Module #4: **Functions**

Rosen 5th ed., §1.8

~44 slides, ~2 lectures

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On to section 1.8… Functions

- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in \mathbf{R}$ a particular value $y = f(x)$, where $y \in \mathbf{R}$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of *any* set to elements of *any* set.

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Function: Formal Definition

- For any sets *A*, *B*, we say that a *function f from (or "mapping") A to B* ($f:A\rightarrow B$) is a particular assignment of exactly one element $f(x)\in B$ to each element $x\in A$.
- Some further generalizations of this idea:
  - A *partial* (non-*total*) function *f* assigns *zero or one* elements of *B* to each element $x\in A$.
  - Functions of *n* arguments; relations (ch. 6).

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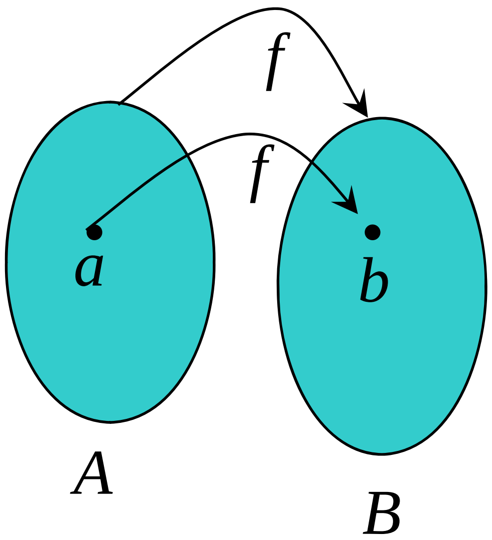
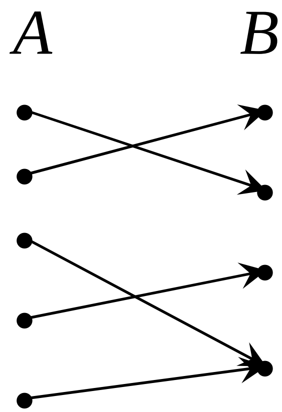
Graphical Representations

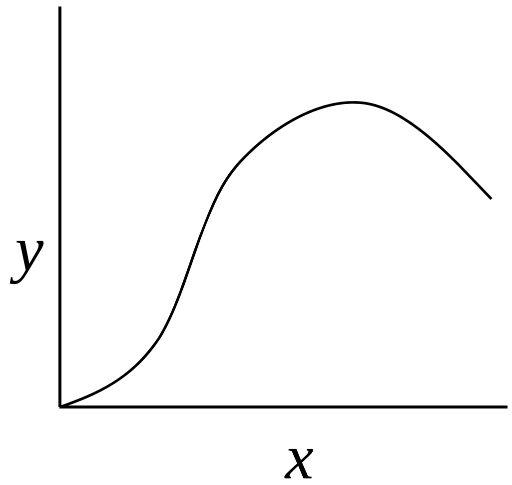
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Functions We've Seen So Far

- A *proposition* can be viewed as a function from "situations" to truth values $\{\mathbf{T},\mathbf{F}\}$
  - A logic system called *situation theory*.
  - p="It is raining."; s=our situation here,now
  - $p(s) \in \{\mathbf{T},\mathbf{F}\}$.
- A *propositional operator* can be viewed as a function from *ordered pairs* of truth values to truth values: $\vee((\mathbf{F},\mathbf{T})) = \mathbf{T}$.

Another example: $\rightarrow((\mathbf{T},\mathbf{F})) = \mathbf{F}$.

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More functions so far…

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values): P :≡ "is 7 feet tall"; P(Mike) = "Mike is 7 feet tall." = **False**.
- A *bit string B of length n* can be viewed as a function from the numbers {1,…,n} (bit *positions*) to the *bits* {0,1}. *E.g.*, B=101 ➔ B(3)=1.

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Still More Functions

- A *set S* over universe *U* can be viewed as a function from the elements of *U* to {**T**, **F**}, saying for each element of *U* whether it is in *S*. S={3}; S(0)=**F**, S(3)=**T**.
- A *set operator* such as $\cap$,$\cup$, $\overline{\ \ }$ can be viewed as a function from pairs of sets to sets.
  - Example: $\cap((\{1,3\},\{3,4\})) = \{3\}$

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A Neat Trick

- Sometimes we write Y^X to denote the set F of *all* possible functions $f:X \rightarrow Y$.
- This notation is especially appropriate, because for finite X, Y, $|F| = |Y|^{|X|}$.
- If we use representations $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{T} \equiv \mathbf{1}$, $\mathbf{2} :\equiv \{\mathbf{0},\mathbf{1}\} = \{\mathbf{F},\mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to $\mathbf{2}$, so the power set of S (set of all such fns.) is $\mathbf{2}^S$ in this notation.

Some Function Terminology

- If f:$A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R\subseteq B$ of f is $\{b \mid \exists a\, f(a)=b\}$.

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Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

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Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown!.
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}!.

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Operators (general definition)

- An n-ary *operator* over the set S is any function from the set of ordered n-tuples of elements of S, to S itself.
- *E.g.*, if $S=\{\mathbf{T},\mathbf{F}\}$, $\neg$ can be seen as a unary operator, and $\wedge,\vee$ are binary operators on S.
- Another example: $\cup$ and $\cap$ are binary operators on the set of all sets.

Constructing Function Operators

- If • ("dot") is any operator over B, then we can extend • to also denote an operator over functions $f:A\rightarrow B$.
- *E.g.*: Given any binary operator $\bullet:B\times B\rightarrow B$, and functions $f,g:A\rightarrow B$, we define $(f \bullet g):A\rightarrow B$ to be the function defined by: $\forall a\in A, (f \bullet g)(a) = f(a)\bullet g(a)$.

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Function Operator Example

- $+, \times$ ("plus","times") are binary operators over **R**. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f,g:\mathbf{R}\rightarrow\mathbf{R}$:
  - $(f+g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f \times g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f \times g)(x) = f(x) \times g(x)$

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Function Composition Operator

- For functions g:$A\rightarrow B$ and f:$B\rightarrow C$, there is a special operator called *compose* ("$\bigcirc$").
  - It composes (creates) a new function out of f,g by applying f to the result of g.
  - $(f\bigcirc g)$:$A\rightarrow C$, where $(f\bigcirc g)(a) = f(g(a))$.
  - Note $g(a)\in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\bigcirc$ (like Cartesian $\times$, but unlike $+,\wedge,\cup$) is non-commuting. (Generally, $f\bigcirc g \neq g\bigcirc f$.)

Images of Sets under Functions

- Given $f{:}A\rightarrow B$, and $S\subseteq A$,
- The *image* of S under f is simply the set of all images (under f) of the elements of S.
  $f(S) :\equiv \{f(s) \mid s\in S\}$
  $:\equiv \{b \mid \exists\, s\in S{:}\ f(s)=b\}$.
- Note the range of f can be defined as simply the image (under f) of f's domain!

One-to-One Functions

- A function is *one-to-one (1-1)*, or *injective*, or *an injection*, iff every element of its range has *only* 1 pre-image.
  - Formally: given $f{:}A \rightarrow B$,
    "x is injective" $:\equiv (\neg\exists x,y{:}\ x \neq y \wedge f(x)=f(y))$.
- Only one element of the domain is mapped to any given one element of the range.
  - Domain & range have same cardinality. What about codomain?

May Be Larger

- Each element of the domain is injected into a different element of the range.
  - Compare "each dose of vaccine is injected into a different patient."

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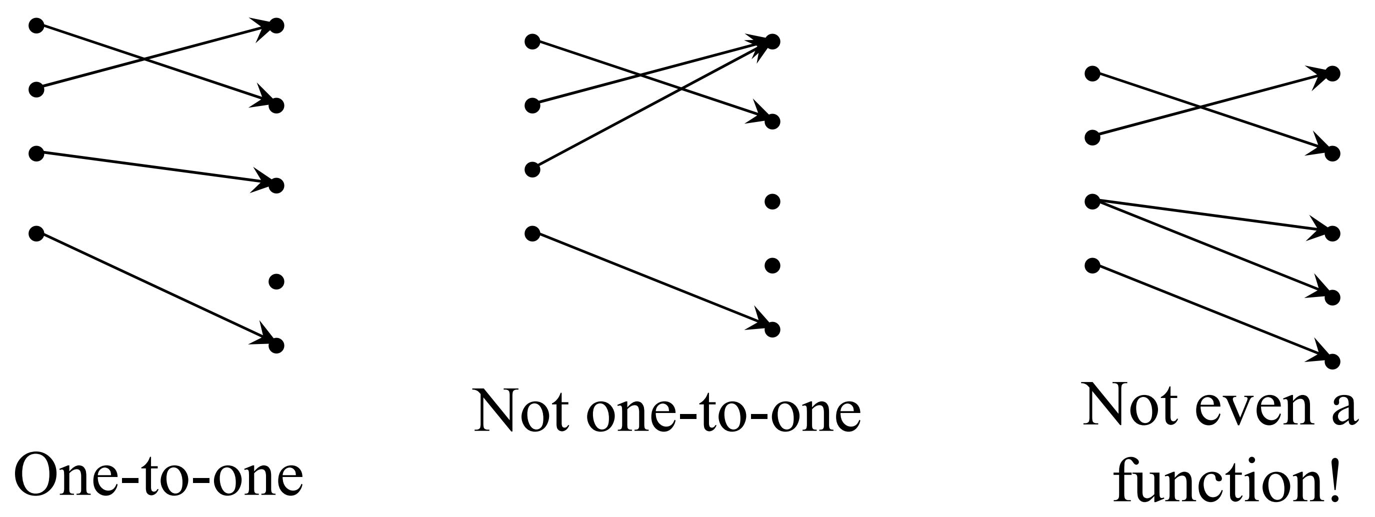
One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



Sufficient Conditions for 1-1ness

- For functions f over numbers,
  - f is *strictly* (or *monotonically*) *increasing* iff $x>y \rightarrow f(x)>f(y)$ for all x,y in domain;
  - f is *strictly* (or *monotonically*) *decreasing* iff $x>y \rightarrow f(x)<f(y)$ for all x,y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. *E.g.* x^3
  - *Converse is not necessarily true. E.g.* $1/x$

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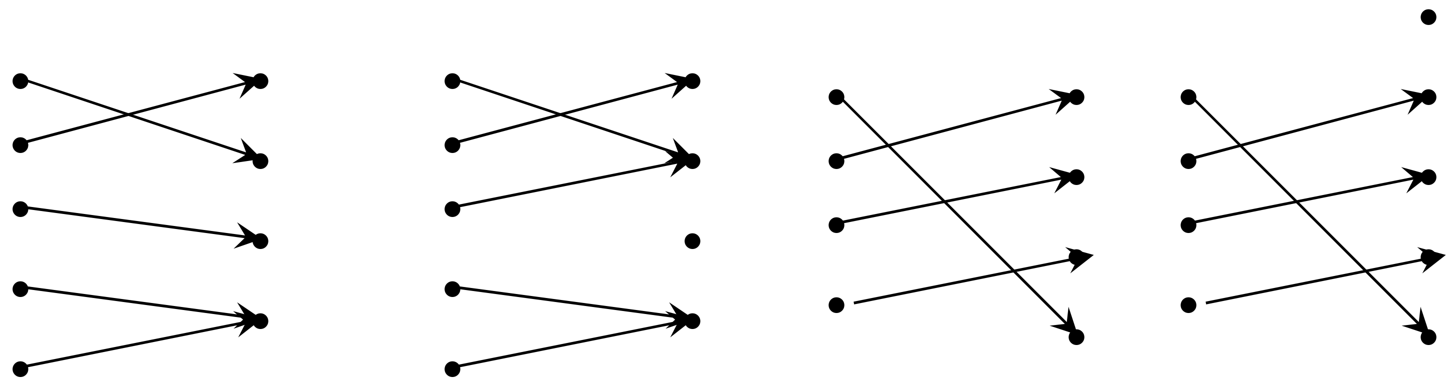
Onto (Surjective) Functions

- A function $f{:}A{\rightarrow}B$ is *onto* or *surjective* or *a surjection* iff its range is equal to its codomain ($\forall b{\in}B, \exists a{\in}A{:}\ f(a){=}b$).
- An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.
- *E.g.*, for domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Onto (but not 1-1) | Not Onto (or 1-1) | Both 1-1 and onto | 1-1 but not onto

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Bijections

- A function f is *a one-to-one correspondence*, or *a bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f:A\rightarrow B$, there exists an *inverse of f*, written $f^{-1}:B\rightarrow A$, which is the unique function such that $f^{-1} \circ f = I$ (the identity function)

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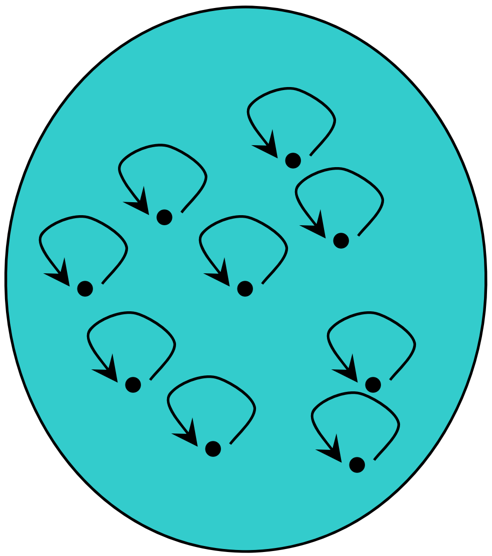
The Identity Function

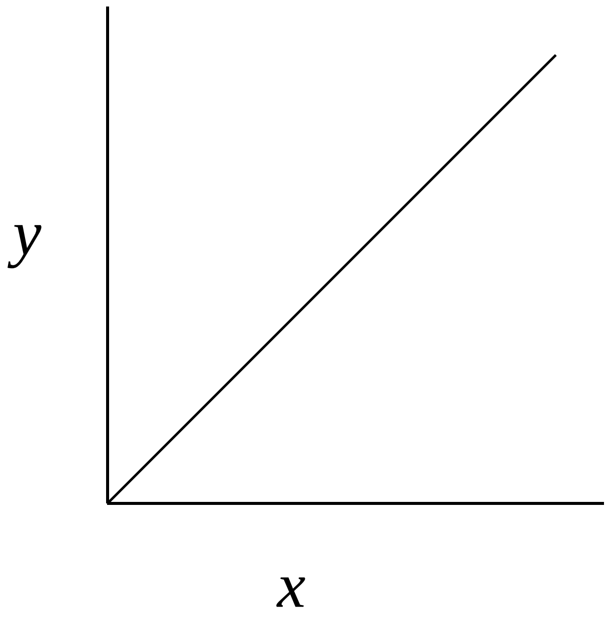
- For any domain A, the *identity function* $I{:}A\rightarrow A$ (variously written, I_A, **1**, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$: $I(a)=a$.
- Some identity functions you've seen:
  - +ing 0, ·ing by 1, ∧ing with **T**, ∨ing with **F**, ∪ing with ∅, ∩ing with *U*.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



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A Fun Application

- In a computer, the function mapping *state at clock cycle #t* to *state at clock cycle #t+1* is called the computer's *transition function*.
- If the transition function is reversible (a bijection), then the computer's operation in theory requires *no energy expenditure*.
- The study of low-power *reversible computing* techniques based on this idea is my primary research area.

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Graphs of Functions

- We can represent a function $f:A \rightarrow B$ as a set of ordered pairs $\{(a,f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair $(a,f(a))$.
  - Later (ch.6): *relations* loosen this restriction.
- For functions over numbers, we can represent an ordered pair (x,y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x.

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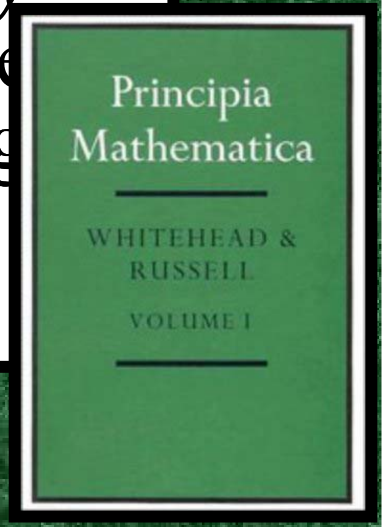
Comment About Representations

- You can represent any type of discrete structure (propositions, bit-strings, numbers, sets, ordered pairs, functions) in terms of virtually any of the other structures (or some combination thereof).
- Probably none of these structures is *truly* more fundamental than the others (whate that would mean). However, strings, log and sets are often used as the foundation for all else. *E.g.* in ➔



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A Couple of Key Functions

- In discrete math, we will frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ ("floor of x") is the largest (most positive) integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of x") is the smallest (most negative) integer $\geq x$.

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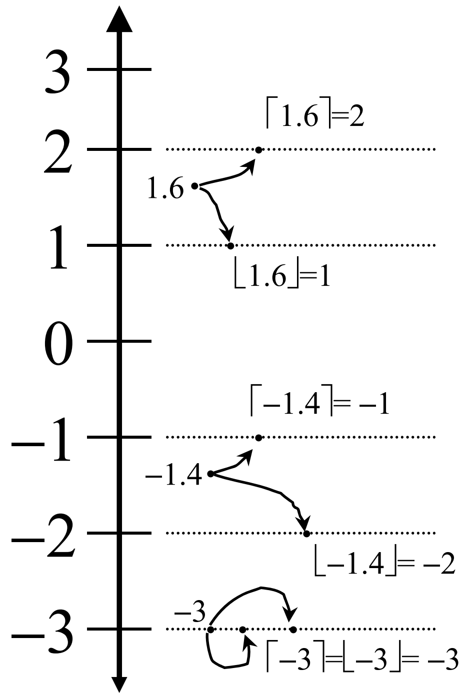
Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin \mathbf{Z}$, $\lfloor -x \rfloor \neq -\lfloor x \rfloor$ & $\lceil -x \rceil \neq -\lceil x \rceil$
- Note that if $x \in \mathbf{Z}$, $\lfloor x \rfloor = \lceil x \rceil = x$.



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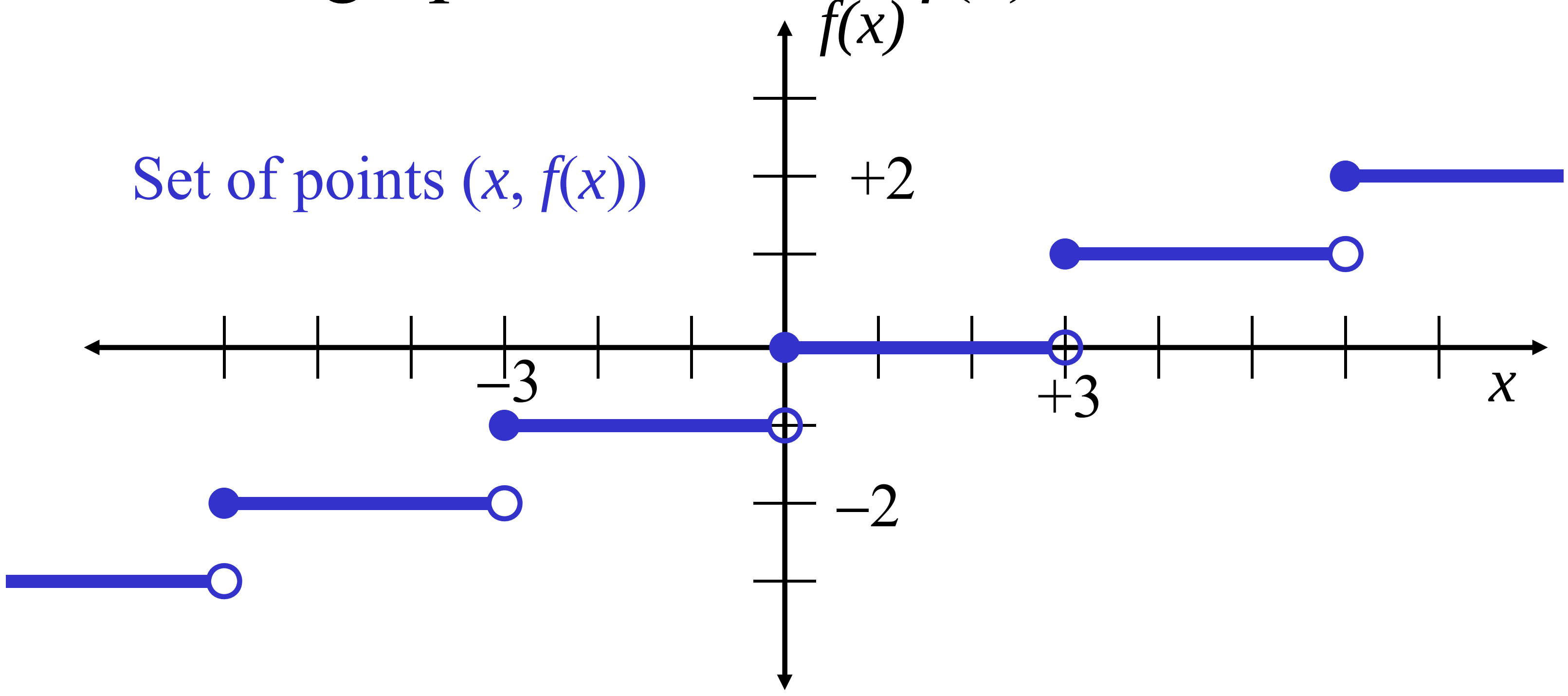
Plots with floor/ceiling

Note that for $f(x)=\lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $a \geq 0$ and $a<1$, but not for $a=1$. We say that the set of points $(a,0)$ that is in f does not include its *limit* or *boundary* point $(a,1)$. Sets that do not include all of their limit points are called *open sets*. In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve.

Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of §1.8 (Functions)

- Function variables f, g, h, …
- Notations: $f:A\rightarrow B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1}, binary operators $+$, $-$, *etc.*, and $\circ$.
- The $\mathbf{R}\rightarrow\mathbf{Z}$ functions $\lfloor x\rfloor$ and $\lceil x\rceil$.

Module #4, Topic #∞: **Cardinality & Infinite Sets**

Rosen 5th ed., §3.2
~1 lecture

Infinite Cardinalities (from §3.2)

- Using what we learned about *functions* in §1.8, it's possible to formally define cardinality for infinite sets.
- We show that infinite sets come in different *sizes* of infinite!
- This gives us some interesting proof examples, in anticipation of chapter 3.

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Cardinality: Formal Definition

- For any two (possibly infinite) sets A and B, we say that A and B *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (bijective function) from A to B.
- When A and B are finite, it is easy to see that such a function exists iff A and B have the same number of elements $n \in \mathbf{N}$.

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Countable versus Uncountable

- For any set *S*, if *S* is finite or $|S|=|\mathbf{N}|$, we say *S* is *countable*. Else, *S* is *uncountable.*
- Intuition behind "**countable**:" we can *enumerate* (generate in series) elements of *S* in such a way that *any* individual element of *S* will eventually be *counted* in the enumeration. Examples: **N**, **Z**.
- **Uncountable**: *No* series of elements of *S* (even an infinite series) can include all of *S*'s elements. Examples: **R**, $\mathbf{R}^2$, P(**N**)

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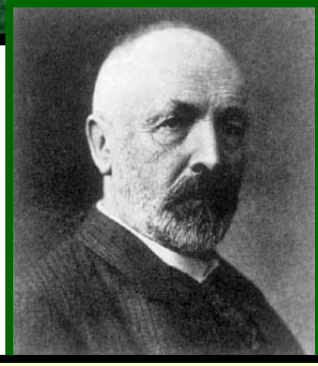
Countable Sets: Examples

- **Theorem:** The set **Z** is countable.
  - **Proof:** Consider f:**Z**→**N** where $f(i)=2i$ for $i \geq 0$ and $f(i) = -2i-1$ for $i<0$. Note f is bijective.
- **Theorem:** The set of all ordered pairs of natural numbers (n,m) is countable.
  - Consider listing the pairs in order by their sum $s=n+m$, then by n. Every pair appears once in this series; the generating function is bijective.

Uncountable Sets: Example

- **Theorem:** The open interval $[0,1) :\equiv \{r\in\mathbf{R}|\ 0 \le r < 1\}$ is uncountable.
- **Proof** by *diagonalization*: (Cantor, 1891)
  - Assume there is a series $\{r_i\} = r_1, r_2, \ldots$ containing *all* elements $r\in[0,1)$.
  - Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index: ... *(continued on next slide)*



Georg Cantor
1845-1918

Uncountability of Reals, cont'd

A postulated enumeration of the reals:

$$r_1 = 0.d_{1,1}\,d_{1,2}\,d_{1,3}\,d_{1,4}\,d_{1,5}\,d_{1,6}\,d_{1,7}\,d_{1,8}\ldots$$
$$r_2 = 0.d_{2,1}\,d_{2,2}\,d_{2,3}\,d_{2,4}\,d_{2,5}\,d_{2,6}\,d_{2,7}\,d_{2,8}\ldots$$
$$r_3 = 0.d_{3,1}\,d_{3,2}\,d_{3,3}\,d_{3,4}\,d_{3,5}\,d_{3,6}\,d_{3,7}\,d_{3,8}\ldots$$
$$r_4 = 0.d_{4,1}\,d_{4,2}\,d_{4,3}\,d_{4,4}\,d_{4,5}\,d_{4,6}\,d_{4,7}\,d_{4,8}\ldots$$

- Now, consider a real number generated by taking all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.

That real doesn't appear in the list!

Uncountability of Reals, fin.

- *E.g.*, a postulated enumeration of the reals:
  $r_1 =$ `0.301948571`…
  $r_2 =$ `0.103918481`…
  $r_3 =$ `0.039194193`…
  $r_4 =$ `0.918237461`…
- OK, now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103… can't be on the list anywhere!

Transfinite Numbers

- The cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of the natural numbers, $\aleph_0 :\equiv |\mathbf{N}|$, is the *first transfinite cardinal* number. (There are none smaller.)
- The *continuum hypothesis* claims that $|\mathbf{R}| = \aleph_1$, the *second transfinite cardinal*.

Proven impossible to prove or disprove!

Do Uncountable Sets Really Exist?

- The set of objects that can be defined using finite-length strings of symbols ("descriptions") is only *countable*.
- Therefore, any uncountable set must consist primarily of elements which individually have *no* finite description.
- Löwenheim-Skolem theorem: No consistent theory can ever *force* an interpretation involving uncountables.
- The "constructivist school" asserts that only objects constructible from finite descriptions exist. (*e.g.* $\neg\exists\mathbf{R}$)
- Most mathematicians are happy to use uncountable sets anyway, because postulating their existence has not led to any demonstrated contradictions (so far).

Countable vs. Uncountable

- You should:
  - Know how to define "same cardinality" in the case of infinite sets.
  - Know the definitions of *countable* and *uncountable*.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.

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