Module #3:
The Theory of Sets

Rosen 5th ed., §§1.6-1.7

~43 slides, ~2 lectures

Introduction to Set Theory (§1.6)

- A *set* is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous in computer software systems.
- *All* of mathematics can be defined in terms of some form of set theory (using predicate logic).

Naïve set theory

- Basic premise: Any collection or class of objects (*elements*) that we can describe (by any means whatsoever) constitutes a set.
- But, the resulting theory turns out to be *logically inconsistent*!
  - This means, there exist naïve set theory propositions p such that you can prove that both p and $\neg p$ follow logically from the postulates of the theory!
  - $\therefore$ The conjunction of the postulates is a contradiction!
  - This theory is fundamentally uninteresting, because any possible statement in it can be (very trivially) "proved" by contradiction!
- More sophisticated set theories fix this problem.

Basic notations for sets

- For sets, we'll use variables *S*, *T*, *U*, …
- We can denote a set *S* in writing by listing all of its elements in curly braces:
  - {a, b, c} is the set of whatever 3 objects are denoted by a, b, c.
- *Set builder notation*: For any proposition $P(x)$ over any universe of discourse, $\{x|P(x)\}$ is *the set of all x such that P(x).*

Basic properties of sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are *distinct* (unequal); multiple listings make no difference!
  - If a=b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains at most 2 elements!

Definition of Set Equality

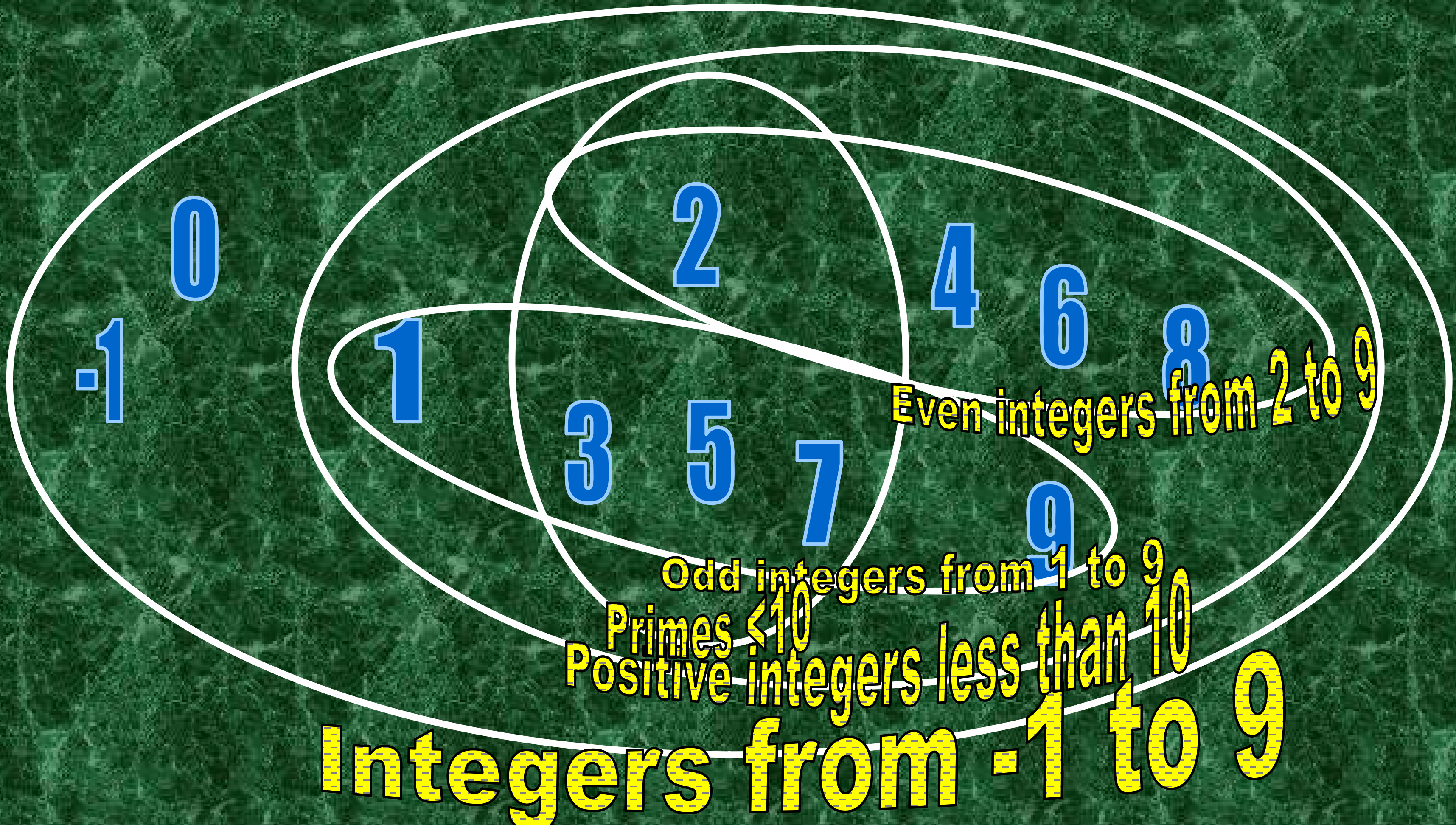
- Two sets are declared to be equal *if and only if* they contain exactly the same elements.
- In particular, it does not matter *how the set is defined or denoted.*
- For example: The set $\{1, 2, 3, 4\}$ =
  $\{x \mid x$ is an integer where $x>0$ and $x<5\}$ =
  $\{x \mid x$ is a positive integer whose square is >0 and $<25\}$

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:
  N = {0, 1, 2, …} The **N**atural numbers.
  Z = {…, -2, -1, 0, 1, 2, …} The *Z*ntegers.
  R = The "**R**eal" numbers, such as 374.1828471929498181917281943125…
- Infinite sets come in different sizes!

More on this after module #4 (functions).

Venn Diagrams



Basic Set Relations: Member of

- $x \in S$ ("x is in S") is the proposition that object x is an $\in$*lement* or *member* of set S.
  - *e.g.* $3 \in \mathbf{N}$, "a" $\in \{x \mid x$ is a letter of the alphabet$\}$
  - Can define set equality in terms of $\in$ relation:
    $\forall S,T: S=T \leftrightarrow (\forall x: x \in S \leftrightarrow x \in T)$
    "Two sets are equal iff they have all the same members."
- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"

The Empty Set

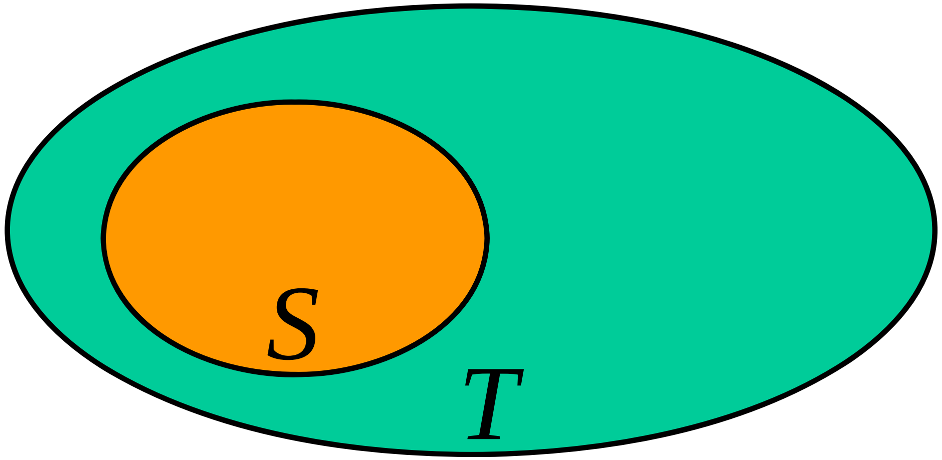
- $\varnothing$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
- $\varnothing = \{\} = \{x|\textbf{False}\}$
- No matter the domain of discourse, we have the axiom $\neg\exists x\colon x\in\varnothing$.

Subset and Superset Relations

- $S \subseteq T$ ("S is a subset of T") means that every element of S is also an element of T.
- $S \subseteq T \Leftrightarrow \forall x\ (x \in S \rightarrow x \in T)$
- $\emptyset \subseteq S$, $S \subseteq S$.
- $S \supseteq T$ ("S is a superset of T") means $T \subseteq S$.
- Note $S = T \Leftrightarrow S \subseteq T \wedge S \supseteq T$.
- $S \not\subseteq T$ means $\neg(S \subseteq T)$, *i.e.* $\exists x(x \in S \wedge x \notin T)$

Proper (Strict) Subsets & Supersets

- $S \subset T$ ("S is a proper subset of T") means that $S \subseteq T$ but $T \nsubseteq S$. Similar for $S \supset T$.



Example:
$\{1,2\} \subset \{1,2,3\}$

Venn Diagram equivalent of $S \subset T$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- *E.g.* let $S=\{x \mid x \subseteq \{1,2,3\}\}$ then $S=\{\varnothing,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Very Important!

Cardinality and Finiteness

- $|S|$ (read "the *cardinality* of S") is a measure of how many different elements S has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$, $|\{\{1,2,3\},\{4,5\}\}|$ = _2_
- If $|S| \in \mathbf{N}$, then we say S is *finite*. Otherwise, we say S is *infinite*.
- What are some infinite sets we've seen?

N Z R

The *Power Set* Operation

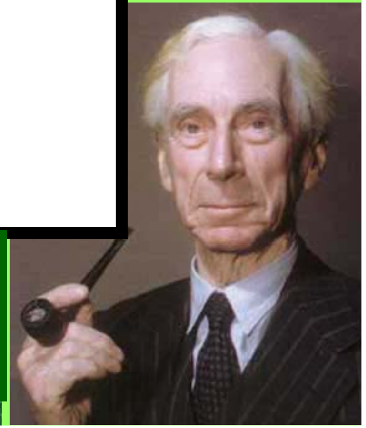
- The *power set* P(S) of a set S is the set of all subsets of S. P(S) = $\{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Sometimes P(S) is written $\mathbf{2}^S$.
  Note that for finite S, $|P(S)| = 2^{|S|}$.
- It turns out that $|P(\mathbf{N})| > |\mathbf{N}|$.
  There are different sizes of infinite sets!

Review: Set Notations So Far

- Variable objects x, y, z; sets S, T, U.
- Literal set {a, b, c} and set-builder $\{x|P(x)\}$.
- $\in$ relational operator, and the empty set $\varnothing$.
- Set relations $=$, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $\not\subset$, etc.
- Venn diagrams.
- Cardinality $|S|$ and infinite sets **N**, **Z**, **R**.
- Power sets $\mathrm{P}(S)$.

Naïve Set Theory is Inconsistent

- There are some naïve set *descriptions* that lead pathologically to structures that are not *well-defined*. (That do not have consistent properties.)
- These "sets" mathematically *cannot* exist.
- *E.g.* let $S = \{x \mid x \notin x\}$. Is $S \in S$?
- Therefore, consistent set theories must restrict the language that can be used to describe sets.
- For purposes of this class, don't worry about it!



Bertrand Russell
1872-1970

Ordered *n*-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence of length n* is written $(a_1, a_2, \ldots, a_n)$. The *first* element is a_1, *etc.*
- Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.
- Empty sequence, singlets, pairs, triples, quadruples, quin<u>tuples</u>, …, *n*-tuples.

Cartesian Products of Sets

- For sets A, B, their *Cartesian product* $A \times B :\equiv \{(a, b) \mid a \in A \wedge b \in B\}$.
- *E.g.* $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Note that for finite A, B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is *not* commutative: $\neg\forall AB: A \times B = B \times A$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

René Descartes
(1596-1650)

Review of §1.6

- Sets S, T, U… Special sets **N**, **Z**, **R**.
- Set notations {a,b,...}, $\{x|P(x)\}$…
- Set relation operators $x\in S$, $S\subseteq T$, $S\supseteq T$, $S=T$, $S\subset T$, $S\supset T$. (These form propositions.)
- Finite vs. infinite sets.
- Set operations $|S|$, P(S), $S\times T$.
- Next up: §1.5: More set ops: $\cup$, $\cap$, $-$.

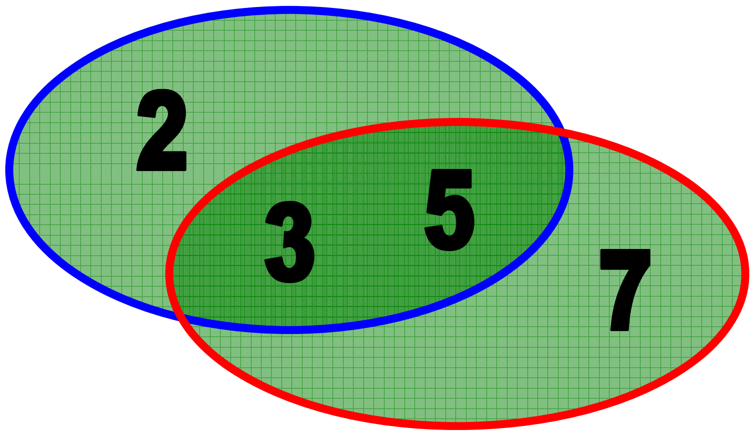
Start §1.7: The Union Operator

- For sets A, B, their $\cup$ *nion* $A\cup B$ is the set containing all elements that are either in A, **or** ("$\vee$") in B (or, of course, in both).
- Formally, $\forall A,B$: $A\cup B = \{x \mid x\in A \vee x\in B\}$.
- Note that $A\cup B$ contains all the elements of A **and** it contains all the elements of B: $\forall A, B$: $(A\cup B \supseteq A) \wedge (A\cup B \supseteq B)$

Union Examples

- $\{a,b,c\}\cup\{2,3\} = \{a,b,c,2,3\}$
- $\{2,3,5\}\cup\{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$

Required Form



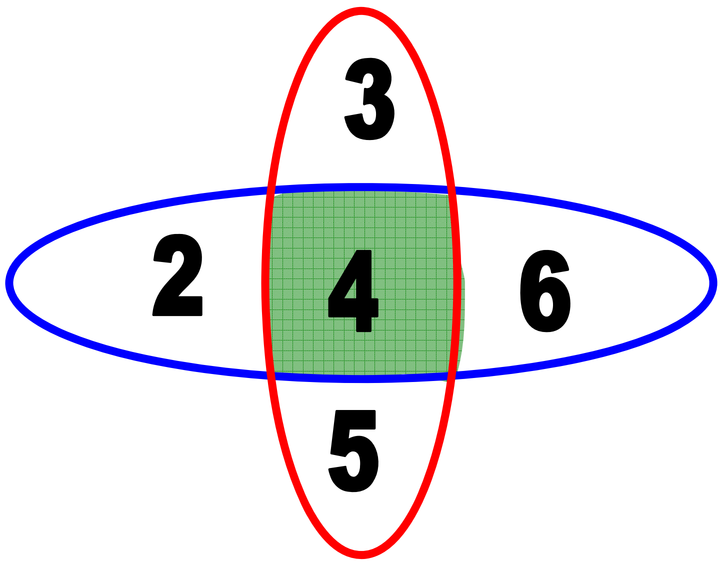
Think "The **Uni**ted States of America includes every person who worked in any U.S. state last year." (This is how the IRS sees it...)

The Intersection Operator

- For sets A, B, their *intersection* $A\cap B$ is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B.
- Formally, $\forall A,B$: $A\cap B \equiv \{x \mid x\in A \wedge x\in B\}$.
- Note that $A\cap B$ is a subset of A **and** it is a subset of B:
$\forall A, B$: $(A\cap B \subseteq A) \wedge (A\cap B \subseteq B)$

Intersection Examples

- $\{a,b,c\} \cap \{2,3\} = \underline{\ \emptyset\ }$
- $\{2,4,6\} \cap \{3,4,5\} = \underline{\ \{4\}\ }$



Think "The **<u>intersection</u>** of University Ave. and W 13th St. is just that part of the road surface that lies on *both* streets."

Disjointedness

- Two sets A, B are called *disjoint* (*i.e.*, unjoined) iff their intersection is empty. ($A \cap B = \varnothing$)
- Example: the set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?
  $|$ ... $B|$
- Example: How many ...
  cl... ...nsider set $E = I \cup M$,
  $I = \{s \mid s$ turned in ... ...eet$\}$
  $M =$ ... ...TAs their ema...
- Some students ...
  ... $|I| + |M| - |I \cap M|$

Subtract out items in intersection, to compensate for double-counting them!

Set Difference

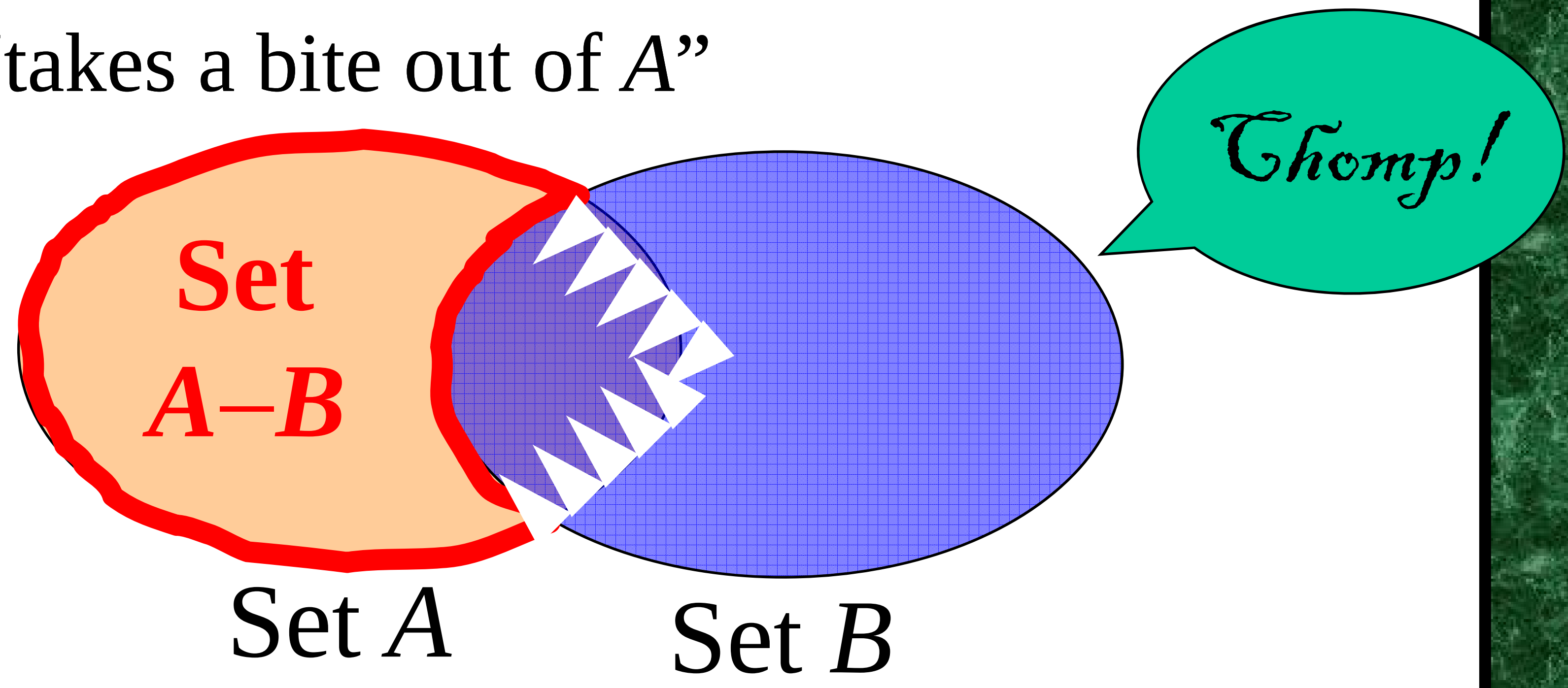
- For sets *A*, *B*, the *difference of A and B*, written *A*–*B*, is the set of all elements that are in *A* but not *B*.
- $A - B :\equiv \{x \mid x\in A \wedge \mathrm{x}\notin B\}$
  $= \{x \mid \neg(x\in A \rightarrow x\in B) \}$
- Also called:
  The *complement of B with respect to A.*

Set Difference Examples

- $\{1,\cancel{2},\cancel{3},4,\cancel{5},6\} - \{2,3,5,7,9,11\} =$ $\{1,4,6\}$
- $\mathbf{Z} - \mathbf{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{0, 1, \ldots\}$
  $= \{x \mid x \text{ is an integer but not a nat. \#}\}$
  $= \{x \mid x \text{ is a negative integer}\}$
  $= \{\ldots, -3, -2, -1\}$

Set Difference - Venn Diagram

- *A*-*B* is what's left after *B* "takes a bite out of *A*"



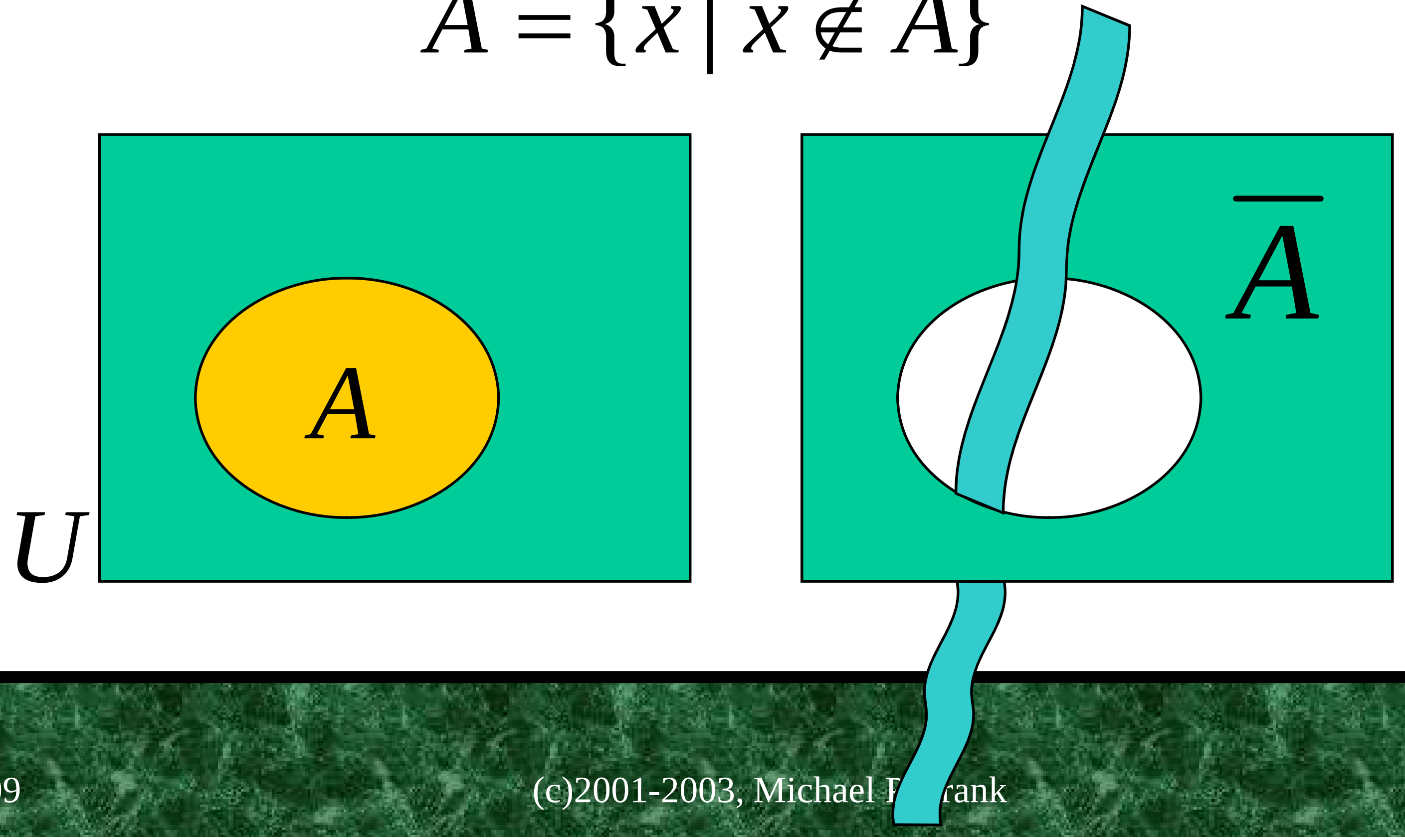
Set Complements

- The *universe of discourse* can itself be considered a set, call it U.
- When the context clearly defines U, we say that for any set $A \subseteq U$, the *complement* of A, written $\overline{A}$, is the complement of A w.r.t. U, *i.e.*, it is $U - A$.
- *E.g.*, If $U=\mathbf{N}$, $\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$

More on Set Complements

- An equivalent definition, when *U* is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities

- Identity: $A \cup \emptyset = A \quad A \cap U = A$
- Domination: $A \cup U = U \quad A \cap \emptyset = \emptyset$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A \quad A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where *E*s are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- Use set builder notation & logical equivalences.
- Use a *membership table*.

Method 1: Mutual subsets

Example: Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

- Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
  - We know that $x\in A$, and either $x\in B$ or $x\in C$.
    - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
    - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
  - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- Show $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$. …

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A\cup B)-B = A-B$.

A	B	$A\cup B$	$(A\cup B)-B$	$A-B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Review of §1.6-1.7

- Sets S, T, U… Special sets **N**, **Z**, **R**.
- Set notations {a,b,...}, $\{x|P(x)\}$…
- Relations $x\in S$, $S\subseteq T$, $S\supseteq T$, $S=T$, $S\subset T$, $S\supset T$.
- Operations $|S|$, P(S), $\times$, $\cup$, $\cap$, $-$, $\overline{S}$
- Set equality proof techniques:
  - Mutual subsets.
  - Derivation using logical equivalences.

Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on *ordered pairs* of sets (A,B) to operating on sequences of sets $(A_1,\ldots,A_n)$, or even unordered *sets* of sets, $X=\{A \mid P(A)\}$.

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union:
  $A \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection:
  $A \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcap_{A \in X} A$

Representations

- A frequent theme of this course will be methods of *representing* one discrete structure using another discrete structure of a different type.
- *E.g.*, one can represent natural numbers as
  - Sets: **0**:≡∅, **1**:≡{**0**}, **2**:≡{**0**,**1**}, **3**:≡{**0**,**1**,**2**}, …
  - Bit strings: **0**:≡`0`, **1**:≡`1`, **2**:≡`10`, **3**:≡`11`, **4**:≡`100`, …

Representing Sets with Bit Strings

For an enumerable u.d. U with ordering $\{x_1, x_2, \ldots\}$, represent a finite set $S \subseteq U$ as the finite bit string $B = b_1 b_2 \ldots b_n$ where $\forall i: x_i \in S \leftrightarrow (i < n \wedge b_i = 1)$.

E.g. $U = \mathbf{N}$, $S = \{2,3,5,7,11\}$, B=001101010001.

In this representation, the set operators "$\cup$", "$\cap$", "$\overline{\quad}$" are implemented directly by bitwise OR, AND, NOT!