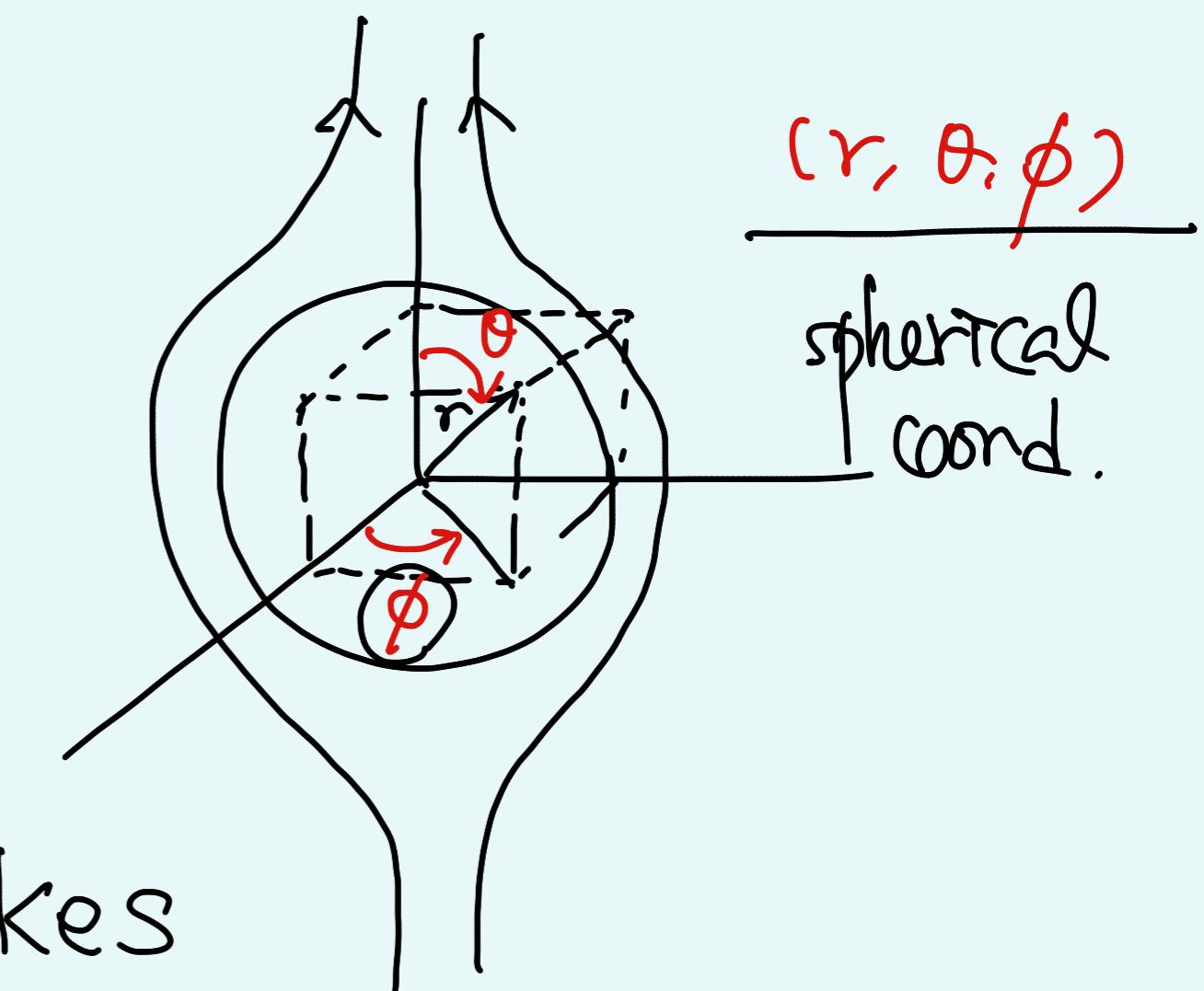
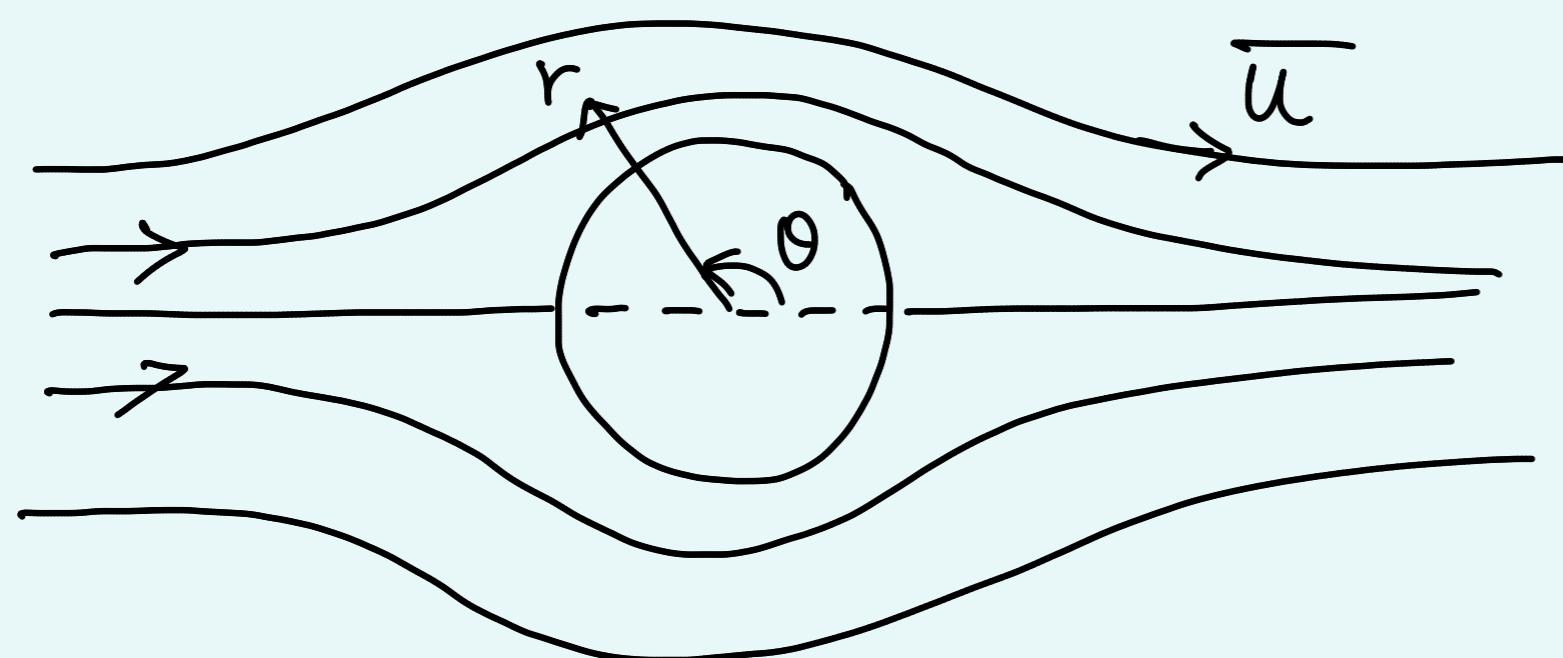


"Low Reynolds number" flow past a sphere



Stokes

axisymmetric flow in spherical polar coordinates

$$\bar{u} = [u_r(r, \theta), u_\theta(r, \theta), 0]$$

stream function $\psi(r, \theta)$

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

$\nabla \cdot \bar{u} = 0$ automatically.

$$\left[\begin{array}{l} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{array} \right] \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = -\nabla p + \mu \nabla^2 \bar{u}$$

What is the drag D ?

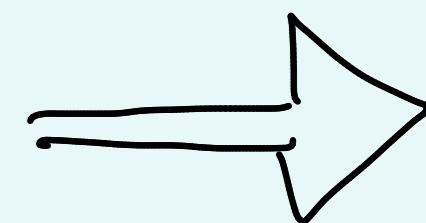
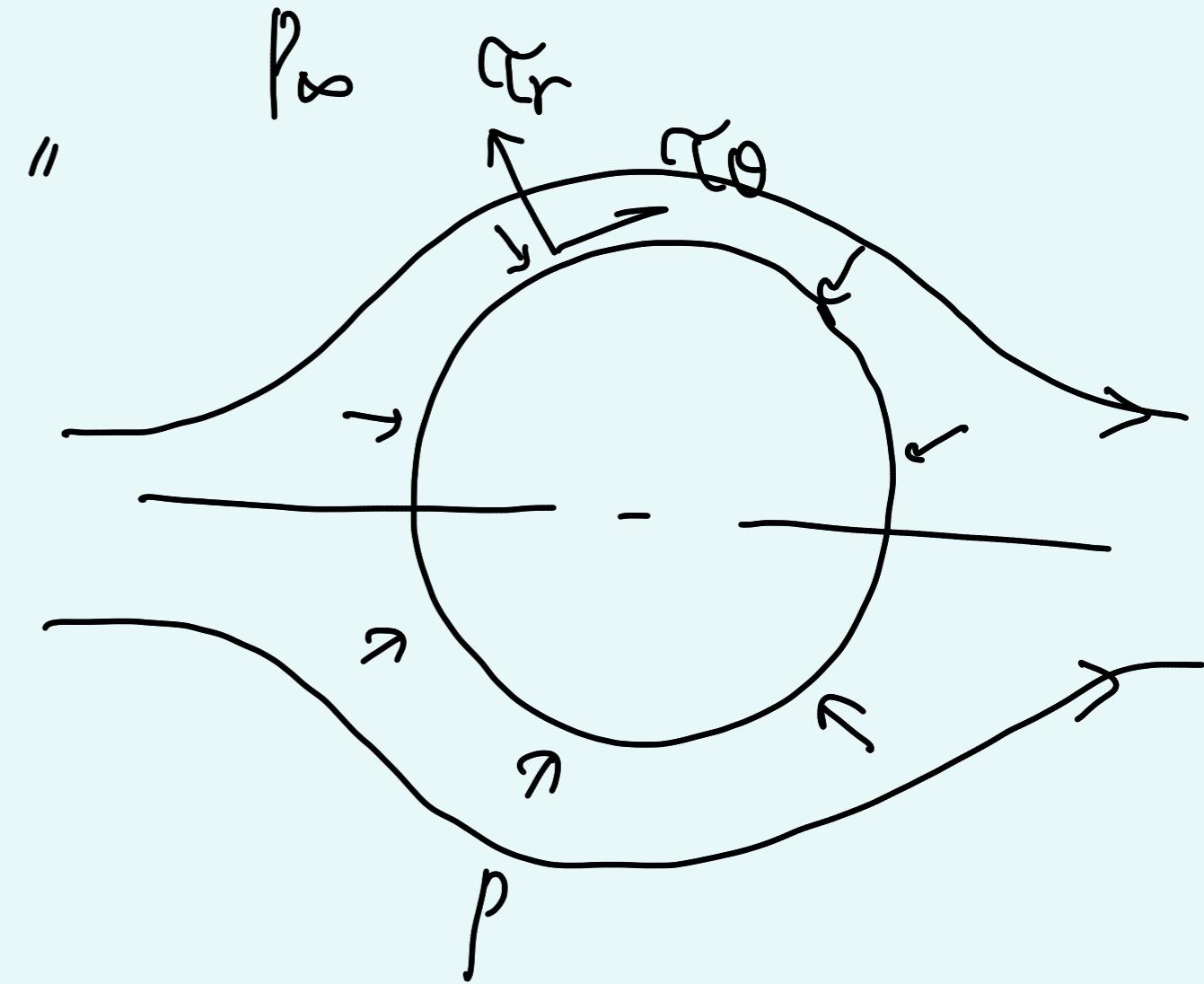
Using $E^2 \phi = \frac{3}{2} U a r^{-1} \sin^2 \theta$

$$p = p_\infty - \frac{3}{2} \frac{\mu U a}{r^2} \cos \theta$$

stress component on the sphere:

$$\begin{cases} \tau_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} \\ \tau_{r\theta} = \mu r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{\mu}{r} \frac{\partial u_r}{\partial \theta} \\ \tau_{r\phi} = 0 \end{cases}$$

stress component in the direction of net force



$$\tau = \tau_r \cos \theta - \tau_\theta \sin \theta$$

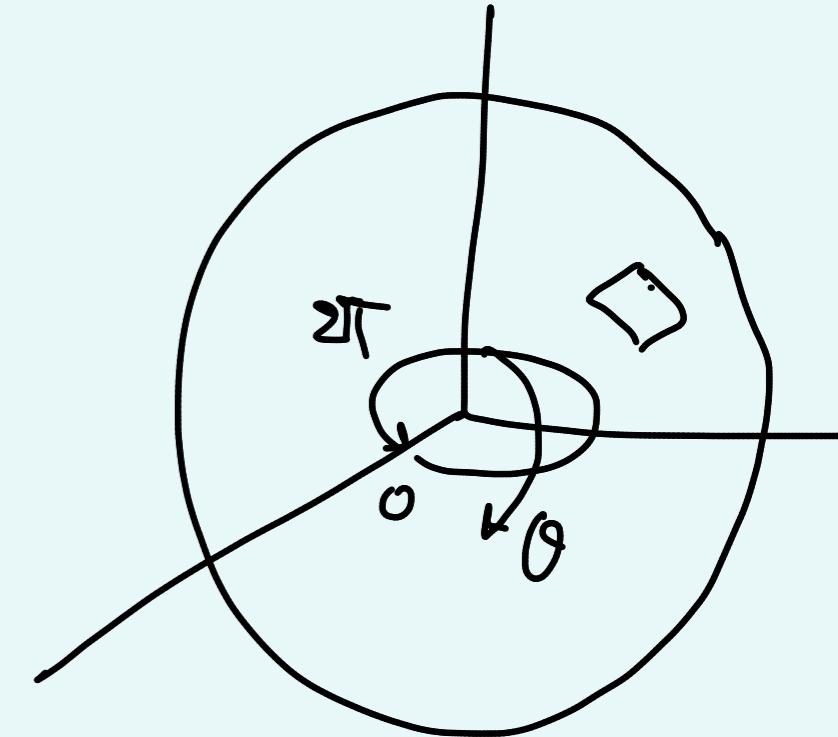
$$= -p_\infty \cos \theta + \frac{3}{2} \frac{\mu U}{a}$$

drag $D = \int_0^{2\pi} \int_0^\pi \tau a^2 \sin \theta d\theta d\phi$

$D = 6\pi \mu U a$

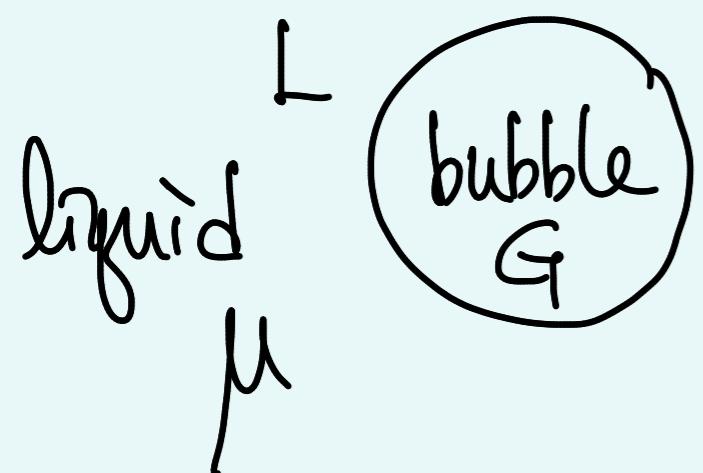
Stokes' law

$$|_{r=a}$$



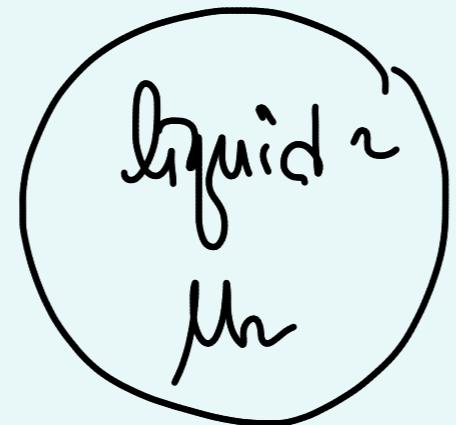
~~Stokes law~~
~~Reynolds~~

Other external flows ($Re \ll 1$)



$$D = 6\pi \mu U a$$

- liquid 1
 μ_1

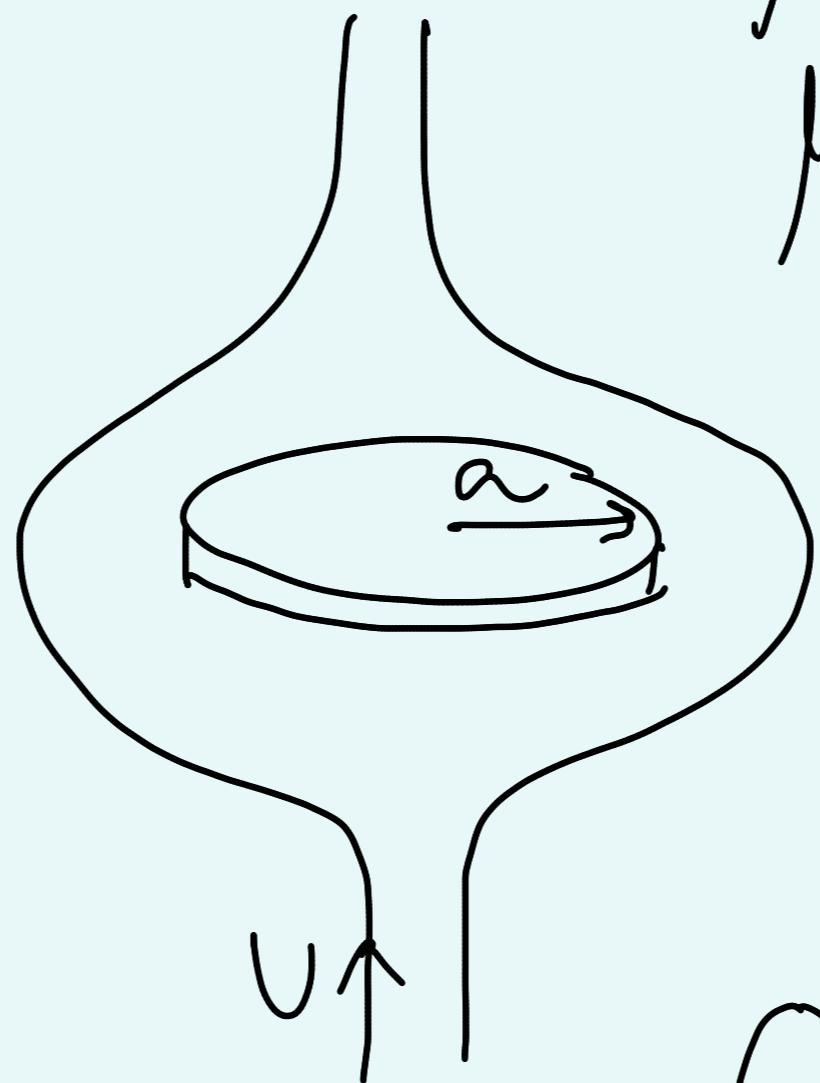


$$D = 4\pi \mu_1 \nu a \left(\frac{\mu_1 + \frac{3}{2} \mu_2}{\mu_1 + \mu_2} \right)$$

$\mu_1 \gg \mu_2 \rightarrow \text{bubble } (4\pi)$

$\mu_1 \ll \mu_2 \rightarrow \text{drop } (6\pi)$

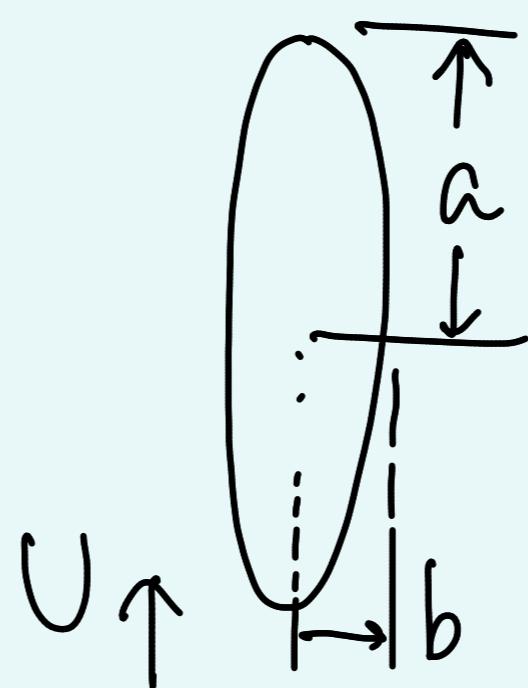
- circular disk



$$D = 16 \mu \nu a$$

$$\frac{a}{b} \gg 1$$

- Elongated rod



$$D = \frac{4\pi \mu \nu a^2}{\ln\left(\frac{a}{b}\right) + 0.19315}$$

General procedure to solve low-Re flows.

Revisit the governing eq.

$$\nabla \times (\nabla p) = \mu \nabla^2 \bar{u}$$

$\nabla \cdot \bar{u} = 0$

If B.C involve \bar{u} alone

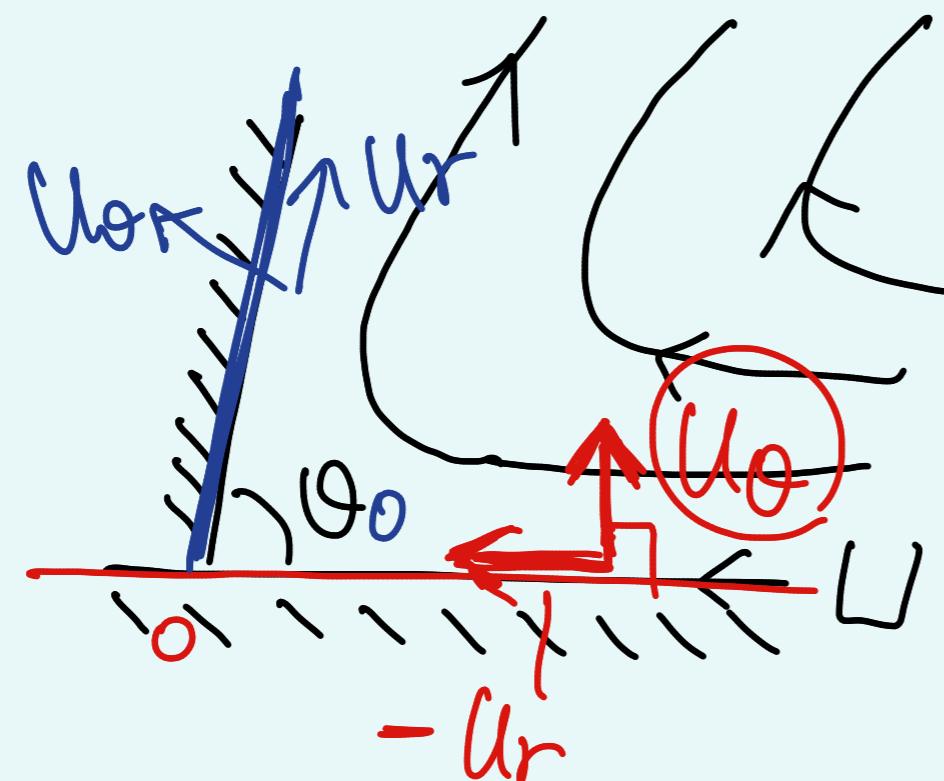
$$\nabla^2(\nabla \times \bar{u}) = 0, \quad \nabla \cdot \bar{u} = 0$$

2D

$$\psi \text{ (stream fn)}, \quad \nabla \times \bar{u} = -\hat{k} \nabla^2 \psi$$
$$\nabla^2(\nabla^2 \psi) = 0$$

biharmonic eq.

Corner flow



cyl.

$$\bar{u} = (u_r, u_\theta)$$

stream fn. $u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$

$$u_\theta = - \frac{\partial \psi}{\partial r}$$

$$-u_r = U$$

$$\nabla^4 \psi = \nabla^2 (\nabla^2 \psi) = 0$$

$$\nabla^2 \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \psi = 0. \quad \frac{\partial \psi}{\partial \theta}^{(r)} = -rU$$

B.C., at $\theta = 0$. $\frac{\partial \psi}{\partial r} = 0, \frac{1}{r} \frac{\partial \psi}{\partial \theta} = -U$

at $\theta = \theta_0$, $\frac{\partial \psi}{\partial r} = 0, \frac{1}{r} \frac{\partial \psi}{\partial \theta} = 0$

$$\psi = R(r) \Theta(\theta) = r f(\theta)$$

$$f(\theta) = A \underline{\sin \theta} + B \underline{\cos \theta} + C \underline{\theta \sin \theta} + D \underline{\theta \cos \theta}$$

B.C: $f(0) = f(\theta_0) = f'(\theta_0) = 0$

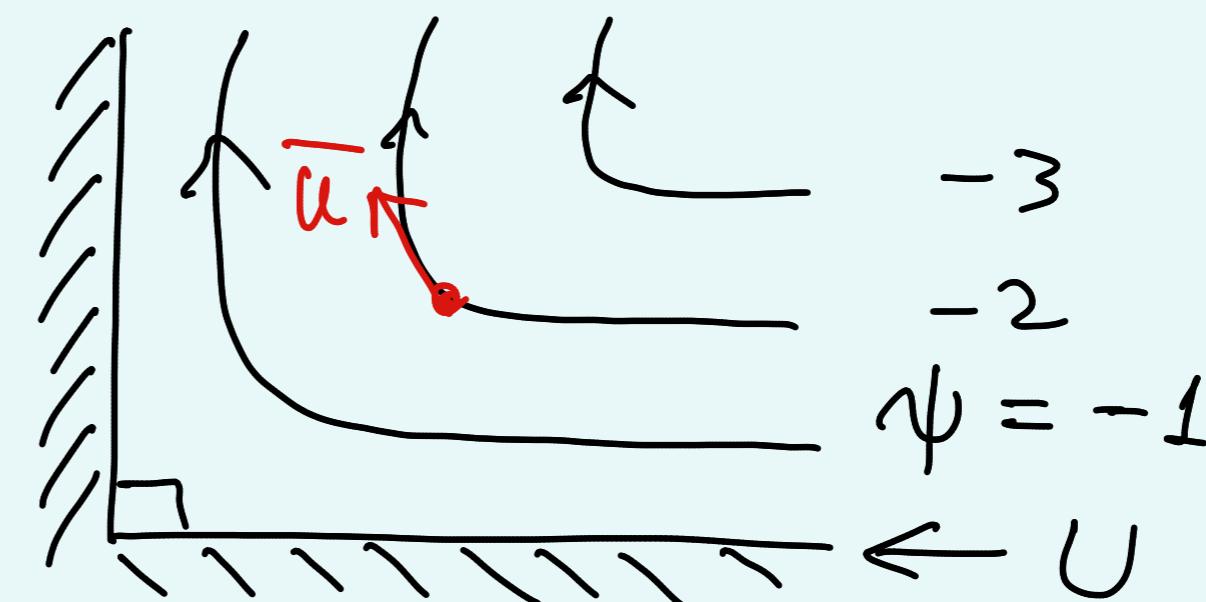
$$f'(0) = -U.$$

$$[A, B, C, D] = [-\theta_0^2, 0, \theta_0 - \sin \theta_0 \cos \theta_0, \sin^2 \theta_0] \frac{U}{\theta_0^2 - \sin^2 \theta_0}$$

If $\theta_0 = \frac{\pi}{2}$

$$\psi = \frac{rU}{\frac{1}{4}\pi^2 - 1} \left(-\frac{\pi^2}{4} \underline{\sin \theta} + \frac{\pi}{2} \underline{\theta \sin \theta} + \underline{\theta \cos \theta} \right)$$

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$$



Tangential stress

$$\tau_{r\theta} = \mu \left[r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] ,$$

$$r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) = - r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) , \quad \psi = r f(\theta)$$

$$\sim r \frac{\partial}{\partial r} \left(\frac{1}{r} f \right)$$

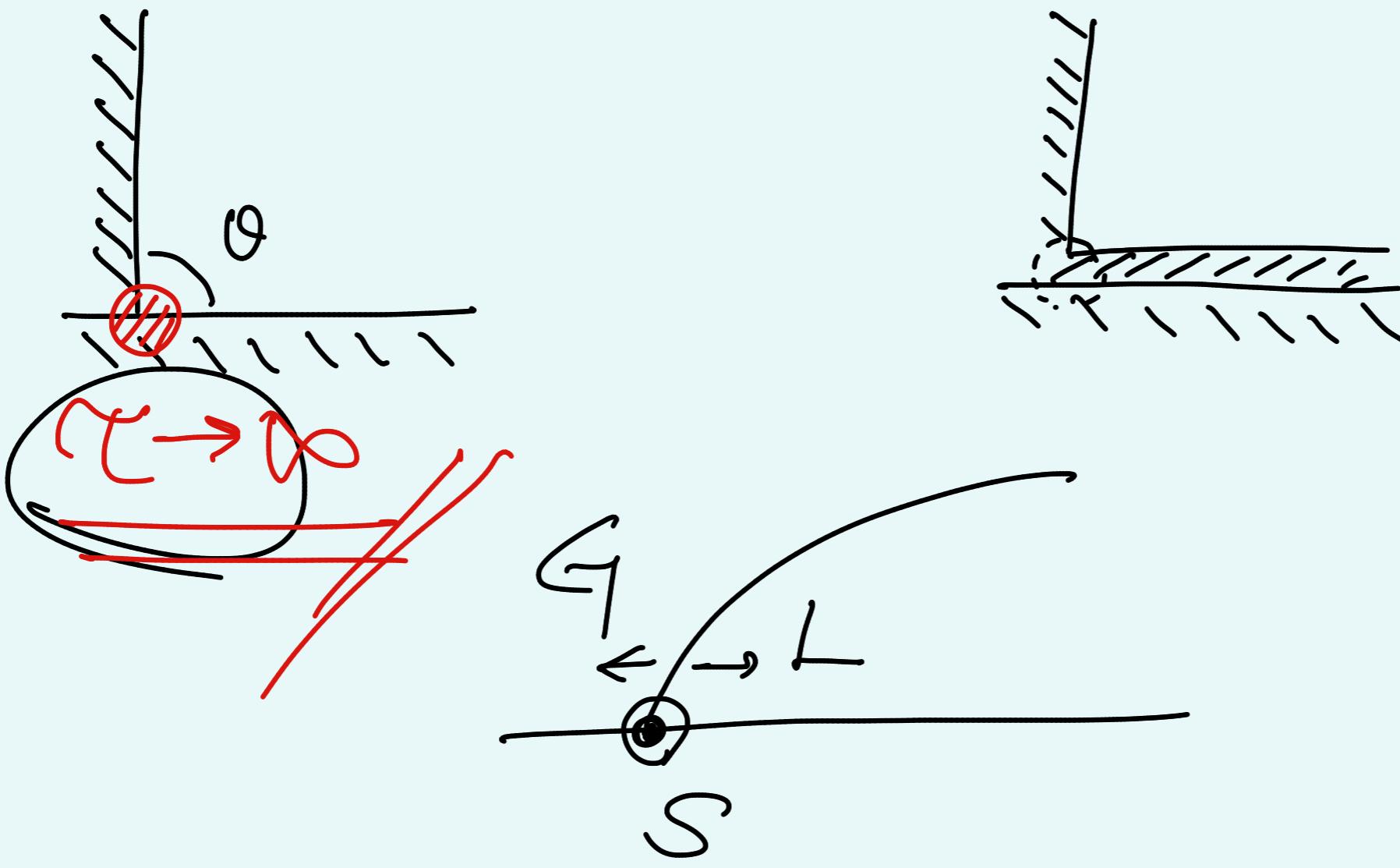
$$\sim \frac{r}{r^2} f \sim \frac{f}{r}$$

$$\frac{1}{r} \frac{\partial u_r}{\partial \theta} \sim \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \right)$$

$$\sim \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{1}{r} r f' \right)$$

$$\sim \frac{1}{r} f''$$

$\text{as } r \rightarrow 0 , \quad \gamma_{r0} \rightarrow \infty$

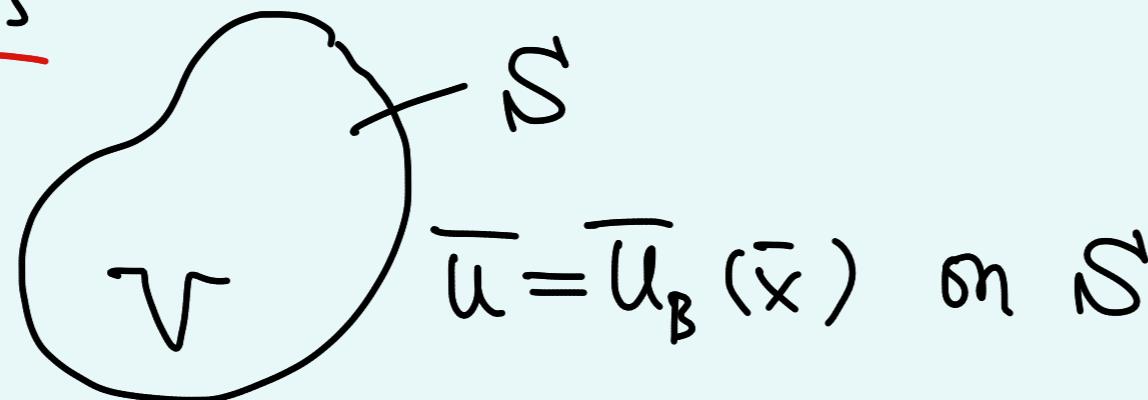


Uniqueness and reversibility of slow flows

$$0 = -\nabla p + \mu \nabla^2 \bar{u}$$

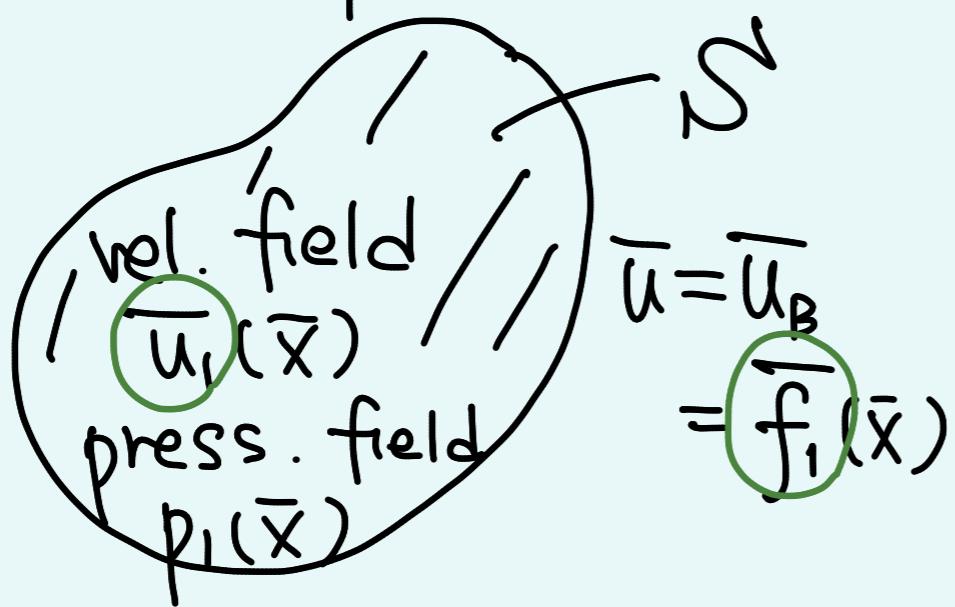
$$\nabla \cdot \bar{u} = 0$$

- uniqueness



There is at most one soln of eq (*) which satisfies the B.C.

- reversibility



change
B.C.

$$\bar{u}_B = -\bar{f}_1(\bar{x})$$

soln of (*) = $-\bar{u}_1(\bar{x})$

pressure = $C - p_1(\bar{x})$

→ only soln.

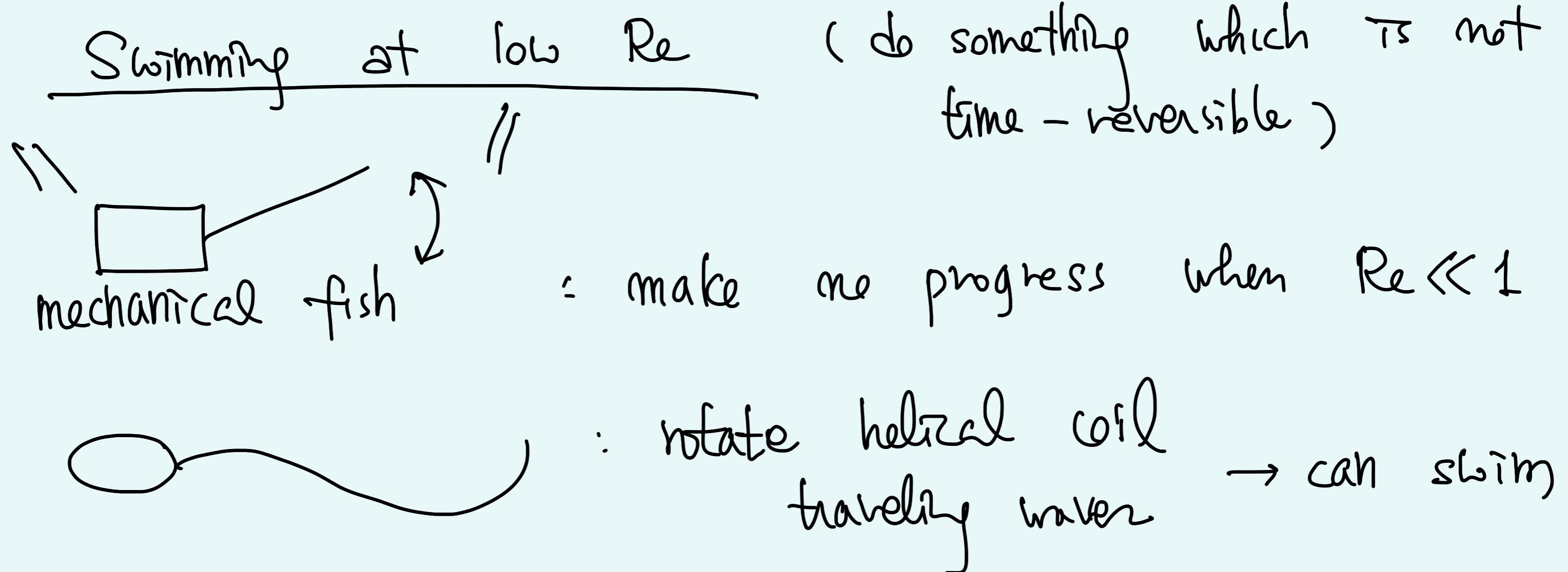
(*)

$$0 = -\nabla p + \mu \nabla^2 \bar{u}_1$$

$$\nabla \cdot \bar{u}_1 = 0$$

$$-\nabla \cdot \bar{u}_1 = 0.$$

∴ Inasmuch as the slow flow eqns hold,
'reversed' boundary conditions lead to reversed flow



G. K. Batchelor

An Introduction to Fluid Dynamics