Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers
SECOND EDITION

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Definitions, Theorems, Proofs, Examples, Quizzes, Problems, Solutions

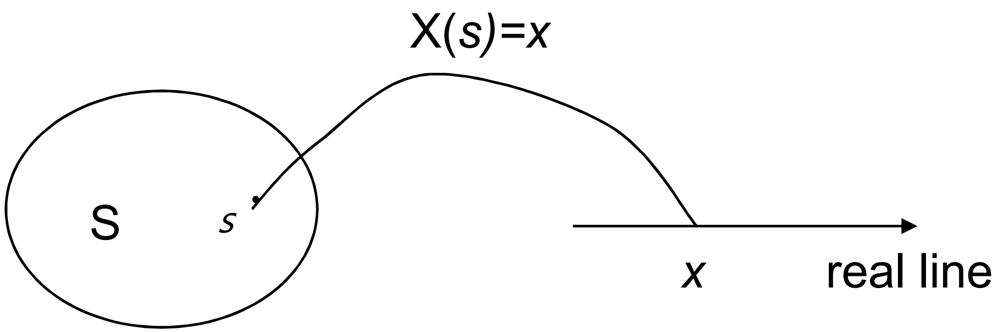
Chapter 2

Section 2.1

Definitions

Definition 2.1 Random Variable

A random variable *consists of an experiment with a probability measure $P[\cdot]$ defined on a sample space S and a function that assigns a real number to each outcome in the sample space of the experiment.*



Random Variable and Event Space

$A = \{s : X(s) \le x\}$ for every set $A \subset S$, A = event = the set of s in S such that the values assumed by the random variable function $X(\cdot)$, for those as its argument, $X(\cdot) = \zeta$, are less than or equal to the given number on the real line.

◆ Note : A should be identified for all $x \in R^1$.

Valid and Invalid Random Variable

Example (Valid and Invalid Random Variable)

Consider throwing a dice once and reading the face value.

$$S=\{f_1.f_2,f_3,f_4,f_5,f_6\} \qquad E=\{\phi,S,\{even\},\{odd\}\}$$

(1)Define

$$\mathrm{x}(s)=\begin{cases}1 & \text{for } s=f_1,f_3,f_5\\ 2 & \text{for } s=f_2,f_4,f_6\end{cases}$$

(Note that a sample has to be mapped to a unique value the same as a function.)

Then

$$\text{(i) for } x<1,\ A=\{s:\mathrm{x}(s)\le x\}=\phi$$

$$\text{(ii) for } 1\le x<2,\ A=\{s:\mathrm{x}(s)\le x\}=\{f_1,f_3,f_5\}=\{odd\}$$

$$\text{(iii) for } x\ge 2,\ A=\{s:\mathrm{x}(s)\le x\}=\{f_1,f_2,f_3,f_4,f_5,f_6\}=S$$

$\mathrm{x}(\cdot)$ is a valid random variable.

Example (continued)

(2) Define y such that, $y(s) = i$ for $s = f_i$

(i) for $y < 1$, $A = \{y(s) \le y\} = \phi$

(ii) for $1 \le y < 2$, $A = \{y(s) \le y\} = \{f_1\} \ne event$

$y(\cdot)$ is an invalid random variable.

Definition 2.2 Discrete Random Variable

X *is a* discrete *random variable if the range of* X *is a countable set*

$$S_X = \{x_1, x_2, \ldots\}.$$

Definition 2.3 Finite Random Variable

X *is a* finite random variable *if the range is a finite set*

$$S_X = \{x_1, x_2, \cdots, x_n\}.$$

Section 2.2

Probability Mass Function

Definition 2.4 Probability Mass Function (PMF)

The probability mass function *(PMF) of the discrete random variable* X *is*

$$P_X(x) = P[X = x]$$

Example 2.5

Suppose we observe three calls at a telephone switch where voice calls (v) and data calls (d) are equally likely. Let X denote the number of voice calls, Y the number of data calls, and let $R = XY$. The sample space of the experiment and the corresponding values of the random variables X, Y, and R are

	Outcomes	ddd	ddv	dvd	dvv	vdd	vdv	vvd	vvv
	$P[\cdot]$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$	$1/8$
Random	X	0	1	1	2	1	2	2	3
Variables	Y	3	2	2	1	2	1	1	0
	R	0	2	2	2	2	2	2	0

Example 2.6 Problem

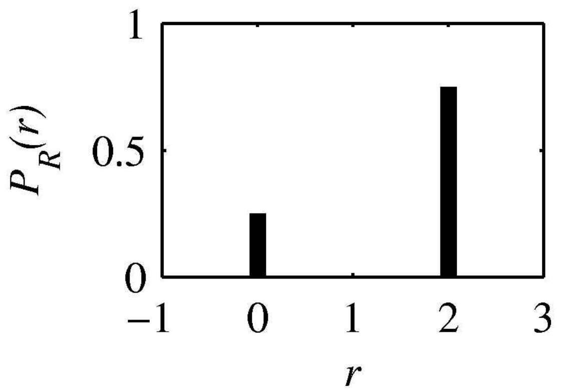
From Example 2.5, what is the PMF of R?

Example 2.6 Solution

From Example 2.5, we see that $R = 0$ if either outcome, DDD or VVV, occurs so that

$$P[R = 0] = P[DDD] + P[VVV] = 1/4.$$

For the other six outcomes of the experiment, $R = 2$ so that $P[R = 2] = 6/8$. The PMF of R is



$$P_R(r) = \begin{cases} 1/4 & r = 0, \\ 3/4 & r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.1

For a discrete random variable X with PMF $P_X(x)$ and range S_X:

(a) For any x, $P_X(x) \geq 0$.

(b) $\sum_{x \in S_X} P_X(x) = 1$.

(c) For any event $B \subset S_X$, the probability that X is in the set B is

$$P[B] = \sum_{x \in B} P_X(x).$$

Proof: Theorem 2.1

All three properties are consequences of the axioms of probability (Section 1.3). First, $P_X(x) \geq 0$ since $P_X(x) = P[X = x]$. Next, we observe that every outcome $s \in S$ is associated with a number $x \in S_X$. Therefore, $P[x \in S_X] = \sum_{x \in S_X} P_X(x) = P[s \in S] = P[S] = 1$. Since the events $\{X = x\}$ and $\{X = y\}$ are disjoint when $x \neq y$, B can be written as the union of disjoint events $B = \bigcup_{x \in B} \{X = x\}$. Thus we can use Axiom 3 (if B is countably infinite) or Theorem 1.4 (if B is finite) to write

$$P[B] = \sum_{x \in B} P[X = x] = \sum_{x \in B} P_X(x).$$

Quiz 2.2

The random variable N has PMF

$$P_N(n) = \begin{cases} c/n & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find

(1) The value of the constant c

(2) $P[N = 1]$

(3) $P[N \geq 2]$

(4) $P[N > 3]$

Section 2.3

Families of Discrete Random Variables

Example 2.8

Consider the following experiments:

- Flip a coin and let it land on a table. Observe whether the side facing up is heads or tails. Let X be the number of heads observed.
- Select a student at random and find out her telephone number. Let $X = 0$ if the last digit is even. Otherwise, let $X = 1$.
- Observe one bit transmitted by a modem that is downloading a file from the Internet. Let X be the value of the bit (0 or 1).

Definition 2.5 Bernoulli (p) Random Variable

X *is a* Bernoulli (p) *random variable if the PMF of* X *has the form*

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \textit{otherwise} \end{cases}$$

where the parameter p *is in the range* $0 < p < 1$.

Example 2.9 Problem

Suppose you test one circuit. With probability p, the circuit is rejected. Let X be the number of rejected circuits in one test. What is $P_X(x)$?

Example 2.9 Solution

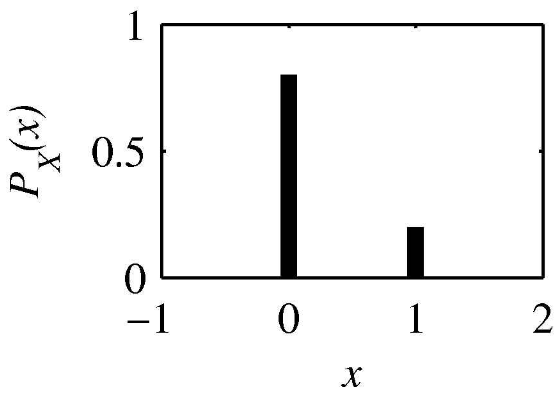
Because there are only two outcomes in the sample space, $X = 1$ with probability p and $X = 0$ with probability $1 - p$.

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the number of circuits rejected in one test is a Bernoulli (p) random variable.

Example 2.10

If there is a 0.2 probability of a reject,



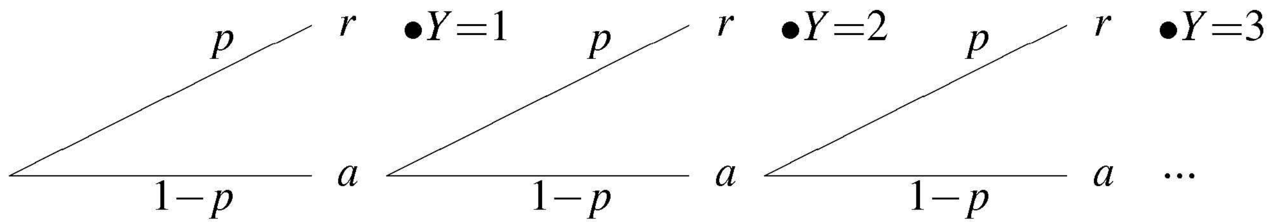
$$P_X(x) = \begin{cases} 0.8 & x = 0 \\ 0.2 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2.11 Problem

In a test of integrated circuits there is a probability p that each circuit is rejected. Let Y equal the number of tests up to and including the first test that discovers a reject. What is the PMF of Y?

Example 2.11 Solution

The procedure is to keep testing circuits until a reject appears. Using a to denote an accepted circuit and r to denote a reject, the tree is



From the tree, we see that $P[Y=1]=p$, $P[Y=2]=p(1-p)$, $P[Y=3]=p(1-p)^2$, and, in general, $P[Y=y]=p(1-p)^{y-1}$. Therefore,

$$P_Y(y)=\begin{cases} p(1-p)^{y-1} & y=1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Y is referred to as a *geometric random variable* because the probabilities in the PMF constitute a geometric series.

Definition 2.6 Geometric (p) Random Variable

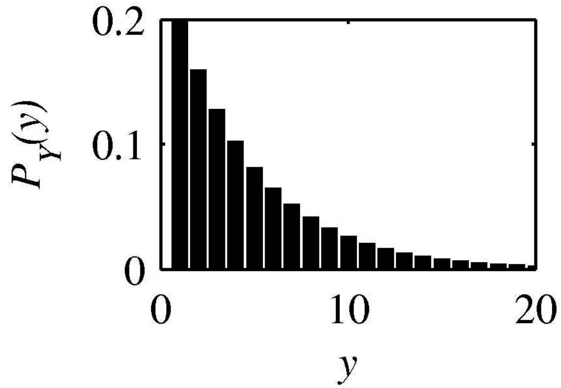
X *is a* geometric (p) *random variable if the PMF of* X *has the form*

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \textit{otherwise.} \end{cases}$$

where the parameter p *is in the range* $0 < p < 1$*.*

Example 2.12

If there is a 0.2 probability of a reject,



$$P_Y(y) = \begin{cases} (0.2)(0.8)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Example 2.13 Problem

Suppose we test n circuits and each circuit is rejected with probability p independent of the results of other tests. Let K equal the number of rejects in the n tests. Find the PMF $P_K(k)$.

Example 2.13 Solution

Adopting the vocabulary of Section 1.9, we call each discovery of a defective circuit a *success*, and each test is an independent trial with success probability p. The event $K = k$ corresponds to k successes in n trials, which we have already found, in Equation (1.18), to be the binomial probability

$$P_K(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

K is an example of a *binomial random variable*.

Remember $P(AB) = P(A)P(B)$ if A and B are independent!

Definition 2.7 Binomial (n, p) Random Variable

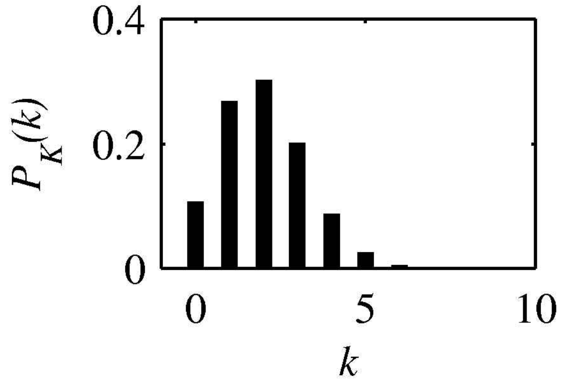
X is a binomial (n, p) *random variable if the PMF of X has the form*

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 < p < 1$ and n is an integer such that $n \geq 1$.

Example 2.14

If there is a 0.2 probability of a reject and we perform 10 tests,



$$P_K(k) = \binom{10}{k}(0.2)^k(0.8)^{10-k}.$$

Example 2.15 Problem

Suppose you test circuits until you find k rejects. Let L equal the number of tests. What is the PMF of L?

Example 2.15 Solution

For large values of k, the tree becomes difficult to draw. Once again, we view the tests as a sequence of independent trials where finding a reject is a success. In this case, $L = l$ if and only if there are $k-1$ successes in the first $l-1$ trials, *and* there is a success on trial l so that

$$P[L = l] = P\left[\underbrace{k-1 \text{ rejects in } l-1 \text{ attempts}}_{A}, \underbrace{\text{success on attempt } l}_{B}\right]$$

The events A and B are independent since the outcome of attempt l is not affected by the previous $l-1$ attempts. Note that $P[A]$ is the binomial probability of $k-1$ successes in $l-1$ trials so that

$$P[A] = \binom{l-1}{k-1} p^{k-1}(1-p)^{l-1-(k-1)}$$

Finally, since $P[B] = p$,

$$P_L(l) = P[A]\,P[B] = \binom{l-1}{k-1} p^{k}(1-p)^{l-k}$$

L is an example of a *Pascal* random variable.

Definition 2.8 Pascal (k, p) Random Variable

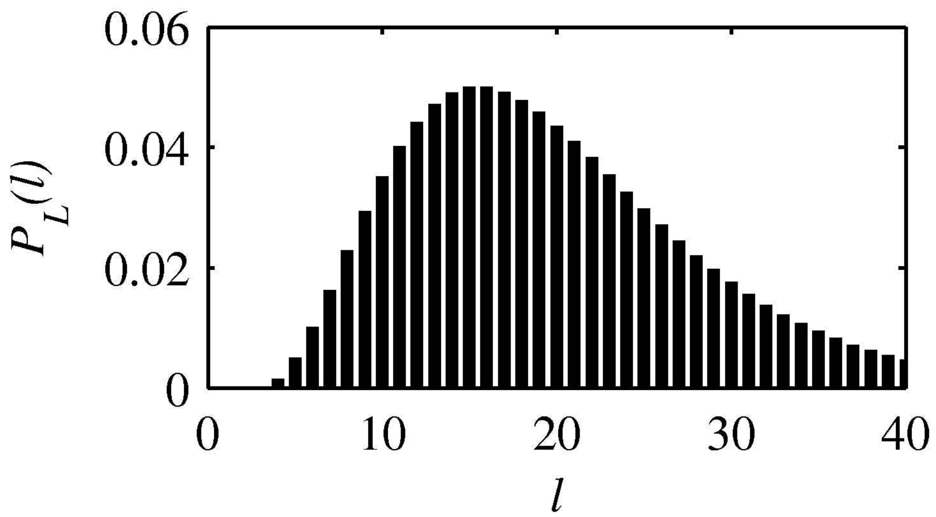
X is a Pascal (k, p) *random variable if the PMF of X has the form*

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

where $0 < p < 1$ and k is an integer such that $k \geq 1$.

Example 2.16

If there is a 0.2 probability of a reject and we seek four defective circuits, the random variable L is the number of tests necessary to find the four circuits. The PMF is



$$P_L(l) = \binom{l-1}{3}(0.2)^4(0.8)^{l-4}.$$

Example 2.17

In an experiment with equiprobable outcomes, the random variable N has the range $S_N = \{k, k+1, k+2, \cdots, l\}$, where k and l are integers with $k < l$. The range contains $l-k+1$ numbers, each with probability $1/(l-k+1)$. Therefore, the PMF of N is

$$P_N(n) = \begin{cases} 1/(l-k+1) & n = k, k+1, k+2, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

N is an example of a *discrete uniform* random variable.

Definition 2.9 Discrete Uniform (k, l) Random Variable

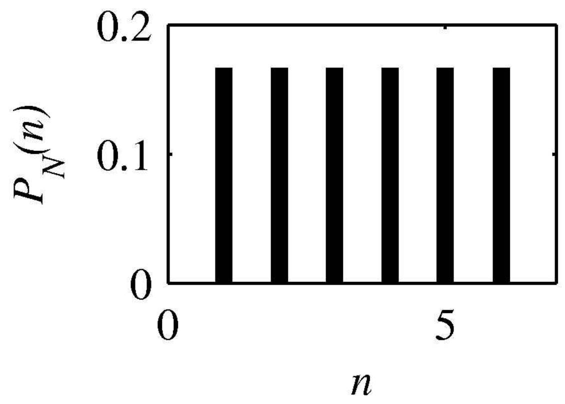
X *is a* discrete uniform (k, l) *random variable if the PMF of* X *has the form*

$$P_X(x) = \begin{cases} 1/(l-k+1) & x = k, k+1, k+2, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

where the parameters k *and* l *are integers such that* $k < l$.

Example 2.18

Roll a fair die. The random variable N is the number of spots that appears on the side facing up. Therefore, N is a discrete uniform $(1, 6)$ random variable and



$$P_N(n) = \begin{cases} 1/6 & n = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10 Poisson (α) Random Variable

X is a Poisson (α) *random variable if the PMF of X has the form*

$$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x = 0, 1, 2, \ldots, \\ 0 & \textit{otherwise}, \end{cases}$$

where the parameter α is in the range $\alpha > 0$.

Example 2.19 Problem

The number of hits at a Web site in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? What is the probability that there are no more than two hits in an interval of one second?

Example 2.19 Solution

In an interval of 0.25 seconds, the number of hits H is a Poisson random variable with $\alpha = \lambda T = (2 \text{ hits/s}) \times (0.25 \text{ s}) = 0.5$ hits. The PMF of H is

$$P_H(h) = \begin{cases} 0.5^h e^{-0.5}/h! & h = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The probability of no hits is

$$P[H = 0] = P_H(0) = (0.5)^0 e^{-0.5}/0! = 0.607.$$

In an interval of 1 second, $\alpha = \lambda T = (2 \text{ hits/s}) \times (1\text{s}) = 2$ hits. Letting J denote the number of hits in one second, the PMF of J is

$$P_J(j) = \begin{cases} 2^j e^{-2}/j! & j = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

To find the probability of no more than two hits, we note that $\{J \le 2\} = \{J = 0\} \cup \{J = 1\} \cup \{J = 2\}$ is the union of three mutually exclusive events. Therefore,

$$\begin{aligned} P[J \le 2] &= P[J = 0] + P[J = 1] + P[J = 2] \\ &= P_J(0) + P_J(1) + P_J(2) \\ &= e^{-2} + 2^1 e^{-2}/1! + 2^2 e^{-2}/2! = 0.677. \end{aligned}$$

Poisson Random Variable as a Limit of Binomial RV

We place "at random" n points in the (0,T) interval. What is the probability that k of these points will lie in the interval (t_1, t_2)?

Define

A(event) = { When we place a single point, it is placed in the (t_1, t_2) interval }

$$P(\mathrm{A}) = \frac{t_2 - t_1}{T} \equiv p \quad , \quad q = 1 - p = P(\mathrm{A}^C)$$

Then,

P{ k points in the interval } = P { A occurs k times } = $\binom{n}{k} p^k q^{n-k}$.

If $n >> 1, p << 1$

$$\binom{n}{k} p^k q^{n-k} \approx e^{-np} \frac{(np)^k}{k!} \quad \text{(Poisson Theorem)}$$

Poisson Theorem

If $n \to \infty,\ p \to 0,\ np \to \alpha$, then

$$\lim_{n\to\infty}\binom{n}{k}p^k(1-p)^{n-k} = \frac{e^{-\alpha}\alpha^k}{k!}$$

(Proof)

$$P(x=k)=\binom{n}{k}p^k(1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!}p^k(1-p)^{n-k}$$

$np=\alpha$, hence $p=\dfrac{\alpha}{n}$ and $1-p=\dfrac{n-\alpha}{n}$

$$P(x=k)=\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\alpha}{n}\right)^k\left(\frac{n-\alpha}{n}\right)^{n-k}$$

$$=\frac{\alpha^k}{k!}\left[(1)(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})\right]\left[1-\frac{\alpha}{n}\right]^{n-k}$$

$$=\frac{\alpha^k}{k!}\left[(1)(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})\right]\left(1-\frac{\alpha}{n}\right)^n\left(1-\frac{\alpha}{n}\right)^{-k}$$

Poisson Theorem (continued)

as $n \to \infty,$ $\lim_{n\to\infty}(1-\frac{\alpha}{n})^n = e^{-\alpha}$

Therefore, when $n \to \infty$

$$P(x=k) = e^{-\alpha}\frac{\alpha^k}{k!}.$$

Exercise (Poisson Theorem)

Problem: In a large hotel it is known that 99% of all guests return room keys when checking out. If 250 engineers check out after a large conference, what is the probability that not more than three will fail to return their keys?

Solution:

Let N = total number of engineers = 250;

p = probability to fail to return keys = 0.01;

k = number of engineer to fail to return keys.

Conditions are met to apply the Poisson Theorem, i.e.,

$$N = 250 \gg 1$$

$$p = 0.01 \ll 1$$

$$Np = \alpha = 2.5.$$

Exercise (continued)

$$p(0 \text{ fail to return key}) = \binom{N}{k} p^k (1-p)^{N-k} \approx \frac{\alpha^k e^{-\alpha}}{k!} = \frac{2.5^0 e^{-2.5}}{0!} = 0.0821$$

$$p(1 \text{ fail to return key}) = \frac{2.5^1 e^{-2.5}}{1!} = 0.2052$$

$$p(2 \text{ fail to return key}) = \frac{2.5^2 e^{-2.5}}{2!} = 0.2565$$

$$p(3 \text{ fail to return keys}) = \frac{2.5^3 e^{-2.5}}{3!} = 0.2138$$

$$p(\text{no more than } 3 \text{ fail to return keys}) = 0.0821 + 0.2052 + 0.2565 + 0.2138 = 0.7576.$$