Module #4.5, Topic #∞: **Cardinality & Infinite Sets**

Rosen 5th ed., last part of §3.2
~10 slides, ½ lecture

Infinite Cardinalities (from §3.2)

- Using what we learned about *functions* in §1.8, it's possible to formally define cardinality for infinite sets.
- We show that infinite sets come in different *sizes* of infinite!
- This also gives us some interesting proof examples.

Cardinality: Formal Definition

- For any two (possibly infinite) sets A and B, we say that A and B *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (bijective function) from A to B.
- When A and B are finite, it is easy to see that such a function exists iff A and B have the same number of elements $n \in \mathbf{N}$.

Countable versus Uncountable

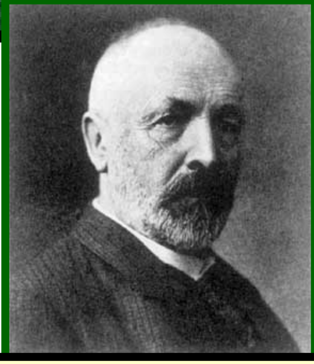
- For any set *S*, if *S* is finite or if |*S*|=|**N**|, we say *S* is *countable*. Else, *S* is *uncountable*.
- Intuition behind "**countable**:" we can *enumerate* (generate in series) elements of *S* in such a way that *any* individual element of *S* will eventually be *counted* in the enumeration. Examples: **N**, **Z**.
- **Uncountable**: *No* series of elements of *S* (even an infinite series) can include all of *S*'s elements. Examples: **R**, $\mathbf{R}^2$, P(**N**)

Countable Sets: Examples

- **Theorem:** The set **Z** is countable.
  - **Proof:** Consider f:**Z**→**N** where $f(i)=2i$ for $i\geq 0$ and $f(i) = -2i-1$ for $i<0$. Note f is bijective.
- **Theorem:** The set of all ordered pairs of natural numbers (n,m) is countable.
  - Consider listing the pairs in order by their sum $s=n+m$, then by n. Every pair appears once in this series; the generating function is bijective.

Uncountable Sets: Example

- **Theorem:** The open interval $[0,1) :\equiv \{r \in \mathbf{R} | \ 0 \leq r < 1\}$ is uncountable.
- **Proof** by *diagonalization*: (Cantor, 1891)
  - Assume there is a series $\{r_i\} = r_1, r_2, \ldots$ containing *all* elements $r \in [0,1)$.
  - Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index: ... *(continued on next slide)*



Georg Cantor
1845-1918

Uncountability of Reals, cont'd

A postulated enumeration of the reals:

$$r_1 = 0.d_{1,1}\,d_{1,2}\,d_{1,3}\,d_{1,4}\,d_{1,5}\,d_{1,6}\,d_{1,7}\,d_{1,8}\ldots$$
$$r_2 = 0.d_{2,1}\,d_{2,2}\,d_{2,3}\,d_{2,4}\,d_{2,5}\,d_{2,6}\,d_{2,7}\,d_{2,8}\ldots$$
$$r_3 = 0.d_{3,1}\,d_{3,2}\,d_{3,3}\,d_{3,4}\,d_{3,5}\,d_{3,6}\,d_{3,7}\,d_{3,8}\ldots$$
$$r_4 = 0.d_{4,1}\,d_{4,2}\,d_{4,3}\,d_{4,4}\,d_{4,5}\,d_{4,6}\,d_{4,7}\,d_{4,8}\ldots$$

- Now, consider a real number generated by taking all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.

That real doesn't appear in the list!

Uncountability of Reals, fin.

- *E.g.*, a postulated enumeration of the reals:
  $r_1 = \texttt{0.301948571}\ldots$
  $r_2 = \texttt{0.103918481}\ldots$
  $r_3 = \texttt{0.039194193}\ldots$
  $r_4 = \texttt{0.918237461}\ldots$
- OK, now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103… can't be on the list anywhere!

Transfinite Numbers

- The cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of the natural numbers, $\aleph_0 :\equiv |\mathbf{N}|$, is the *first transfinite cardinal* number. (There are none smaller.)
- The *continuum hypothesis* claims that $|\mathbf{R}| = \aleph_1$, the *second transfinite cardinal*.

Proven impossible to prove or disprove!

Do Uncountable Sets Really Exist?

- The set of objects that can be defined using finite-length strings of symbols ("descriptions") is only *countable*.
- Therefore, any uncountable set must consist primarily of elements which individually have *no* finite description.
- Löwenheim-Skolem theorem: No consistent theory can ever *force* an interpretation involving uncountables.
- The "constructivist school" asserts that only objects constructible from finite descriptions exist. (*e.g.* $\neg\exists\mathbf{R}$)
- Most mathematicians are happy to use uncountable sets anyway, because postulating their existence has not led to any demonstrated contradictions (so far).

Countable vs. Uncountable

- You should:
  - Know how to define “same cardinality” in the case of infinite sets.
  - Know the definitions of *countable* and *uncountable*.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.