457.309 Hydraulics & Laboratory
.04 Review on Dimensional Analysis

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Today's objectives

- Understanding hydraulic system components
- Reminding dimensional analysis

Various dimensional representations:

- How about Slope, strain, angle, Reynolds number?

- We discussed the categorization of hydraulic components
  - Geometric group : only in length
  - Kinematic group: no mass nor force
  - Dynamic group: all

Empirical Formulation of General Flow Equation

- Dimensional analysis is powerful tool to determine which parameters are important in certain condition
  - Open channel: gravity and friction,
  - Pipe: pressure and viscous forces
  - Shear stress: boundary roughness and viscosity.

Empirical Formulation of General Flow Equation

- Let's assume *f* is a function of hydraulic event and express in the terms defined in the previous slide, then

$$f(D, V, \rho, L, \Delta(p+\gamma h), \mu, \gamma, \sigma, E, z, s, r, A, B, C, \ldots) = 0$$

- The variables in this equation can be grouped in a number of dimensionless groups or terms. In the above equation, there is no temperature
- So, we can describe all terms with 3 parameters of length, time and force (or mass)
- Which could be the basic variables?
  - Usually we choose, geometry, kinematic, & dynamic variables (not parameters)
  - *D* (geometry), *V* (kinematic), *ρ* (dynamic) are most basic variables in this equation.

Empirical Formulation of General Flow Equation

- Therefore,

(basic parameters)

$$f(D, V, \rho, L, \Delta(p+\gamma h), \mu, \gamma, \sigma, E, z, s, r, A, B, C, \ldots) = 0$$

$$f\left(\frac{D^{a_1} V^{b_1} \rho^{c_1}}{L}, \frac{D^{a_2} V^{b_2} \rho^{c_2}}{\Delta(p+\gamma h)}, \frac{D^{a_3} V^{b_3} \rho^{c_3}}{\mu}, \frac{D^{a_4} V^{b_4} \rho^{c_4}}{\gamma}, \ldots\right) = 0$$

- Our purpose is to eliminate the dimensions. How about the first term?
  - $a_1 = 1$ why?
  - And $b_1 = 0$, why?,
  - $C_1 = 0$
- This is pretty straight forward and simple

Empirical Formulation of General Flow Equation

$$f\left(\frac{D^{a_1}V^{b_1}\rho^{c_1}}{L}, \frac{D^{a_2}V^{b_2}\rho^{c_2}}{\Delta(p+\gamma h)}, \frac{D^{a_3}V^{b_3}\rho^{c_3}}{\mu}, \frac{D^{a_4}V^{b_4}\rho^{c_4}}{\gamma}, \ldots\right) = 0$$

- The from second term, we need to do in more complicate and general method
- Using, MLT expression (mass, length, time)
- What is the dimension of the second term?

Pressure = Force/Area

[pressure]=[MLT^{-2}]

Empirical Formulation of General Flow Equation

$$\frac{D^{a_2}V^{b_2}\rho^{c_2}}{\Delta(p+\gamma h)} = (L)^{a_2}\left(LT^{-1}\right)^{b_2}\left(ML^{-3}\right)^{c_2}\left(ML^{-1}T^{-2}\right)^{-1}$$

- If this equation is dimensionless, then

$$(L)^{a_2}\left(LT^{-1}\right)^{b_2}\left(ML^{-3}\right)^{c_2}\left(ML^{-1}T^{-2}\right)^{-1} = M^0L^0T^0$$

For M, c_2-1=0, This yields c_2=1

For T, -b_2+2=0, This yields b_2=2

For L, a_2-b_2-c_3+1=0, This yields a_2=0

$$\frac{D^{a_2}V^{b_2}\rho^{c_2}}{\Delta(p+\gamma h)} = \frac{D^0V^2\rho^1}{\Delta(p+\gamma h)} = \frac{\rho V^2}{\Delta(p+\gamma h)}$$

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Empirical Formulation of General Flow Equation

$$f\left(\frac{D}{L}, \frac{\rho V^2}{\Delta(p+\gamma h)}, \frac{\rho VD}{\mu}, \frac{\rho V^2}{\gamma D}, \frac{\rho V^2 D}{\sigma}, \frac{\rho V^2}{\gamma D}, \frac{\rho V^2}{E}, \frac{D}{z}, \frac{D}{s}, \frac{D}{r} \ldots\right) = 0$$

- This equation involves all of the parameters of the previous equation but is in a dimensionless form and with less terms (-3).
- This indicates that each element does not tell the real physics of flow, but the relative values of each term is crucial for explaining the flow patterns.

Empirical Formulation of General Flow Equation

$$f\left(\frac{D}{L},\frac{\rho V^2}{\Delta(p+\gamma h)},\frac{\rho VD}{\mu},\frac{\rho V^2}{\gamma D},\frac{\rho V^2 D}{\sigma},\frac{\rho V^2}{\gamma D},\frac{\rho V^2}{E},\frac{D}{z},\frac{D}{s},\frac{D}{r}\ldots\right)=0$$

$$\frac{\rho V^2}{\Delta(p+\gamma h)},\frac{\rho V^2}{\mu v/D},\frac{\rho V^2}{\gamma D},\frac{\rho V^2}{\sigma/D},\frac{\rho V^2}{\gamma D},\frac{\rho V^2}{E}$$

- The numerators and denominators are dimensionally equivalent to force per unit area (stress)
- The numerator is inertia force (inherent in fluid's motion)
- The denominators are "pressure, viscous, gravity, surface tension, elasticity)

Empirical Formulation of General Flow Equation

$$\frac{\rho V^2}{\Delta(p+\gamma h)},\ \frac{\rho V^2}{\mu v / D},\ \frac{\rho V^2}{\gamma D},\ \frac{\rho V^2}{\sigma / D},\ \frac{\rho V^2}{\gamma D},\ \frac{\rho V^2}{E}$$

$$\frac{V}{\sqrt{\Delta(p+\gamma h)/\rho}},\ \frac{V}{\sqrt{\mu v / D/\rho}},\ \frac{V}{\sqrt{\gamma D/\rho}},\ \frac{V}{\sqrt{\sigma / D/\rho}},\ \frac{V}{\sqrt{\gamma D/\rho}},\ \frac{V}{\sqrt{E/\rho}}$$

- Euler number
- Reynolds number
- Froude number
- Weber number
- Mach number

$$E_n = 2\left[\frac{V}{\sqrt{\Delta(p+\gamma h)/\rho}}\right]^{-2} = \frac{\Delta(p+\gamma h)}{\frac{1}{2}\rho V^2}$$

$$R_n = \left[\frac{V}{\sqrt{\mu v / D/\rho}}\right]^{2} = \frac{DV\rho}{\mu} = \frac{DV}{\nu}$$

$$F_n = \frac{V}{\sqrt{\gamma D/\rho}} = \frac{V}{\sqrt{gD}}$$

$$W_n = \left[\frac{V}{\sqrt{\sigma / D/\rho}}\right]^{2} = \frac{\rho D V^2}{\sigma}$$

$$M_n = \frac{V}{\sqrt{E/\rho}}$$

Methods of Dimensional Analysis

- There are two methods for dimensional analysis
  - Rayleigh method
    - 5-6 variables are OK
    - Over this number, then difficult
  - Buckingham-Pi (method)
    - Dealing with any number of variables
    - Widely numbers of variable are OK
- I suggest for you to use Buckingham-Pi rather than Rayleigh

Buckingham-Pi method

- It expresses the *n* number of variables into (n-m) number of dimensionless groups (in the last derivation).
- m may be 3 or 2 (*L, T, M* or *F* (in general hydraulics)
- If a phenomenon involves *n* variables a_1, a_2, a_3, a_4,… a_n, and one of these variables, say a_1 is dependent on the remaining independent variables, we may express the relationship between the variable in general form

$$a_1 = \phi\left(a_2, a_3, a_4, \cdots, a_n\right)$$

Buckingham-Pi method

- Steps for solution

1. Identify the significant parameters, and let the number of these parameters be *n*.
2. Define the set of fundamental dimensions (*M, L* and *T or FLT sys.* Sometimes *q*.)
3. Find number of m.
   - Kinetic problem *m* must be 2
   - General hydraulic problems *m* is 2 or 3
   - We call this as “recurring variables”
4. After determining the number of recurring variables, select them from the entire set of parameters:
   - The first repeated variable: geometric variable. (*L* or manipulation)
   - The second repeated variable: Kinematics of problem. Should contain *T*. (no mass no force)
   - The third repeated variable: Dynamics of the problem. Should contain mass or force
5. The number of π terms is equal to (*n-m*). Solve
6. Check the dimensions

Buckingham-Pi method

- Using the Buckingham method, derive an expression for the shear stress τ, in a fluid flowing in a pipe assuming that it is a function of the diameter, D, pipe roughness ε, fluid density ρ, dynamic viscosity μ, and fluid velocity v.

Buckingham-Pi method (Example)

- Variables: τ, D, e, ρ, μ, v.
  - The number of variables n=6
- The variable contains F and M (m=3) and $n\text{-}m=6\text{-}3=3$.
- Select the following repeated variables:
  1. D, Geometry
  2. V, kinematics (including T)
  3. ρ, dynamics (including M or F)
- Each of the pi terms should contain the above three repeated variables plus another new variable.
- Since $n\text{-}m=6\text{-}3=3$, there are three π s. If $n\text{-}m=7\text{-}3=4$, there are four π s

The first pi (π_1)

$$\pi_1 = (D)^{a_1}(V)^{b_1}(\rho)^{c_1}(\tau)^{-1}$$

$$M^0T^0L^0 = (L)^{a_1}\left(LT^{-1}\right)^{b_1}\left(ML^{-3}\right)^{c_1}\left(ML^{-1}T^{-2}\right)^{-1}$$

- For M, $0=c_1-1$. $c_1=1$
- For T, $0=-b_1+2$. $b_1=2$
- For L, $0=a_1+b_1-3c_1+1=a_1+2-3+1$, $a_1=0$
- Therefore,

$$\pi_1 = V^2\rho\tau^{-1} = \frac{\rho V^2}{\tau} \quad or \quad \pi_1 = \frac{\tau}{\rho V^2}$$

The second pi (π_2)

$$\pi_2 = (D)^{a_2} (V)^{b_2} (\rho)^{c_2} (e)^{-1}$$

$$M^0 T^0 L^0 = (L)^{a_2} \left(LT^{-1}\right)^{b_2} \left(ML^{-3}\right)^{c_2} (L)^{-1}$$

- For *M*, $0=c_2$. $c_1=0$
- For *T*, $0=-b_2$. $b_1=0$
- For *L*, $0=a_2+b_2-3c_2-1$, $a_2=1$
- Therefore,

$$\pi_2 = \frac{D}{e} \quad or \quad \pi_2 = \frac{e}{D}$$

The third pi (π_3)

$$\pi_3 = (D)^{a_3}(V)^{b_3}(\rho)^{c_3}(\mu)^{-1}$$

$$M^0T^0L^0 = (L)^{a_3}(LT^{-1})^{b_3}(ML^{-3})^{c_3}(ML^{-1}T^{-1})^{-1}$$

- For *M*, $0=c_3-1$. $c_3=1$
- For *T*, $0=-b_3+1$. $b_3=1$
- For *L*, $0=a_3+b_3-3c_3+1$, $a_3=1$
- Therefore,

$$\pi_3 = \frac{\rho V D}{\mu} = R_n$$

Now, we collect all pi terms

$$\varphi(\pi_1, \pi_2, \pi_3) = 0$$

Or

$$\pi_1 = \varphi(\pi_2, \pi_3)$$

Hence

$$\frac{\tau}{\rho V^2} = \varphi_1\left(\frac{D}{e}, R_n\right)$$

or

$$\tau = \rho V^2 \varphi_1\left(\frac{D}{e}, R_n\right)$$

Example 2.

The efficiency of a propeller η is believed to be dependent on its diameter, *D*, the fluid density, ρ and dynamic viscosity μ, angular velocity ω and discharge Q. Using dimensional analysis, develop a relation between η and these variables.

Solution:

The variables are η, *D*, ρ, μ, ω, *Q.*

The number of variables

$n=6$.

The number of repeated variables

$m=3$.

The number of π*-terms* =

$6-3=3$.

The repeated variables are selected as

D, ω, ρ.

Example 2.

- The first π- term is

$$\pi_1 = (D)^{a_1} (\varpi)^{b_1} (\rho)^{c_1} (\eta)^{-1}$$

$$M^0 T^0 L^0 = (L)^{a_1} \left(T^{-1}\right)^{b_1} \left(ML^{-3}\right)^{c_1} \left(M^0 T^0 L^0\right)^{-1}$$

- Keep in mind that, efficiency is dimensionless and angular velocity's unit is 1/T

- For *M*, $0=c_1$. $c_1=0$
- For *T*, $0=-b_1$. $b_1=0$
- For *L*, $0=a_1-3c_1$, $a_1=0$

Therefore, $$\pi_1 = \frac{1}{\eta} \quad or \quad \pi_1 = \eta$$

Example 2.

The second π- term is

$$\pi_2 = (D)^{a_2} (\varpi)^{b_2} (\rho)^{c_2} (\mu)^{-1}$$

$$M^0 T^0 L^0 = (L)^{a_2} \left(T^{-1}\right)^{b_2} \left(ML^{-3}\right)^{c_2} \left(MT^{-1}L^{-1}\right)^{-1}$$

- For M, $0=c_2-1$. $c_2=1$
- For T, $0=-b_2+1$. $b_2=1$
- For L, $0=a_2-3c_2+1$ $a_2=2$

Therefore,

$$\pi_2 = \frac{D^2 \varpi \rho}{\mu}$$

Example 2.

The third π-term is

$$\pi_3 = (D)^{a_3} (\varpi)^{b_3} (\rho)^{c_3} (Q)^{-1}$$

$$M^0 T^0 L^0 = (L)^{a_3} \left(T^{-1}\right)^{b_3} \left(ML^{-3}\right)^{c_3} \left(T^{-1} L^3\right)^{-1}$$

- For M, $0=c_3$. $c_3=0$.
- For T, $0=-b_3+1$. $b_3=1$.
- For L, $0=a_3-3c_3-3$ $a_3=3$

Therefore,

$$\pi_3 = \frac{\varpi D^3}{Q}$$

Relating the three π-terms, we obtain

$$\pi_1 = \varphi(\pi_2, \pi_3)$$

or

$$\eta = \varphi\left(\frac{D^2 \varpi \rho}{\mu}, \frac{\varpi D^3}{Q}\right)$$

Useful hints

$$\pi_1 = \varphi_1\left(\pi_1{}^2\right) = \varphi_2\left(\sqrt{\pi_1}\right) = \varphi_3\left(\pi_1\left(\frac{\pi_2}{\pi_3}\right)\right) = \varphi_4\left(\pi_1\sqrt{\frac{\pi_3}{\pi_1}}\right)$$

$$\pi_1 = \varphi_1(\pi_2, \pi_3, \pi_4) \quad or$$

$$\pi_3 = \varphi_2(\pi_1, \pi_2, \pi_4) \quad or$$

$$\frac{\pi_3}{\pi_2} = \varphi_3(\pi_1, \pi_4) \quad or$$

$$\frac{\pi_1}{\pi_3} = \varphi_4\left(\pi_2, \frac{\pi_4}{\pi_1}\right) \quad or$$

one variable : geometry,

one variable : kinematic (fluid)

one variable : dynamic (flow)

The dependent variable should never be selected as a repeated variable.

Example 3.

Prove that the power, *p*, developed by a Kaplan turbine can be expressed as

$$P = \rho D^5 N^3 \varphi\left(\frac{H}{D}, \frac{\rho D^2 N}{\mu}, \frac{DN^2}{g}\right)$$

Where ρ is the fluid density, *D* is the average diameter of the vanes, *N* is the speed in rpm, *H* is the operation head, μ is the dynamic viscosity, and g is the acceleration of gravity.

Comment:

1. There are 7 variables
2. 3 main parameters (geometry, kinematic, dynamic), and these are repeatable parameters
3. Therefore, *H,* μ*, g* and *p* must be used for constructing π.