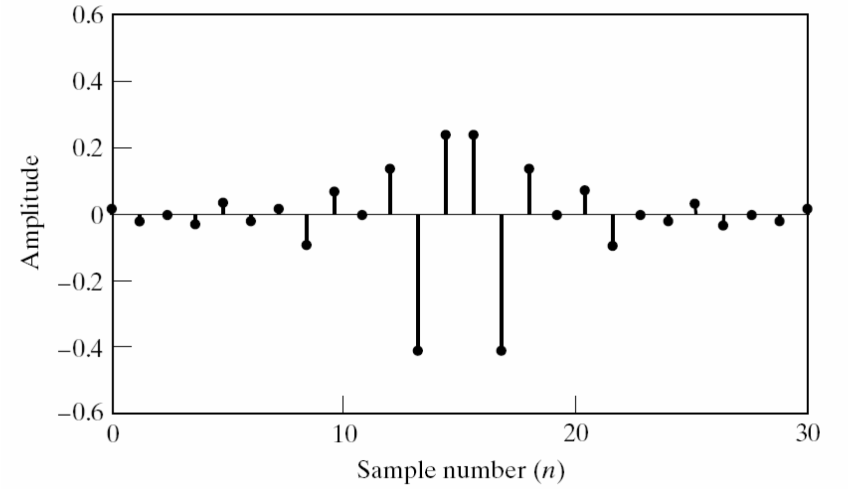
(b)

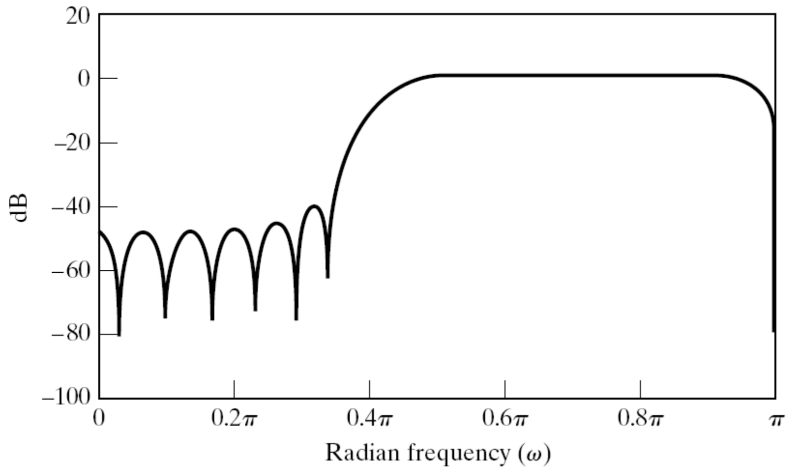




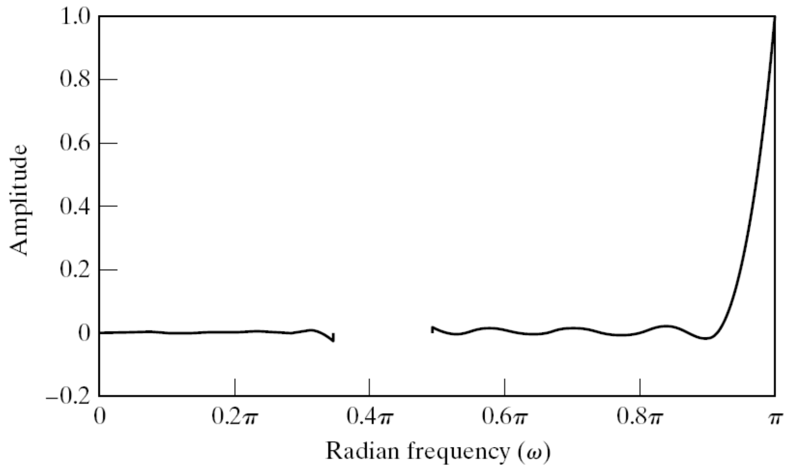
- High-pass Kaiser high-pass filter example with M=25 (type-II)
  - Use of a type-II Kaiser HPF

    may not be appropriate

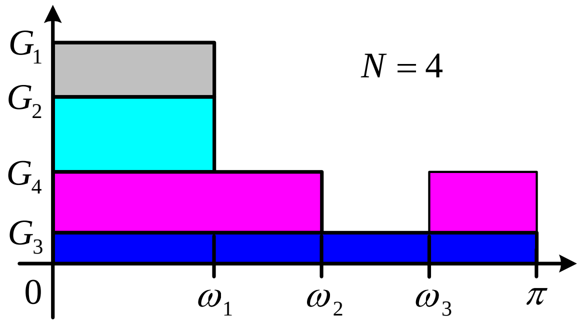




(b)



– FIR filter with generalized response



$$\Rightarrow h_d[n] = \sum_{k=1}^{N} (G_k - G_{k+1}) \frac{\sin \omega_k \left(n - \frac{M}{2}\right)}{\pi \left(n - \frac{M}{2}\right)}$$

where $G_{N+1} = 0$

$$\Rightarrow h[n] = h_d[n]\omega[n]$$

– Differentiator

$$H_d\left(e^{j\omega}\right) = j\omega e^{-j\frac{\omega M}{2}} \quad -\pi < \omega < \pi$$

$$\Rightarrow h_d[n] = \frac{\cos\left[\pi\left(n - \frac{M}{2}\right)\right]}{n - \frac{M}{2}} - \frac{\sin\left[\pi\left(n - \frac{M}{2}\right)\right]}{\pi\left(n - \frac{M}{2}\right)^2}$$

In the case of using a symmetric window of size $M+1$, $h[n] = h_d[n] \cdot \omega[n]$ results in $h[M-n] = -h[n]$

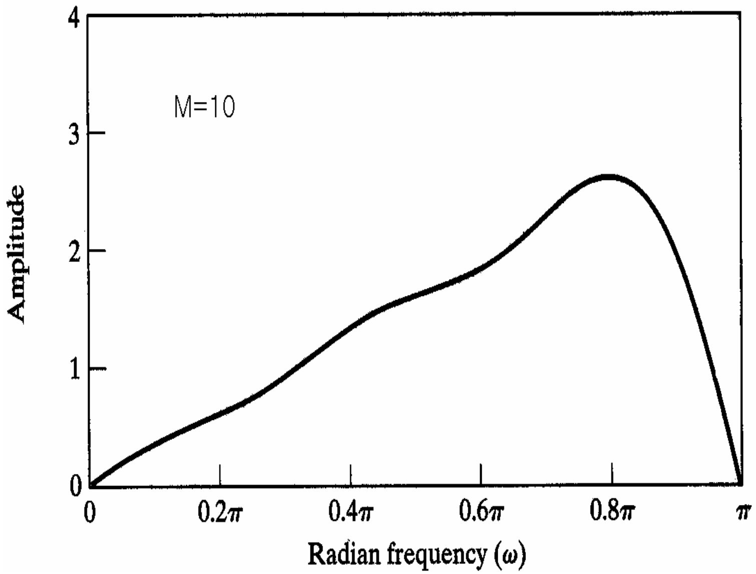
Kaiser formulae were developed for frequency response with simple magnitude discontinuities, but still works on differentiator

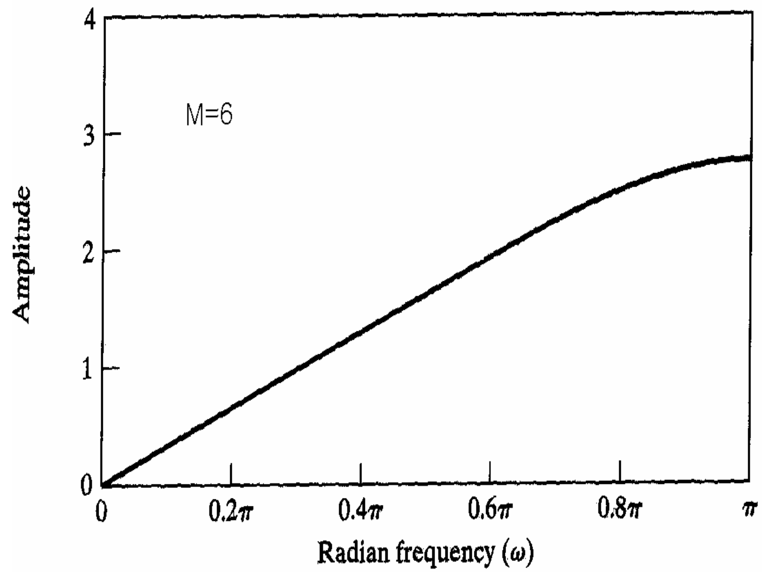
- Kaiser window design of a differentiator

The amplitude error: $E(\omega)=\omega-A_o(e^{j\omega});\ \ 0\le\omega\le\pi$

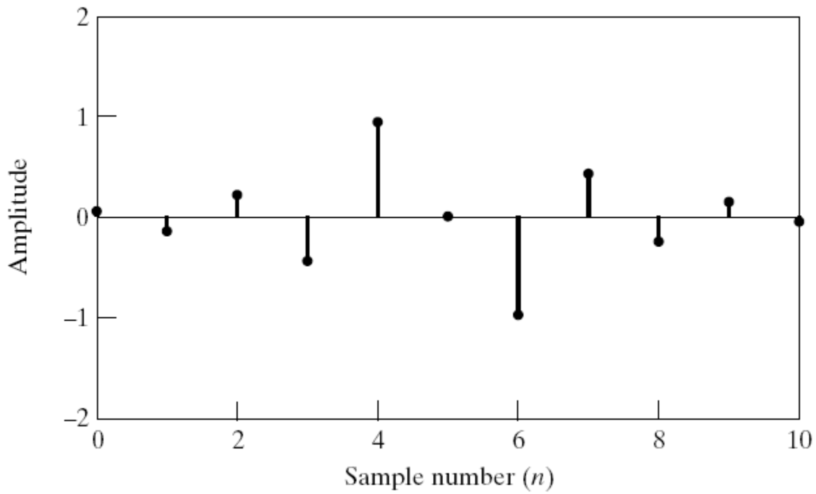
where $A_o(e^{j\omega})$ is the amplitude of the filter $H(z)$ and has zeros at $z=\pm1$

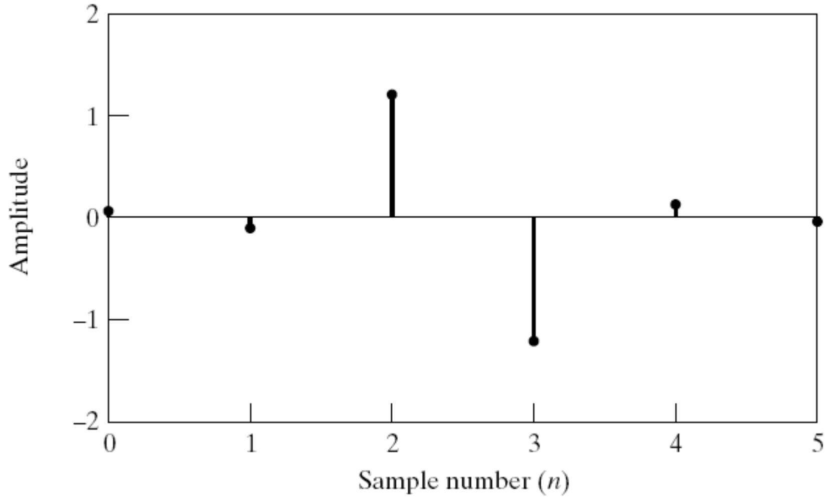
By not imposing constraint *H(z)* to have a zero at *z=-1*, it is possible to design a filter having better approximation to the desired response, while using a less number of filter taps.





- Example: Impulse response of Kaiser-windowed differentiatore





- Optimum approximation of FIR filters
  - Design a filter that best approximates the desired response for a given *M*
  - Rectangular window provides the best-mean squared approximation:

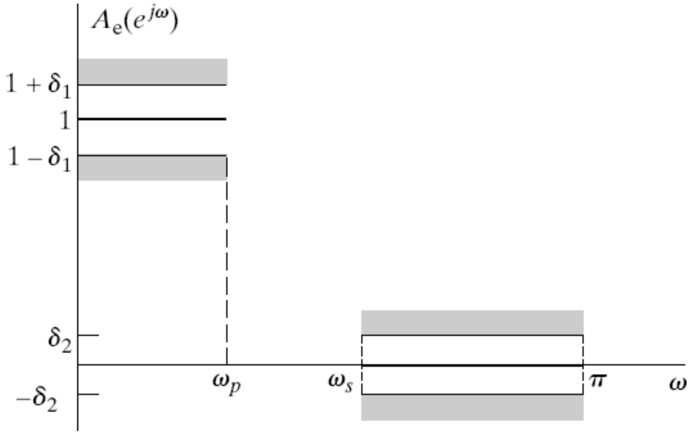
$$h[n]=\begin{cases} h_d[n], & 0\le n\le M \\ 0\ \ , & \text{otherwise}\end{cases}$$

minimizes

$$\varepsilon^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right)-H\left(e^{j\omega}\right)\right|^2 d\omega$$

$$=\sum_{n=-\infty}^{\infty}\left|e^2[n]\right|$$

$$e[n]\triangleq\begin{cases} h_d[n]-h[n]; & 0\le n\le M \\ h_d[n]; & \text{otherwisw}\end{cases}$$



- Do not permit individual control errors in different bands

- Consider the design of a type-I FIR filter with zero phase, i.e., $h_e[n] = h_e[-n]$
  - Corresponding frequency response

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n} = h_e[0] + \sum_{n=0}^{L} 2h_e[n]\cos(\omega n)$$

    is a real, even and periodic function of ω

    - A causal system can be obtained by delaying it by *L* samples

$$h[n] \triangleq h_e[n-L] = h_e[2L-n]$$

  - From a polynomial approximation

$$\cos\omega n = T_n(\cos\omega); \quad T_n(x) = \cos\left(n\cos^{-1}x\right)$$

    $A(e^{j\omega})$ can be represented as an *L*-th degree trigonometric polynomial

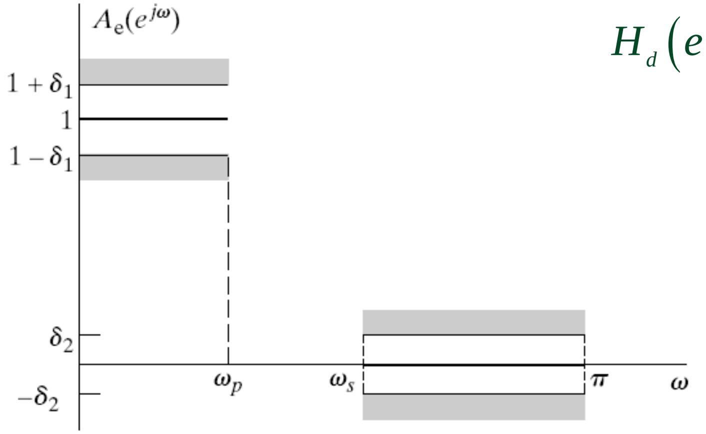
$$A(e^{j\omega}) = \sum_{k=0}^{L} a_k \cos^k \omega = P(x)\big|_{x=\cos\omega}; \quad P(x) = \sum_{k=0}^{L} a_k x^k$$

  - Define an error function by

$$E(\omega) = W(\omega)\left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2$$

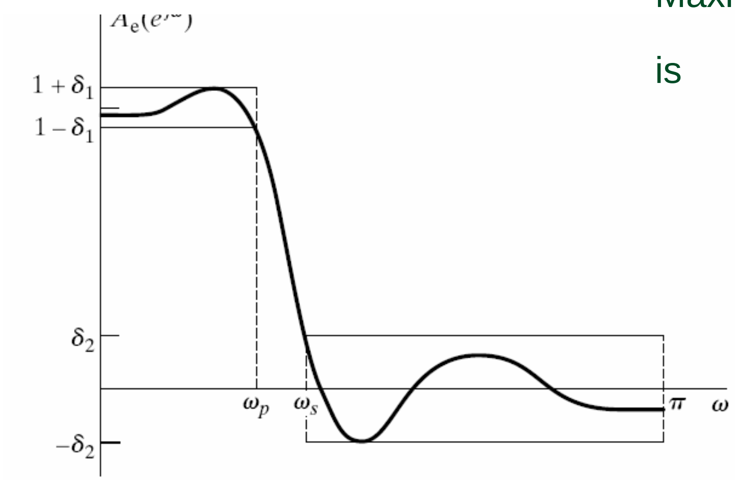
    where $W(\omega)$ is a weighting function

– Example



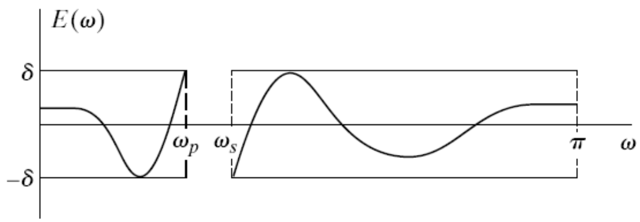
$$H_d\left(e^{j\omega}\right)=\begin{cases}1, & 0\le\omega\le\omega_p\\ 0, & \omega_s\le\omega\le\pi\end{cases}$$

$$W(\omega)=\begin{cases}\dfrac{1}{K}; & 0\le\omega\le\omega_p\\ 1, & \omega_s\le\omega\le\pi\end{cases};\quad K=\frac{\delta_1}{\delta_2}$$



Maximum weighted absolute approximation error

is $\delta=\delta_2$



– Minimax or Chevyshev criterion

$$\min_{\{h_e[n];0\le n\le L\}}\left(\max_{\omega\in F}\left|E(\omega)\right|\right)$$

$$F=\{\omega:0\le\omega\le\omega_p,\ \omega_s\le\omega\le\pi\}$$

- Find a set of impulse response that minimizes δ

– Alternative theorem:

- Let denote a closed subset comprising the disjoint union of closed subsets of real axis *x*. Then

$$P(x)=\sum_{k=0}^{m}a_k x^k$$

  is an *m*-th order polynomial

- For desired function $D_P(x)$ and positive function $W_P(x)$ which are continuous on F_P , and weighted error defined by

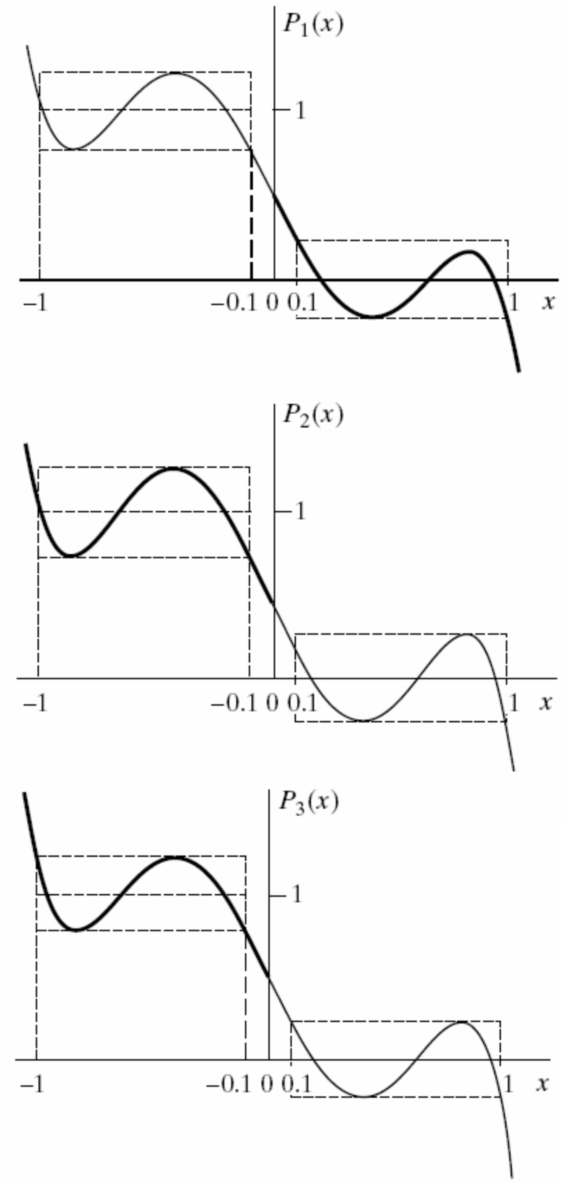
$$E_P(x)=W_P(x)\left[D_P(x)-P(x)\right]$$

  a necessary and sufficient condition that be a unique r-th order polynomial minimizing $\|E\|=\max_{x\in F_P}\left|E_P(x)\right|$ is that $E_P(x)$ has at least $r+2$ alternations At least $r+2$ values x_i in F_P such that $x_1<x_2<\cdots<x_{r+2}$ and

$$E_P(x_i)=-E_P(x_{i+1})=\pm\|E\|,\quad i=1,2,\cdots,r+2$$

– Example: Consider 5-th order polynomials $P_j(x)$ that approximate unity for $-1 \le x \le -0.1$ and zero for $0.1 \le x \le 1$

- Assume that $W_P(x) = 1$ for these two regions
- The optimum 5-th order polynomial has at least 7 alternations of the error in the region in F_P
- $P_1(x)$ has 3 alternations in $-1 \le x \le -0.1$ and 2 alternations in $0.1 \le x \le 1$
- $P_2(x)$ also has 5 alternations; 3 in $-1 \le x \le -0.1$ and 2 in $0.1 \le x \le 1$
- $P_3(x)$ has 8 alternations in $-1 \le x \le -0.1$ and $0.1 \le x \le 1$ $\Rightarrow$ optimum polynomial



- Optimum type-I lowpass filters
  - Consider polynomial $P(x)$ defined by

$$P(\cos\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k$$

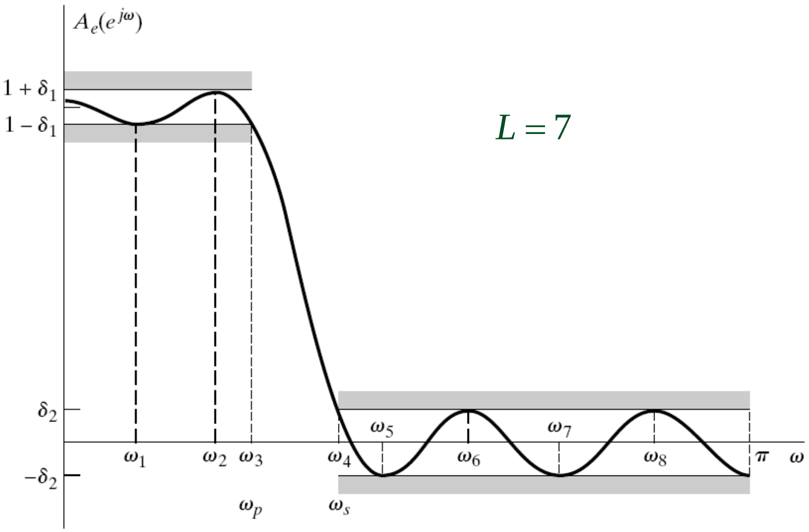
  - Assume that

$$D_P(\cos\omega) = \begin{cases} 1; & \cos\omega_p \le \cos\omega \le 1 \\ 0; & -1 \le \cos\omega \le \cos\omega_s \end{cases}$$

$$W_P(\cos\omega) = \begin{cases} 1/K; & \cos\omega_p \le \cos\omega \le 1 \\ 0; & -1 \le \cos\omega \le \cos\omega_s \end{cases}$$

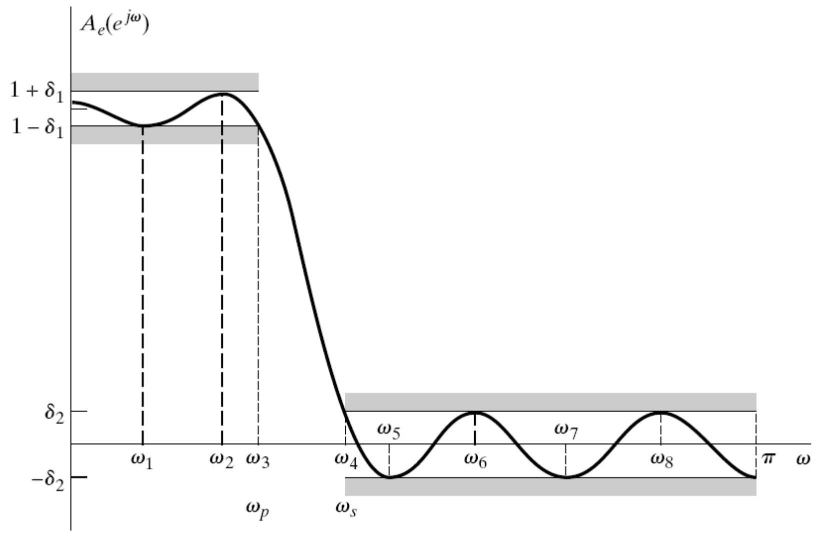
$$E_P(\cos\omega) = W_P(\cos\omega)\left[D_P(\cos\omega) - P(\cos\omega)\right]$$

$$F_P = \left\{\omega : 0 \le \omega \le \omega_p,\ \omega_s \le \omega \le \pi\right\}$$

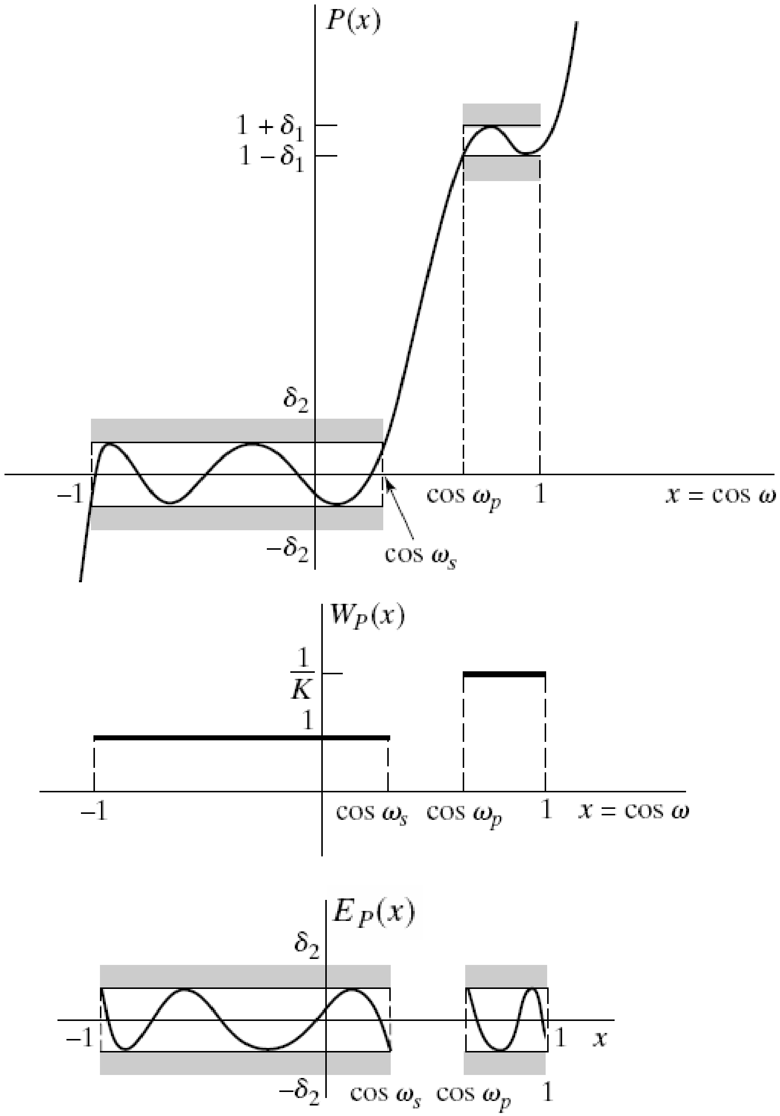


  - Equi-ripple approximation
    - A set of coefficients $\{a_k\}$ is designed to make $E_P(\cos\omega)$ have at least $(L+2)$ alternations on F_P

- Optimum type-I lowpass filters (cont.)
  - Redraw $A\left(e^{j\omega}\right)$ in terms of $x = \cos\omega$



$L = 7$

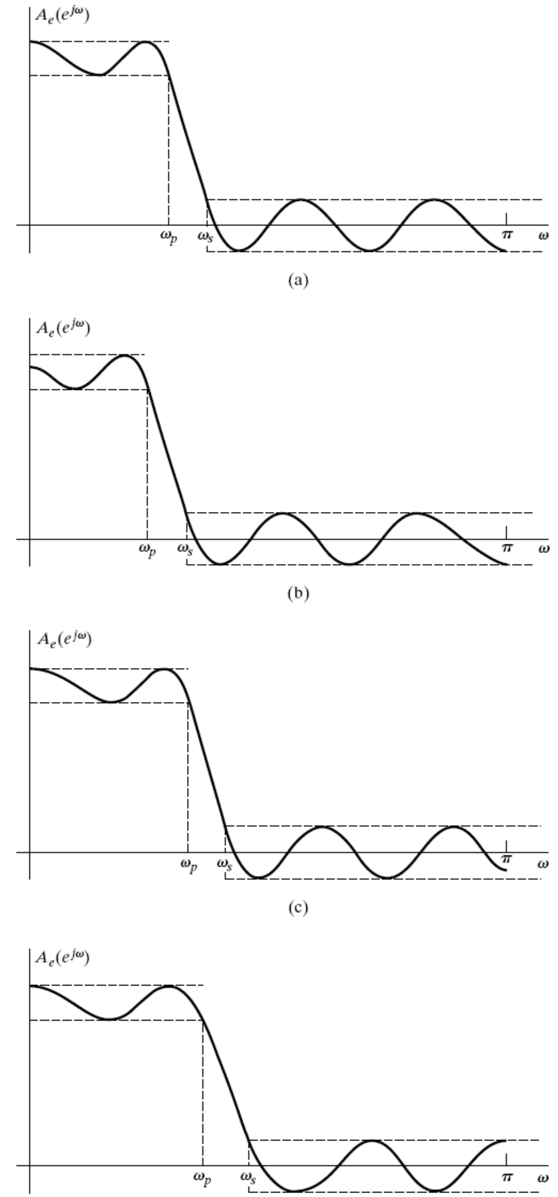


- Optimum type-I lowpass filters (cont.)
  - Max. possible number of alternations of the error is $(L+3)$ since an L-th degree polynomial can have at most $(L-1)$ points with zero slope

$$\frac{\partial}{\partial \omega} P(\cos\omega) = -\sin\omega \sum_{k=0}^{L} k a_k (\cos\omega)^{k-1}$$

$$= -\sin\omega \sum_{k=0}^{L-1} (k+1) a_{k+1} (\cos\omega)^{k}$$

    - Has zeros at $\omega = 0,\ \omega = \pi$ and roots of $(L-1)$-th order polynomial
  - Alternations always occur at $\omega = \omega_p$ and $\omega = \omega_s$
  - The filter will be equi-ripple except possibly at $\omega = 0$ or $\omega = \pi$



- Parks-McClellan algorithm
  - The alternation theorem indicates that the optimum filter $A_e(e^{j\omega})$ will satisfies

$$W(\omega_i)\left[H_d(e^{j\omega_i}) - A_e(e^{j\omega_i})\right] = (-1)^{i+1}\delta; \quad i = 1, 2, \cdots, L+2$$

$$\Rightarrow H_d(e^{j\omega_i}) = A_e(e^{j\omega_i}) + (-1)^{i+1}\delta / W(\omega_i)$$

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n} = h_e[0] + \sum_{n=0}^{L} 2h_e[n]\cos(\omega n) = \sum_{k=0}^{L} a_k \cos^k \omega = \sum_{k=0}^{L} a_k x^k$$

  - $H_d(e^{j\omega_i})$ can be rewritten as

$$\begin{bmatrix} H_d(e^{j\omega_1}) \\ H_d(e^{j\omega_2}) \\ \vdots \\ H_d(e^{j\omega_{L+2}}) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix}$$

  - Solution for a_k and δ can be found by means of polynomial interpolation

- Parks-McClellan algorithm (cont.)

  – For a given set of extremal frequencies, the P-M algorithm determines

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d\left(e^{j\omega_k}\right)}{\sum_{k=1}^{L+2} \frac{b_k (-1)^{k+1}}{W(\omega_k)}}; \qquad b_k = \prod_{\substack{i=1\\ i\neq k}}^{L+2} \frac{1}{x_k - x_i}$$

  – For given a_k and δ, the error function has magnitude $\pm\delta$ at $L+2$ frequencies ω_i

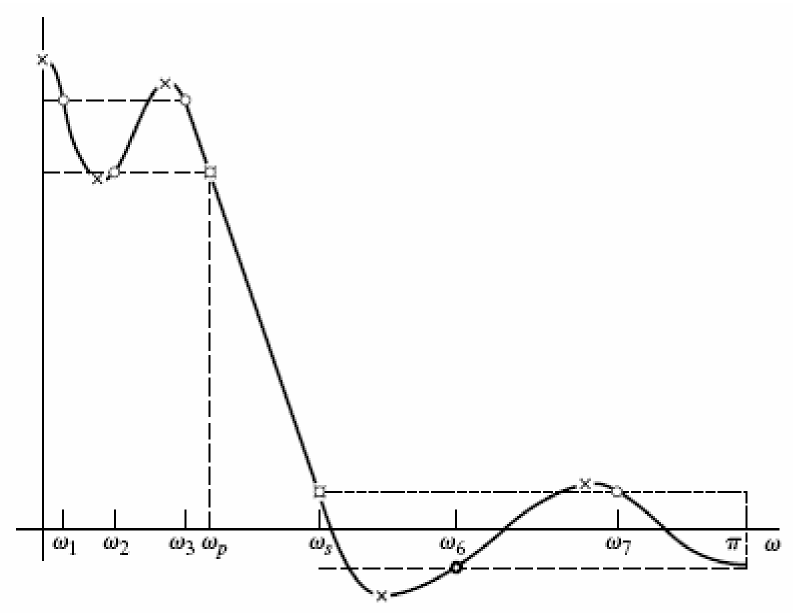
  – $A_e\left(e^{j\omega}\right)$ has values $1 \pm K\delta$ for $0 \le \omega \le \omega_p$ and $\pm\delta$ for $\omega_s \le \omega \le \pi$

  – An *L*-th order trigonometric polynimial $A_e\left(e^{j\omega}\right)$ can be obtained by interpolating $E(\omega_i)$

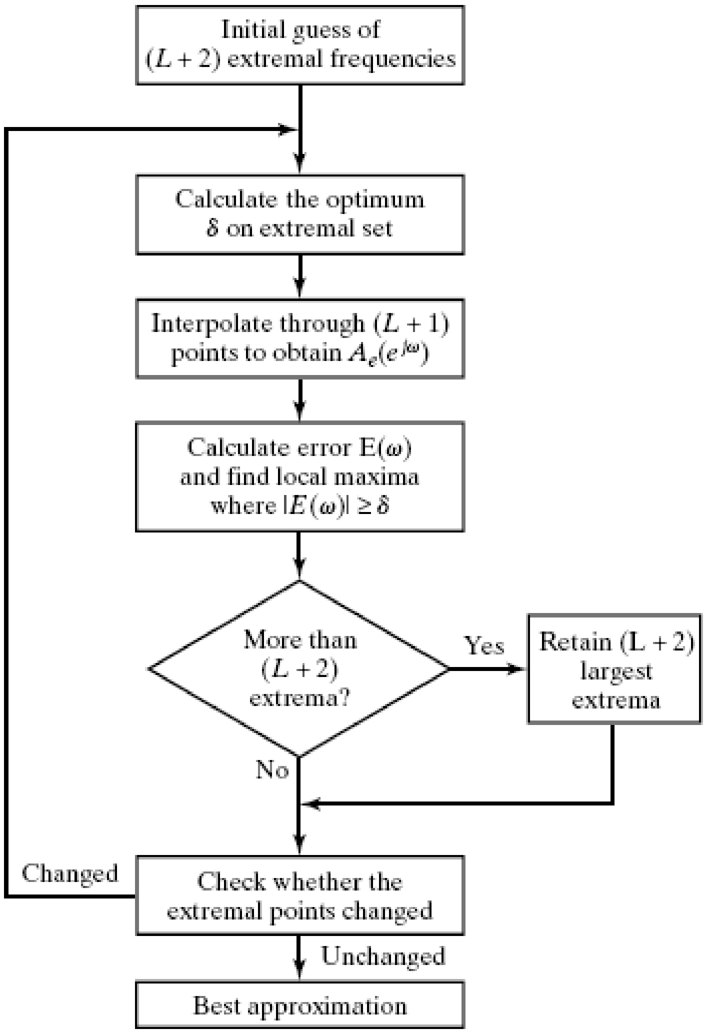
$$A_e\left(e^{j\omega}\right) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1} \frac{d_k C_k}{x - x_k}}{\sum_{k=1}^{L+1} \frac{d_k}{x - x_k}}$$

$$x = \cos\omega, \quad C_k = H_d\left(e^{j\omega_k}\right) - \frac{(-1)^{k+1}\delta}{W(\omega_k)}, \quad d_k = \prod_{\substack{i=1\\ i\neq k}}^{L+1} \frac{1}{x_k - x_i} = b_k\left(x_k - x_{L+2}\right)$$

- Parks-McClellan algorithm (cont.)

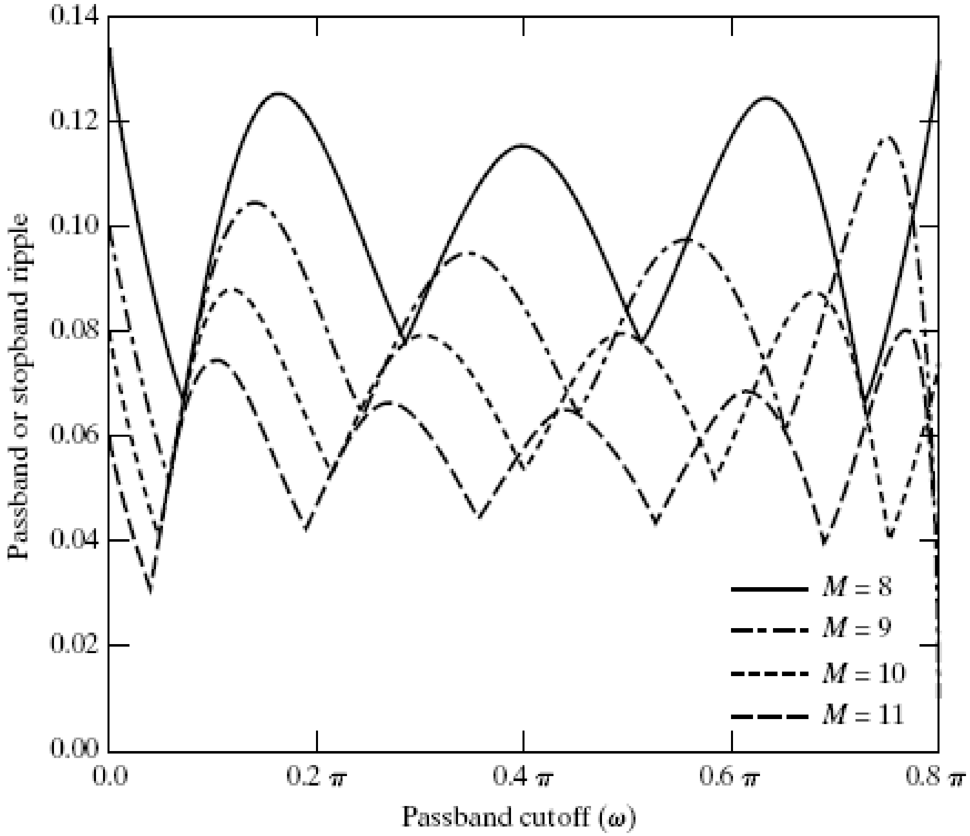


Example: $L = 7$



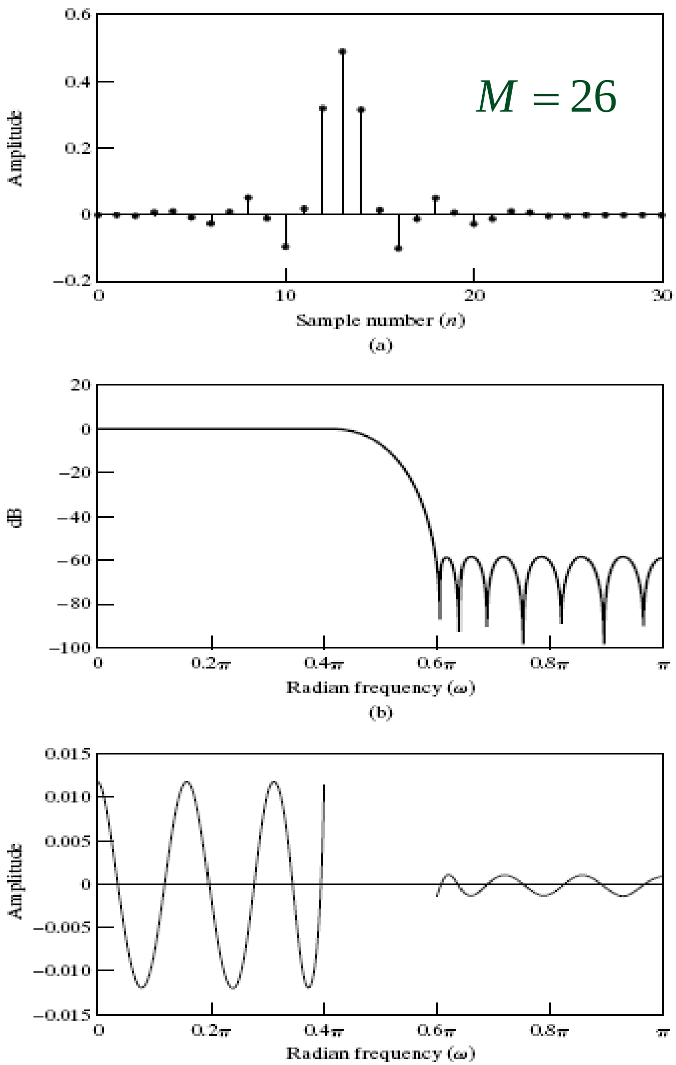
- Characteristics of optimum FIR filters
  - Size of the filter tap

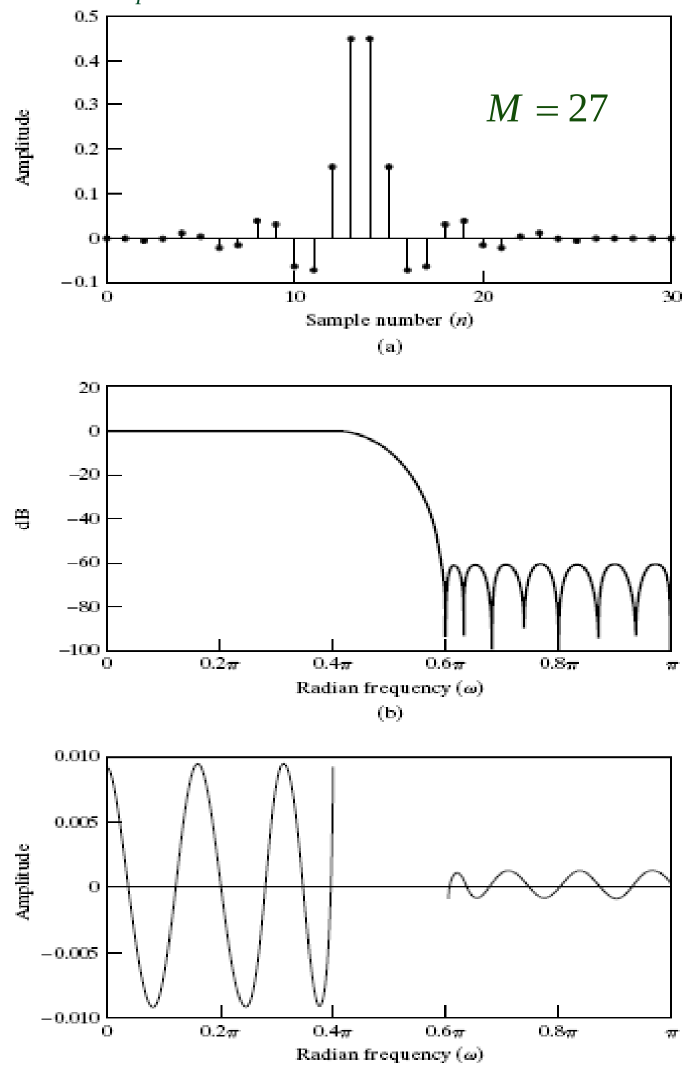
$$M = \frac{-10\log(\delta_1\delta_2) - 13}{2.324(\omega_s - \omega_p)}$$



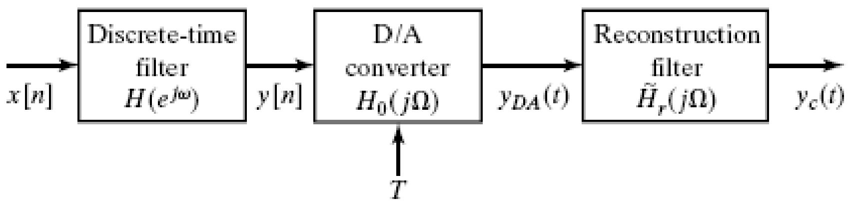
- Filter design examples by Parks-McClellan algorithm

$$\delta_1 = 10^2;\ \delta_2 = 10^{-3}$$
$$\omega_p = 0.4\pi;\ \omega_s = 0.6\pi$$





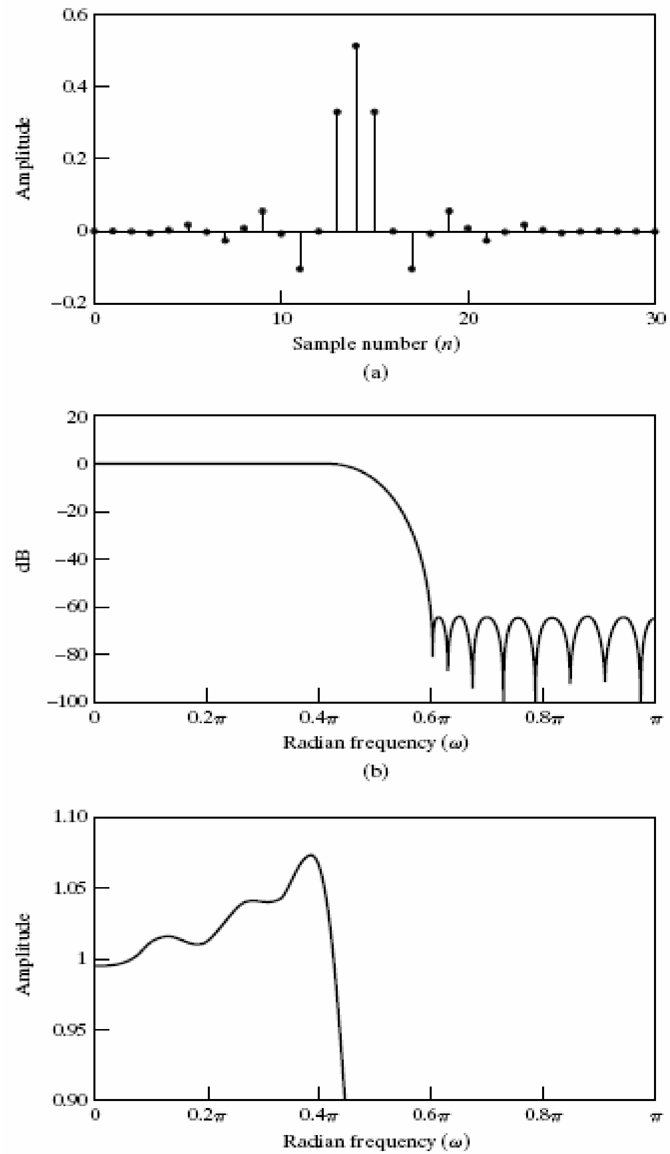
- Filter design example (cont.)
  - Compensation of zero-order holder



$$H_d\left(e^{j\omega}\right)=\begin{cases}\dfrac{\omega/2}{\sin\left(\omega/2\right)}; & 0\le\omega\le\omega_p\\ 0; & \omega_s\le\omega\le\pi\end{cases}$$

$$\delta_1=10^{2};\ \delta_2=10^{-3};\ \omega_p=0.4\pi;\ \omega_s=0.6\pi$$

$$M=28\Rightarrow L=14$$



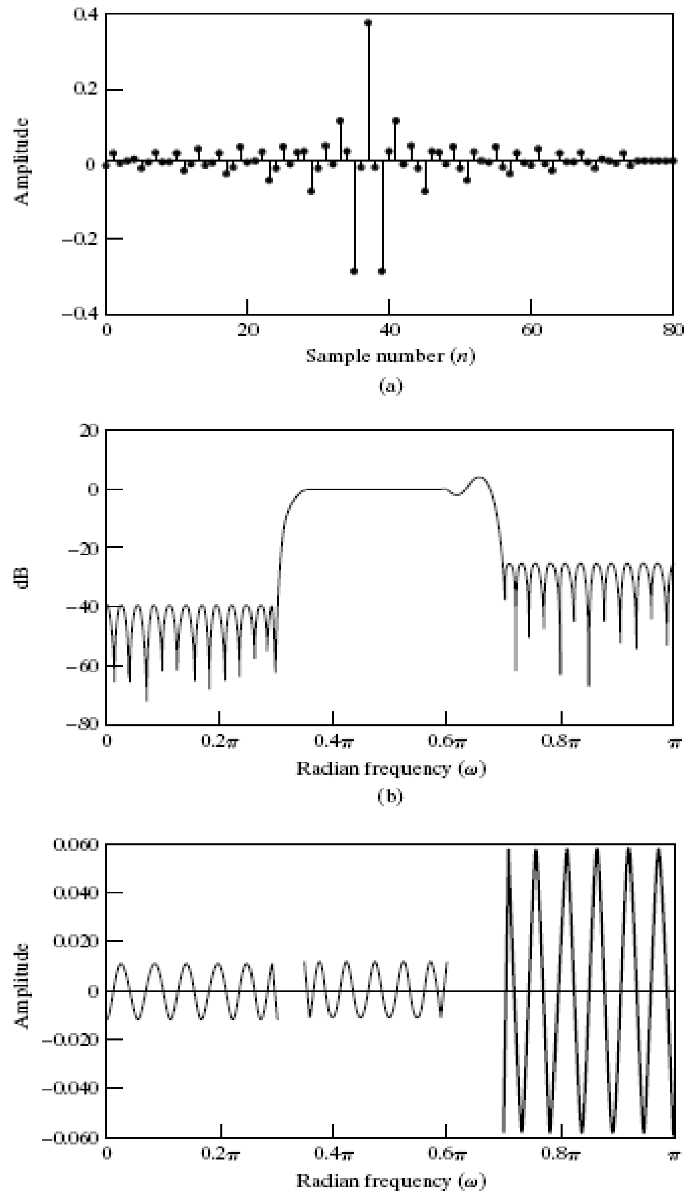
- Optimum bandpass filter design

$$H_d\left(e^{j\omega}\right)=\begin{cases}0; & 0\le\omega\le0.3\pi\\ 1; & 0.35\pi\le\omega\le0.3\pi\\ 0; & 0.7\pi\le\omega\le\pi\end{cases}$$

$$W\left(e^{j\omega}\right)=\begin{cases}1; & 0\le\omega\le0.3\pi\\ 1; & 0.35\pi\le\omega\le0.3\pi\\ 0.2; & 0.7\pi\le\omega\le\pi\end{cases}$$

$$M=74\Rightarrow L=37$$

  – Local extrema may occur in the transition regions
  – The approximation need not be equi-ripple in the approximation regions



- Optimum approximation of FIR filters
  - Design a filter that best approximates the desired response for a given *M*
  - Rectangular window provides the best-mean squared approximation:

$$h[n]=\begin{cases} h_d[n], & 0\le n\le M \\ 0, & \text{otherwise}\end{cases}$$

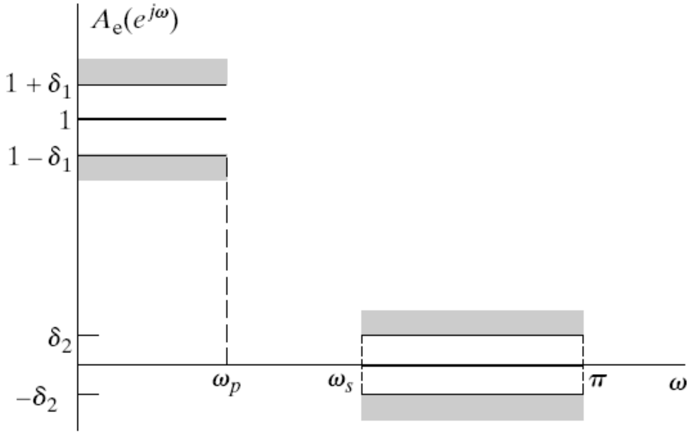
minimizes

$$\varepsilon^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right)-H\left(e^{j\omega}\right)\right|^2 d\omega$$

$$=\sum_{n=-\infty}^{\infty}\left|e^2[n]\right|$$

$$e[n]\triangleq\begin{cases} h_d[n]-h[n]; & 0\le n\le M \\ h_d[n]; & \text{otherwisw}\end{cases}$$

- Do not permit individual control errors in different bands



- Consider the design of a type-I FIR filter with zero phase, i.e., $h_e[n] = h_e[-n]$
  - Corresponding frequency response

$$A_e(e^{j\omega}) = \sum_{n=-L}^{L} h_e[n] e^{-j\omega n} = h_e[0] + \sum_{n=0}^{L} 2h_e[n]\cos(\omega n)$$

is a real, even and periodic function of ω

  - A causal system can be obtained by delaying it by *L* samples

$$h[n] \triangleq h_e[n-L] = h_e[2L-n]$$

  - From a polynomial approximation

$$\cos\omega n = T_n(\cos\omega); \quad T_n(x) = \cos(n\cos^{-1}x)$$

$A(e^{j\omega})$ can be represented as an *L*-th degree trigonometric polynomial

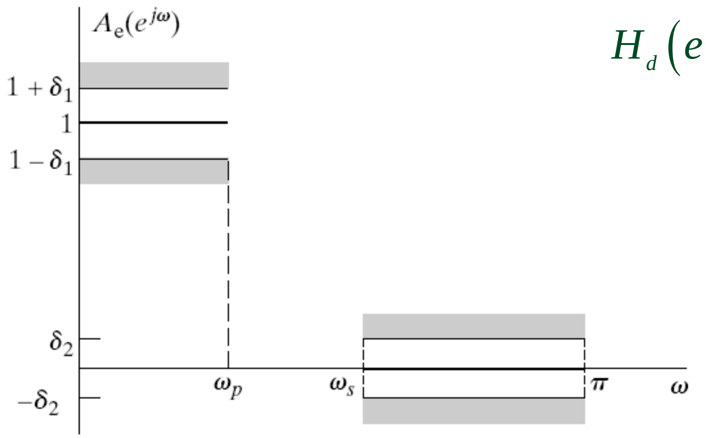
$$A(e^{j\omega}) = \sum_{k=0}^{L} a_k \cos^k \omega = P(x)\big|_{x=\cos\omega}; \quad P(x) = \sum_{k=0}^{L} a_k x^k$$

  - Define an error function by

$$E(\omega) = W(\omega)\left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2$$

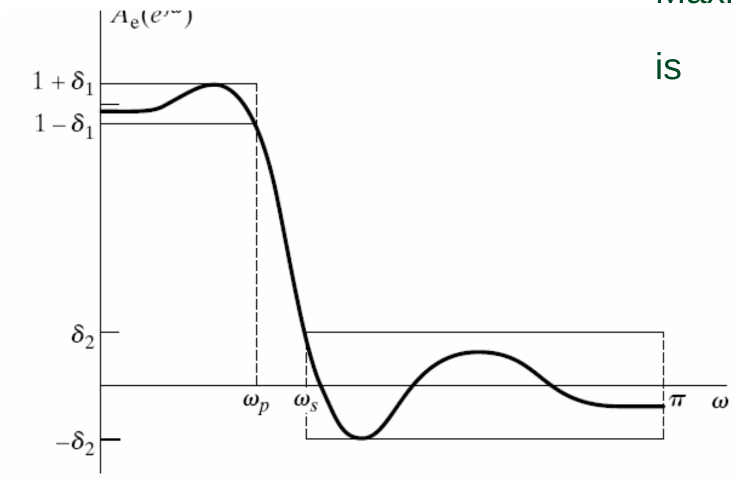
where $W(\omega)$ is a weighting function

– Example



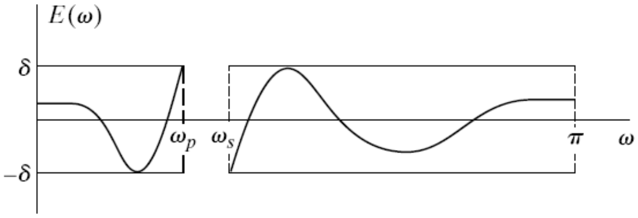
$$H_d\left(e^{j\omega}\right)=\begin{cases}1, & 0\le\omega\le\omega_p\\ 0, & \omega_s\le\omega\le\pi\end{cases}$$

$$W(\omega)=\begin{cases}\dfrac{1}{K}; & 0\le\omega\le\omega_p\\ 1, & \omega_s\le\omega\le\pi\end{cases};\quad K=\frac{\delta_1}{\delta_2}$$



Maximum weighted absolute approximation error

is $\delta=\delta_2$



– Minimax or Chevyshev criterion

$$\min_{\{h_e[n];0\le n\le L\}}\left(\max_{\omega\in F}\left|E(\omega)\right|\right)$$

$$F=\{\omega:0\le\omega\le\omega_p,\ \omega_s\le\omega\le\pi\}$$

- Find a set of impulse response that minimizes δ

– Alternative theorem:

- Let denote a closed subset comprising the disjoint union of closed subsets of real axis *x*. Then

$$P(x)=\sum_{k=0}^{m}a_k x^k$$

is an *m*-th order polynomial

- For desired function $D_P(x)$ and positive function $W_P(x)$ which are continuous on F_P , and weighted error defined by

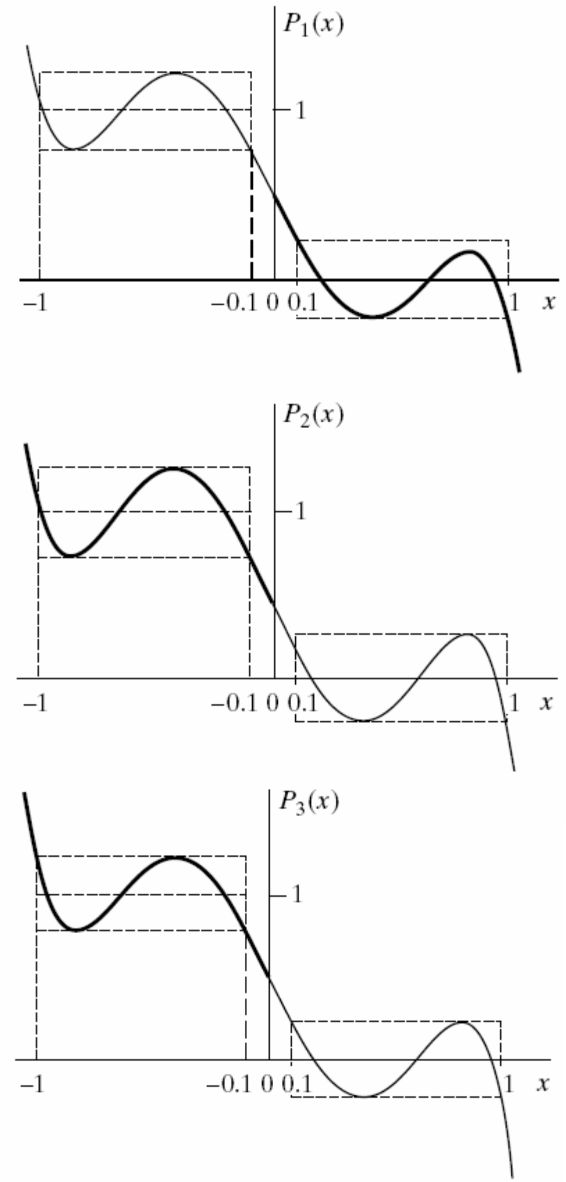
$$E_P(x)=W_P(x)\left[D_P(x)-P(x)\right]$$

a necessary and sufficient condition that be a unique r-th order polynomial minimizing $\|E\|=\max_{x\in F_P}|E_P(x)|$ is that $E_P(x)$ has at least $r+2$ alternations At least $r+2$ values x_i in F_P such that $x_1<x_2<\cdots<x_{r+2}$ and

$$E_P(x_i)=-E_P(x_{i+1})=\pm\|E\|,\quad i=1,2,\cdots,r+2$$

– Example: Consider 5-th order polynomials $P_j(x)$ that approximate unity for $-1 \le x \le -0.1$ and zero for $0.1 \le x \le 1$

- Assume that $W_P(x) = 1$ for these two regions
- The optimum 5-th order polynomial has at least 7 alternations of the error in the region in F_P
- $P_1(x)$ has 3 alternations in $-1 \le x \le -0.1$ and 2 alternations in $0.1 \le x \le 1$
- $P_2(x)$ also has 5 alternations; 3 in $-1 \le x \le -0.1$ and 2 in $0.1 \le x \le 1$
- $P_3(x)$ has 8 alternations in $-1 \le x \le -0.1$ and $0.1 \le x \le 1$ $\Rightarrow$ optimum polynomial



- Optimum type-I lowpass filters
  - Consider polynomial $P(x)$ defined by

$$P(\cos\omega)=\sum_{k=0}^{L} a_k(\cos\omega)^k$$

  - Assume that

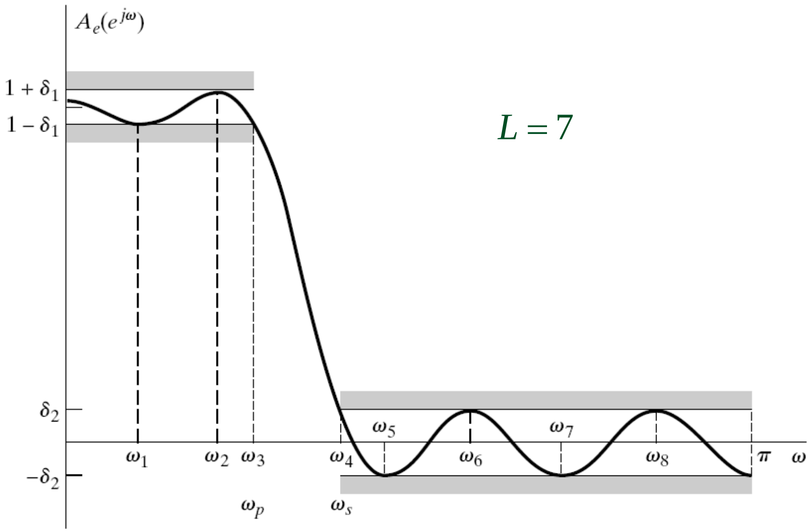
$$D_P(\cos\omega)=\begin{cases}1; & \cos\omega_p \le \cos\omega \le 1\\ 0; & -1\le \cos\omega \le \cos\omega_s\end{cases}$$

$$W_P(\cos\omega)=\begin{cases}1/K; & \cos\omega_p \le \cos\omega \le 1\\ 0; & -1\le \cos\omega \le \cos\omega_s\end{cases}$$

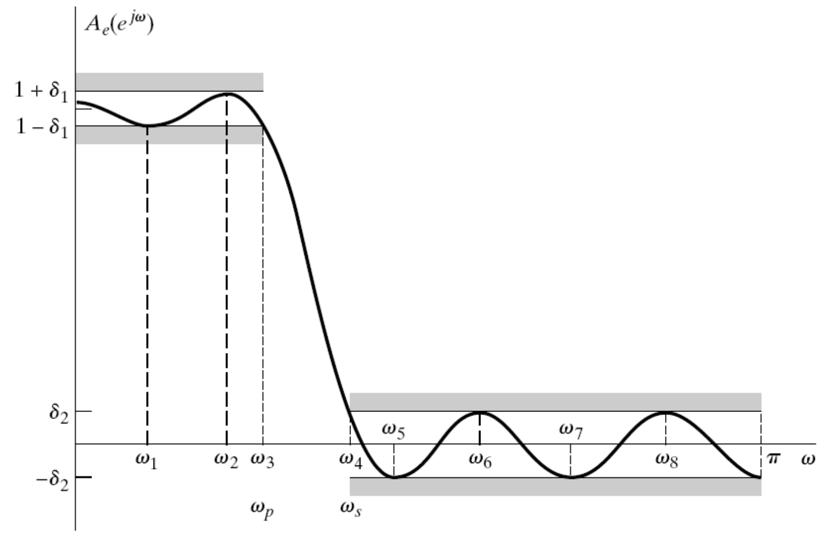
$$E_P(\cos\omega)=W_P(\cos\omega)\left[D_P(\cos\omega)-P(\cos\omega)\right]$$

$$F_P=\{\omega : 0\le\omega\le\omega_p,\ \omega_s\le\omega\le\pi\}$$

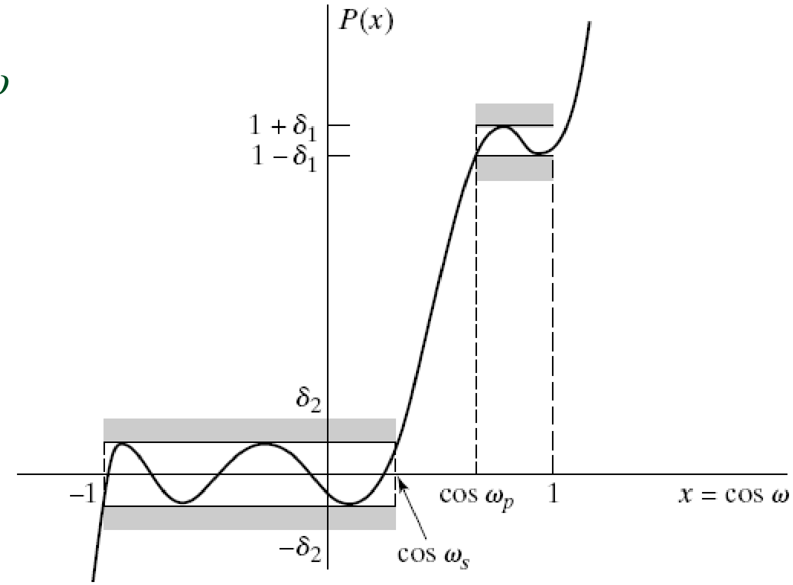
  - Equi-ripple approximation
    - A set of coefficients $\{a_k\}$ is designed to make $E_P(\cos\omega)$ have at least $(L+2)$ alternations on F_P

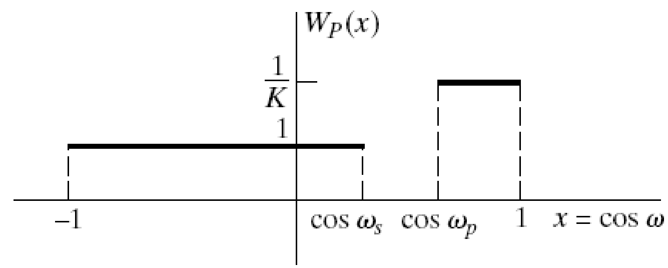


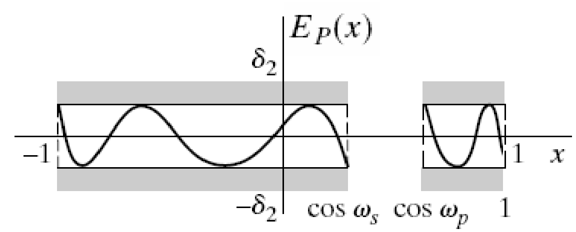
- Optimum type-I lowpass filters (cont.)
  - Redraw $A\left(e^{j\omega}\right)$ in terms of $x = \cos\omega$



$L = 7$



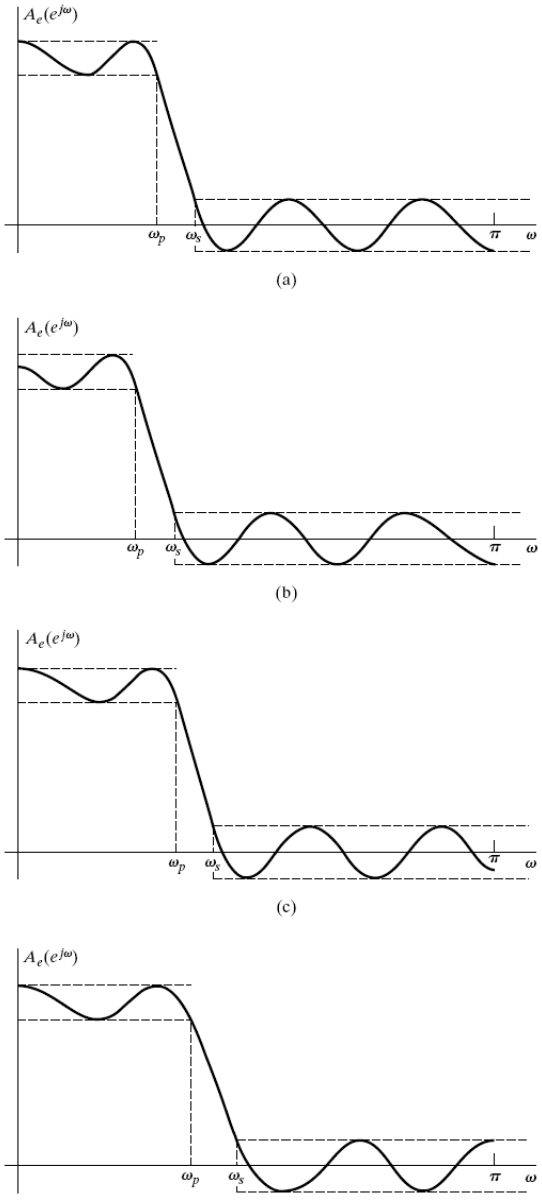




- Optimum type-I lowpass filters (cont.)
  - Max. possible number of alternations of the error is $(L+3)$ since an L-th degree polynomial can have at most $(L-1)$ points with zero slope

$$\frac{\partial}{\partial\omega}P(\cos\omega) = -\sin\omega\sum_{k=0}^{L} k a_k (\cos\omega)^{k-1}$$
$$= -\sin\omega\sum_{k=0}^{L-1}(k+1)a_{k+1}(\cos\omega)^{k}$$

    - Has zeros at $\omega=0,\ \omega=\pi$ and roots of $(L-1)$ -th order polynomial
  - Alternations always occur at $\omega=\omega_p$ and $\omega=\omega_s$
  - The filter will be equi-ripple except possibly at $\omega=0$ or $\omega=\pi$



- Parks-McClellan algorithm
  - The alternation theorem indicates that the optimum filter $A_e\left(e^{j\omega}\right)$ will satisfies

$$W(\omega_i)\left[H_d\left(e^{j\omega_i}\right)-A_e\left(e^{j\omega_i}\right)\right]=(-1)^{i+1}\delta; \quad i=1,2,\cdots,L+2$$

$$\Rightarrow H_d\left(e^{j\omega_i}\right)=A_e\left(e^{j\omega_i}\right)+(-1)^{i+1}\delta/W(\omega_i)$$

$$A_e\left(e^{j\omega}\right)=\sum_{n=-L}^{L}h_e[n]e^{-j\omega n}=h_e[0]+\sum_{n=0}^{L}2h_e[n]\cos(\omega n)=\sum_{k=0}^{L}a_k\cos^k\omega=\sum_{k=0}^{L}a_k x^k$$

  - $H_d\left(e^{j\omega_i}\right)$ can be rewritten as

$$\begin{bmatrix} H_d\left(e^{j\omega_1}\right) \\ H_d\left(e^{j\omega_2}\right) \\ \vdots \\ H_d\left(e^{j\omega_{L+2}}\right) \end{bmatrix}=\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \frac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \frac{-1}{W(\omega_2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \frac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix}$$

  - Solution for a_k and δ can be found by means of polynomial interpolation

- Parks-McClellan algorithm (cont.)
  - For a given set of extremal frequencies, the P-M algorithm determines

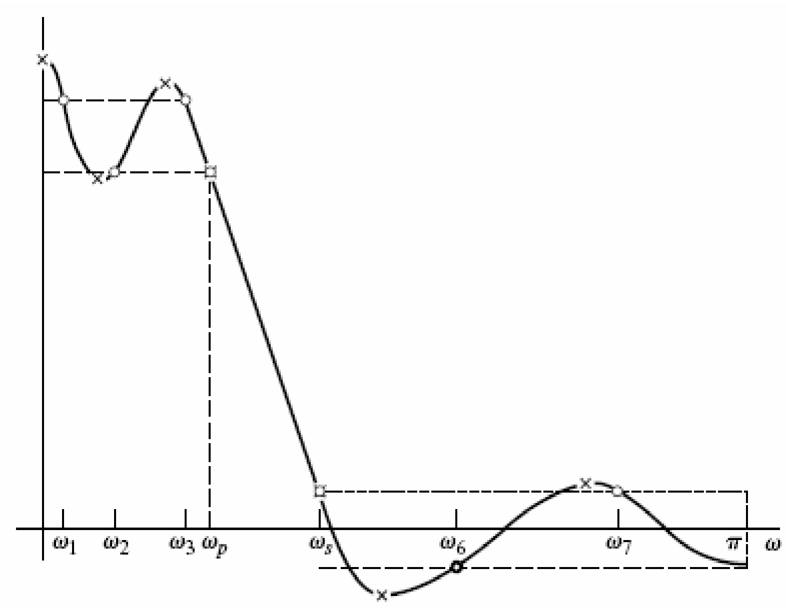
$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d\left(e^{j\omega_k}\right)}{\sum_{k=1}^{L+2} \frac{b_k (-1)^{k+1}}{W(\omega_k)}}; \qquad b_k = \prod_{\substack{i=1 \\ i \neq k}}^{L+2} \frac{1}{x_k - x_i}$$

  - For given a_k and δ, the error function has magnitude $\pm\delta$ at $L+2$ frequencies ω_i
  - $A_e\left(e^{j\omega}\right)$ has values $1 \pm K\delta$ for $0 \le \omega \le \omega_p$ and $\pm\delta$ for $\omega_s \le \omega \le \pi$
  - An *L*-th order trigonometric polynimial $A_e\left(e^{j\omega}\right)$ can be obtained by interpolating $E(\omega_i)$

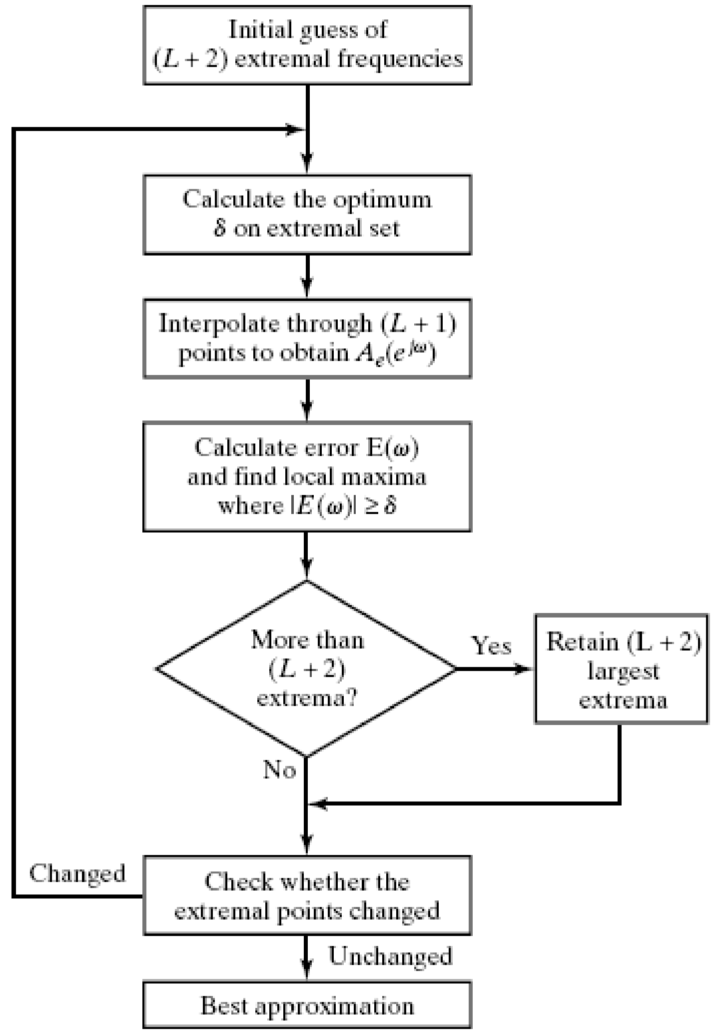
$$A_e\left(e^{j\omega}\right) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1} \frac{d_k C_k}{x - x_k}}{\sum_{k=1}^{L+1} \frac{d_k}{x - x_k}}$$

$$x = \cos\omega, \quad C_k = H_d\left(e^{j\omega_k}\right) - \frac{(-1)^{k+1}\delta}{W(\omega_k)}, \quad d_k = \prod_{\substack{i=1 \\ i \neq k}}^{L+1} \frac{1}{x_k - x_i} = b_k\left(x_k - x_{L+2}\right)$$

- Parks-McClellan algorithm (cont.)

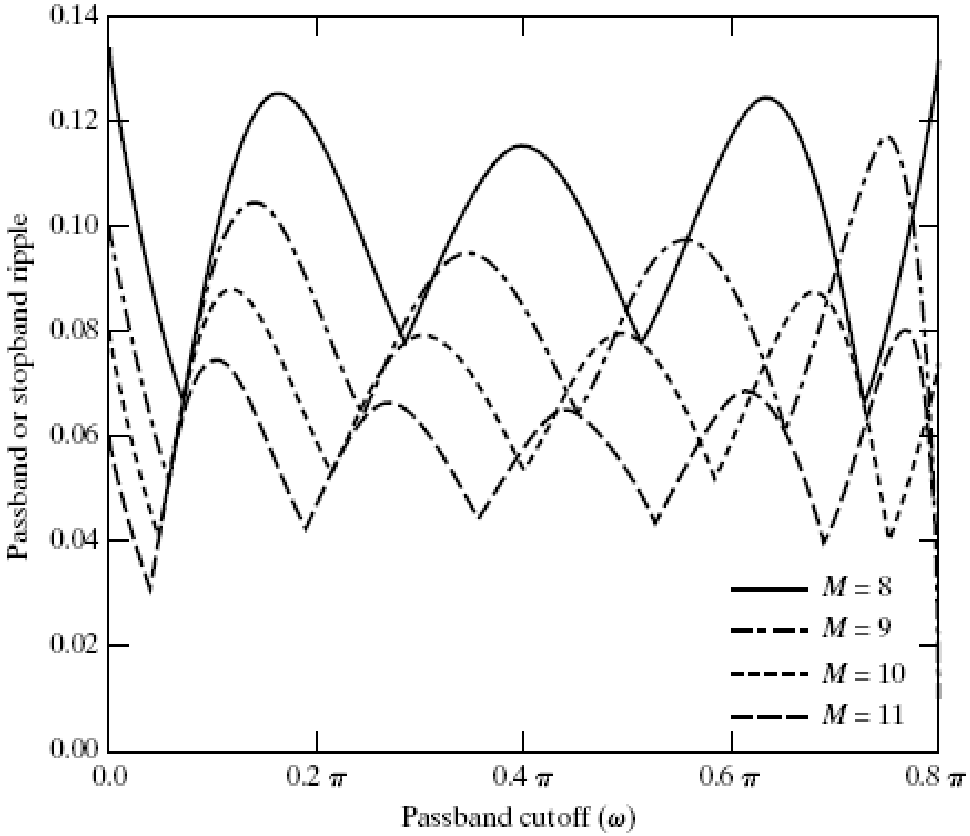


Example: $L = 7$



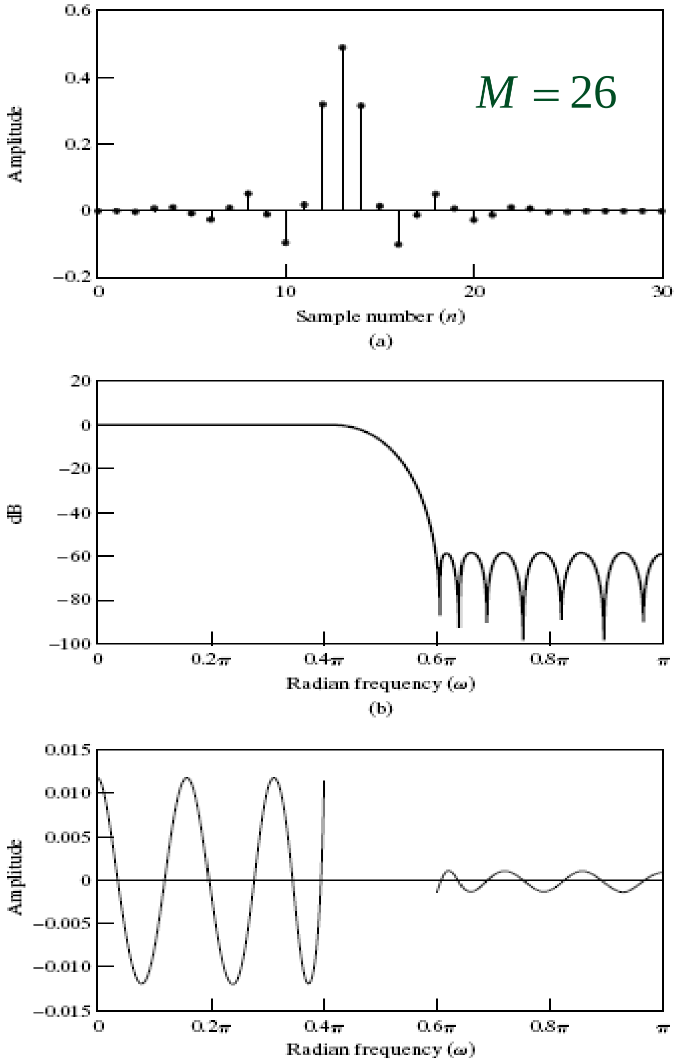
- Characteristics of optimum FIR filters
  - Size of the filter tap

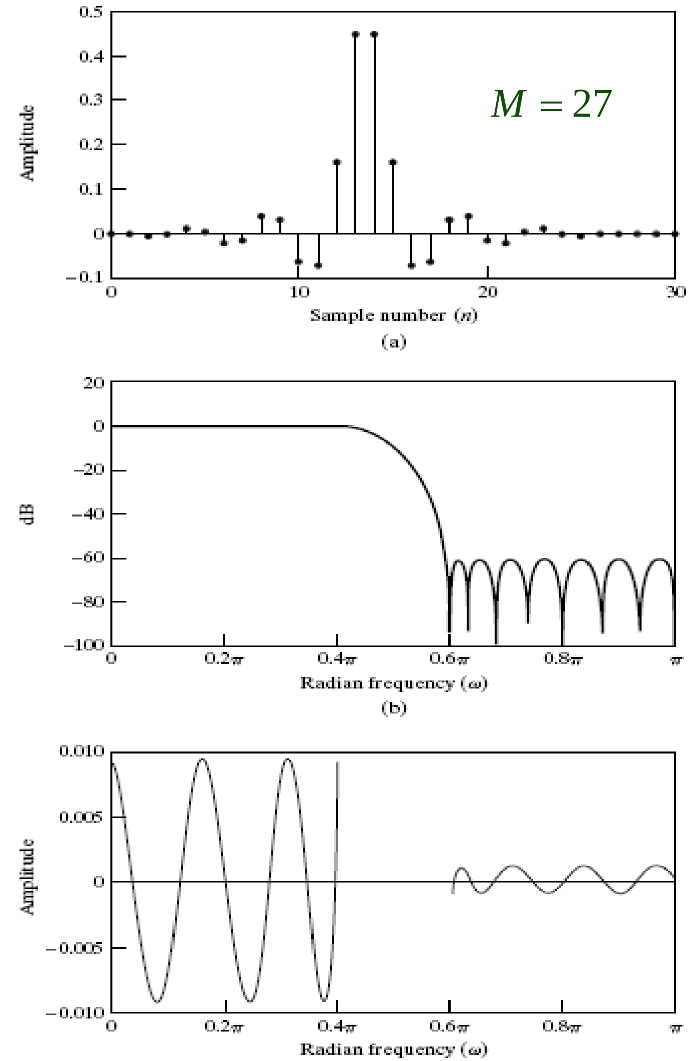
$$M = \frac{-10\log(\delta_1\delta_2) - 13}{2.324(\omega_s - \omega_p)}$$



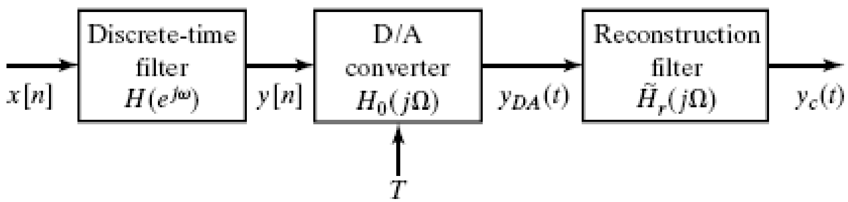
- Filter design examples by Parks-McClellan algorithm

$$\delta_1 = 10^2;\ \delta_2 = 10^{-3}$$
$$\omega_p = 0.4\pi;\ \omega_s = 0.6\pi$$





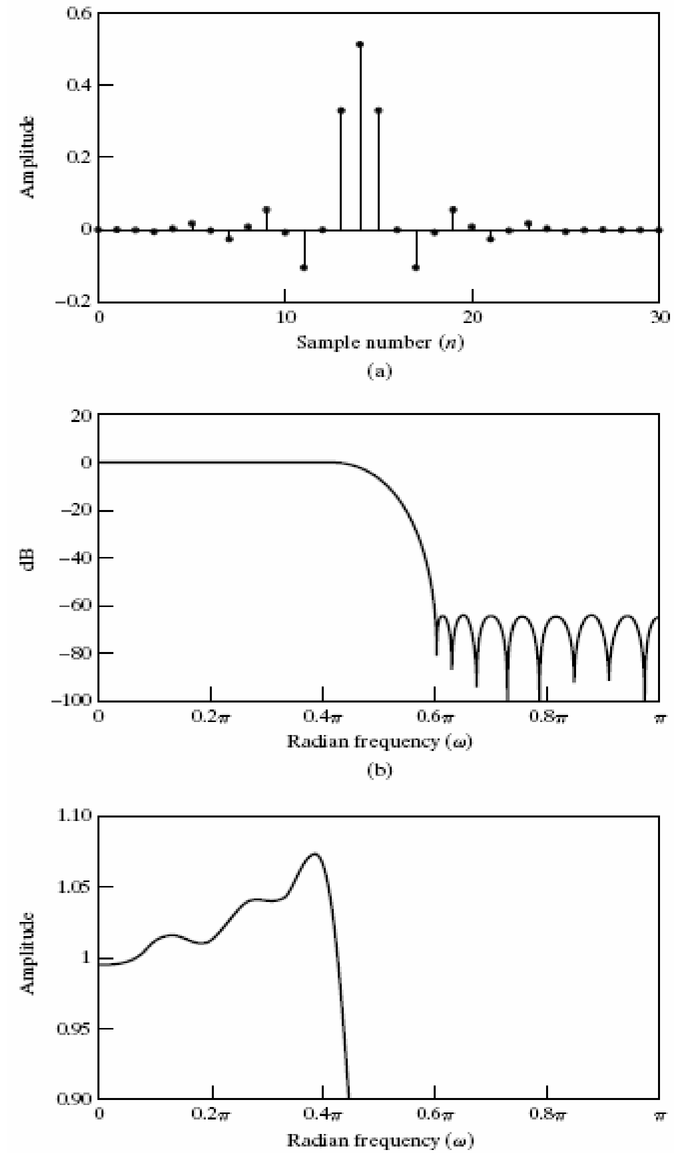
- Filter design example (cont.)
  - Compensation of zero-order holder



$$H_d\left(e^{j\omega}\right)=\begin{cases}\dfrac{\omega/2}{\sin(\omega/2)}; & 0\le\omega\le\omega_p\\ 0; & \omega_s\le\omega\le\pi\end{cases}$$

$$\delta_1=10^{2};\ \delta_2=10^{-3};\ \omega_p=0.4\pi;\ \omega_s=0.6\pi$$

$$M=28\Rightarrow L=14$$



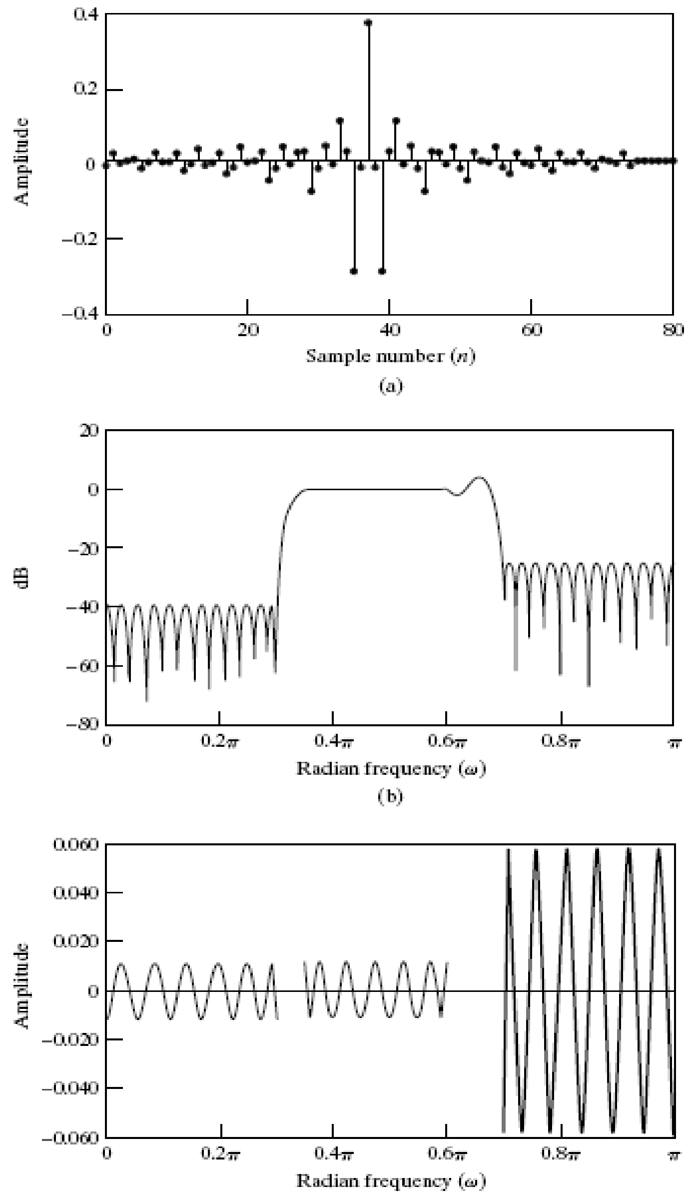
- Optimum bandpass filter design

$$H_d\left(e^{j\omega}\right)=\begin{cases}0; & 0\le\omega\le 0.3\pi\\ 1; & 0.35\pi\le\omega\le 0.3\pi\\ 0; & 0.7\pi\le\omega\le\pi\end{cases}$$

$$W\left(e^{j\omega}\right)=\begin{cases}1; & 0\le\omega\le 0.3\pi\\ 1; & 0.35\pi\le\omega\le 0.3\pi\\ 0.2; & 0.7\pi\le\omega\le\pi\end{cases}$$

$$M=74\Rightarrow L=37$$

 – Local extrema may occur in the transition regions, which may not be acceptable
 – The approximation need not be equi-ripple in the approximation regions



Discrete Fourier transform

- Discrete Fourier series representation of periodic sequences
  - Let $\tilde{x}[n]$ be a periodic sequence with period *N* such that $\tilde{x}[n]=\tilde{x}[n+mN]$ for any integer *m*

    Example A periodic complex exponential

    $$e_k[n]=e^{j\frac{2\pi kn}{N}}=e_k[n+mN]; \quad k \text{ is an interger}$$

    Note that $e_k[n]=e_{k+\ell N}[n]$ for any integer ℓ
  - A set of *N* periodic complex exponentials $\{e_k[n]\},\ k=0,1,2,\ldots N-1,$ defines all the frequency components

    $$\Rightarrow \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}[k]e^{j\frac{2\pi kn}{N}}$$

    $$\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi mn}{N}}$$
  - Fourier series representation

    $$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}[k]e^{j\frac{2\pi kn}{N}}$$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi rn}{N}}=\sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi kn}{N}}e^{-j\frac{2\pi rn}{N}}$$

$$=\sum_{k=0}^{N-1}\tilde{X}[k]\frac{1}{N}\sum_{k=0}^{N-1}e^{j\frac{2\pi(k-r)n}{N}}$$

Since $\frac{1}{N}\sum_{n=0}^{N-1}e^{j\frac{2\pi(k-r)n}{N}}=\begin{cases}1, & k-r=\ell N\\ 0, & \text{otherwise}\end{cases}$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{j\frac{2\pi rn}{N}}=\tilde{X}[r]$$

- The Fourier series coefficient can be obtained by

$$\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi kn}{N}}$$

$$\tilde{X}[k+N]=\sum_{n-0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi(k+N)n}{N}}=\sum_{n-0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi kn}{N}}e^{-j2\pi n}=\tilde{X}[k]$$

- The Fourier series coefficients is a periodic sequence

– Discrete Fourier series (DFS)

- Define by $\tilde{x}[n]\xleftrightarrow{DFS}\tilde{X}[k]$

Analysis equation: $\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]w_N^{kn}; \qquad w_N=e^{-j\frac{2\pi}{N}}$

Synthesis equation: $\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]w_N^{-kn}$

Example Periodic pulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - mN] = \begin{cases} 1, & n = mM \\ 0, & \text{otherwise} \end{cases}$$

Since $\tilde{x}[n] = \delta[n], \quad 0 \le n \le N-1$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] w_N^{-kn} = 1$$

$$\Rightarrow \tilde{x}[n] = \frac{1}{N} \sum \tilde{X}[k] w_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} w_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}$$

Example Duality in DFS

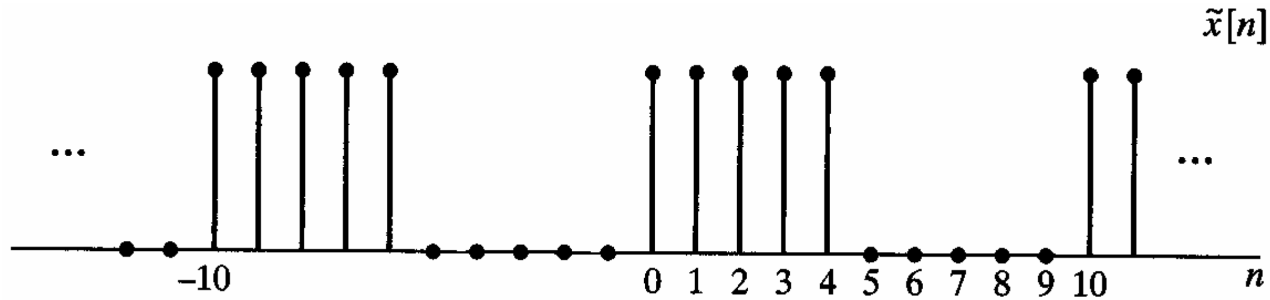
Let the DFS coefficient be a periodic impulse train

$$\tilde{Y}[k] = N \sum_{r=-\infty}^{\infty} \delta[k - rN]$$

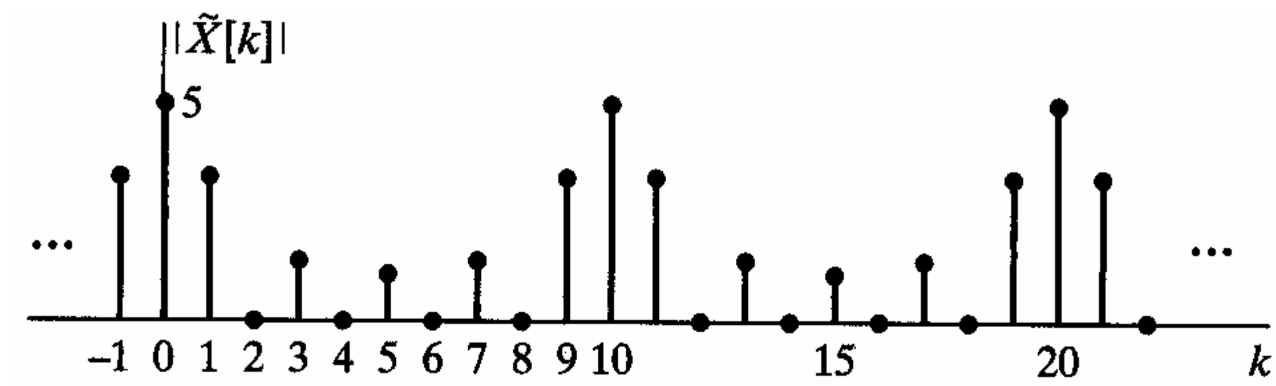
$$\Rightarrow \tilde{y}[n] = \sum_{k=0}^{N-1} N\delta[k] w_N^{-kn} = w_N^{-0} = 1, \quad \forall n$$

$$\tilde{Y}[k] = N\tilde{x}[n] \leftrightarrow \tilde{y}[n] = \tilde{X}[n]$$

Example Periodic rectangular pulse train



$$X[k] = \sum_{n=0}^{4} W_{10}^{nk} = \sum_{n=0}^{4} e^{-j\frac{2\pi}{10}kn} = \frac{1-W_{10}^{5k}}{1-W_{10}^{k}} = \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi k}{10}} e^{-j\frac{4\pi k}{10}}$$



- Any periodic sequence can be represented as a sum of complex exponential sequences

- Properties of DFS

  – Linearity

$$a\tilde{x}_1[n]+b\tilde{x}_2[n]\overset{DFS}{\longleftrightarrow}a\tilde{X}[k]+b\tilde{X}[k]$$

  – Shift of a sequence

$$\sum_{n=0}^{N-1}\tilde{x}[n-m]e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}\tilde{x}[n-m]e^{-j\frac{2\pi}{N}(n-m)}e^{-j\frac{2\pi}{N}}=\tilde{X}[k]e^{-j\frac{2\pi}{N}km}=\tilde{X}[k]W_N^{km}$$

$$\Rightarrow\quad \tilde{x}[n-m]\overset{DFS}{\longleftrightarrow}\tilde{X}[k]W_N^{km}$$

$$\tilde{x}[n]W_N^{-n\ell}\overset{DFS}{\longleftrightarrow}\tilde{X}[k-\ell]$$

  – Duality

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]w_N^{-kn}\Rightarrow N\tilde{x}[-n]=\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{kn}$$

$$\Rightarrow N\tilde{x}[-k]=\sum_{k=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

If $\tilde{x}[n]\overset{DFS}{\longleftrightarrow}\tilde{X}[k],\ \tilde{X}[n]\overset{DFS}{\longleftrightarrow}N\tilde{x}[-k]$

  – Symmetry properties

$$\tilde{x}^*[n]\overset{DFS}{\longleftrightarrow}\tilde{X}^*[-k]$$

$$\tilde{x}^*[-n]\overset{DFS}{\longleftrightarrow}\tilde{X}^*[k]$$

$\Rightarrow$ If $\tilde{x}[n]$ is real, $\tilde{X}[k] = \tilde{X}^*[-k]$

$$\left|\tilde{X}[k]\right| = \left|\tilde{X}[-k]\right|$$

– Periodic convolution

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]$$

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]\right)W_N^{kn}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m]\sum_{n=0}^{N-1} \tilde{x}_2[n-m]W_N^{kn}$$

$$= \tilde{X}_2[k]\sum_{m=0}^{N-1} \tilde{x}_1[m]W_N^{km} = \tilde{X}_1[k]\tilde{X}_2[k]$$

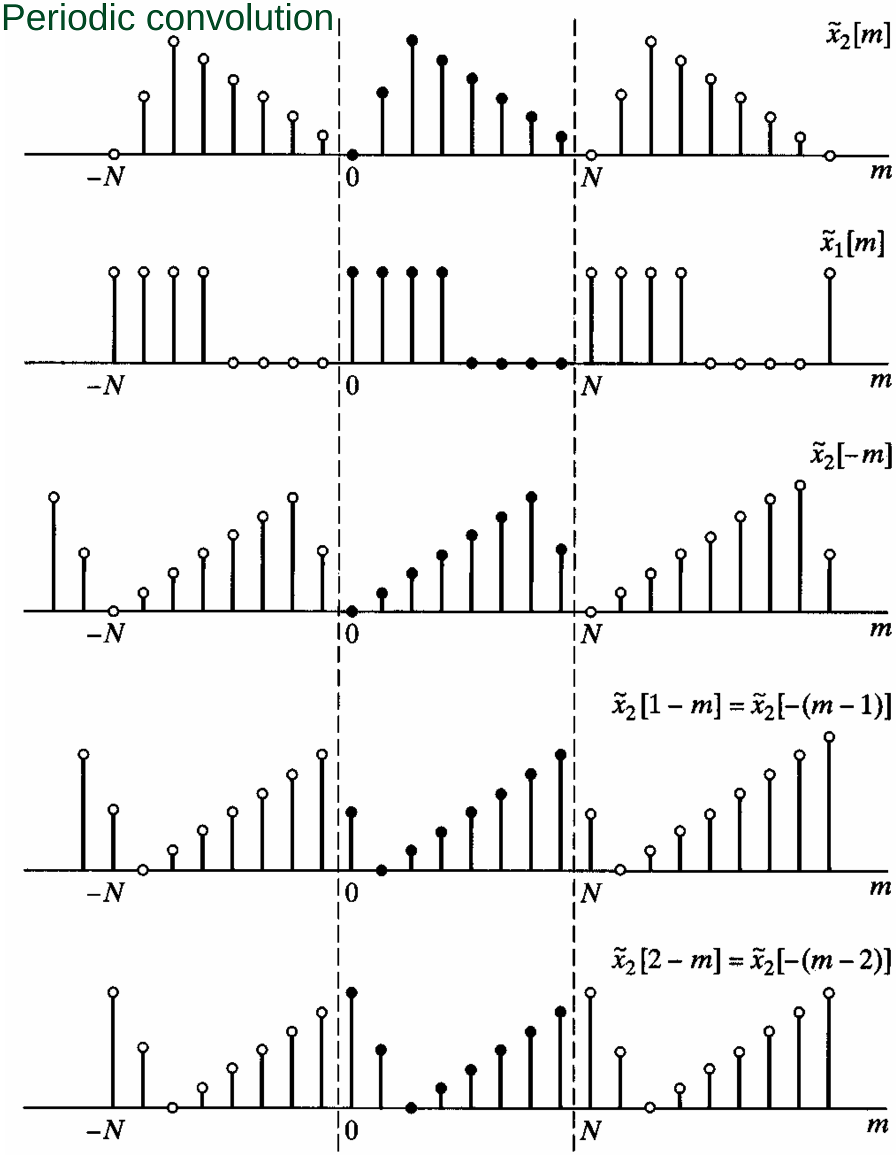
$$\Rightarrow \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \xleftrightarrow{DFS} \tilde{X}_1[k]\tilde{X}_2[k]$$

- Duality

$$\tilde{x}_1[n]\tilde{x}_2[n] \xleftrightarrow{DFS} \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\tilde{X}_2[k-m]$$

- Difference between aperiodic convolution and periodic convolution
  - The sum is over an interval $0 \le m \le N-1$
  - $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$

Example Periodic convolution



– Properties of DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

(*continued*)

– Properties of DFS (cont.)

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
10.	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11.	$\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12.	$j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14.	$\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16.	$\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17.	$\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

- Fourier transform (F.T.) of periodic signals
  - A sequence should be absolutely summable to guarantee the uniform convergence of its Fourier transform
  - Periodic signals are not absolutely summable
  - The F.T. of a periodic signal can be interpreted as an impulse train in the frequency domain with impulse values proportional to the DFS coefficients for the sequence
  - If $\tilde{x}[n]$ is periodic with period *N*, the Fourier transform of $\tilde{x}[n]$ is defined by

$$\tilde{x}[n] \Leftrightarrow \tilde{X}\left(e^{j\omega}\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

    - Although the F.T. of a periodic sequence does not converge in the normal sense, the introduction of impulses enables to use the F. T.
    - Note: $\tilde{X}\left(e^{j\omega}\right)$ has a necessary periodicity with period 2π

$$\frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\tilde{X}\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\frac{2\pi}{N}\sum_{k}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega; \quad 0 < \varepsilon < 2\pi / N$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\varepsilon}^{2\pi-\varepsilon}\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \tilde{x}[n]$$

Example: Fourier transform of a periodic impulse train

$$\tilde{p}[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] \Rightarrow \tilde{P}[k] = 1, \forall k$$

$$\Rightarrow \tilde{P}(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left[\omega - \frac{2\pi k}{N}\right]$$

– For a finite-length sequence $x[n]$, $(i.e., x[n] = 0 \text{ for } n < 0 \text{ and } n \geq N)$

$$\tilde{x}[n] = x[n] * \tilde{p}[n]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \delta[\ell - mN] x[n-\ell] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega}) \tilde{P}(e^{j\omega})$$

$$= X(e^{j\omega}) \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

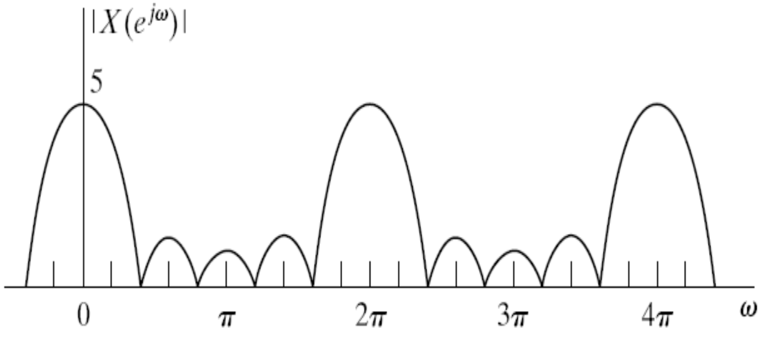
$$= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X\left(e^{j\frac{2\pi k}{N}}\right) \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$\Rightarrow \tilde{X}[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} = X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}k}$$

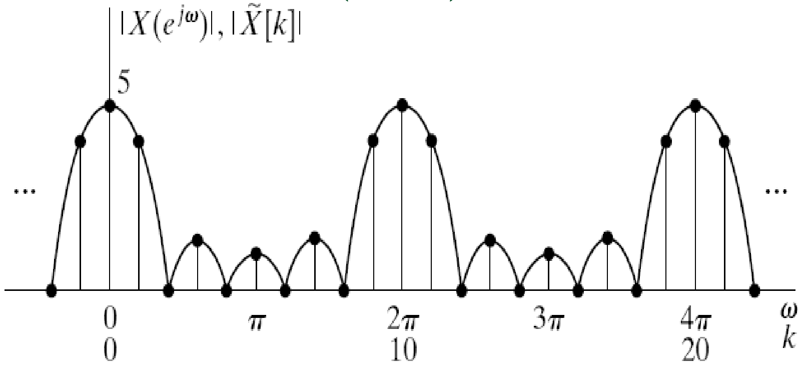
Thus, $\tilde{X}[k]$ is obtained by sampling the Fourier transform of a finite sequence $x[n]$

Example: Relationship between the DFS coefficients and the FT of one period

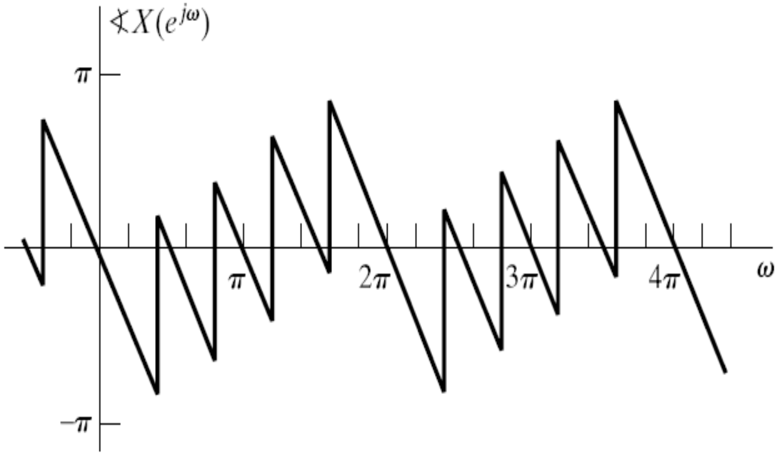
$$x[n]=\begin{cases}1; & 0\le n\le 4\\ 0; & \text{otherwise}\end{cases} \Rightarrow X\left(e^{j\omega}\right)=\sum_{n=0}^{4}e^{-j\omega n}=\frac{\sin\left(5\omega/2\right)}{\sin\left(\omega/2\right)}e^{-j2\omega}$$

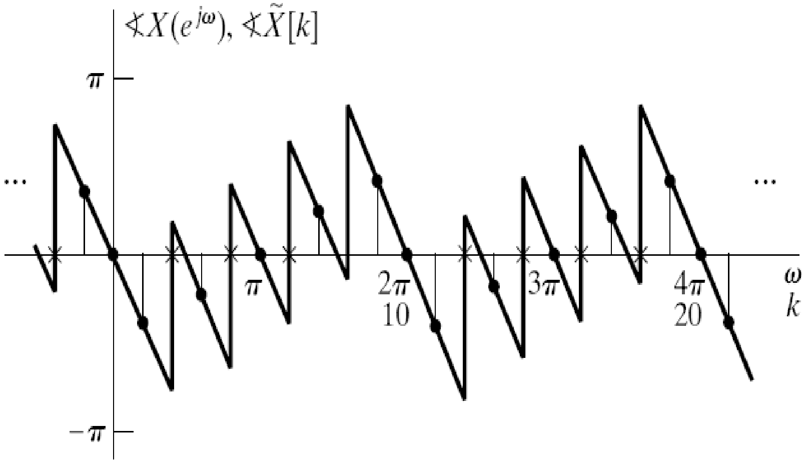


(a)



(a)





- Sampling the Fourier transform
  - Since $X(e^{j\omega})$ is periodic with period 2π, $\tilde{X}[k]$ is also periodic in *k* with period *N*.
  - Since $\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} = X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}k}$,

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi}{N}mk}\right] e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

$$= \sum_{m=-\infty}^{\infty} x[m] \sum_{\ell=-\infty}^{\infty} \delta[n-m-\ell N]$$

$$= x[n] * \sum_{\ell=-\infty}^{\infty} \delta[n-\ell N]$$

$$= \sum_{\ell=-\infty}^{\infty} x[n-\ell N]$$

The samples of the F.T. of $x[n]$ can be thought of as DFS coefficients of a periodic sequence $\tilde{x}[n]$ obtained through periodic replication of $x[n]$

– A periodic sequence $\tilde{X}[k]$ can be interpreted as equally spaced samples of the Fourier transform of one period of $\tilde{x}[n]$

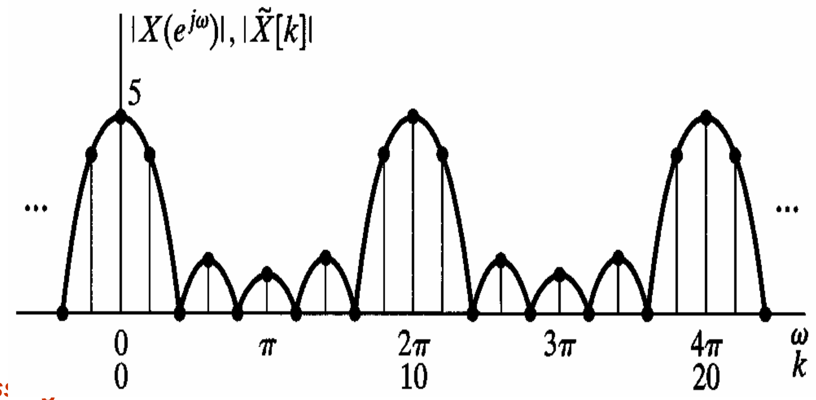
$$x[n]=\begin{cases}\tilde{x}[n], & 0\le n\le N-1\\ 0\ , & \text{otherwise}\end{cases}$$

$$X\left(e^{j\omega}\right)=\sum_{n=0}^{N-1}x[n]e^{-j\omega n}=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\omega n}$$

$$\Rightarrow \tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}} \Leftarrow \text{Sampling of the Fourier transform at a rate of } \frac{2\pi}{\omega}$$

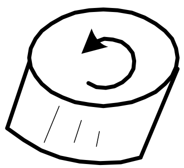
Example
$$x[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & \text{otherwise}\end{cases}$$

$$X\left(e^{j\omega}\right)=\sum_{n=0}^{4}e^{-j\omega n}=\frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}e^{-j2\omega}\Rightarrow \tilde{X}[k]=\frac{\sin(k\pi/2)}{\sin(k\pi/10)}e^{-j4k\pi/10}$$



- Fourier representation of finite-duration sequences: Discrete Fourier transform (DFT)

$$x[n]=\begin{cases}\tilde{x}[n], & 0\le n\le N-1\\ 0\ , & \text{otherwise}\end{cases} \Leftrightarrow \tilde{x}[n]=\sum_{\ell=-\infty}^{\infty} x[n-\ell N]=x\left[((n))_N\right]$$



where $x\left[((n))_N\right]\triangleq x\left[(n \text{ modulo } N)\right]$

– $\tilde{x}[n]$ can be visualized as warping the finite sequence around the cylinder

– Since the Fourier series coefficients $\tilde{X}[k]$ of $\tilde{x}[n]$ is also a periodic sequence with a period of *N*, the DFT of $x[n]$ can be obtained by

$$X[k]=\begin{cases}\tilde{X}[k], & 0\le k\le N-1\\ 0\ , & \text{otherwise}\end{cases}$$

$$\tilde{X}[k]=X\left[((k))_N\right]\Leftrightarrow \tilde{X}[k]=\sum_{m=-\infty}^{\infty} X[k-mN]$$

– Since $\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{nk}$ and $\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-nk}$,

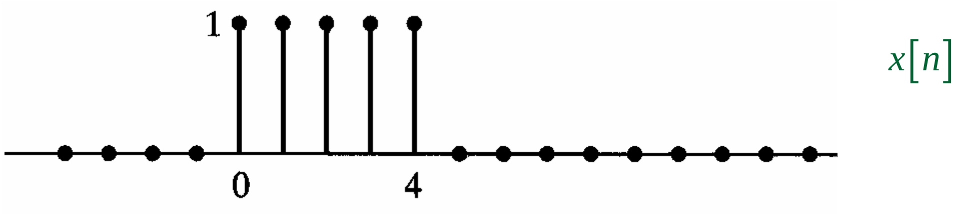
$$X[k]=\begin{cases}\sum_{k=0}^{N-1}x[n]W_N^{nk}, & 0\le n\le N-1\\ 0\ , & \text{otherwise}\end{cases} \text{ and } x[n]=\begin{cases}\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-nk}, & 0\le n\le N-1\\ 0\ , & \text{otherwise}\end{cases}$$

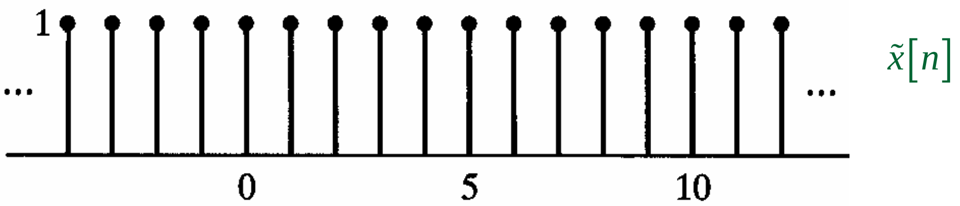
$$x[n]\xleftrightarrow{DFT}X[k]$$

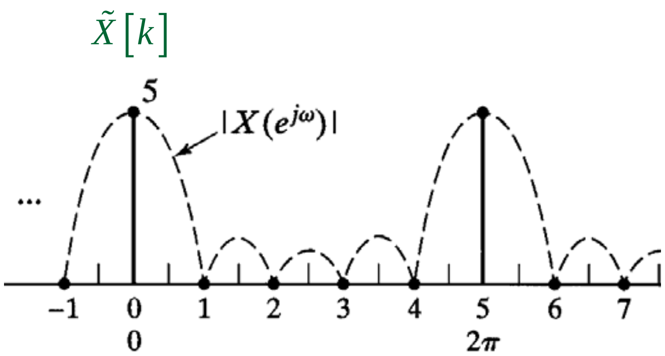
– Example

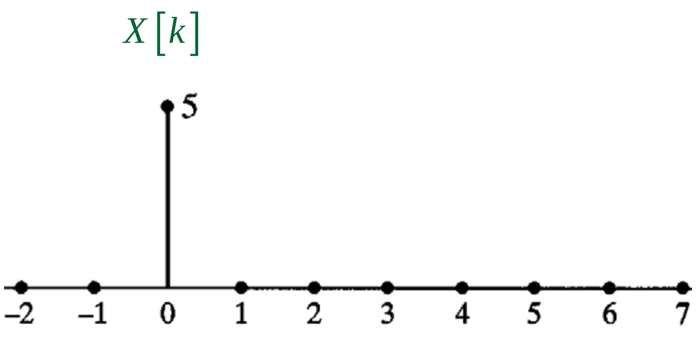
$$\tilde{X}[k]=\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}} = \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{2\pi k}{N}}} = \begin{cases} N, & k=0,\pm N,\pm 2N,\cdots \\ 0, & \text{otherwise} \end{cases}$$

N=5;

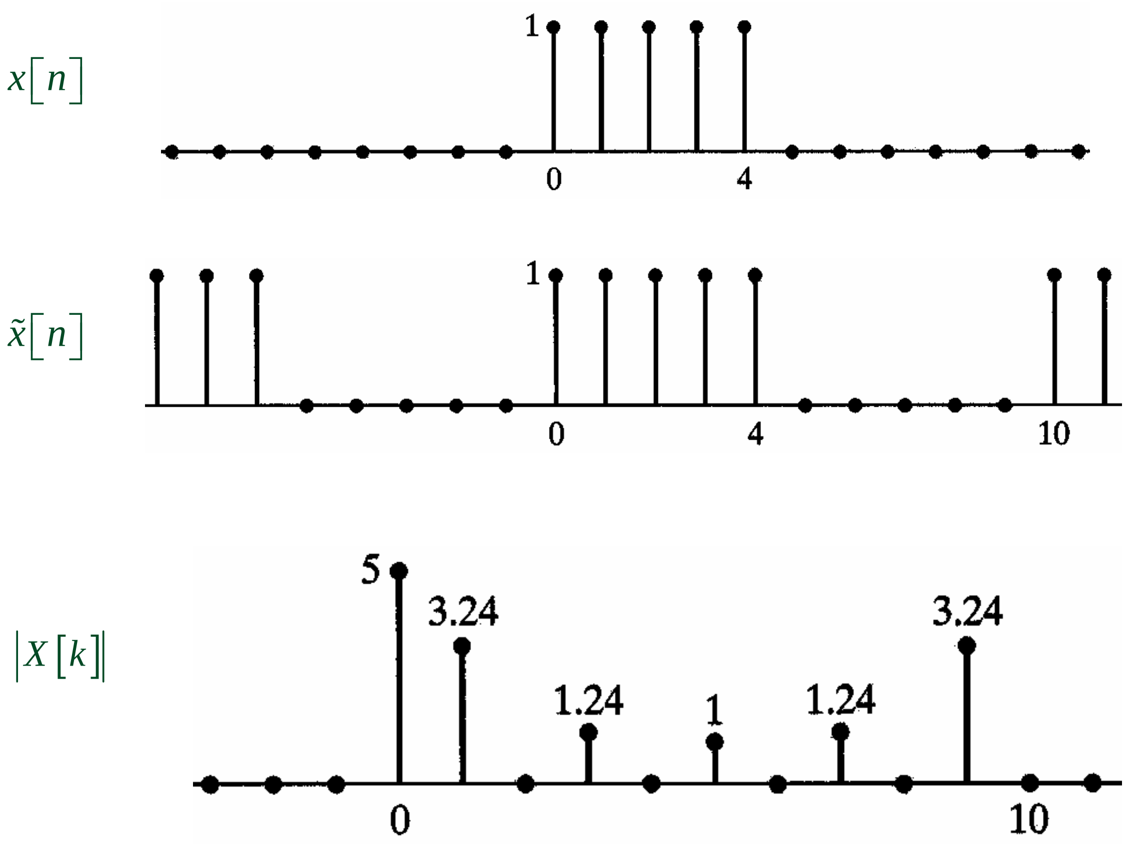








– Example: N=10



- Properties of the DFS
  - Linearity

$$x[n] \xleftrightarrow{DFT} X[k] \Rightarrow \alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DFT} \alpha X_1[k] + \beta X_2[k]$$

N_1 N_2 $N \geq \max[N_1, N_2]$ with zero padding

  - Circular shift

$$x_1[n] = x[n-m] \xleftrightarrow{DFT} X(e^{j\omega})e^{-j\omega m}$$

    - Let $x_1[n] \xleftrightarrow{DFT} X_1[k] = X[k]e^{-j\frac{2\pi}{N}km}$
    - Since $x_1[n] = 0$ outside $0 \leq n \leq N-1$, $x_1[n]$ cannot be obtained by simply time shifting $x[n]$

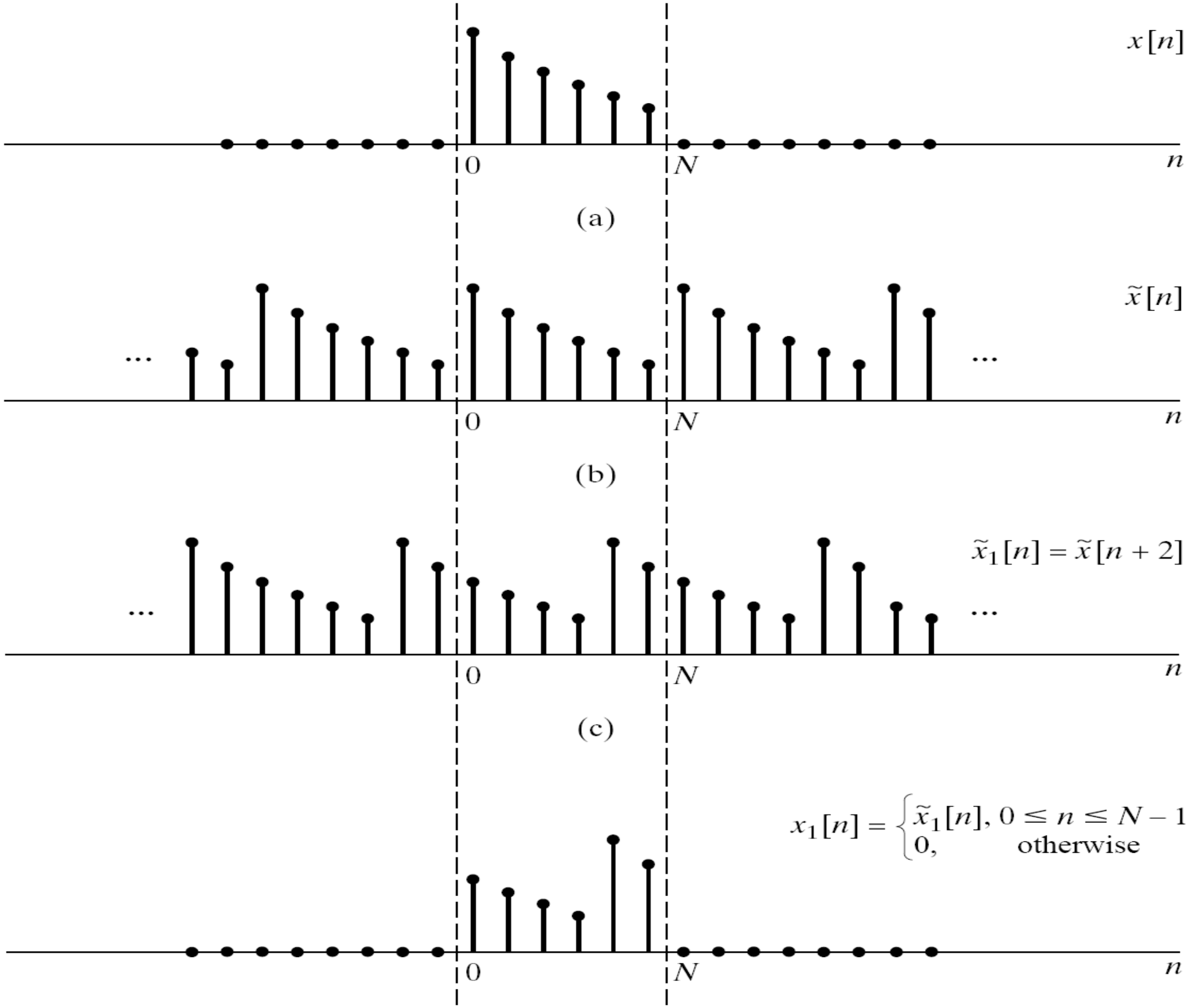
$$\Rightarrow \tilde{x}_1[n] = x_1[((n))_N] \xleftrightarrow{DFT} \tilde{X}_1[k] = X_1[((k))_N]$$

$$\Rightarrow \tilde{X}_1[k] = X[((k))_N]e^{-j\frac{2\pi m((k))_N}{N}} = X[((k))_N]e^{-j\frac{2\pi km}{N}} = \tilde{X}[k]e^{-j\frac{2\pi km}{N}}$$

$$\Rightarrow x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-m))_N], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow x[((n-m))_N] \xleftrightarrow{DFT} X[k]e^{-j\frac{2\pi km}{N}}, \quad 0 \leq n \leq N-1: \text{ circular shift}$$

– Example: circular shift



– Duality

$$\tilde{x}[n] = X\left[((n))_N\right] \xleftrightarrow{DFT} \tilde{X}[k] = X\left[((k))_N\right]$$

$$\Rightarrow \tilde{X}[k] \xleftrightarrow{DFT} N\tilde{x}[-k]$$

Let $x_1[n] = X[n] \Rightarrow \tilde{x}_1[n] \xleftrightarrow{DFT} \tilde{X}_1[k] = N\tilde{x}[-k]$

$$X_1[k] = \begin{cases} N\tilde{x}[-k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} N\, x\left[((-k))_N\right], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad X[n] \xleftrightarrow{DFT} N\, x\left[((-k))_N\right], \quad 0 \le k \le N-1$$

– Symmetry

DFT of $x[n]$ = DFS coefficients of $\tilde{x}[n] = x\left[((n))_N\right]$

$$\Rightarrow x^*[n] \xleftrightarrow{DFT} X^*\left[((-k))_N\right], \quad 0 \le n \le N-1$$

$$x^*\left[((-n))_N\right] \xleftrightarrow{DFT} X^*[k]$$

$$\tilde{x}_e[n] = \frac{1}{2}\left\{\tilde{x}[n] + \tilde{x}^*[-n]\right\}; \quad \tilde{x}[n] = x\left[((n))_N\right]$$

$$\tilde{x}_0[n] = \frac{1}{2}\left\{\tilde{x}[n] - \tilde{x}^*[-n]\right\}$$

$$x_e[n] = \tilde{x}_e[n], \quad 0 \le n \le N-1$$
$$= \frac{1}{2}\left\{x\left[((n))_N\right] + x^*\left[((n))_N\right]\right\}$$
$$= \frac{1}{2}\left\{x[n] + x^*[N-n]\right\} \quad \Leftarrow ((-n))_N = (N-n); \quad ((n))_N = n; \quad 0 \le n \le N-1$$
$$x_e[n] \overset{DFT}{\longleftrightarrow} X_e[k] = \frac{1}{2}\left\{X[k] + X^*[k]\right\} = \mathrm{Re}\left\{X[k]\right\}$$

Similarly,

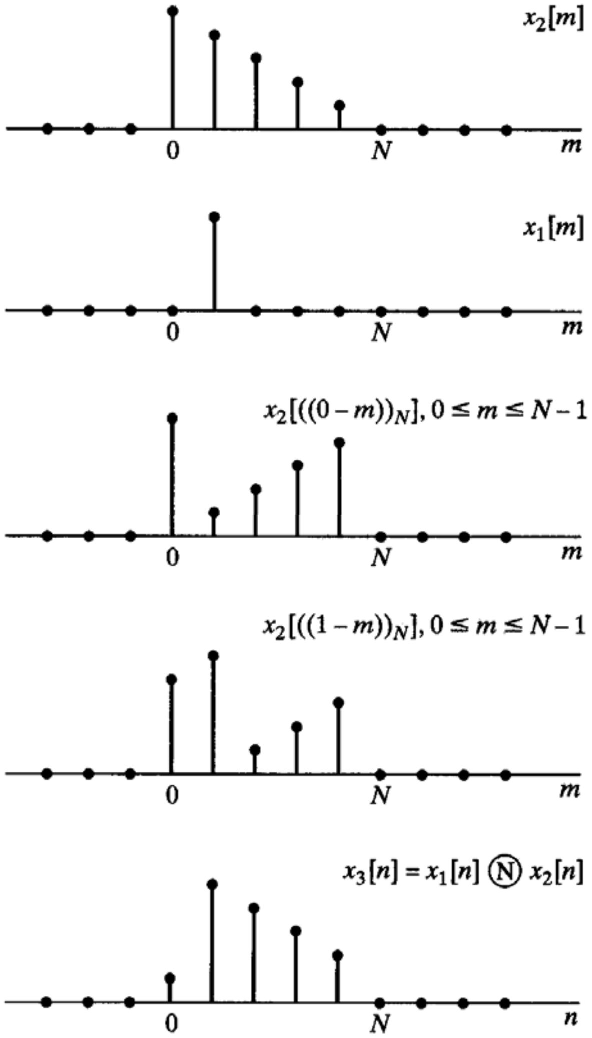
$$x_o[n] = \tilde{x}_o[n], \quad 0 \le n \le N-1$$
$$\Rightarrow x_0[n] \overset{DFT}{\longleftrightarrow} j\,\mathrm{Im}\left\{X[k]\right\}$$

– *N*-point Circular convolution

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \quad 0 \le n \le N-1$$
$$= \sum_{m=0}^{N-1} x_1\left[((m))_N\right] x_2\left[((n-m))_N\right]$$
$$= \sum_{m=0}^{N-1} x_1[m] x_2\left[((n-m))_N\right]$$
$$\triangleq x_1[n] \; \textcircled{N} \; x_2[n]$$
$$= \sum_{m=0}^{N-1} x_1\left[((n-m))_N\right] x_2[m]$$
$$= x_2[n] \; \textcircled{N} \; x_1[n]$$

Example

$x_1[n] = \delta[n-1]$



Example: *N*-point circular convolution

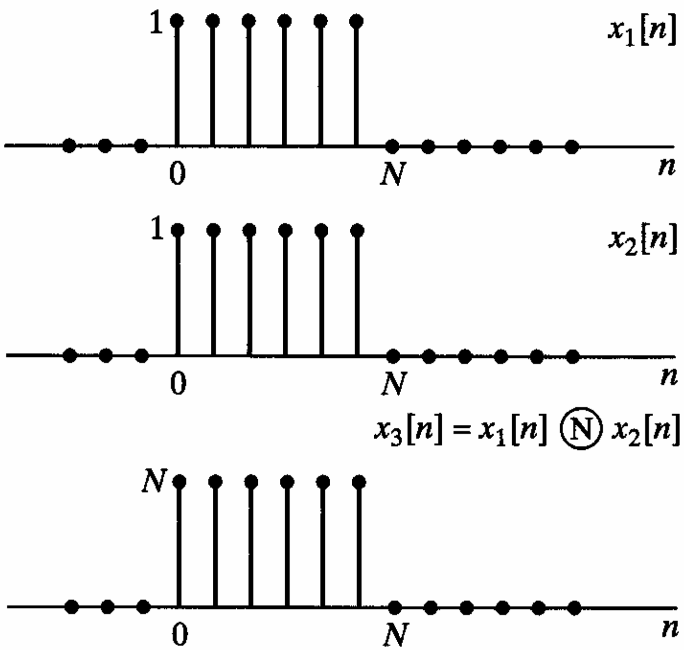
$$x_1[n] = x_2[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

- *N*=*L*:

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} 1 \cdot W_N^{kn} = \begin{cases} N, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow X_3[k] = X_1[k] X_2[k] = \begin{cases} N^2, & k=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow x_3[n] = N, \quad 0 \le n \le N-1$$



- *N* =2 *L*: Linear convolution ?

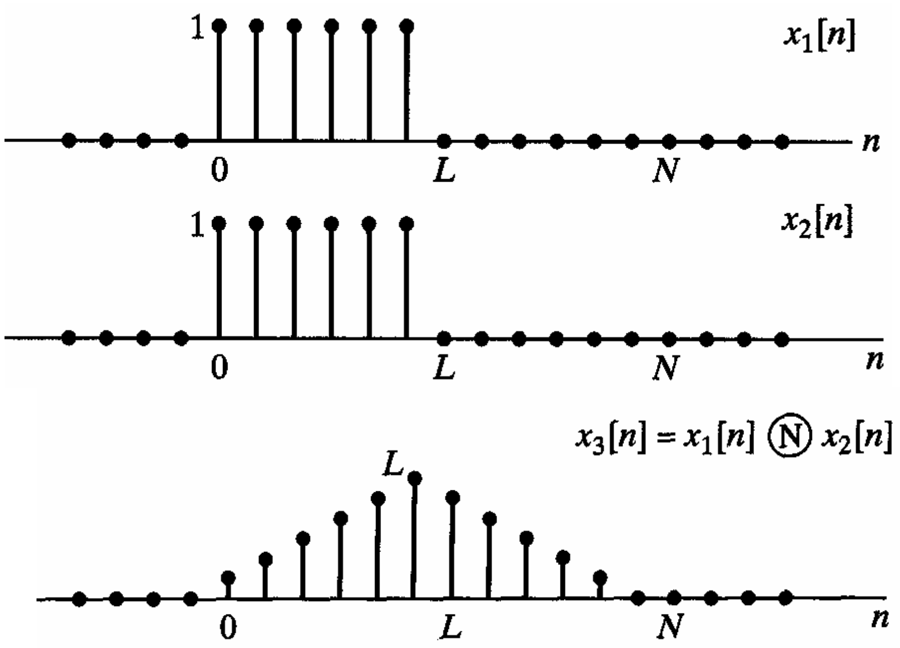
$$x_1[n] \, \textcircled{N} \, x_2[n] \xleftrightarrow{DFS} X_1[k] X_2[k]$$

$$\Leftrightarrow x_1[n] x_2[n] \xleftrightarrow{DFS} X_3[k] = \frac{1}{N} \sum_{\ell=0}^{N-1} X_1[\ell] X_2\left[((k-\ell))_N\right]$$

$$\frac{1}{N} X_1[k] \, \textcircled{N} \, X_2[k]$$

Example: *N(=2L)*-point circular convolution

$$x_1[n] = x_2[n] = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

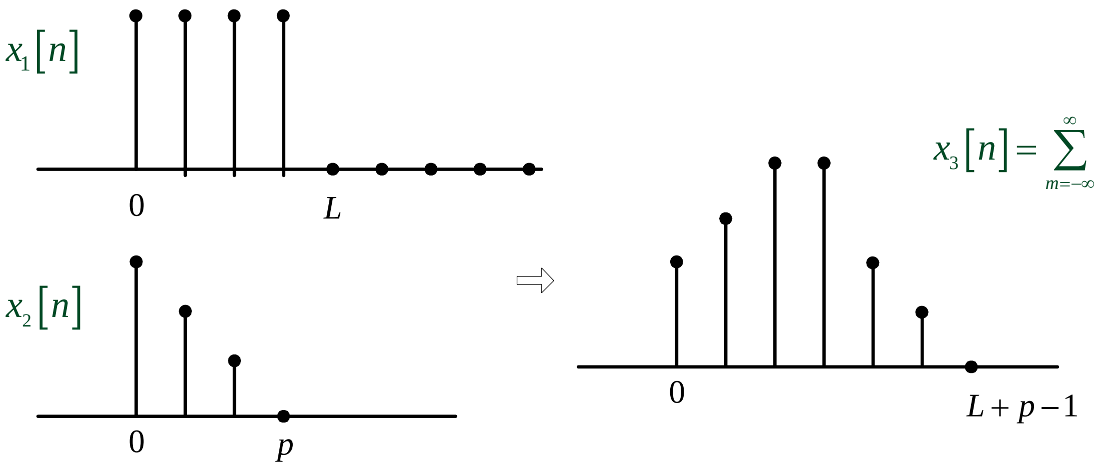


$$x_1[n] \circledN x_2[n] \overset{DFT}{\longleftrightarrow} X_1[k]X_2[k]$$

$$x_1[n]x_2[n] \overset{DFT}{\longleftrightarrow} X_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2\left[((k-\ell))_N\right]$$

$$= \frac{1}{N} X_1[k] \circledN X_2[k]$$

- Linear convolution using the DFT



$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

– $N \triangleq (L+p-1)$-point circular convolution with zero padding

$$x_p[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}; \qquad \tilde{x}[n] = \sum_m x[n-mN]$$

$$X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega}) \Rightarrow X_3[k] = X_3(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}, \quad 0 \le k \le N-1$$

$$= X_1(e^{j\omega})X_2(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$$

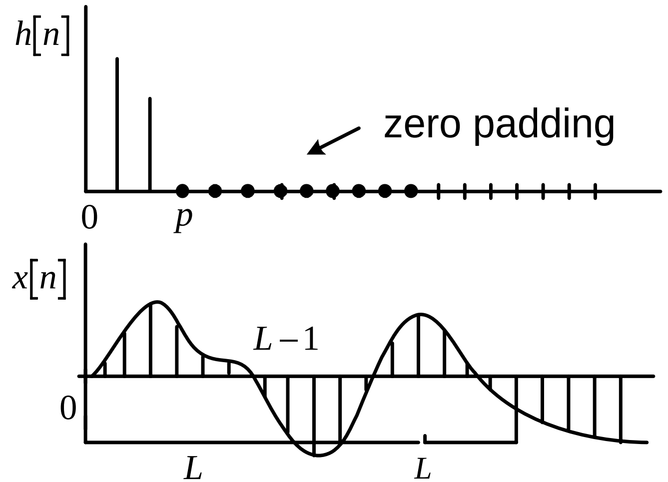
$$= X_1[k]X_2[k]$$

$$\Rightarrow x_{3p}[n] = x_1[n] \, \textcircled{N} \, x_2[n]$$

- The circular convolution of two finite-length sequences can be equivalent to linear convolution of the two sequences.

$$x_{3p}[n] = \begin{cases} \sum_m x_3[n - mN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- Calculation of the output of an LTI using the DFT

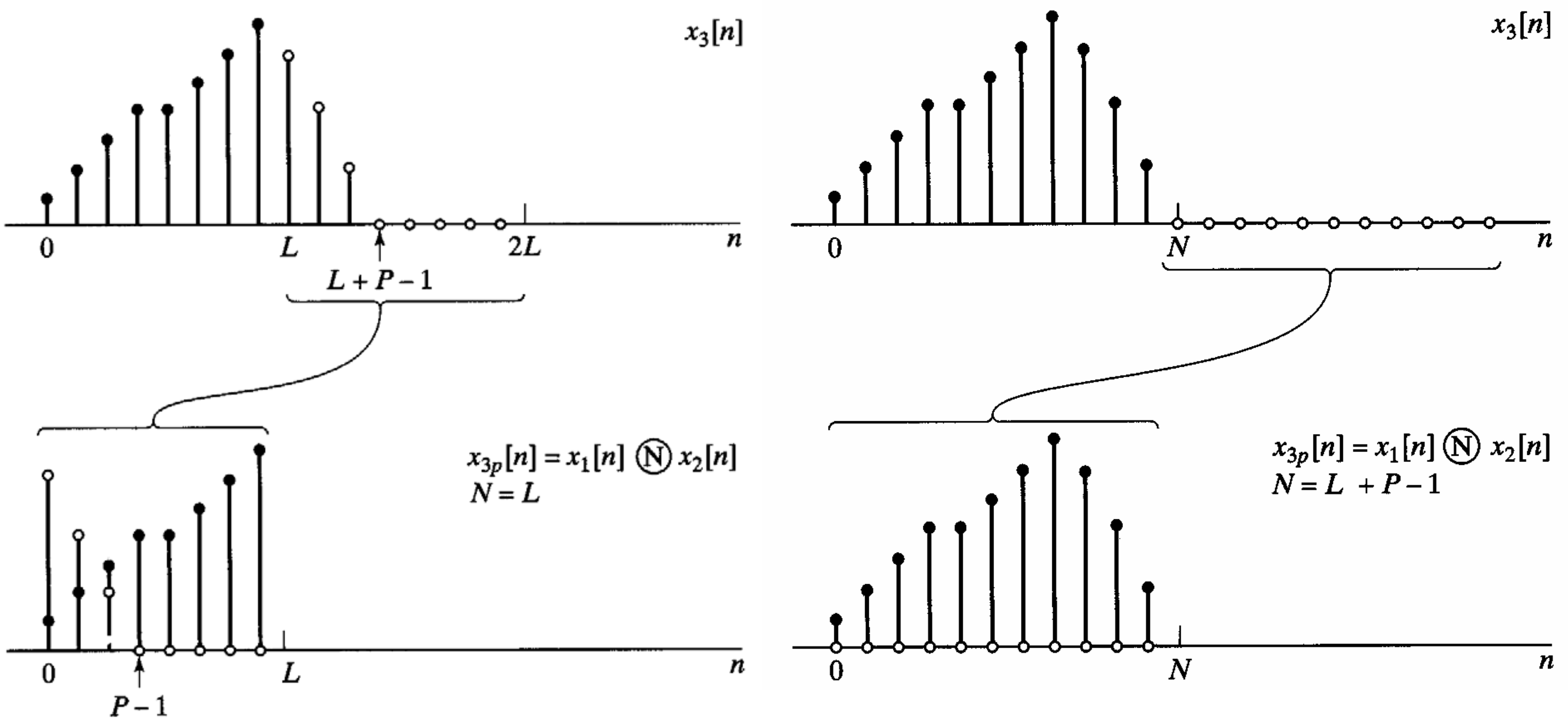


$$y[n] = h[n] * x[n]$$

- How to resolve the issue when $x[n]$ is infinite?
  - Assume $x[n] = 0$ for $n < 0$
  - The sequence $x[n]$ can be decomposed in to a sum of shifted finite-length segments of length *L*

$$x[n] = \sum_{m=0}^{\infty} x_m[n - mL]; \quad x_m[n] = \begin{cases} x[n + mL], & 0 \le n \le N-1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

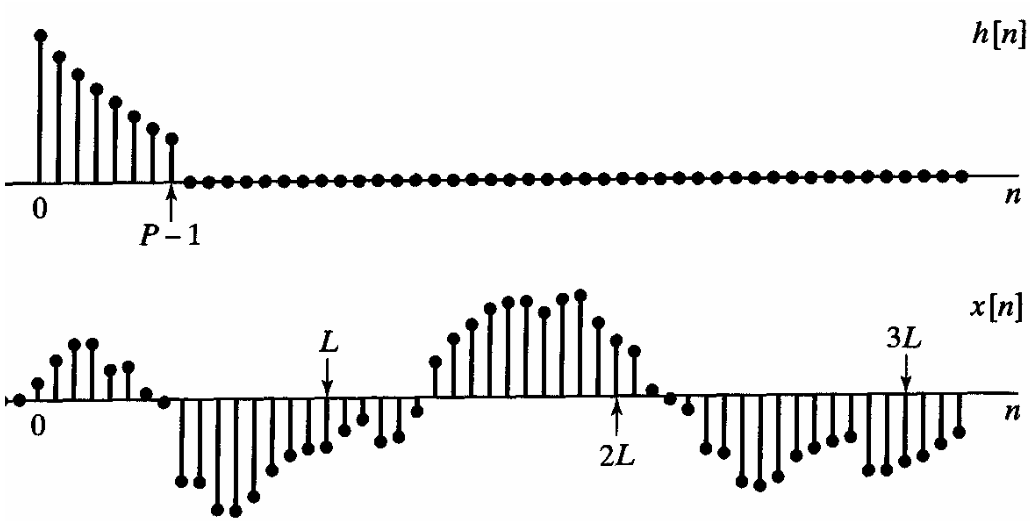
– Illustration of warping around due to improper circular convolution



$N = L$

$N = L + p - 1$

- Linear convolution using the discrete Fourier transform



– How to resolve the issue when $x[n]$ is infinite?
  - Assume $x[n]=0$ for $n<0$
  - The sequence $x[n]$ can be decomposed in to a sum of shifted finite-length segments of length *L*

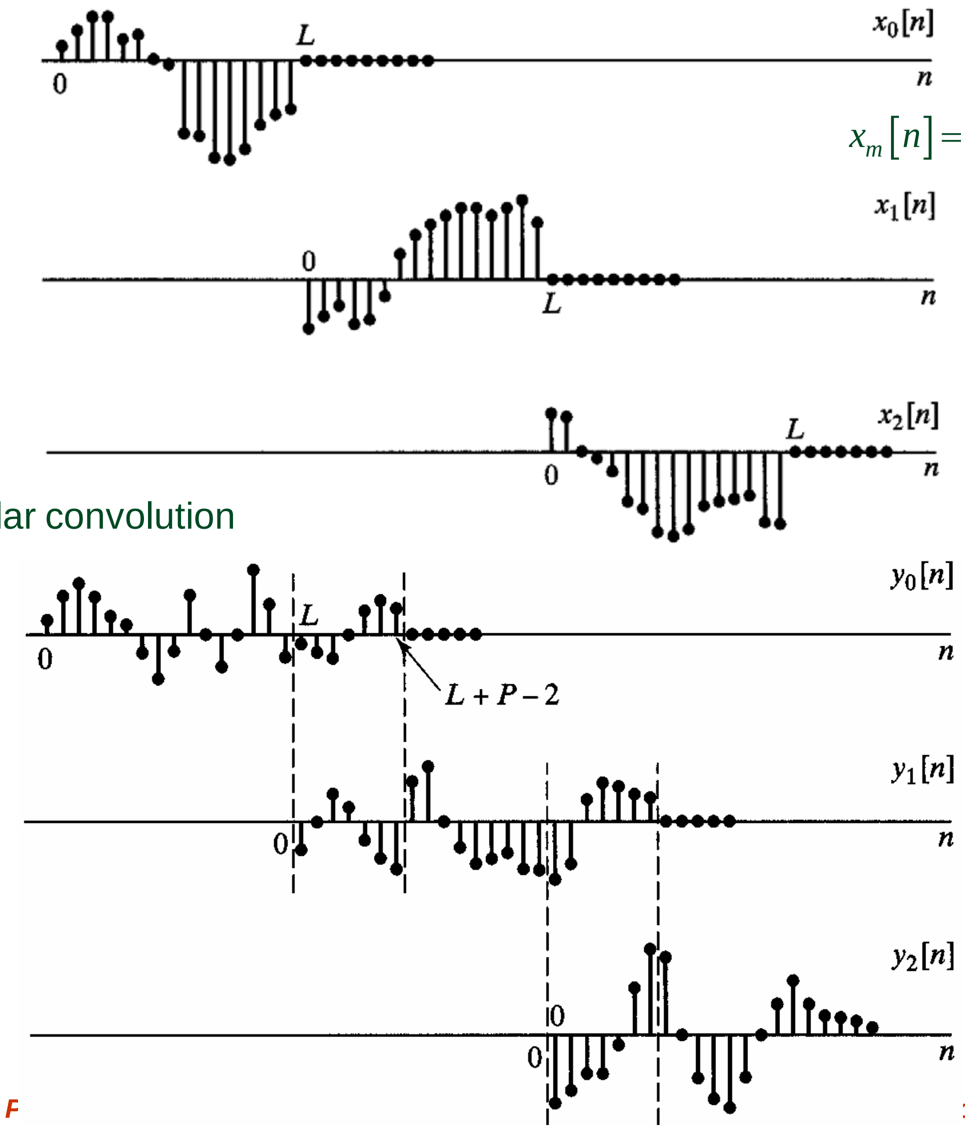
$$x[n]=\sum_{m=0}^{\infty} x_m[n-mL];$$

where $x_m[n]=\begin{cases} x[n+mL], & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$

– Overlap-add method $y[n] = h[n] * x[n]$

$x_m[n] = x[n + mL],\ 0 \le n \le L-1$

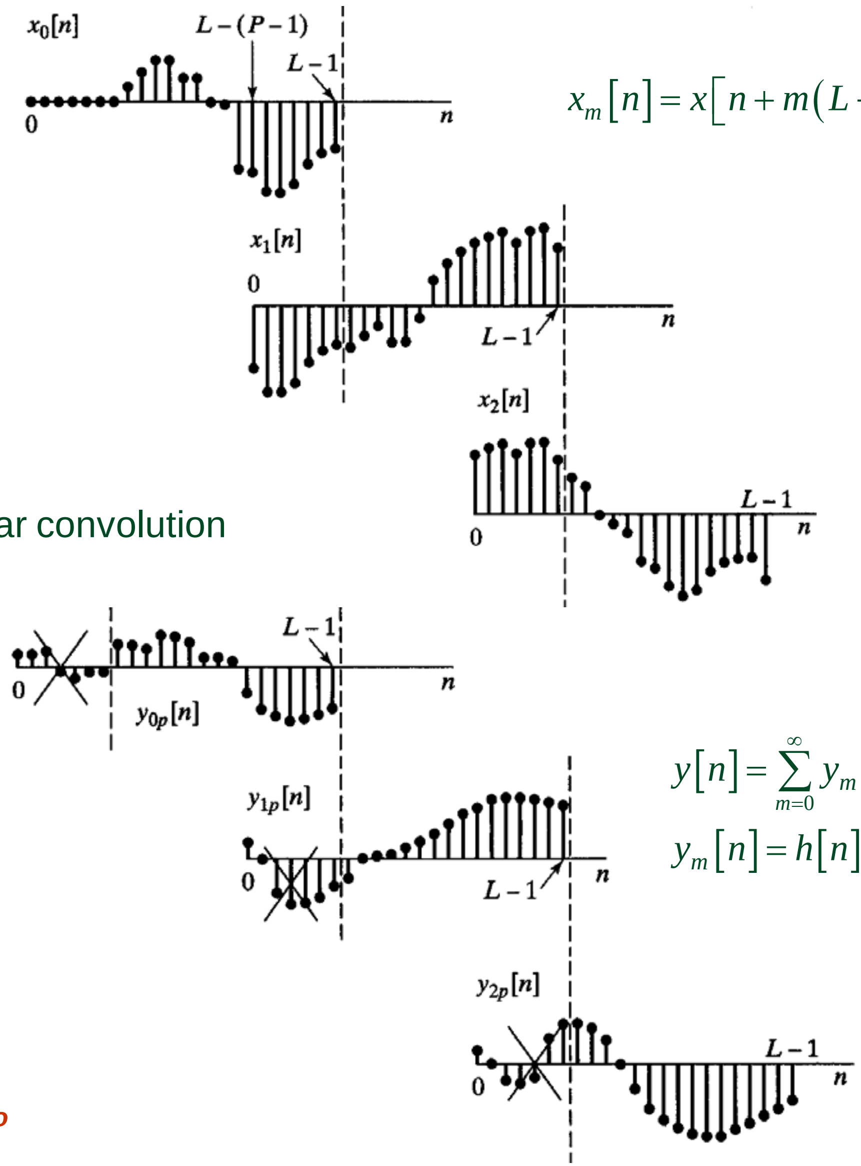
N-point circular convolution



$$y[n] = \sum_{m=0}^{\infty} y_m[n - mL]$$

$$y_m[n] = h[n] * x_m[n]$$

– Overlap-save method $y[n]=h[n]*x[n]$



$$x_m[n]=x[n+m(L-p+1)-p+1]$$
$$0\le n\le L-1$$

N-point circular convolution

$$y[n]=\sum_{m=0}^{\infty} y_m[n-m(L-p+1)+p-1]$$
$$y_m[n]=h[n]*x_m[n]$$

Homework #7 :
8 - 29, - 31, - 34, - 36, - 39, - 49, & - 64
Due by June 9

- The discrete Cosine transform (DCT)
  - General class of finite-length transform

$$A[k]=\sum_{n=0}^{N-1}x[n]\,\phi_k^*[n]$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}A[k]\,\phi_k[n]$$

where $\{\phi_k[n]\}$ are referred to the basis sequences s.t. $\frac{1}{N}\sum_{n=0}^{N-1}\phi_k[n]\phi_m^*[n]=\begin{cases}1, & m=k\\ 0, & m\neq k\end{cases}$

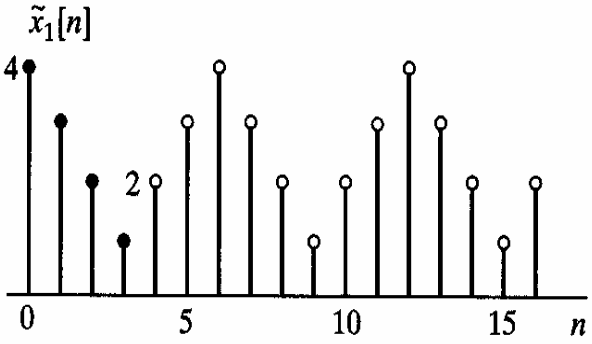
e.g. DFT: $\phi_k[n]=e^{j\frac{2\pi kn}{N}}$

  - DCT definition
    - The basis sequences $\{\phi_k[n]\}$ are cosine functions $\cos\frac{\pi kn}{N}$
    - Assume that $x[n]$ are periodic and even symmetric
  - Type-I periodic sequence: even periodic symmetry about *n=0, N-1, 2(N-1),* · · ·

$$\tilde{x}_1[n]=x_\alpha\left[((n))_{2N-2}\right]+x_\alpha\left[((-n))_{2N-2}\right]$$

$$x_\alpha[n]=\alpha[n]x[n]$$

$$\alpha[n]=\begin{cases}\frac{1}{2} & n=0 \text{ and } N-1\\ 1 & 1\le n\le N-2\end{cases}$$



- DCT-1 is defined by

$$X_1^c[k] = 2\sum_{n=0}^{N-1} \alpha[n] x_1[n] \cos\frac{\pi k n}{N-1}, \quad 0 \le k \le N-1$$

$$x_1[n] = \frac{1}{N-1}\sum_{k=0}^{N-1} \alpha[k] X_1^c[k] \cos\frac{\pi k n}{N-1}, \quad 0 \le n \le N-1$$

– Type-II periodic sequence: periodic with period *2N*

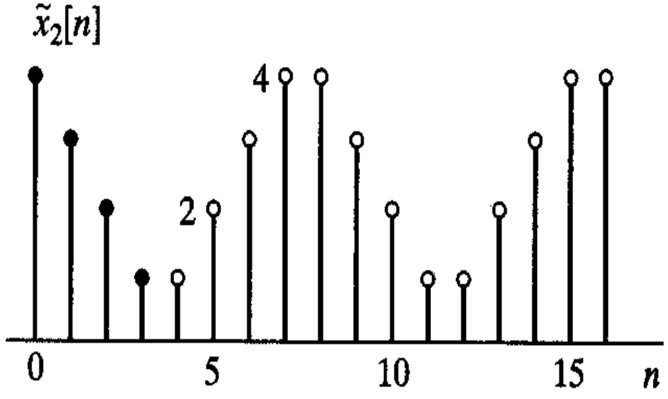
$$\tilde{x}_2[n] = x\left[((n))_{2N}\right] + x\left[((-n-1))_{2N}\right]$$

- DCT-2 is defined by

$$X_2^c[k] = 2\sum_{n=0}^{N-1} x_2[n] \cos\frac{\pi k(2n+1)}{2N}, \quad 0 \le k \le N-1$$

$$x_2[n] = \frac{1}{N}\sum_{k=0}^{N-1} \beta[k] X_2^c[k] \cos\frac{\pi k(2n+1)}{2N}, \quad 0 \le n \le N-1$$

$$\beta[k] = \begin{cases} \frac{1}{2}, & k = 0 \\ 1, & 1 \le k \le N-1 \end{cases}$$



– Relationship between DFT and DCT-1

$$x_1[n] = x_\alpha\left[((n))_{2N-2}\right] + x_\alpha\left[((-n))_{2N-2}\right] = \tilde{x}_1[n],\ 0 \le n \le 2N-3$$

- *2(N-1)*-point DFT of $x_1[n]$:

$$X_1[k] = X_\alpha[k] + X_\alpha^*[k] = 2\,\mathrm{Re}\{X_\alpha[k]\},\quad 0 \le k \le 2N-3$$

$$X_\alpha[k] = 2\sum_{n=0}^{N-1}\alpha[n]x[n]\cos\frac{2\pi kn}{2(N-1)}$$

$$= X_1^c[k]$$

- The DCT-1 of *N*-point sequence is identical to *2(N-1)*-point DFT of $x_1[n]$.

$$x_1[n] = \frac{1}{2(N-1)}\sum_{k=0}^{2N-3} X_1[k]e^{j\frac{2\pi kn}{2(N-1)}}$$

$$= \frac{1}{2(N-1)}\left[\sum_{k=0}^{N-1} X_1^c[k]e^{j\frac{\pi kn}{(N-1)}} + \sum_{k=N}^{2N-3} X_1^c[2(N-1)-k]e^{j\frac{\pi kn}{(N-1)}}\right]$$

$$= \frac{1}{2(N-1)}\left[\sum_{k=0}^{N-1} X_1^c[k]e^{j\frac{\pi kn}{N-1}} + \sum_{k=1}^{N-2} X_1^c[k]e^{j\frac{\pi kn}{N-1}}\right]$$

$$= \frac{1}{2(N-1)}\left[\sum_{k=1}^{N-2} X_1^c[k]\left(e^{j\frac{\pi kn}{N-1}} + e^{-j\frac{\pi kn}{N-1}}\right) + X_1^c[0] + X_1^c[N-1]e^{j\frac{\pi(N-1)n}{N-1}}\right]$$

$$= \frac{1}{N-1}\sum_{k=0}^{N-1}\alpha[k]X_1^c[k]\cos\frac{\pi nk}{N-1},\ 0 \le n \le N-1;\quad \alpha[k] = \begin{cases}\frac{1}{2}, & k = 0 \text{ and } N-1\\ 1, & 1 \le k \le N-2\end{cases}$$

$$\sum_{k=N}^{2N-3} X_1^c[2N-2-k]e^{j\frac{\pi kn}{(N-1)}}$$

$$= \sum_{m=1}^{N-2} X_1^c[m]e^{j\frac{(2N-2-m)\pi n}{N-1}}$$

$$= \sum_{m=1}^{N-2} X_1^c[m]e^{j\frac{-nm\pi}{N-1}}$$

- Note : DCT-1 involves only real-valued coefficients

  $\Rightarrow$ reduce the implementation complexity.

– Relationship between DFT and DCT-2

$$x_2[n] = x\left[((n))_{2N}\right] + x\left[((-n-1))_{2N}\right] = \tilde{x}_2[n],\ 0 \le n \le 2N-1$$

- *2N*-point DFT of *2N*-point sequence $x_2[n]$

$$X_2[k] = X[k] + X^*[k]e^{j2\pi k/2N}$$
$$= e^{j\pi k/2N}\left(X[k]e^{-j\pi k/2N} + X^*[k]e^{j\pi k/2N}\right)$$
$$= 2e^{j\pi k/2N}\,\mathrm{Re}\left\{X[k]e^{-j\pi k/2N}\right\},\quad 0 \le k \le 2N-1$$

$$\mathrm{Re}\left\{X[k]e^{-j\pi k/2N}\right\} = \sum_{k=0}^{N-1} x[n]\cos\frac{\pi k(2n+1)}{2N}$$

$$\Rightarrow X_2^c[k] = 2\,\mathrm{Re}\left\{X[k]e^{-j\pi k/2N}\right\};\quad 0 \le k \le N-1$$

$$\Rightarrow X_2^c[k] = X_2[k]e^{-j\pi k/2N};\quad 0 \le k \le N-1$$

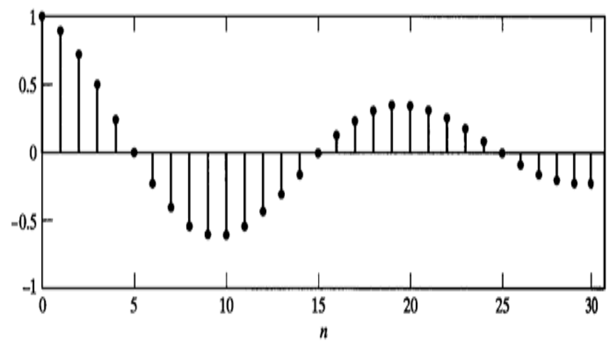
$$X_2^c[2N-k] = 2\sum_{k=0}^{N-1} x[n]\cos\frac{\pi(2N-k)(2n+1)}{2N} = -X_2^c[k];\quad 0 \le k \le 2N-1$$

$$\Rightarrow X_2[k] = \begin{cases} X_2^c[0], & k=0 \\ X_2^c[k]e^{j\pi k/2N}, & k=1,\cdots,N-1 \\ 0, & k=N \\ -X_2^c[2N-k]e^{j\pi k/2N}, & k=N+1,\cdots,2N-1 \end{cases} \quad \Leftrightarrow \quad x_2[n] = \frac{1}{2N}\sum_{k=0}^{2N-1} X_2[k]e^{j2\pi k/2N},\quad n=0,1,\cdots,2N-1$$

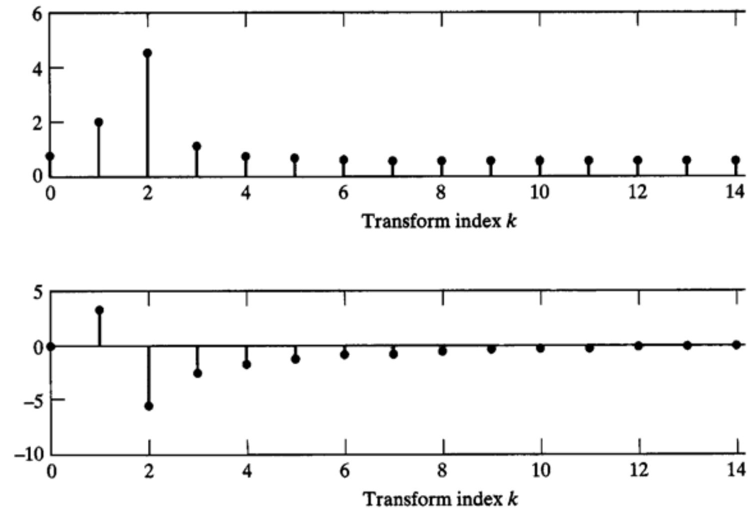
- Energy compaction property of DCT-2
  - Since DCT-2 coefficients of a finite-length sequence are concentrated in the low indices than DFT, DCT-2 is often used for data compression applications in preference to DFT

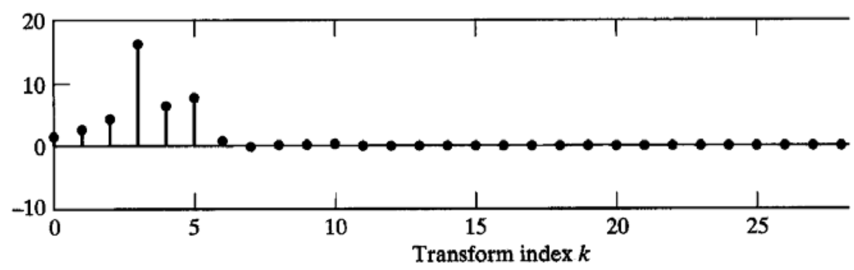
$$\sum_{n=0}^{N-1}\left|x[n]\right|^2=\frac{1}{N}\sum_{k=0}^{N-1}\beta[k]\left|X_2^c[k]\right|^2$$

Example $x[n]=a^n\cos(\omega_0 n+\phi)$



32 - point DFT





$\Leftarrow$ 32 – point DCT

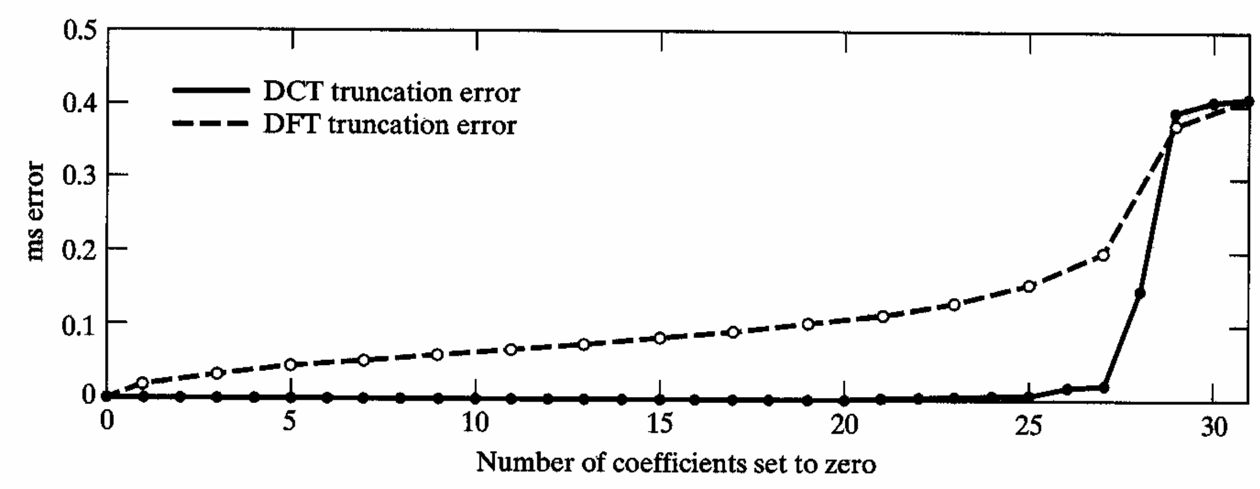
- Energy concentration can be examined by measuring the truncation error of the transformations.

$$x_m[n]=\frac{1}{N}\sum_{k=0}^{N-1}T_m[k]X[k]e^{j2\pi k/2N},\quad n=0,1,\cdots,2N-1;\quad m=1,3,5,\cdots,N-1$$

$$T_m[k]=\begin{cases}1, & 0\le k\le (N-1-m)/2\\ 0, & (N+1-m)/2\le k\le (N-1+m)/2\\ 1, & (N+1+m)/2\le k\le N-1\end{cases}$$

$$x_m^c[n]=\frac{1}{N}\sum_{k=0}^{N-1-m}\beta[k]X_2^c[k]\cos\frac{\pi k(2n+1)}{2N};\quad 0\le n\le N-1$$

MS truncation error : $E_{DFT}[m]=\frac{1}{N}\sum_{n=0}^{N-1}\left|x[n]-x_m[n]\right|^2$



Discrete Fourier transform

- Discrete Fourier series representation of periodic sequences

  – Let $\tilde{x}[n]$ be a periodic sequence with period *N* such that $\tilde{x}[n]=\tilde{x}[n+mN]$ for any integer *m*

  Example A periodic complex exponential

  $$e_k[n]=e^{j\frac{2\pi kn}{N}}=e_k[n+mN]; \quad k \text{ is an interger}$$

  Note that $e_k[n]=e_{k+\ell N}[n]$ for any integer ℓ

  – A set of *N* periodic complex exponentials $\{e_k[n]\}$, $k=0,1,2,...N-1,$ defines all the frequency components

  $$\Rightarrow \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

  $$\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi mn}{N}}$$

  – Fourier series representation

  $$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\widetilde{X}[k]e^{j\frac{2\pi}{N}kn}$$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j\frac{2\pi kn}{N}}e^{-j\frac{2\pi}{N}rn}$$

$$= \sum_{k=0}^{N-1}\tilde{X}[k]\frac{1}{N}\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}(k-r)n}$$

Since $\frac{1}{N}\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases}1, & k-r=\ell N\\ 0, & \text{otherwise}\end{cases}$

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{j\frac{2\pi}{N}rn} = \tilde{X}[r]$$

- The Fourier series coefficient can be obtained by

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{X}[k+N] = \sum_{n-0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}(k+N)n} = \sum_{n-0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}kn}e^{-j2\pi n} = \tilde{X}[k]$$

- The Fourier series coefficients is a periodic sequence

– Discrete Fourier series (DFS)

- Define by $\tilde{x}[n] \xleftrightarrow{DFS} \tilde{X}[k]$

Analysis equation: $\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]w_N^{kn}; \qquad w_N = e^{-j\frac{2\pi}{N}}$

Synthesis equation: $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]w_N^{-kn}$

Example Periodic pulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - mN] = \begin{cases} 1, & n = mM \\ 0, & \text{otherwise} \end{cases}$$

Since $\tilde{x}[n] = \delta[n], \quad 0 \le n \le N-1$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] w_N^{-kn} = 1$$

$$\Rightarrow \tilde{x}[n] = \frac{1}{N} \sum \tilde{X}[k] w_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} w_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn}$$

Example Duality in DFS

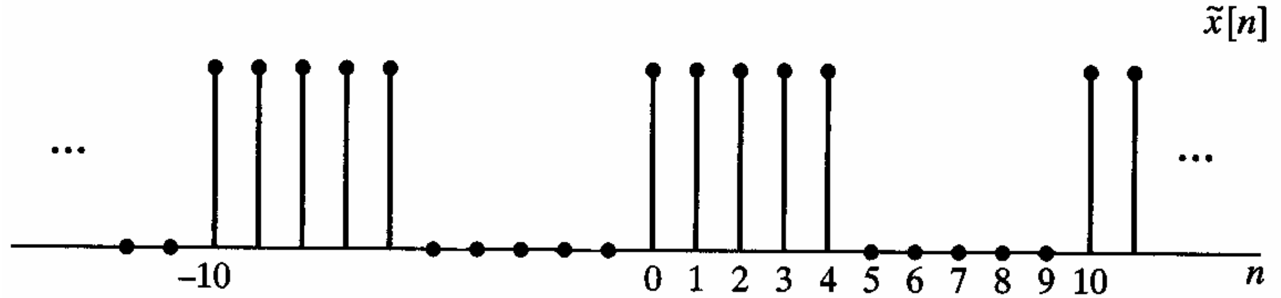
Let the DFS coefficient be a periodic impulse train

$$\tilde{Y}[k] = N \sum_{r=-\infty}^{\infty} \delta[k - rN]$$

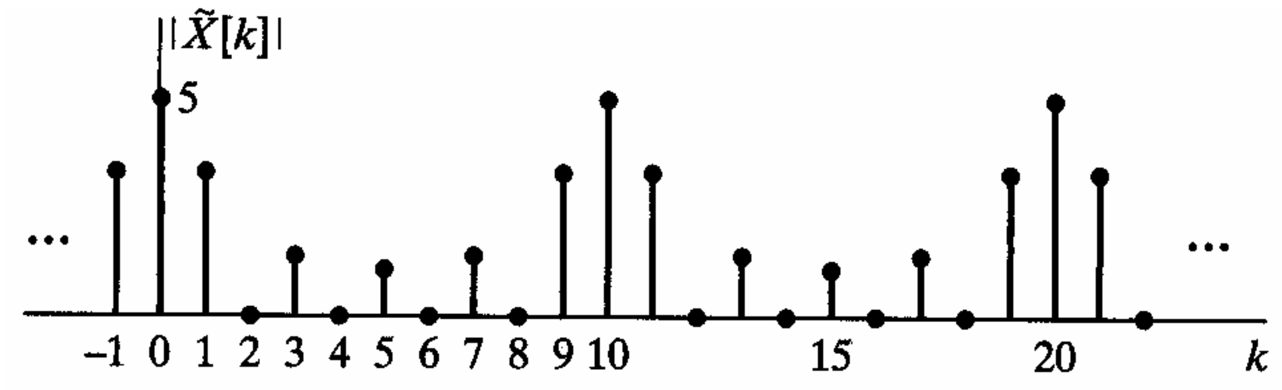
$$\Rightarrow \tilde{y}[n] = \sum_{k=0}^{N-1} N\delta[k] w_N^{-kn} = w_N^{-0} = 1, \quad \forall n$$

$$\tilde{Y}[k] = N\tilde{x}[n] \leftrightarrow \tilde{y}[n] = \tilde{X}[n]$$

Example Periodic rectangular pulse train



$$X[k] = \sum_{n=0}^{4} W_{10}^{nk} = \sum_{n=0}^{4} e^{-j\frac{2\pi}{10}kn} = \frac{1-W_{10}^{5k}}{1-W_{10}^{k}} = \frac{\sin\frac{\pi k}{2}}{\sin\frac{\pi k}{10}} e^{-j\frac{4\pi k}{10}}$$



- Any periodic sequence can be represented as a sum of complex exponential sequences

- Properties of DFS

  - Linearity

$$a\tilde{x}_1[n]+b\tilde{x}_2[n] \overset{DFS}{\longleftrightarrow} a\tilde{X}[k]+b\tilde{X}[k]$$

  - Shift of a sequence

$$\sum_{n=0}^{N-1}\tilde{x}[n-m]e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}\tilde{x}[n-m]e^{-j\frac{2\pi}{N}(n-m)}e^{-j\frac{2\pi}{N}}=\tilde{X}[k]e^{-j\frac{2\pi}{N}km}=\tilde{X}[k]W_N^{km}$$

$$\Rightarrow \quad \tilde{x}[n-m] \overset{DFS}{\longleftrightarrow} \tilde{X}[k]W_N^{km}$$

$$\tilde{x}[n]W_N^{-n\ell} \overset{DFS}{\longleftrightarrow} \tilde{X}[k-\ell]$$

  - Duality

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]w_N^{-kn} \Rightarrow N\tilde{x}[-n]=\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{kn}$$

$$\Rightarrow N\tilde{x}[-k]=\sum_{k=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

If $\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k],\ \tilde{X}[n] \overset{DFS}{\longleftrightarrow} N\tilde{x}[-k]$

  - Symmetry properties

$$\tilde{x}^*[n] \overset{DFS}{\longleftrightarrow} \tilde{X}^*[-k]$$

$$\tilde{x}^*[-n] \overset{DFS}{\longleftrightarrow} \tilde{X}^*[k]$$

$\Rightarrow$ If $\tilde{x}[n]$ is real, $\tilde{X}[k] = \tilde{X}^*[-k]$

$$\left|\tilde{X}[k]\right| = \left|\tilde{X}[-k]\right|$$

– Periodic convolution

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]$$

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]\right)W_N^{kn}$$

$$= \sum_{m=0}^{N-1} \tilde{x}_1[m]\sum_{n=0}^{N-1} \tilde{x}_2[n-m]W_N^{kn}$$

$$= \tilde{X}_2[k]\sum_{m=0}^{N-1} \tilde{x}_1[m]W_N^{km} = \tilde{X}_1[k]\tilde{X}_2[k]$$

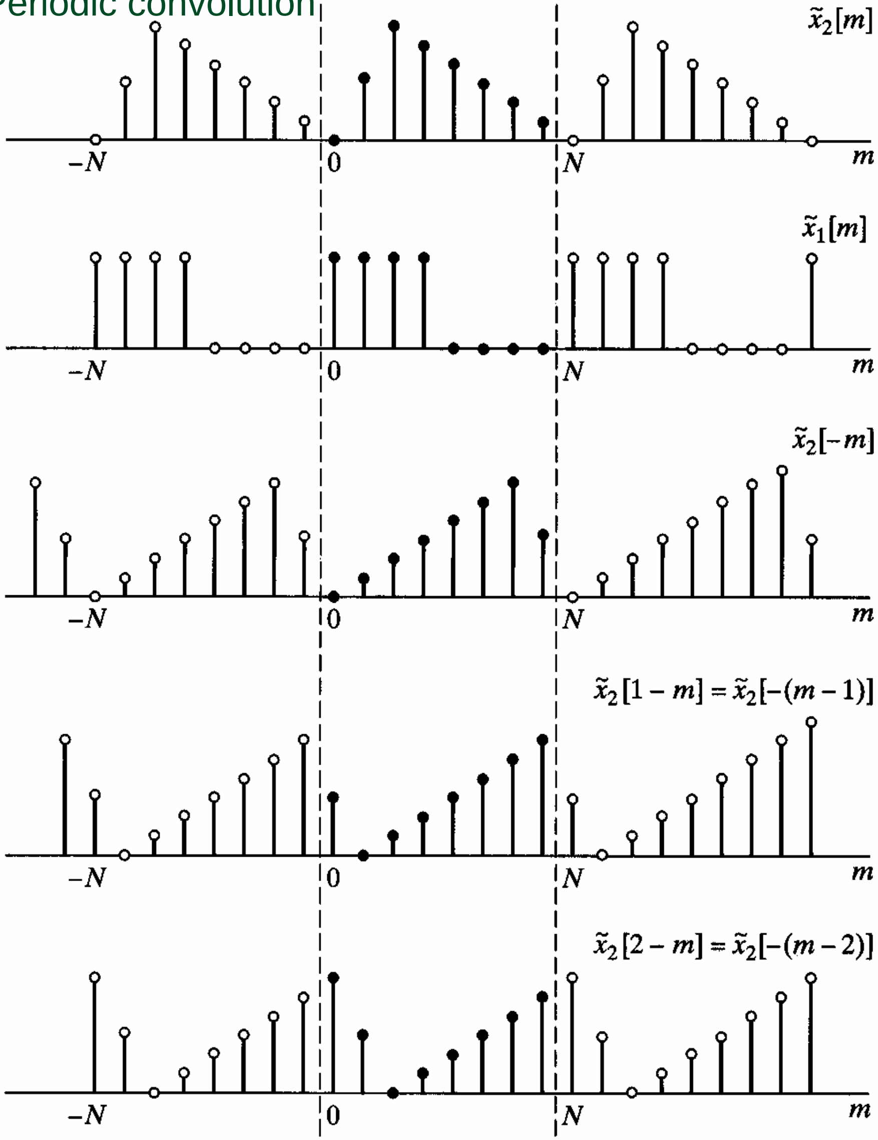
$$\Rightarrow \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \xleftrightarrow{DFS} \tilde{X}_1[k]\tilde{X}_2[k]$$

- Duality

$$\tilde{x}_1[n]\tilde{x}_2[n] \xleftrightarrow{DFS} \frac{1}{N}\sum_{m=0}^{N-1} \tilde{X}_1[m]\tilde{X}_2[k-m]$$

- Difference between aperiodic convolution and periodic convolution
  - The sum is over an interval $0 \le m \le N-1$
  - $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$

Example   Periodic convolution



– Properties of DFS

	Periodic Sequence (Period N)	DFS Coefficients (Period N)
1.	$\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2.	$\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3.	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4.	$\tilde{X}[n]$	$N\tilde{x}[-k]$
5.	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6.	$W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7.	$\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8.	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9.	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$

(*continued*)

– Properties of DFS (cont.)

Periodic Sequence (Period N)	DFS Coefficients (Period N)
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{Re}\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{Im}\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{Re}\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{Im}\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{Re}\{\tilde{X}[k]\} = \mathcal{Re}\{\tilde{X}[-k]\} \\ \mathcal{Im}\{\tilde{X}[k]\} = -\mathcal{Im}\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \sphericalangle\tilde{X}[k] = -\sphericalangle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{Re}\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{Im}\{\tilde{X}[k]\}$

- Fourier transform of periodic signals
  - A sequence should be absolutely summable to guarantee the uniform convergence of its Fourier transform
  - Periodic signals are not absolutely summable
  - The F.T. of a periodic signal can be interpreted to be an impulse train in the frequency domain with the impulse values proportional to the DFS coefficients for the sequence
  - If $\tilde{x}[n]$ is periodic with period *N*, the Fourier transform of $\tilde{x}[n]$ is defined as

$$\tilde{x}[n] \Leftrightarrow \tilde{X}\left(e^{j\omega}\right) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

    - Although the F.T. of a periodic sequence does not converge in the normal sense, the introduction of impulses enables to use the F. T.
    - Note: $\tilde{X}\left(e^{j\omega}\right)$ has the necessary periodicity with period 2π

$$\frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\tilde{X}\left(e^{j\omega}\right)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{0-\varepsilon}^{2\pi-\varepsilon}\frac{2\pi}{N}\sum_{k}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega; \quad 0 < \varepsilon < 2\pi / N$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\varepsilon}^{2\pi-\varepsilon}\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]e^{j\frac{2\pi}{N}kn} = \tilde{x}[n]$$

Example: Fourier transform of a periodic impulse train

$$\tilde{p}[n]=\sum_{m=-\infty}^{\infty}\delta[n-mN]\Rightarrow \tilde{P}[k]=1,\forall k$$

$$\Rightarrow \tilde{P}\left(e^{j\omega}\right)=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left[\omega-\frac{2\pi k}{N}\right]$$

– For a finite-length sequence $x[n]$, $\left(i.e., x[n]=0 \text{ for } n<0 \text{ and } n\geq N\right)$

$$\tilde{x}[n]=x[n]*\tilde{p}[n]$$

$$=\sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\delta[\ell-mN]x[n-\ell]=\sum_{m=-\infty}^{\infty}x[n-mN]$$

$$\tilde{X}\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)\tilde{P}\left(e^{j\omega}\right)$$

$$=X\left(e^{j\omega}\right)\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$$

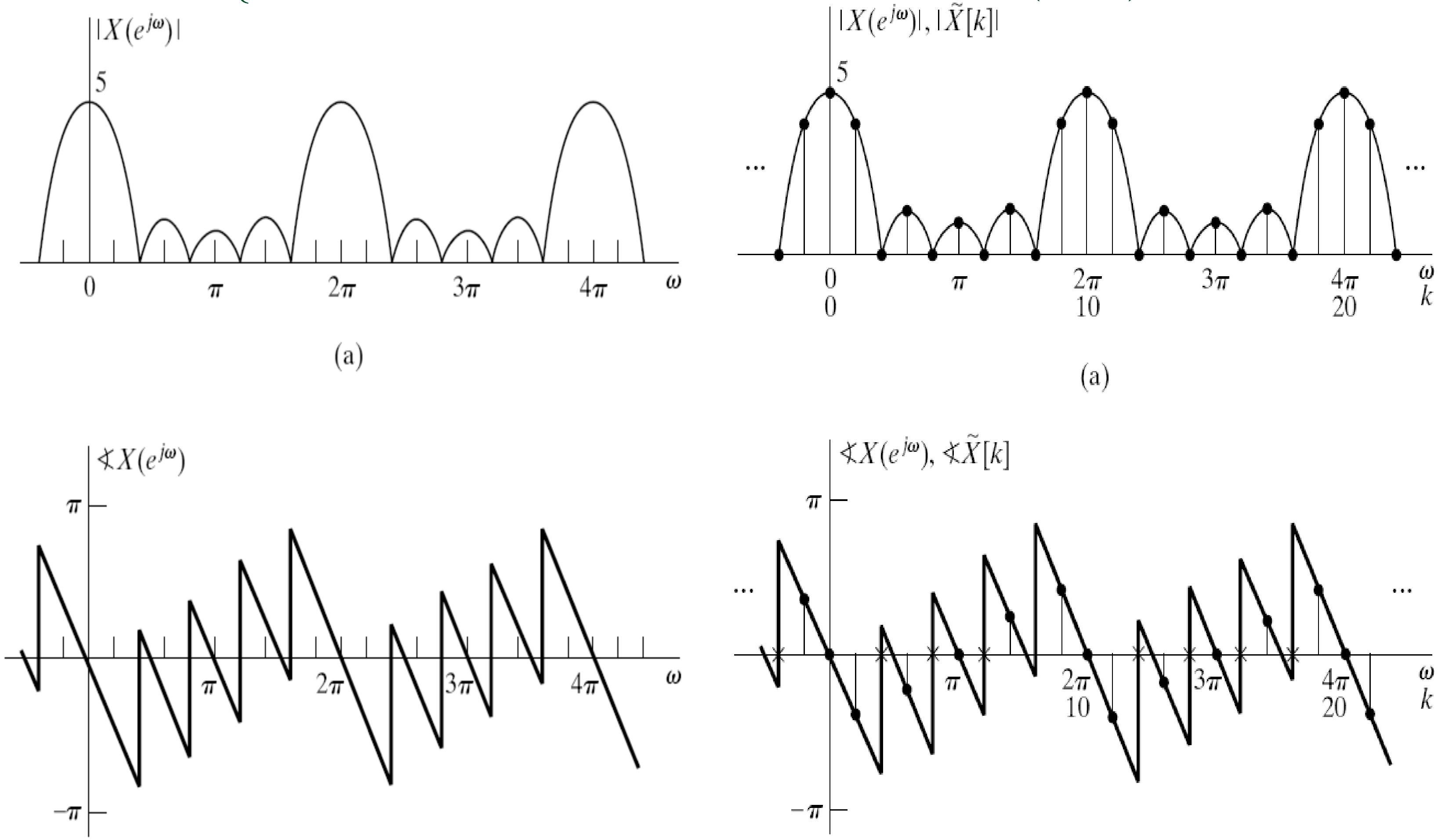
$$=\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}X\left(e^{j\frac{2\pi k}{N}}\right)\delta\left(\omega-\frac{2\pi k}{N}\right)$$

$$\Rightarrow \tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi}{N}k}=X\left(e^{j\frac{2\pi}{N}k}\right)=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi}{N}k}$$

Thus, $\tilde{X}[k]$ is obtained by sampling the Fourier transform of the finite sequence $x[n]$

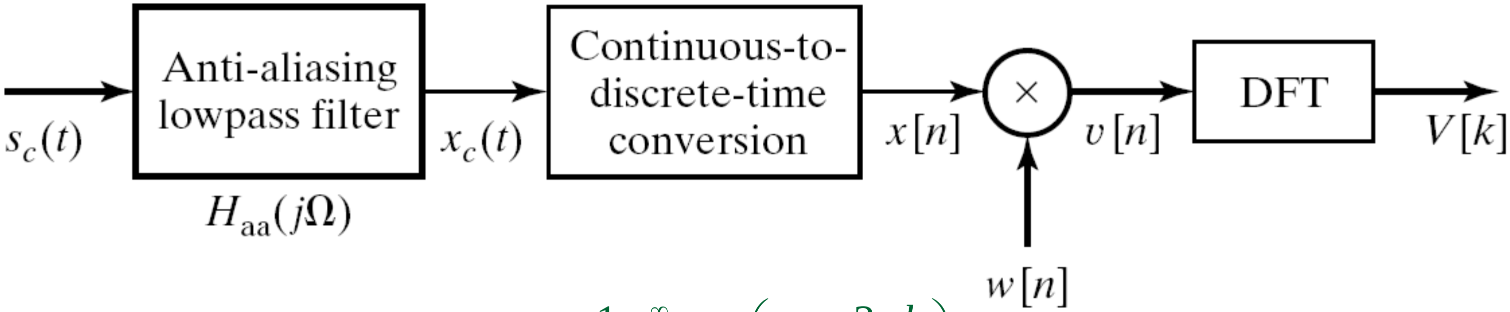
Example: Relationship between the DFS coefficients and the FT of one period

$$x[n]=\begin{cases}1; & 0\le n\le 4\\ 0; & \text{otherwise}\end{cases} \Rightarrow X\left(e^{j\omega}\right)=\sum_{n=0}^{4}e^{-j\omega n}=\frac{\sin(5\omega/2)}{\sin(\omega/2)}e^{-j2\omega}$$



Fourier Analysis of Signals Using DFT

- Discrete Fourier transform (DFT) can analyze the frequency content of continuous-time signal



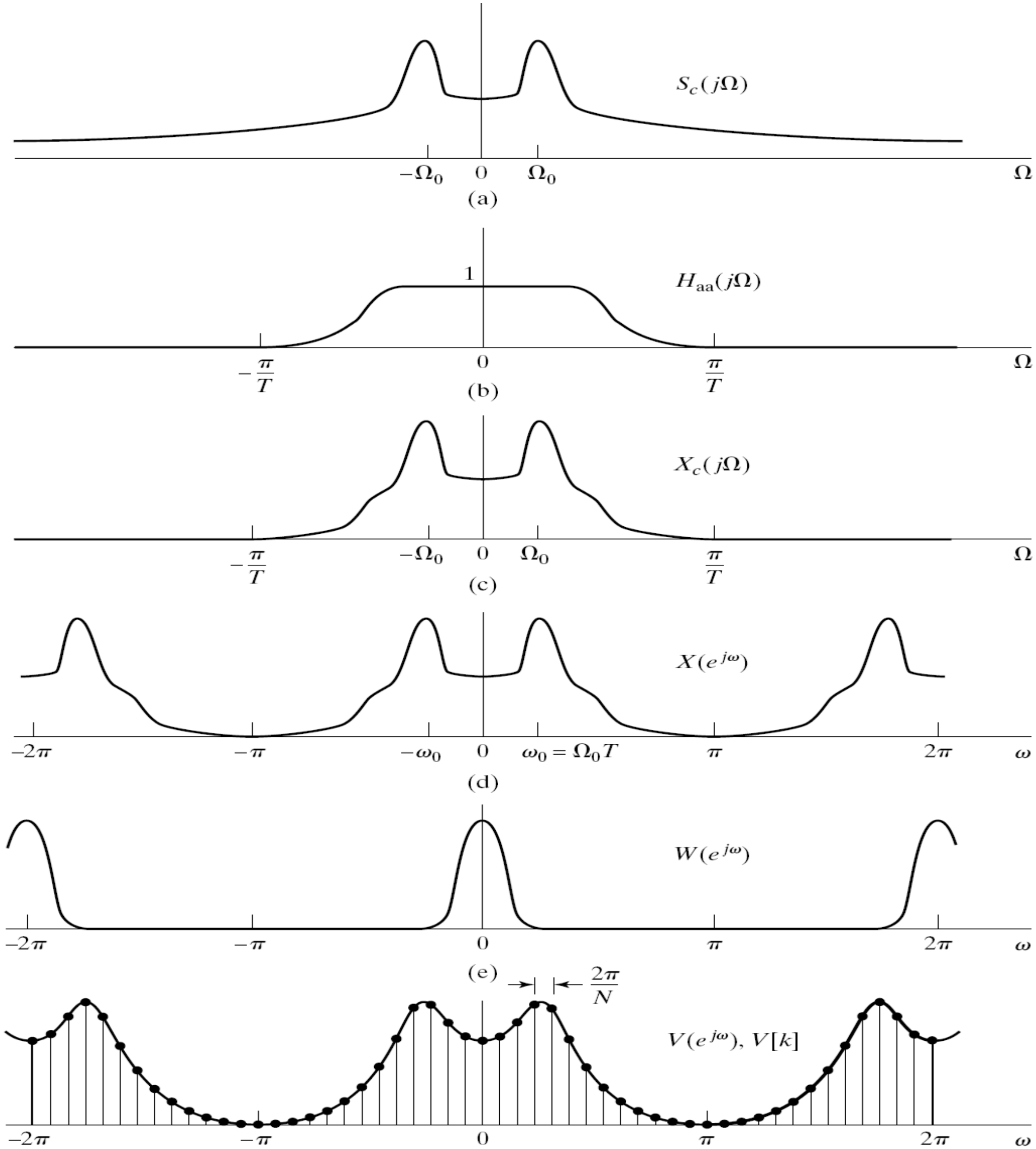
$$X\left(e^{j\omega}\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(\frac{\omega}{T}+\frac{2\pi k}{T}\right)$$

$$V\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\theta}\right)W\left(e^{j(\omega-\theta)}\right)d\theta$$

$$v[n]=x[n]w[n]$$

$$V[k]=\sum_{n=0}^{N-1}v[n]e^{-j\frac{2\pi kn}{N}}=V\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}};\quad \omega=\Omega T$$

– Example


$S_c(j\Omega)$
$-\Omega_0$
0
Ω_0
Ω
(a)
$H_{aa}(j\Omega)$
1
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
(b)
$X_c(j\Omega)$
(c)
$X(e^{j\omega})$
-2π
$-\pi$
$-\omega_0$
$\omega_0 = \Omega_0 T$
π
2π
ω
(d)
$W(e^{j\omega})$
(e)
$\frac{2\pi}{N}$
$V(e^{j\omega})$, $V[k]$

– Length of the DFT

$$V[k]=\sum_{n=0}^{N-1} v[n]\, e^{-j\frac{2\pi kn}{N}} = V\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}}; \quad \omega=\Omega T$$

$$\Delta\Omega \le \frac{2\pi}{NT} \Rightarrow N \ge \frac{2\pi}{\Delta\Omega T}$$

- Example: T=1/5000 and 10Hz resolution

$$10\cdot 2\pi \le \frac{2\pi}{NT}\Big|_{T=1/5000} \Rightarrow N \ge \frac{5000}{10} = 500 \Rightarrow N = 512 = 2^9$$

– Relationship between DFT values

$$N=512;\ \ T=1/5000$$

$$\text{If}\ \ V[11]=2000(1+j);$$

$$V[k]=V^*[((-k))_N] \Rightarrow V[N-k]=V^*[k]$$

$$\Rightarrow V[512-11]=V[501]=V^*[11]=2000(1-j)$$

$$\Omega_{11}=2\pi\cdot 11\cdot 5000/512 = 2\pi\cdot 107.4$$

$$\Rightarrow X_c(\Omega_{11}) \approx T\cdot V[11] = 0.4(1+j)$$

- DFT analysis of sinusoidal signals
  - Windowing and spectral sampling have an important effect on the analysis of sinusoidal signals using the DFT
  - Effect of windowing

$$s_c(t) = A_0\cos(\Omega_0 t + \theta_0) + A_1\cos(\Omega_1 t + \theta_1)$$
$$\Rightarrow x[n] = A_0\cos(\omega_0 n + \theta_0) + A_1\cos(\omega_1 n + \theta_1)$$

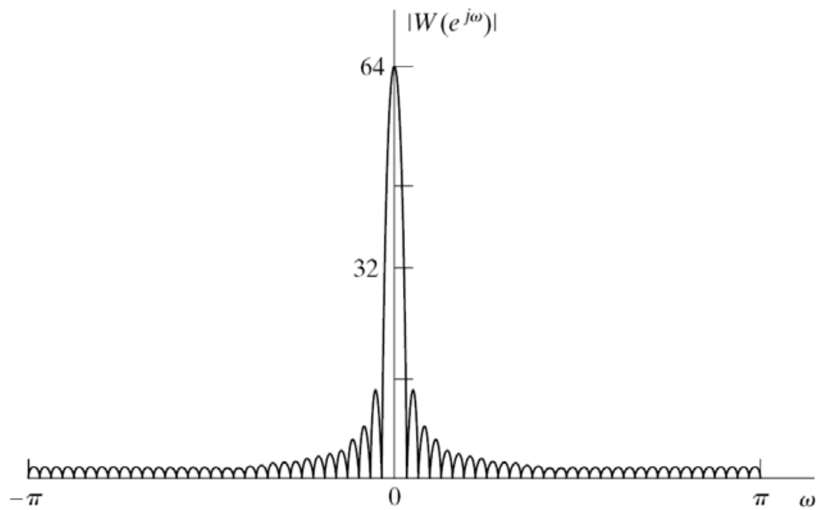
    - After the windowing

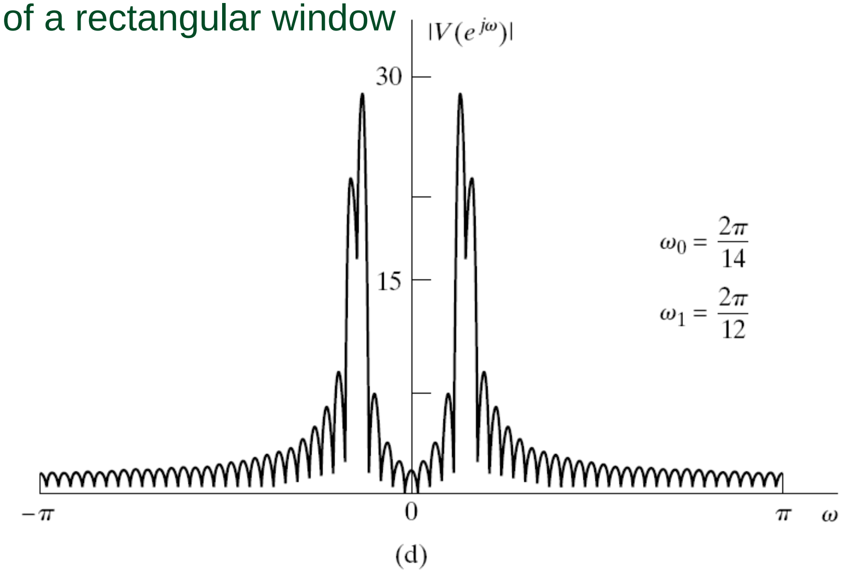
$$\begin{aligned} v[n] &= x[n]w[n] \\ &= A_0 w[n]\cos(\omega_0 n + \theta_0) + A_1 w[n]\cos(\omega_1 n + \theta_1) \\ &= \frac{A_0}{2}\left(w[n]e^{j\omega_0 n+\theta_0} + w[n]e^{-j(\omega_0 n+\theta_0)}\right) + \frac{A_1}{2}\left(w[n]e^{j\omega_1 n+\theta_1} + w[n]e^{-j(\omega_1 n+\theta_1}\right) \end{aligned}$$

$$V\left(e^{j\omega}\right) = \frac{A_0}{2}\left[W\left(e^{j(\omega-\omega_0)}\right)e^{j\theta_0} + W\left(e^{j(\omega+\omega_0)}\right)e^{-j\theta_0}\right] + \frac{A_1}{2}\left[W\left(e^{j(\omega-\omega_1)}\right)e^{j\theta_1} + W\left(e^{j(\omega+\omega_1)}\right)e^{-j\theta_1}\right]$$

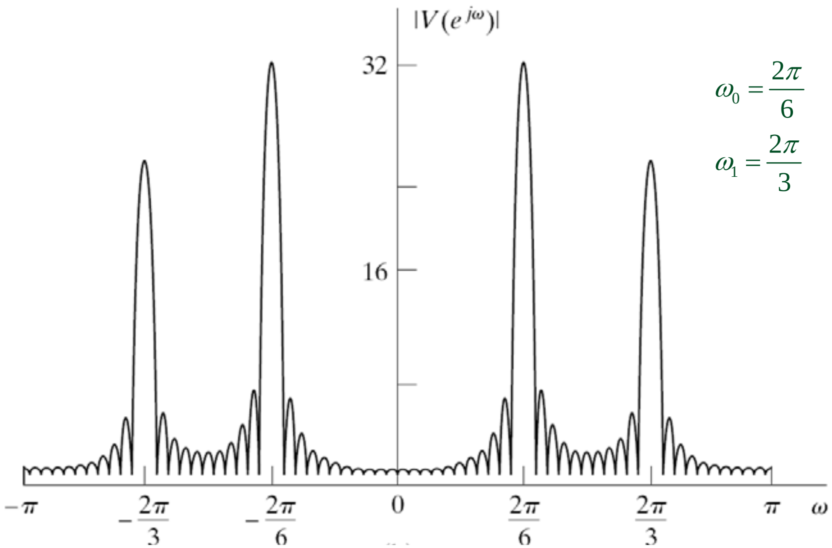
it can be suffered from reduced resolution and leakage of spectrum

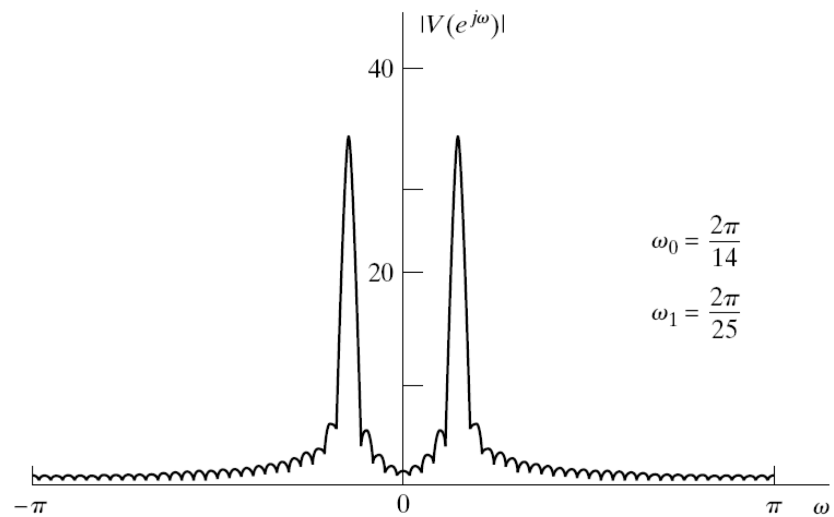
Example Leakage due to the use of a rectangular window



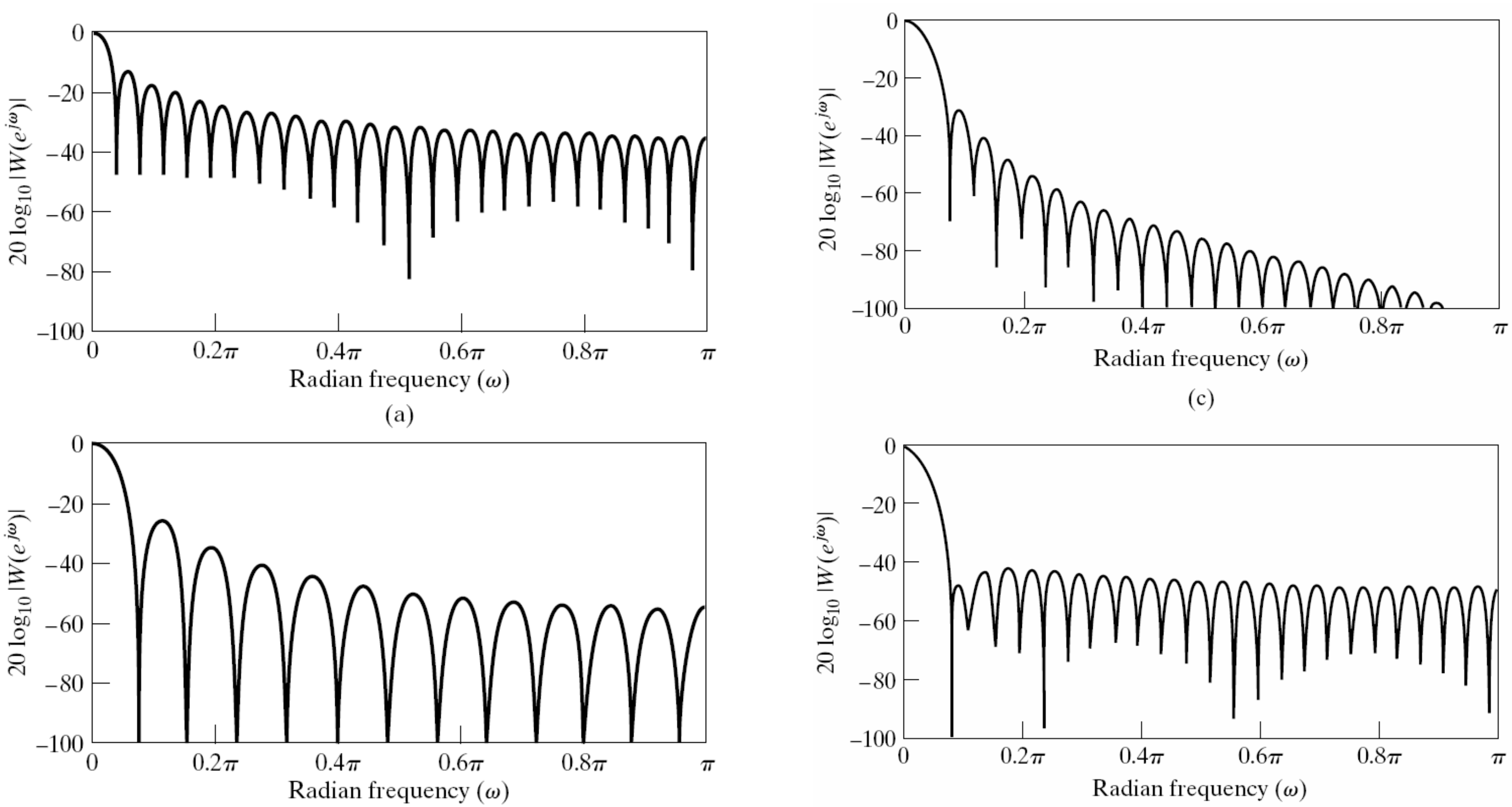


(d)



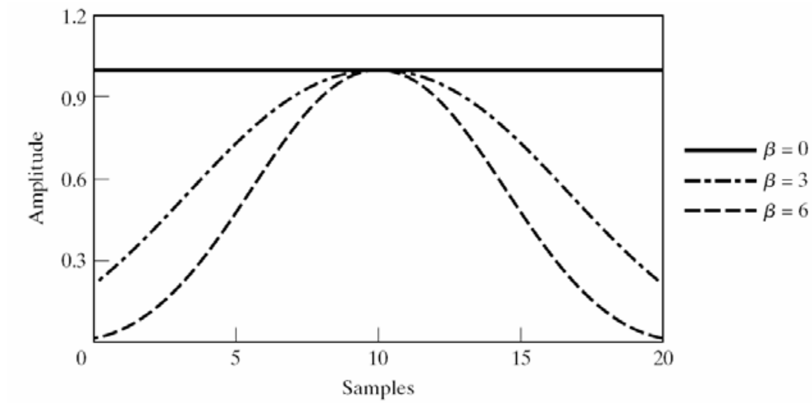


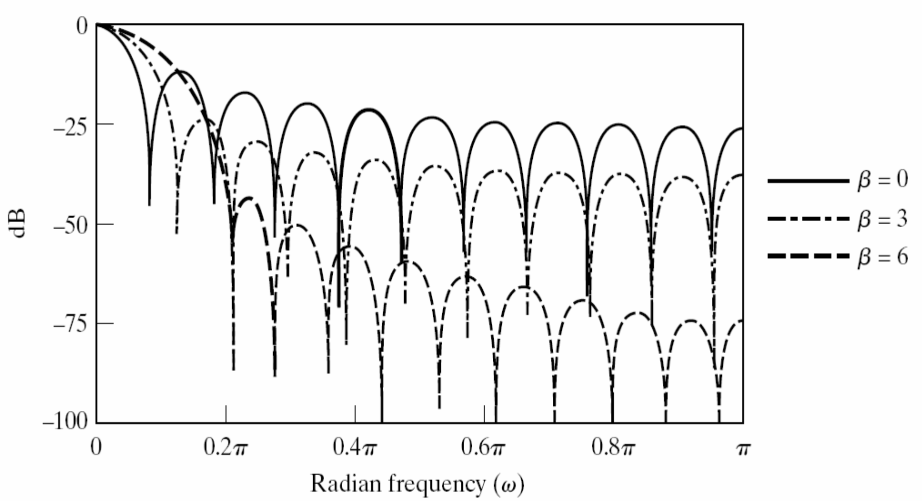
- The resolution is primarily affected by the width of the main lobe of $W\left(e^{j\omega}\right)$
- The degree of leakage depends on the relative amplitude of the main and side lobe of $W\left(e^{j\omega}\right)$
- These are associated with the window length *L* and the shape of the window

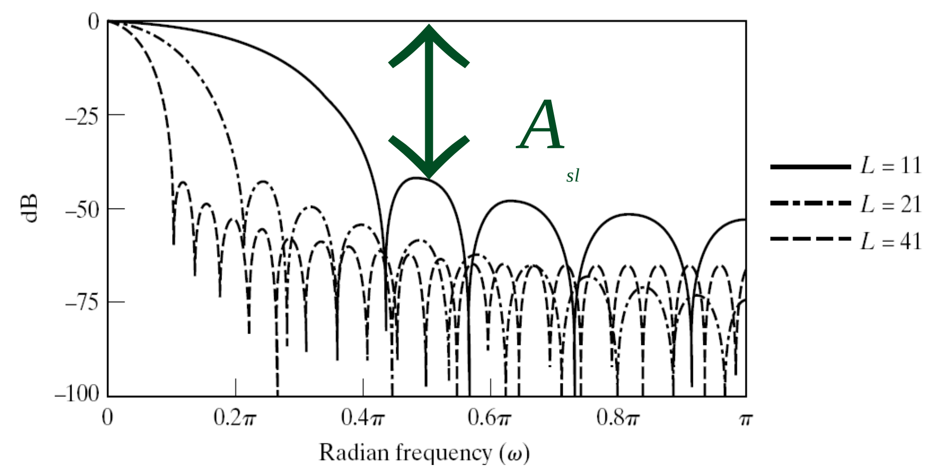


Example: Kaiser window

$$w[n]=\begin{cases}\dfrac{I_0\left[\beta\sqrt{1-\left[\dfrac{n-\alpha}{\alpha}\right]^2}\right]}{I_0(\beta)}, & 0\le n\le L-1,\ \ \alpha=\dfrac{L-1}{2}\\ 0, & \text{otherwise}\end{cases}$$







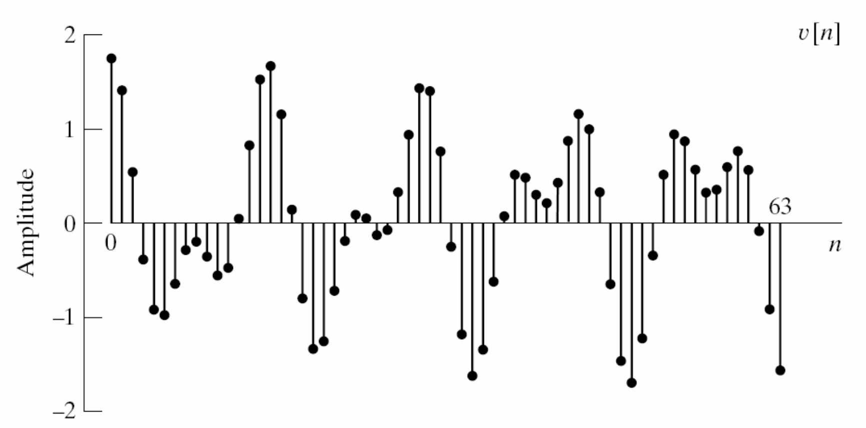
$$L\simeq\frac{24\pi\left(A_{sl}+12\right)}{155\Delta_{ml}}+1$$

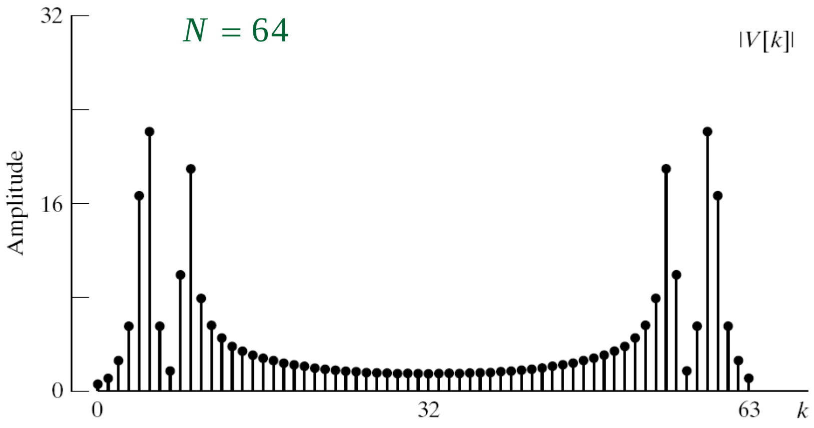
– Effect of spectral sampling

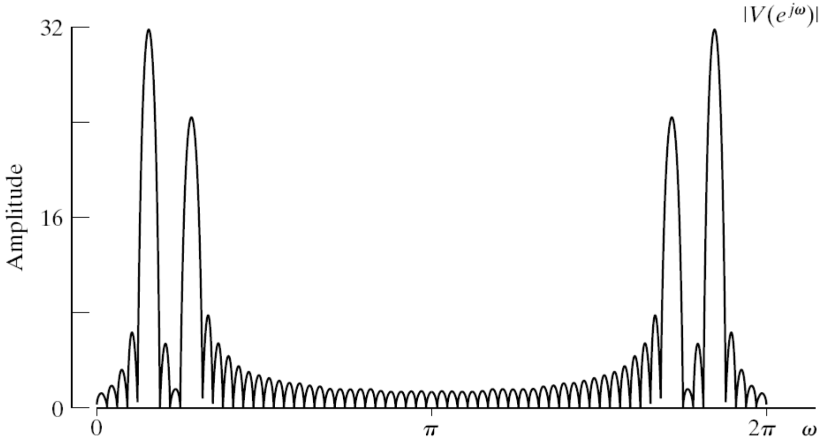
$$V[k]=V\left(e^{j\omega}\right)_{\omega=\frac{2\pi k}{N}};\quad \Omega_k=2\pi k/(NT)$$

- Example

$$v[n]=\cos\frac{2\pi n}{14}+0.75\cos\frac{4\pi n}{15};\quad 0\le n\le 63$$

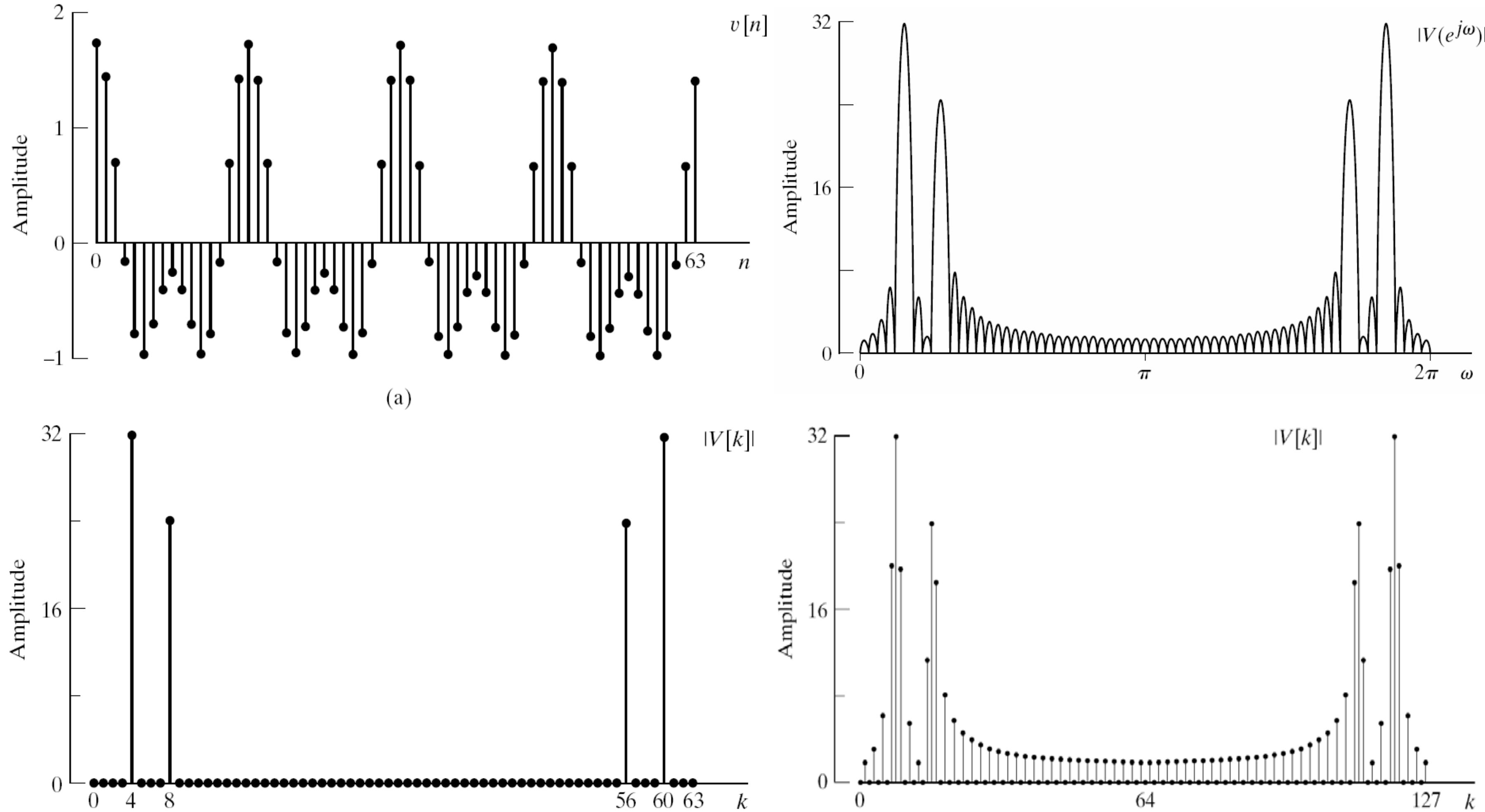






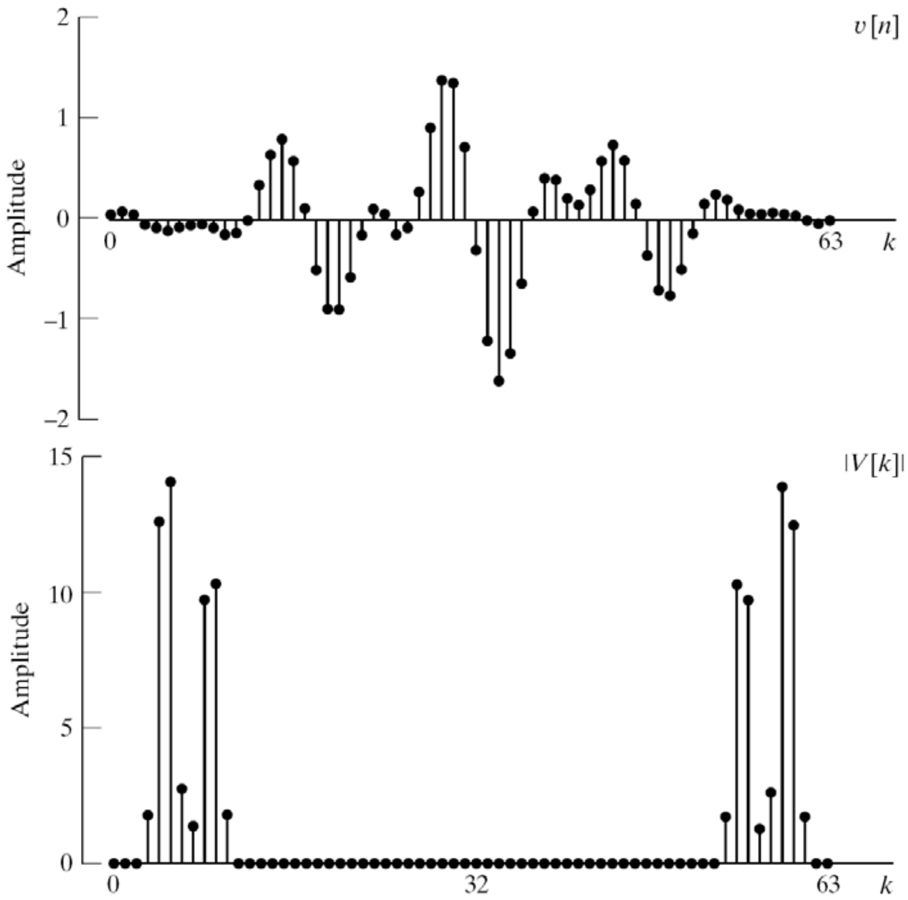
– Spectral sampling with frequencies matching DFT frequencies

- Example $v[n] = \cos\frac{2\pi n}{16} + 0.75\cos\frac{2\pi n}{8}; \quad 0 \le n \le 63$

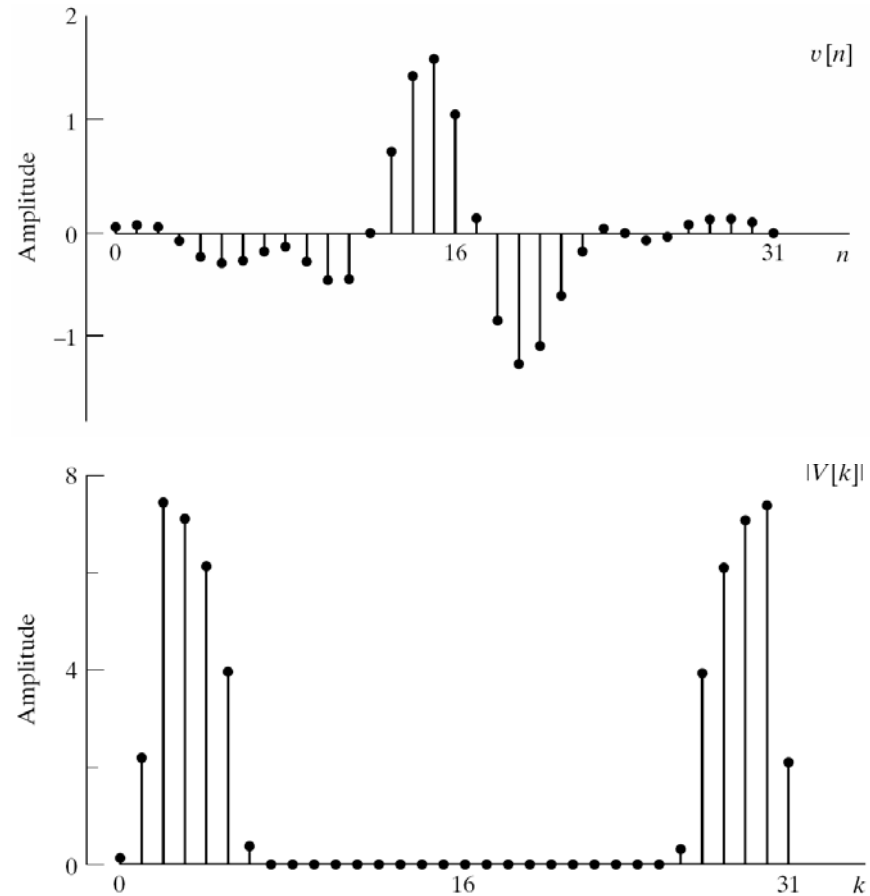


– DFT analysis by Kaiser window

- Example $v[n] = w_K[n]\cos\frac{2\pi n}{14} + 0.75 w_K[n]\cos\frac{4\pi n}{15}; \quad 0 \le n \le 63$



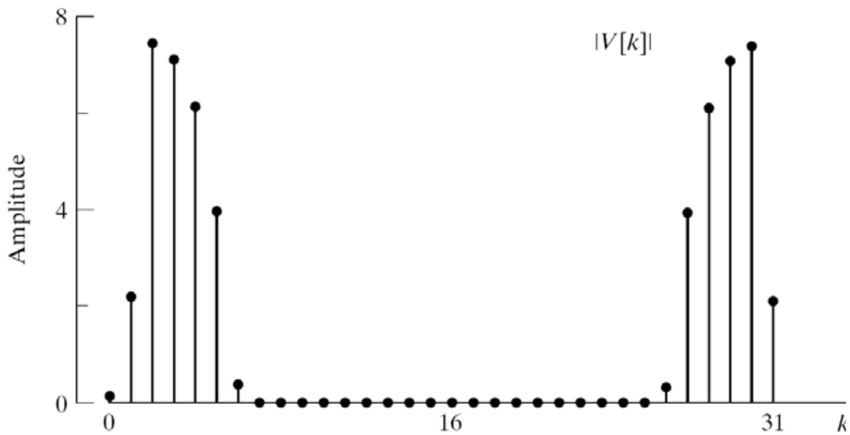
$L = 64$



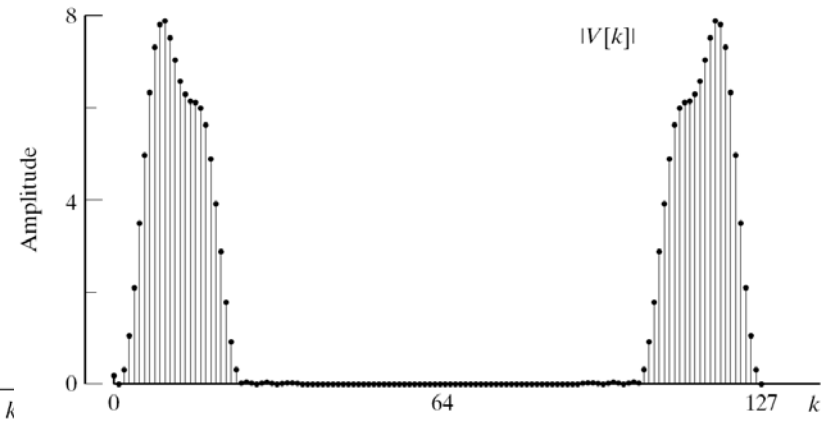
$L = 32$

– DFT analysis with 32-point Kaiser window and zero padding

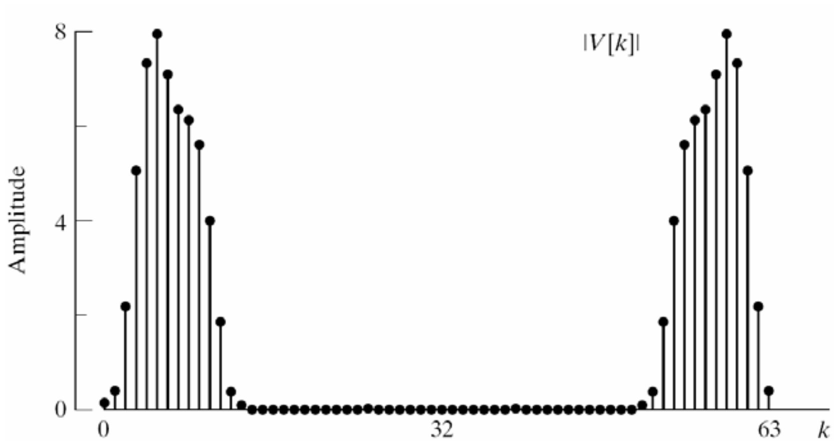
- Example $v[n] = \cos\frac{2\pi n}{14} + 0.75\cos\frac{4\pi n}{15};\ \beta = 5.48,\ L = 32$ Kaiser window



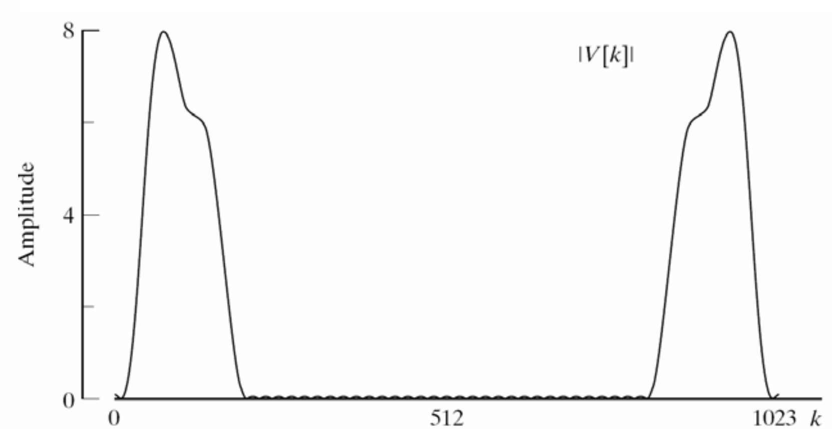
$N = 32$



$N = 128$



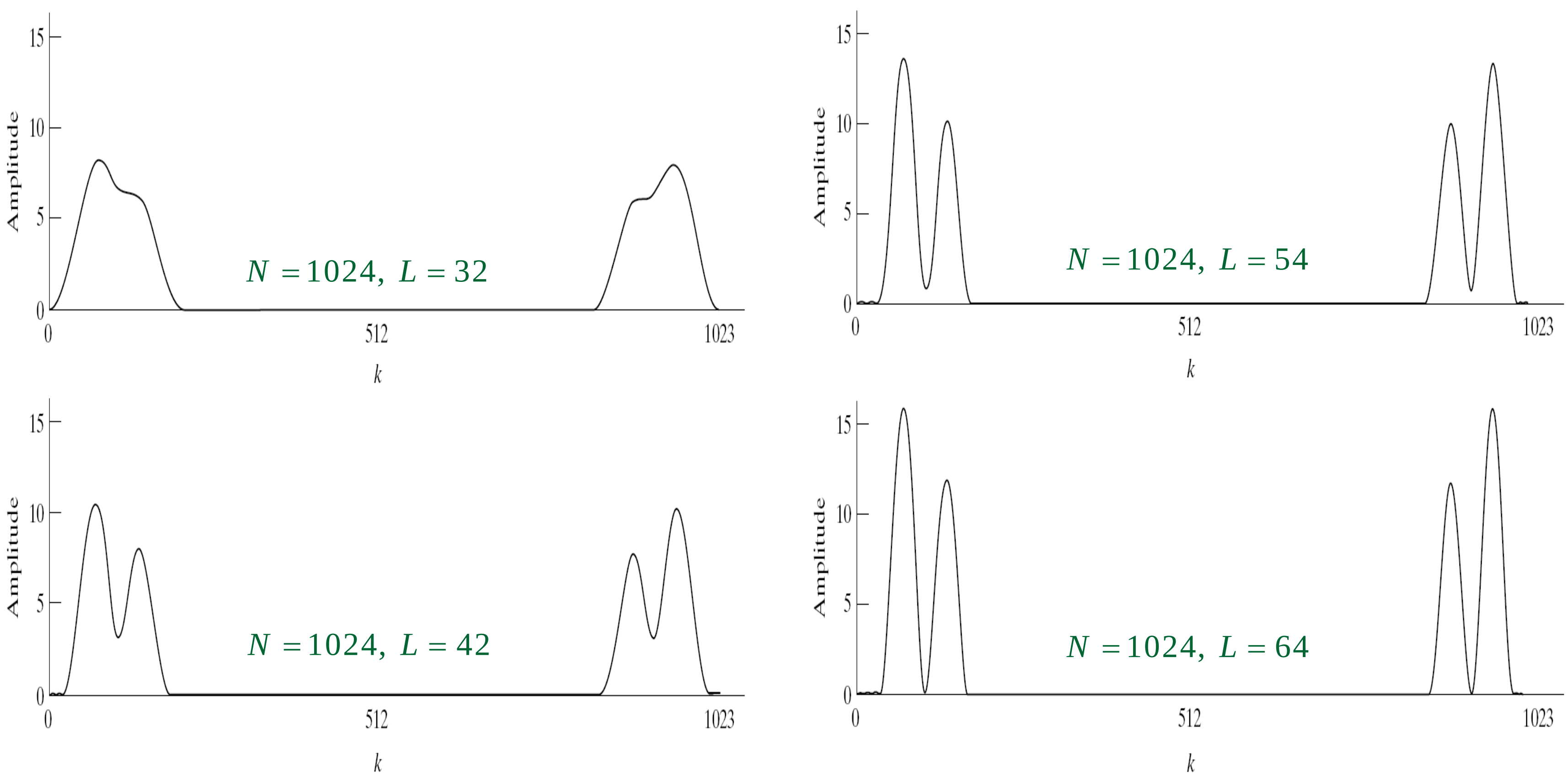
$N = 64$



$N = 1024$

– DFT analysis with Kaiser window and zero padding

- Example $v[n] = \cos\frac{2\pi n}{14} + 0.75\cos\frac{4\pi n}{15}$; $\beta = 5.48$ Kaiser window

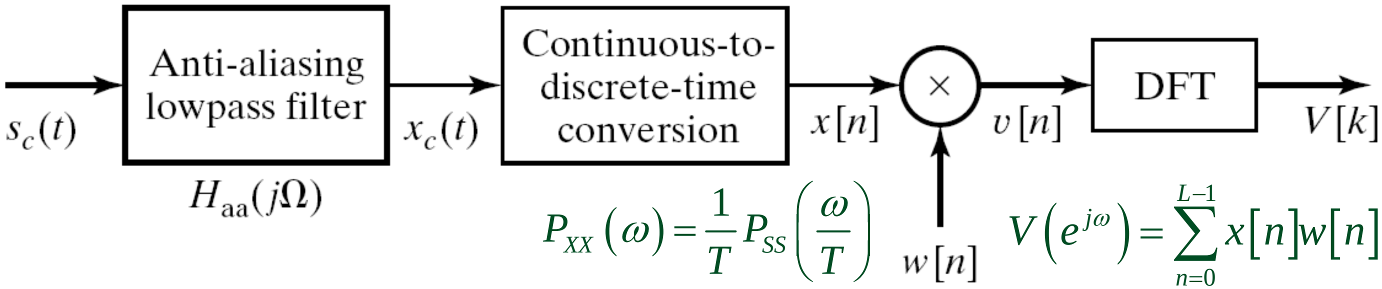


- Fourier analysis of stationary random signals: Periodogram
  - The sample mean and sample variance respectively defined by

$$\hat{m}_X = \frac{1}{L}\sum_{n=0}^{L-1} x[n]; \quad \hat{\sigma}_X^2 = \frac{1}{L}\sum_{n=0}^{L-1}\left(x[n]-\hat{m}_X\right)^2$$

are unbiased and asymptotically unbiased estimators, respectively

  - Periodogram analysis: estimation of the power spectrum



$$P_{XX}(\omega) = \frac{1}{T} P_{SS}\left(\frac{\omega}{T}\right) \qquad V\left(e^{j\omega}\right) = \sum_{n=0}^{L-1} x[n]w[n]e^{-j\omega n}$$

- Periodogram is the F. T. of aperiodic correlation of windowed data sequence

$$I(\omega) = \frac{1}{LU}\left|V\left(e^{j\omega}\right)\right|^2 = \frac{1}{LU}\sum_{m=-L+1}^{L-1} c_{vv}[m]e^{-j\omega m}$$

$$c_{vv}[m] = \sum_{n=0}^{L-1} x[n]w[n]x[n+m]w[n+m]$$

- Modified periodogram: $w[n]$ is not a rectangular window

– Periodogram can be calculated by means of the DFT

$$I(\omega_k) = \frac{1}{LU}|V[k]|^2$$

– Properties of the periodogram:

$$E\{I(\omega)\} = \frac{1}{LU}\sum_{m=-L+1}^{L-1} E\{c_{vv}[m]\}e^{-j\omega m}$$

$$\begin{aligned} E\{c_{vv}[m]\} &= \sum_{n=0}^{L-1} w[n]w[n+m]E\{x[n]x[n+m]\} \\ &= \sum_{n=0}^{L-1} w[n]w[n+m]\phi_{xx}[m] \\ &= c_{ww}[m]\phi_{xx}[m] \end{aligned}$$

$$\phi_{xx}[m] = E\{x[n]x[n+m]\}; \quad c_{ww}[m] = \sum_{n=0}^{L-1} w[n]w[n+m]$$

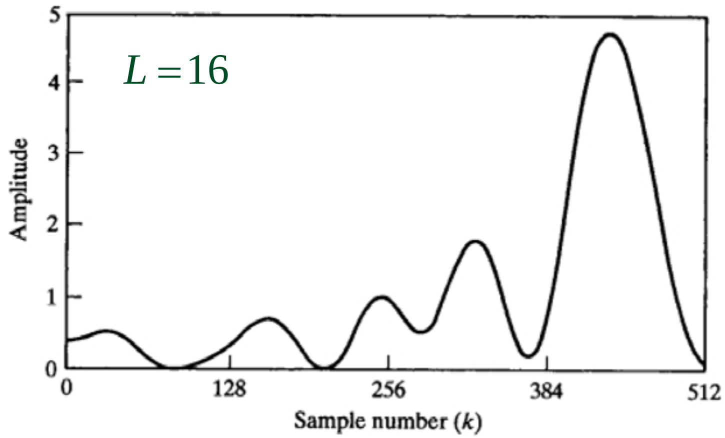
$$E\{I(\omega)\} = \frac{1}{LU}\int_{-\pi}^{\pi} P_{XX}(\theta)C_{WW}\left(e^{j(\omega-\theta)}\right)d\theta; \;\; C_{WW}\left(e^{j\omega}\right) = \left|W\left(e^{j\omega}\right)\right|^2$$

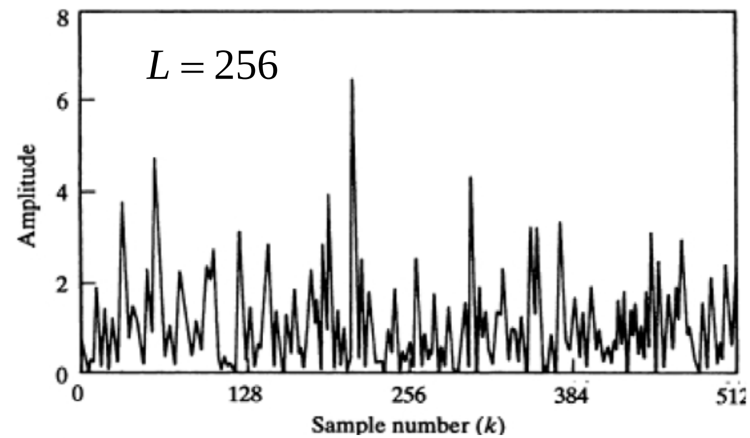
- To make the periodogram unbiased,

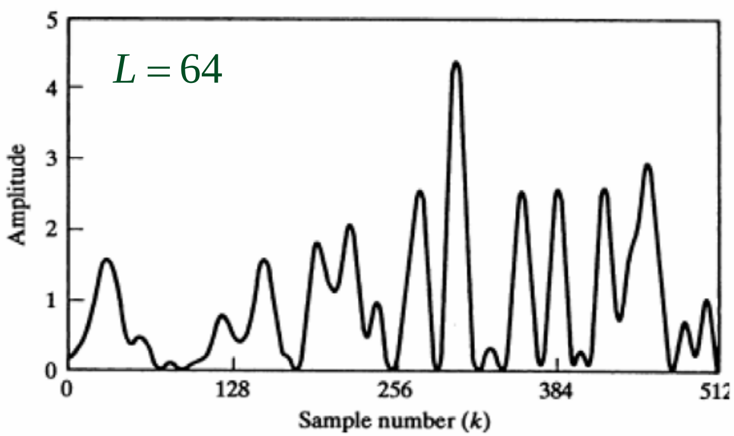
$$\frac{1}{2\pi LU}\int_{-\pi}^{\pi}\left|W\left(e^{j\omega}\right)\right|^2 d\omega = \frac{1}{LU}\sum_{n=0}^{L-1} w^2[n] = 1 \Rightarrow U = \frac{1}{L}\sum_{n=0}^{L-1} w^2[n]$$

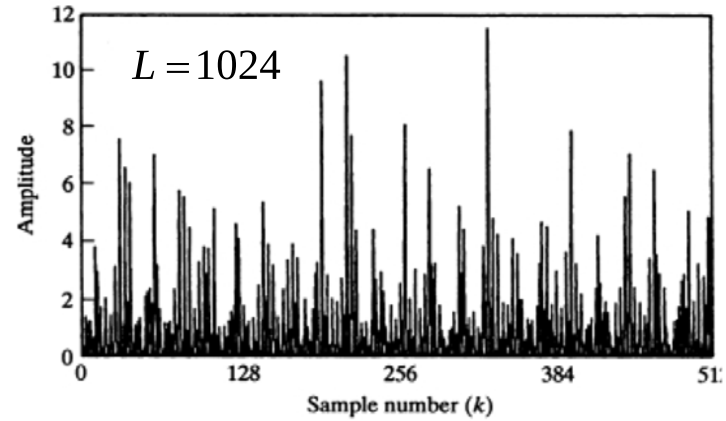
- As the window size increases, the peridogram becomes a consistent estimate
  - Example: white noise generation with rectangular windowing

$$I[k] = I(\omega_k) = \frac{1}{L}\left|V[k]\right|^2 = \frac{1}{L}\left|\sum_{m=-L+1}^{L-1} w[n]x[n]e^{-j2\pi kn/N}\right|^2$$









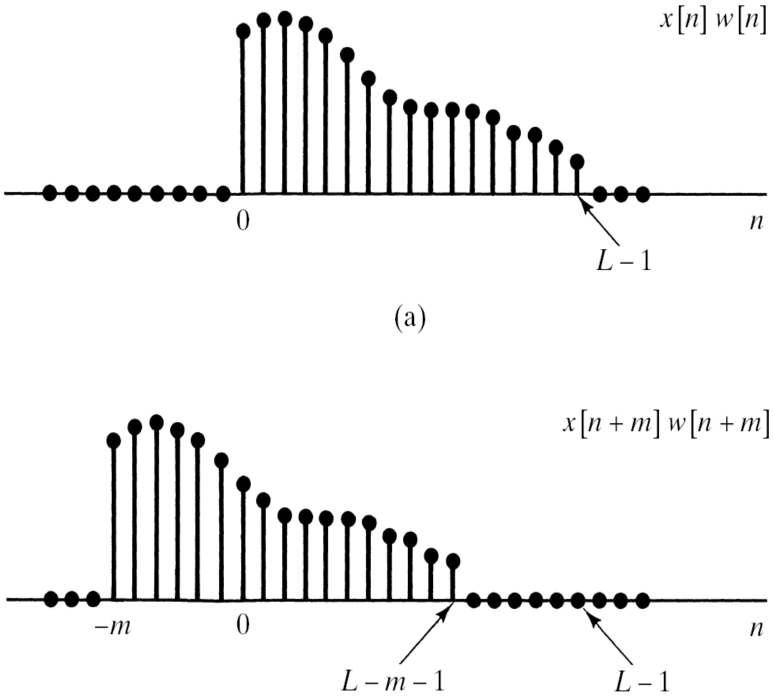
– Edge effect due to covolution of finite sequences

$$c_{vv}[m]=\sum_{n=0}^{L-1} w[n]w[n+m]x[n]x[n+m]$$

- As m becomes close to L, the calculation becomes inconsistent
- This problem can be alleviated by averaging multiple independent periodogram estimates

$$x_r[n]=w[n]x[rR+n];\ 0\le n\le Q-1$$

$$\tilde{I}(\omega)=\frac{1}{K}\sum_{r=0}^{K-1}\frac{1}{LU}\sum_{n=0}^{L-1}\left|X_r\left(e^{j\omega}\right)\right|^2$$



– Example $x[n] = 0.5\cos(2\pi n/21 + \theta) + e[n];\ \ e[n] \sim U\left(-\sqrt{3}, \sqrt{3}\right) \Rightarrow \sigma_e^2 = 1$

$$P_{XX}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] + \sigma_e^2$$

$$Q = 1024;\ \ U = 1$$