457.562 Special Issue on River Mechanics (Sediment Transport)

.03 Review of Fluid Mechanics



Appendix. Cartesian tensors

- Kronecker Delta and Alternating Tensor
  - Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$$

  - Which is written in the matrix form as

$$\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - The most common use (think summation convention)

$$\delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3$$

  - Simply we can write

$$\delta_{ij} u_j = u_i$$

Appendix. Cartesian tensors

- So we call kroneker tensor as “isotropic tensors” of the sec ond order tensors.
- Isotropic tensor for the third order (Alternating tensor)

$$\epsilon_{ijk} = \begin{cases} 1 & if \;\; ijk = 123, 231, or\ 312\ (cyclic\ order) \\ 0 & if\ any\ two\ indices\ are\ equal \\ -1 & if \;\; ijk = 321, 213, or\ 132\ (anticyclic\ order) \end{cases}$$

- Therefore

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} \qquad and \qquad \epsilon_{ijk} = -\epsilon_{ikj}$$

- The epsilon delta relation.

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Appendix. Cartesian tensors

- Dot Products

$$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}=u_1v_1+u_2v_2+u_3v_3=u_iv_i$$

- Cross product

$$\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}=\left(u_2v_3-u_3v_2\right)\mathbf{a}^1+\left(u_3v_1-u_1v_3\right)\mathbf{a}^2+\left(u_1v_2-u_2v_1\right)\mathbf{a}^3$$

  – Symbolic determinant

$$\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{u}=\begin{vmatrix}\mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$$

  – The k-component of uxv can be written as

$$\left(\mathbf{u}\times\mathbf{v}\right)_k=\epsilon_{ijk}u_iv_j=\epsilon_{kij}u_iv_j$$

  – Example, i=1

$$\left(\mathbf{u}\times\mathbf{v}\right)_1=\epsilon_{ij1}u_iv_j=\epsilon_{231}u_2v_3+\epsilon_{321}u_3v_2=u_2v_3-u_3v_2$$

1. Definitions

- Coordinate system
  - Spatial position vector

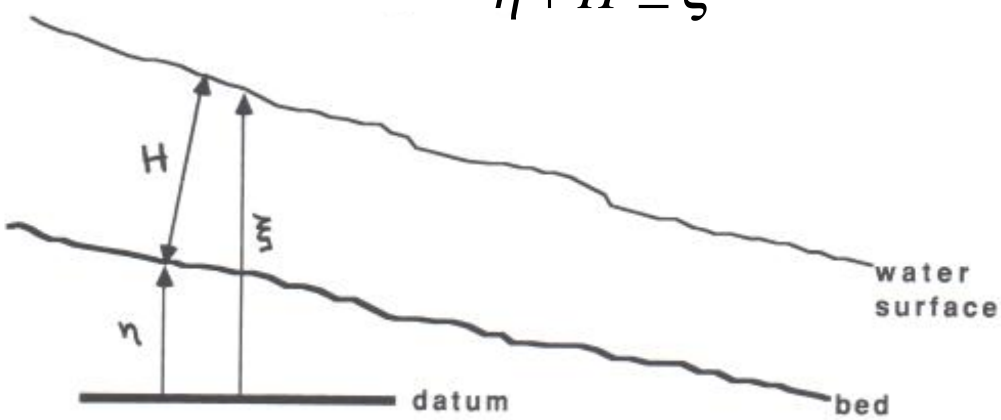
$$x_i = (x_1, x_2, x_3) = (s, n, z)$$

  - Instantaneous flow velocity vector is give

$$u_i = (u_1, u_2, u_3) = (u, v, w)$$

  - Vertical bed and water surface elevation above some datum are given as η and ξ. Channel depth measured normal to the bed is given as H. If z is nearly vertical

$$\eta + H \cong \xi$$



2. Navier-stokes equations

- For flow in a river channel with a dilute concentration of sediment, th ese relations take the form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + g_i$$

- Continuity equation

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}$$

- Gravitational acceleration

$$g_i = \left(g_1, g_2, g_3\right)$$

- Denotes the component of the vector of gravitational acceleration in the x, y, and z directions.

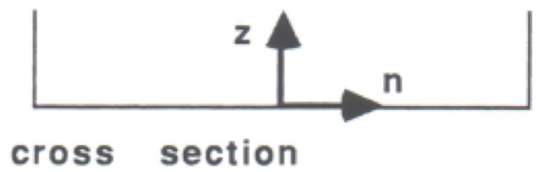
2. Navier-stokes equations

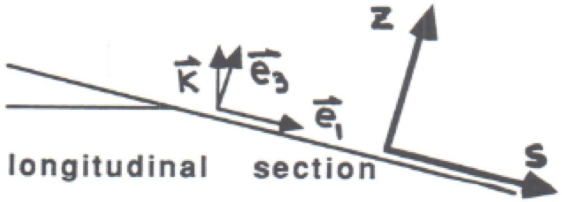
- Let's $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ denote unit vectors in the x_1, x_2, and x_3 directions, and let **k** denote a unit vector in the upward vertical direction.

$$g_i = -g\mathbf{k} \cdot e_i$$

- For example, consider a rectangular channel with a trans versely horizontal bed that is tilted a small angle α in the down stream direction. The bed slope s is given by

$$S = \tan\alpha = -\frac{\partial \eta}{\partial x}$$





2. Navier-stokes equations

- Thus $\vec{e}_1 \cdot \vec{\mathbf{k}} = \cos(90+\alpha) = -\sin\alpha \cong -S,$

  $$\vec{e}_2 \cdot \vec{\mathbf{k}} = 0, \quad \vec{e}_3 \cdot \vec{\mathbf{k}} = \cos\alpha \cong 1$$

- An the gravitational vector can approximated as

  $$g_i = g(S, 0, -1)$$

- gS implies the river flow and represents the downstream force of gravity acting on the flow.

3. Stress tensor

- The Navier-Stokes equations can also be written in the following form:

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} + g_i$$

- Newtonian stress tensor:

$$\tau_{ij} = -p\delta_{ij} + \rho \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

τ_{11}, τ_{22} and τ_{33} denote normal stresses;

the other denote shear stresses.

4. Reynolds Equations

- The effect of turbulence is dominant as regards to river behavior.
- The instantaneous flow field cannot be predicted nor would one be able to process the massive amount of random data were the prediction possible.
- An appropriate technique is to average the Navier-Stokes equations.
- The convective term in N-S equations

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i u_j}{\partial x_j}$$

4. Reynolds Equations

- When the overbar denotes averaging (ensemble) and prime means the fluctuations (or deviation from the mean)

  $$u_i = \overline{u}_i + u_i'$$

- When apply this Reynolds decomposition, then

  $$\overline{u_i u_j} = \overline{u}_i \overline{u}_j + \overline{u_i' u_j'}$$

- Nonlinearity generates a residual term that in fact becomes dominant in the case of turbulence.
- Question: What is the physical meaning of the following term? $\rho \overline{u_i' u_j'}$
- Reynolds stress (tensor): $-\rho \overline{u_i' u_j'}$

4. Reynolds Equations

- The entire N-S equations can be averaged to yield the R eynolds equations.

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \rho\overline{u_i' u_j'}\right) + g_i$$

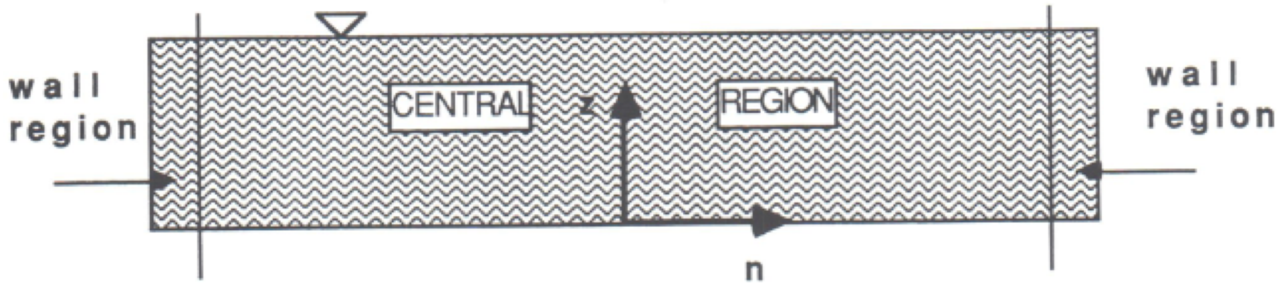
- Here

$$\overline{\tau}_{ij} = -\overline{p}\delta_{ij} + \rho\nu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)$$

$$T_{ij} = \overline{\tau}_{ij} - \rho\overline{u_i' u_j'}$$

5. Boundary shear stress:

(normal flow in a wide, rectangular open channel)



- In a wide rectangular flumes.
- The bed is taken to be horizontal in the transverse directi on but tilted with slope S in the down stream direction.
- Wall effect zone is excluded. Then flow flow is taken to b e steady in time and uniform in the x_1(s) and x_2(n) directi ons.

$$\bar{u}_i = \left(\bar{u}_1(x_3), 0, 0\right) = \left(u(z), 0, 0\right)$$

5. Boundary shear stress:

(normal flow in a wide, rectangular open channel)

- The non-zero components of the mean stress tensor

$$T_{13} = \rho\nu \frac{d\bar{u}_1}{dx_3} - \rho\overline{u_1'u_3'} = \rho\nu\frac{d\bar{u}}{dz} - \rho\overline{u'w'} \equiv \tau$$

$$T_{33} = -\bar{p} - \rho\overline{u_3'u_3'} = -\bar{p} - \rho\overline{w'^2} \equiv -P$$

- The Reynolds equations reduce to

$$(i=1) \quad 0 = \frac{1}{\rho}\frac{d\tau}{dz} + gS \qquad\qquad \tau = \tau_b\left(1 - \frac{z}{H}\right)$$

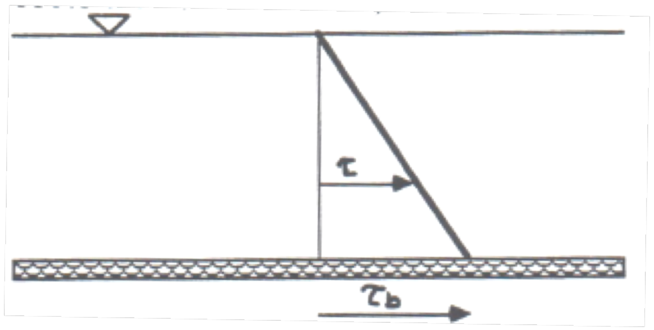
$$(i=3) \quad 0 = -\frac{1}{\rho}\frac{dP}{dz} - g \qquad\qquad P = \rho g H\left(1 - \frac{z}{H}\right)$$

- Integrating at the water surface (z=H), then

$$where \;\; \tau_b = \rho g H S \text{ (bottom shear stress)}$$

5. Boundary shear stress:

(normal flow in a wide, rectangular open channel)



$$\tau = \tau_b \left(1 - \frac{z}{H}\right)$$

$$P = \rho g H \left(1 - \frac{z}{H}\right)$$

- The effective mean pressure obeys the hydrostatic law, a nd the mean shear stress varies linearly, as shown abov e.
- The bottom shear stress drives sediment transport in mo st cases.

$$\tau_b = \rho g H S$$

6. One-dimensional model of varying boundary shear stress
: Gradually varied flow in a wide rectangular channel

- Assumptions:
  - Vary slowly in x and t
  - Wide and rectangular
- One dimensional St. Venant Shallow Water equation
  - If a typical scale of flow variation in the x direction is much larger than the depth H, and the scale of time variation is li kewise much larger than H/u, the Reynolds equations can be approximated by the turbulent boundary layer equation s.
  - Logarithmic law for turbulent flow,

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial s}=-g\frac{\partial}{\partial s}(\eta+H)-\frac{1}{\rho}\tau_b H^{-1}$$

6. One-dimensional model of varying boundary shear stress
: Gradually varied flow in a wide rectangular channel

- One dimensional St. Venant Shallow Water equation

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial s}=-g\frac{\partial}{\partial s}(\eta+H)-\frac{1}{\rho}\tau_b H^{-1}$$

- Continuity equations

$$\frac{\partial H}{\partial t}+\frac{\partial UH}{\partial s}=0$$

$$U=\frac{1}{H}\int_0^H \bar{u}\,dz$$

- With assumptions

$$g\frac{\partial}{\partial s}(\eta+H)=-\frac{1}{\rho}\tau_b H^{-1}$$

$$\tau_b=-\rho gHS-\rho gH\frac{\partial H}{\partial s}$$

6. One-dimensional model of varying boundary shear stress

: Gradually varied flow in a wide rectangular channel

- Velocity decrease, shear decreases, and H increases (d H/ds>0).
- Decreasing bottom shear stress implies declining sediment transport capacity n the down stream direction; this is the mechanism that drives sedimentation in dams.

6. Two dimensional St. Venant Equations

- Depth integrated transverse velocity V is defined as follows:

$$V = \frac{1}{H}\int_0^H \overline{v}\, dz$$

- An integration of the appropriate form of the Reynolds equations yields

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial s} + V\frac{\partial V}{\partial n} = -g\frac{\partial}{\partial s}(\eta + H) - \frac{\tau_{bs}}{\rho H}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial s} + V\frac{\partial V}{\partial n} = -g\frac{\partial}{\partial n}(\eta + H) - \frac{\tau_{bn}}{\rho H}$$

- Continuity $\frac{\partial H}{\partial t} + \frac{\partial UH}{\partial s} + \frac{\partial VH}{\partial n} = 0$