Cycle-by-Cycle Stochastic Error Propagation Model

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Cycle-by-cycle Error Propagation Model

- In the ordinary (or deterministic) power method, the maximum k and the fundamental-mode FSD are calculated by the iterative updates of k and FSD. FSD and k at p-th iteration are calculated by

$$S^p(\mathbf{r}) = \frac{1}{k^{p-1}} \int H(\mathbf{r}' \to \mathbf{r}) S^{p-1}(\mathbf{r}') d\mathbf{r}' \quad \cdots\cdots \quad (C.1)$$

$$k^p = k^{p-1} \frac{\int S^p(\mathbf{r}) d\mathbf{r}}{\int S^{p-1}(\mathbf{r}) d\mathbf{r}} \quad \cdots\cdots \quad (C.2)$$

- In the MC power method, while simulating the fission source neutron at the t-th cycle, the value of $\nu\Sigma_f(\mathbf{r},E)/\Sigma_t(\mathbf{r},E)$ is accumulated in a spatial function $\psi^t(\mathbf{r})$. After M source neutrons have been processed, $\psi^t(\mathbf{r})$ has a probability of generating fission source at $\mathbf{r}$.
- Then, the eigenvalue k is updated by

$$k^t = \frac{\int \psi^t(\mathbf{r}) d\mathbf{r}}{M} \quad \cdots\cdots \quad (C.3)$$

Cycle-by-cycle Error Propagation Model (Contd.)

- From $\psi^t(\mathbf{r})$, the fission source density is updated by

$$S^t(\mathbf{r}) = \frac{\psi^t(\mathbf{r})}{\int \psi^t(\mathbf{r}')d\mathbf{r}'} \qquad \text{(C.4)}$$

- One can define $\varphi^t(\mathbf{r})$ by the number of fission sources per unit source at **r** and cycle *t*. Because $\psi^t(\mathbf{r})$ is determined by processing a fixed number of source neutrons, it has a stochastic error, $\varepsilon_\varphi^t(\mathbf{r})$ at *t*-th cycle. Then, $\varphi^t(\mathbf{r})$ can be expressed by

$$\varphi^t(\mathbf{r}) \equiv \frac{\psi^t(\mathbf{r})}{M} = \int H(\mathbf{r}' \to \mathbf{r})S^{t-1}(\mathbf{r}')d\mathbf{r}' + \varepsilon_\varphi^t(\mathbf{r}) \qquad \text{(C.5)}$$

- The substitutions of Eq. (C.5) into equations (C.3) and (C.4) lead to

$$k^t = \int \varphi^t(\mathbf{r})d\mathbf{r} \qquad \text{(C.6)}$$

$$S^t(\mathbf{r}) = \frac{\varphi^t(\mathbf{r})}{\int \varphi^t(\mathbf{r}')d\mathbf{r}'} \qquad \text{(C.7)}$$

Cycle-by-cycle Error Propagation Model (Contd.)

- The substitution of Eq. (C.7) into Eq. (C.5) leads to

$$\varphi^{t}(\mathbf{r}) = \int H(\mathbf{r}' \to \mathbf{r}) \frac{\varphi^{t-1}(\mathbf{r}')}{\int \varphi^{t-1}(\mathbf{r}'')d\mathbf{r}''} d\mathbf{r}' + \varepsilon_{\varphi}^{t}(\mathbf{r}) \quad \text{(C.8)}$$

- One can define $e_{\varphi}^{t}(\mathbf{r})$ by the difference between $\varphi^{t}(\mathbf{r})$ and the fundamental distribution, $\varphi_0(\mathbf{r})$. It can be expressed by

$$\varphi^{t}(\mathbf{r}) = \varphi_0(\mathbf{r}) + e_{\varphi}^{t}(\mathbf{r}) \quad \text{(C.9)}$$

- And $\varphi_0(\mathbf{r})$ satisfies

$$\varphi_0(\mathbf{r}) = \frac{1}{k_0}\int H(\mathbf{r}' \to \mathbf{r})\varphi_0(\mathbf{r}')d\mathbf{r}', \quad \int \varphi_0(\mathbf{r})d\mathbf{r} = k_0 \quad \text{(C.10)}$$

- The substitution of Eq. (C.9) into Eq. (C.8) leads to

$$\varphi_0(\mathbf{r}) + e_{\varphi}^{t}(\mathbf{r}) = \int H(\mathbf{r}' \to \mathbf{r}) \frac{\varphi_0(\mathbf{r}') + e_{\varphi}^{t-1}(\mathbf{r}')}{\int \left(\varphi_0(\mathbf{r}'') + e_{\varphi}^{t-1}(\mathbf{r}'')\right) d\mathbf{r}''} d\mathbf{r}' + \varepsilon_{\varphi}^{t}(\mathbf{r}) \quad \text{(C.11)}$$

Cycle-by-cycle Error Propagation Model (Contd.)

- From the Taylor's series expansion of Eq. (C.11) to first order , one can find

$$\varphi_0(\mathbf{r}) + e_\varphi^t(\mathbf{r}) \cong \int H(\mathbf{r}' \to \mathbf{r}) \frac{\varphi_0(\mathbf{r}')}{\int \varphi_0(\mathbf{r}'')d\mathbf{r}''} d\mathbf{r}'$$

$$+ \left[\int H(\mathbf{r}' \to \mathbf{r}) \frac{e_\varphi^{t-1}(\mathbf{r}')}{\int \varphi_0(\mathbf{r}'')d\mathbf{r}''} d\mathbf{r}' - \frac{\int H(\mathbf{r}' \to \mathbf{r})\varphi_0(\mathbf{r}')d\mathbf{r}'}{\left(\int \varphi_0(\mathbf{r}'')d\mathbf{r}''\right)^2} \int e_\varphi^{t-1}(\mathbf{r}'')d\mathbf{r}'' \right] + \varepsilon_\varphi^t(\mathbf{r}) \quad \text{(C.12)}$$

- Using Eq. (C.10), Eq. (C.12) can be reduced to

$$e_\varphi^t(\mathbf{r}) \cong \frac{1}{k_0} \int A_0(\mathbf{r}' \to \mathbf{r}) e_\varphi^{t-1}(\mathbf{r}')d\mathbf{r}' + \varepsilon_\varphi^t(\mathbf{r}); \quad \text{(C.13)}$$

$$A_0(\mathbf{r}' \to \mathbf{r}) = \frac{1}{k_0}\left[H(\mathbf{r}' \to \mathbf{r}) - \varphi_0(\mathbf{r}) \right] \quad \text{(C.14)}$$

$$\mathbf{A}_0 = \frac{1}{k_0}\left(\mathbf{H} - \mathbf{S}_0 \cdot \tau\right),$$

Cycle-by-cycle Error Propagation Model (Contd.)

- Eq. (C.13) is the cycle-by-cycle error propagation model of $\varphi^t(\mathbf{r})$ which is the fission source distribution normalized to k^t like Eq. (2.1.7). One may derive the error propagation model for the fission source density normalized to unity.
- The substitution of Eq. (C.5) into Eq. (C.7) leads to

$$S^t(\mathbf{r}) = \frac{\int H(\mathbf{r}' \to \mathbf{r}) S^{t-1}(\mathbf{r}') d\mathbf{r}' + \varepsilon_\varphi^t(\mathbf{r})}{\iint H(\mathbf{r}'' \to \mathbf{r}') S^{t-1}(\mathbf{r}'') d\mathbf{r}'' d\mathbf{r}' + \int \varepsilon_\varphi^t(\mathbf{r}') d\mathbf{r}'} \quad \cdots\cdots \quad (C.15)$$

- From the Taylor's series expansion of Eq. (C.15) by ε_φ^t, one can find

$$S^t(\mathbf{r}) = \frac{\int H(\mathbf{r}' \to \mathbf{r}) S^{t-1}(\mathbf{r}') d\mathbf{r}'}{\iint H(\mathbf{r}'' \to \mathbf{r}') S^{t-1}(\mathbf{r}'') d\mathbf{r}'' d\mathbf{r}'} + \varepsilon^t(\mathbf{r}); \quad \cdots\cdots \quad (C.16)$$

$$\varepsilon^t(\mathbf{r}) \equiv \frac{\varepsilon_\varphi^t(\mathbf{r})}{\iint H(\mathbf{r}'' \to \mathbf{r}') S^{t-1}(\mathbf{r}'') d\mathbf{r}'' d\mathbf{r}'} - \frac{\left(\int H(\mathbf{r}' \to \mathbf{r}) S^{t-1}(\mathbf{r}') d\mathbf{r}'\right)\left(\int \varepsilon_\varphi^t(\mathbf{r}') d\mathbf{r}'\right)}{\left(\iint H(\mathbf{r}'' \to \mathbf{r}') S^{t-1}(\mathbf{r}'') d\mathbf{r}'' d\mathbf{r}'\right)^2} + \cdots$$

Cycle-by-cycle Error Propagation Model (Contd.)

- One can define $e^t(\mathbf{r})$ by the difference between $S^t(\mathbf{r})$ and the fundamental-mode distribution, $S_0(\mathbf{r})$. It can be expressed by

$$S^t(\mathbf{r}) = S_0(\mathbf{r}) + e^t(\mathbf{r}) \qquad \text{(C.17)}$$

- And $S_0(\mathbf{r})$ satisfies

$$S_0(\mathbf{r}) = \frac{1}{k_0}\int H(\mathbf{r}' \to \mathbf{r}) S_0(\mathbf{r}') d\mathbf{r}', \quad \int S_0(\mathbf{r}) d\mathbf{r} = 1 \qquad \text{(C.18)}$$

- The substitution of Eq. (C.17) into Eq. (2.1.16) leads to

$$S_0(\mathbf{r}) + e^t(\mathbf{r}) = \frac{\int H(\mathbf{r}' \to \mathbf{r})\left(S_0(\mathbf{r}') + e^{t-1}(\mathbf{r}')\right) d\mathbf{r}'}{\iint H(\mathbf{r}'' \to \mathbf{r}')\left(S_0(\mathbf{r}'') + e^{t-1}(\mathbf{r}'')\right) d\mathbf{r}'' d\mathbf{r}'} + \varepsilon^t(\mathbf{r}) \qquad \text{(C.18)}$$

Cycle-by-cycle Error Propagation Model (Contd.)

- From the Taylor's series expansion of Eq. (C.19) to first order $e^t(\mathbf{r})$, one can find

$$
\begin{aligned}
S_0(\mathbf{r})+e^t(\mathbf{r}) \cong & \frac{\int H(\mathbf{r}'\to\mathbf{r})S_0(\mathbf{r}')d\mathbf{r}'}{\iint H(\mathbf{r}''\to\mathbf{r}')S_0(\mathbf{r}'')d\mathbf{r}''d\mathbf{r}'} \\
& +\frac{\int H(\mathbf{r}'\to\mathbf{r})e^{t-1}(\mathbf{r}')d\mathbf{r}'}{\iint H(\mathbf{r}''\to\mathbf{r}')S_0(\mathbf{r}'')d\mathbf{r}''d\mathbf{r}'} \\
& -\frac{\int H(\mathbf{r}'\to\mathbf{r})S_0(\mathbf{r}')d\mathbf{r}'}{\left(\iint H(\mathbf{r}''\to\mathbf{r}')S_0(\mathbf{r}'')d\mathbf{r}''d\mathbf{r}'\right)^2}\iint H(\mathbf{r}''\to\mathbf{r}')e^{t-1}(\mathbf{r}'')d\mathbf{r}''d\mathbf{r}' \qquad \text{(C.19)} \\
& +\varepsilon^t(\mathbf{r})
\end{aligned}
$$

- From Eq. (C.18), Eq. (19) can be reduced to

$$e_\varphi^t(\mathbf{r}) \cong \int A_0'(\mathbf{r}'\to\mathbf{r})e^{t-1}(\mathbf{r}')d\mathbf{r}'+\varepsilon^t(\mathbf{r}); \qquad \text{(C.20)}$$

$$A_0'(\mathbf{r}'\to\mathbf{r})=\frac{1}{k_0}\left[H(\mathbf{r}'\to\mathbf{r})-S_0(\mathbf{r})\int H(\mathbf{r}'\to\mathbf{r}'')d\mathbf{r}''\right] \qquad \text{(C.21)}$$

$$\mathbf{A}=\frac{1}{k_0}\left(\mathbf{H}-\mathbf{S}_0\cdot\boldsymbol{\tau}^T\cdot\mathbf{H}\right)$$

Cycle-by-cycle Error Propagation Model (Contd.)

- The operator notation of Eq. (C.20) may be introduced as

$$e^t = A_0' e^{t-1} + \varepsilon^t \qquad \text{(C.22)}$$

- The repeated application of Eq. (C.22) yields

$$e^t = \sum_{t'=0}^{t-1} \left(A_0'\right)^{t'} \varepsilon^{t-t'} + \left(A_0'\right)^{t} e^0 \qquad \text{(C.23)}$$

- It is assumed that the stochastic error generated at cycle t is independent of the accumulated errors of previous cycles and the stochastic errors generated at other cycles:

$$\begin{aligned} E\left[\varepsilon^i e^j\right] &= 0 \ \ (i > j), \\ E\left[\varepsilon^i \varepsilon^j\right] &= 0 \ \ (i \neq j). \end{aligned} \qquad \text{(C.24)}$$

- From Eqs. (C.23) and (C.24), the covariance between fission source densities l cycle apart can be written as

$$\begin{aligned} \text{cov}\left[S^t, S^{t+l}\right] &= E\left[e^t e^{t+l}\right] \\ &= \sum_{t'=0}^{t-1} E\left[\left(A_0'\right)^{t'} \varepsilon^{t-t'} \left(A_0'\right)^{t'+l} \varepsilon^{t-t'}\right] + E\left[\left(A_0'\right)^{t} e^0 \left(A_0'\right)^{t+l} e^0\right] \end{aligned} \qquad \text{(C.25)}$$

Schematic Diagram of Error Propagation Model

$$
\begin{array}{lll}
t=0 & e_m^0 & e_{m'}^0 \\
t=1 & \varepsilon_m^1 \quad \sum_{n=1}^{N} a_{mn}^1 e_n^0 & \varepsilon_{m'}^1 \quad \sum_{n=1}^{N} a_{m'n}^1 e_n^0 \\
t=2 & \varepsilon_m^2 \quad \sum_{n=1}^{N} a_{mn}^1 \varepsilon_n^1 \quad \sum_{n=1}^{N} a_{mn}^2 e_n^0 & \varepsilon_{m'}^2 \quad \sum_{n=1}^{N} a_{m'n}^1 \varepsilon_n^1 \quad \sum_{n=1}^{N} a_{m'n}^2 e_n^0 \\
\vdots & & \\
t=l & \varepsilon_m^l \quad \sum_{n=1}^{N} a_{mn}^1 \varepsilon_n^{l-1} \quad \sum_{n=1}^{N} a_{mn}^2 \varepsilon_n^{l-2} \quad \cdots \quad \sum_{n=1}^{N} a_{mn}^l e_n^0 & \varepsilon_{m'}^l \quad \sum_{n=1}^{N} a_{m'n}^1 \varepsilon_n^{l-1} \quad \sum_{n=1}^{N} a_{m'n}^2 \varepsilon_n^{l-2} \quad \cdots \quad \sum_{n=1}^{N} a_{m'n}^l e_n^0
\end{array}
$$

$$
\boxed{e_m^t = \sum_{t'=0}^{t-1} \sum_n a_{mn}^{t'} \varepsilon_n^{t-t'} + \sum_n a_{mn}^t e_n^0}
$$